The Search for Pseudo Orthogonal Latin Squares of Order Six

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Abstract

We report on the complete computer search for a strongly regular graph with parameters (36,15,6,6) and chromatic number six. The result is that no such graph exists.

1 Introduction

Consider a Latin square S of order n. The *Latin square graph* , of S is defined on the entries of S, where two entries are adjacent whenever they are in the same row, in the same column, or carry the same symbol. It is well-known and easily verified that if $n \geq 3$, , is strongly regular with parameters $(n^2, 3(n-1), n, 6)$. If the Latin square S has an orthogonal mate $S^{\perp}$, the n symbols of $S^{\perp}$ give a partition of the vertex set of , into n cocliques of size n, that is, a colouring of , with n colours. And vice versa, if , can be coloured with n colours, each colour class has to be a transversal of S, so the colour classes produce an orthogonal mate. A strongly regular graph with the same parameters as , is called a *pseudo Latin square graph* and if such a graph has chromatic number n, we speak of a *pseudo orthogonal pair of Latin squares* of order n (for short POLS-n). Only for $n = 6$ and $n = 2$, there exist no pair of orthogonal Latin squares. So POLS-n exist for all $n \geq 3$, except maybe for $n = 6$. The search for POLS-6 is the subject of this report. The result is nagative:

Theorem 1.1 *There exists no pseudo Latin square graph of order six with chromatic number six.*

An n-colouring of a pseudo Latin square graph of order n meets the lower bound of Hoffman (see [5], [10], or [11]) for the chromatic number of a graph. Colourings of strongly regular graphs that meet Hoffman's bound are called Hoffman colourings and have been studied by Haemers and Tonchev [10]. The parameter set (36,15,6,6) is the smallest open case in their Table 1 of feasible parameter sets.

A strongly regular graph with a Hoffman colouring gives rise to an association scheme with three classes (the classes are: adjacent; non-adjacent with the same colour; non-adjacent with different colours). In the list of Van Dam [7] of feasible parameters for a 3-class association scheme the considