Fissioned Triangular Schemes Via the Cross-Ratio
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Document version:
Publisher's PDF, also known as Version of record

Publication date:
1999

Link to publication

Citation for published version (APA):

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Download date: 09. Sep. 2019
Abstract

A construction of association schemes is presented; these are fission schemes of the triangular schemes $T(n)$ where $n = q + 1$ with $q$ any prime power. The key observation is quite elementary, being that the natural action of $PGL(2, q)$ on the 2-element subsets of the projective line $PG(1, q)$ is generously transitive. Also some observations on the intersection parameters and fusion schemes of these association schemes are made.
1 The construction

This paper is a sequel to [4]. In that paper, it was observed that almost all known self-dual classical association schemes have natural fission schemes (fissioning the maximum-distance relation only); whereas in the non-self-dual case there seemed to be no analogous fission schemes. Subsequently, we found that there is at least one such non-self-dual classical association scheme that admits an interesting fission scheme, namely the triangular scheme \( T(n) = J(n, 2) \) where \( n = q + 1 \) with \( q \) any prime power; this is the object of the present work. For terminology and background, we refer to Bannai and Ito [2] for association schemes and Hirschfeld [7] for finite geometry. Recall that the group \( PGL(2, q) \) acts (as Möbius transformations) on the projective line \( PG(1, q) \); this action is (sharply) 3-transitive. There is a natural induced action on the 2-element subsets of the projective line, namely \( M(\{x, y\}) := \{M(x), M(y)\} \) for each \( M \) in \( PGL(2, q) \). In the proof below we apply the basic fact (cf. [7], p. 135) that the cross-ratio

\[
\rho(a, b, c, d) := \frac{(a - c)(b - d)}{(a - d)(b - c)}
\]

is a complete invariant for ordered quadruples of distinct points on the projective line, i.e. one quadruple may be mapped to another quadruple (via a Möbius transformation) if and only if they have the same cross-ratio.

**Theorem.** The action of \( PGL(2, q) \) on the two-element subsets of \( PG(1, q) \) is generously transitive.

**Proof.** Given intersecting 2-sets \( \{a, b\} \) and \( \{a, c\} \), there is some \( M \) in \( PGL(2, q) \) that swaps them, since the group is triply transitive. And given disjoint 2-sets \( \{a, b\} \) and \( \{c, d\} \), there is also some Möbius transformation that interchanges them, because the ordered quadruples \( (a, b, c, d) \) and \( (c, d, a, b) \) have the same cross-ratio. □

Given any transitive permutation group \( G \) acting on a set \( \Omega \), the orbitals are the orbits in \( \Omega \times \Omega \) under the natural action of \( G \) on pairs. If \( G \) is generously transitive, then the orbitals form the relations (associate classes) of a symmetric association scheme (cf. [2], p. 54). In our case, the relations can be described as follows. One relation, say \( R_1 \), is the line-graph of the complete graph (i.e. one relation of the triangular scheme \( T(q + 1) \) has remained unfissioned). Next, for each reciprocal pair \( \{s, s^{-1}\} \) of elements in \( GF(q) \backslash \{0, 1\} \), there is a relation \( R_{\{s, s^{-1}\}} \) where \( \{a, b\} \) and \( \{c, d\} \) are in this relation.
when \( \rho(a,b,c,d) \) equals \( s \) or \( s^{-1} \). Note that \( \rho(b,a,c,d) = \rho(a,b,c,d)^{-1} \) so this makes sense as a definition for unordered pairs \( \{a,b\} \). Henceforth we will write \( R_s \) instead of \( R_{\{s,s^{-1}\}} \) for typographical reasons; note that since the field element 1 cannot occur as a cross-ratio, this notation will not conflict with that of relation \( R_1 \) above.

We now easily find that this fissioned triangular scheme, which we shall denote by \( FT(q+1) \), has \( \frac{1}{2}(q + 1) \) associate classes if \( q \) is odd and \( \frac{1}{2}q \) classes if \( q \) is even. When \( q \) is odd the field element \(-1\) is equal to its own reciprocal; thus the relation \( R_{-1} \) has valency \( \frac{1}{2}(q - 1) \) which is half the valency of the other relations \( R_s \) with \( s \) in \( GF(q) \setminus \{0, 1, -1\} \). The relation \( R_1 \) has valency \( 2(q - 1) \).

We remark that for small odd \( q \) the relation \( R_{-1} \) is a familiar object: for \( q = 5 \) it is the line-graph of Petersen’s graph; for \( q = 7 \) it is the Coxeter graph (this was apparently known to Coxeter himself, cf. p. 122 in [6]); for \( q = 9 \) it is the line-graph of Tutte’s 8-cage. There seem to be some other such “sporadic isomorphisms”: for example when \( q = 11 \) the relation \( R_2 = R_{\{2,6\}} \) is the line-graph of the point-block incidence graph of the (unique) symmetric \((11, 6, 3)\)-design; and when \( q = 9 \) and \( \{s, s^{-1}\} \) is the pair of primitive fourth roots of unity, then \( R_s \) is the second subconstituent of the Gewirtz graph (cf. [5], page 106).

### 2 Intersection parameters

It is possible to give explicit formulas for the intersection parameters \( p_{ij}^k \) of the association scheme \( FT(q+1) \); we now sketch the main points of the derivation. The cases \( q \) odd and \( q \) even are similar, with the latter case being slightly cleaner since the exceptional case “\( \rho = -1 \)” doesn’t occur. So we will only present the case \( q \) even; besides, this case is the more pertinent one in the discussion of fusion schemes in Section 3.

So let \( q = 2^e \) be any power of two. The scheme \( FT(2^e + 1) \) has \( 2^{e-1} \) classes. The relation \( R_1 \) has valency \( 2(q - 1) \) and each of the other relations \( R_s = R_{\{s,s^{-1}\}} \) (for \( s \) in \( GF(q) \setminus \{0,1\} \)) has valency \( q - 1 \). The intersection parameters involving \( R_1 \) are easy to work out and we list them without proof: for distinct \( r \) and \( s \) (and \( s \neq r^{-1} \)) in \( GF(q) \setminus \{0,1\} \), \( p_{11}^r = q - 1 \), \( p_{11}^r = 4 \), \( p_{1r}^r = 2 \), \( p_{rr}^r = 1 \), and \( p_{rs}^r = 2 \).

Now let the symbols \( r, s \) and \( t \) represent three (not necessarily distinct) elements of \( GF(q) \setminus \{0,1\} \); we aim at a formula for \( p_{st}^r \). What one has to do is fix a pair of 2-sets
\{a, b\} and \{c, d\} in relation \(R_r\), and count the number of 2-sets \{x, y\} such that \{a, b\} and \{x, y\} are in relation \(R_s\) and \{c, d\} and \{x, y\} are in relation \(R_t\). The triple transitivity of \(PGL(2, q)\) is useful here, since it implies that we may take, without loss of generality, \(\{a, b\} = \{\infty, 0\}\) and \(\{c, d\} = \{1, r\}\). For the unknown pair \{x, y\} we then get the two equations

\[
s \text{ or } s^{-1} = \frac{(\infty - x)(0 - y)}{(\infty - y)(0 - x)} = \frac{y}{x}
\]

and

\[
t \text{ or } t^{-1} = \frac{(1 - x)(r - y)}{(1 - y)(r - x)}
\]

The equations (1) and (2) together involve two essentially different cases, not four, since \(\{y, x\} = \{x, y\}\); thus we may fix the left-hand side of (1) as being \(s\), and examine the two cases for (2) in turn. In the first case we have \(y = sx\) and

\[
t = \frac{(1 - x)(r - y)}{(1 - y)(r - x)} = \frac{(1 - x)(r - sx)}{(1 - sx)(r - x)}
\]

This leads to the following quadratic for \(x\) (after changing all minus signs to plus signs, as we may since we are in characteristic two):

\[
s(t + 1)x^2 + (rst + r + s + t)x + r(t + 1) = 0
\]

The other case (when the left-hand side of (2) is \(t^{-1}\)) leads to the similar quadratic

\[
s(t + 1)x^2 + (rs + rt + st + 1)x + r(t + 1) = 0
\]

Note that since \(r, s\) and \(t\) are all in \(GF(q) \setminus \{0, 1\}\), the equations (3) and (4) are genuine quadratics, with non-zero quadratic and constant terms. The linear coefficient \((rst + r + s + t)\) in (3) could equal 0, in which case the unique solution for \(x\) is the square root of \(\frac{r}{s}\). If \(rst + r + s + t \neq 0\), then (3) has (two) solutions \(x\) if and only if

\[
Tr \left[ \frac{rs(t + 1)^2}{(rst + r + s + t)^2} \right] = 0
\]
where $Tr(z)$ is the trace map from $GF(2^e)$ onto $GF(2)$. Similarly, if $rs + rt + st + 1 \neq 0$ then (4) has (two) solutions $x$ if and only if

$$Tr \left[ \frac{rs(t+1)^2}{(rs + rt + st + 1)^2} \right] = 0$$

(6)

Thus $p_{st}^r$ has a value of anywhere from 0 to 4. A reasonably concise formula is the following: let $A = A(r,s,t)$ be the expression for the argument of the trace map in (5), and $B = B(r,s,t)$ the one for (6). Then, when $rst + r + s + t \neq 0$ and $rs + rt + st + 1 \neq 0$

$$p_{st}^r = 2 + (-1)^{Tr[A]} + (-1)^{Tr[B]}$$

(7)

with the obvious modifications being made in the other cases. Incidentally, it is easy to check that $(rst + r + s + t)$ and $(rs + rt + st + 1)$ cannot simultaneously equal 0.

We make one more remark concerning the form of the intersection parameters. The expressions $A(r,s,t)$ and $B(r,s,t)$ are not symmetric in $s$ and $t$, hence the formula (7) for $p_{st}^r$ appears not to be symmetric either. This may seem strange, since we know from general principles that $p_{st}^r = p_{ts}^r$. An explanation for this is the following. $A(r,s,t)$ has the same trace as $C(r,s,t) := \frac{rs + rt + st}{(rst + r + s + t)}$ since their sum is of the form $\frac{xy}{x^2 + y^2}$ and such field elements, in characteristic two, must have trace 0 (exercise for the reader). Similarly $B(r,s,t)$ has the same trace as $D(r,s,t) := \frac{rst(r + s + t)}{(rs + rt + st + 1)x}$. Thus we may replace $A$ by $C$ and $B$ by $D$ in (7) without changing the value of the right-hand side; and $C$ and $D$ are both symmetric functions of the three variables $r, s$ and $t$. This confirms the fact that, since the valencies $n_r$ are the same for all $r$ in $GF(q) \setminus \{0, 1\}$, the intersection parameter $p_{st}^r$ is symmetric in all three variables.

It would be interesting to find explicit formulas for the entries of the eigenmatrix (character table) of $FT(q + 1)$. One strategy for doing this (used by Bannai and his co-workers in several papers; see [1] for a survey) is the following. First calculate all of the intersection parameters; it is usually feasible to do this, at least in some reasonable algebraic form perhaps involving character sums. This tells us what the intersection matrices $B_i(k,j) := p_{ij}^k$ are. Secondly, from these $B_i$’s (at small values of $q$) it may be possible to guess what the eigenmatrix $P$ should be. Once the right guess has been made it is usually straightforward to actually prove the result, using Theorem II.4.1 in [2]. Unfortunately, we have been unable so far to guess the general shape of $P$ for our
schemes $FT(q+1)$; we generated by computer these character tables for all prime powers $q$ less than 40, and they seem to have a very complicated form.

## 3 Fusion schemes

Given any association scheme, it is of interest to determine all of its fusion schemes (also called subschemes). This is in general a very hard problem that has not been worked out completely even for quite classical examples such as the Johnson schemes (cf. [8]). In the case of the schemes $FT(q+1)$, there is of course the original two-class triangular scheme $T(q+1)$. Observe also that if $q = p^e$ is a proper power of a prime $p$, then the Frobenius map $x \mapsto x^p$ (and its iterates) gives a fusion scheme. In other words $PTL(2, q)$ is an overgroup of $PGL(2, q)$, and the orbitals under $PTL(2, q)$ constitute a fusion scheme of $FT(q+1)$.

Limited computational evidence suggests that $FT(q + 1)$ has no other nontrivial fusions, except maybe in some sporadic cases, and when $q = 4^f$ ($f$ any integer at least 2) where there seems to be an interesting 4-class fusion scheme. We say “seems” because we are lacking a proof that this is indeed an association scheme. To describe this (putative) scheme, let the ground-set be all 2-element subsets of the projective line $PG(1, 4^f)$; the four possible relations for two distinct 2-sets $\{a, b\}$ and $\{c, d\}$ are:

- $S_1 : \{a, b\} \cap \{c, d\} \neq \emptyset$, i.e. $R_1$ in the earlier notation.
- $S_2 : \{a, b\} \cap \{c, d\} = \emptyset$ and the cross-ratio $\rho = \rho(a, b, c, d)$ satisfies $\rho^{2^f-1} = 1$,
  i.e. $\rho$ lies in the subfield $GF(2^f)$.
- $S_3 : \{a, b\} \cap \{c, d\} = \emptyset$ and the cross-ratio $\rho = \rho(a, b, c, d)$ satisfies $\rho^{2^f+1} = 1$.
- $S_4 :$ The remainder.

We have been able to show by computer that these four relations do indeed form a scheme when $f$ is less that or equal to 6. Also we can prove in general that some of the intersection parameters, such as $p_{23}^3$, are well defined; but certain other parameters such as $p_{33}^2$ have left us baffled. An explicit knowledge of the eigenmatrix of $FT(4^f + 1)$ would theoretically settle this question (cf. [8], Lemma 1), which is partly why we earlier raised the issue of computing it.

**Conjecture.** The above relations $S_i$ on the 2-subsets of $PG(1, 4^f)$ do form a 4-class association scheme for all $f \geq 2$. The corresponding eigenmatrix is given by
We note finally that, granting this conjecture, one can merge $S_2$ and $S_3$ to get a 3-class scheme, and then further merge $S_1$ with $S_2$ and $S_3$ to get a 2-class scheme. The resulting graph $G = S_1 \cup S_2 \cup S_3$ is strongly regular with parameters $v = 2^{2f-1}(2^{2f} + 1)$, $k = (2^f + 1)(2^{2f} - 1)$, $\lambda = (2^f - 1)(3 \cdot 2^f + 2)$, $\mu = 2^{f+1}(2^f + 1)$. Graphs with these parameters have already been constructed by Brouwer and Wilbrink (cf. [3], 7B); it was checked that in the smallest case $f = 2$ ($v = 136$) the two constructions yield isomorphic strongly regular graphs. We know nothing for larger values; but the two constructions look totally different, so that it is a reasonable guess that they are not isomorphic in general.

$$P = \begin{bmatrix}
1 & 2(4^f - 1) & (2^{f-1} - 1)(4^f - 1) & 2^{f-1}(4^f - 1) & 2^f(2^{f-1} - 1)(4^f - 1) \\
1 & 4^f - 3 & 2 - 2^f & -2^f & -2^f(2^f - 2) \\
1 & -2 & 1 - 2^f & 0 & 2^f \\
1 & -2 & (2^{f-1} - 1)(2^f - 1) & 2^{f-1}(2^f - 1) & -2^f(2^f - 2) \\
1 & -2 & 2^{f-1}(2^f - 1) + 1 & -2^{f-1}(2^f + 1) & 2^f
\end{bmatrix}$$
References


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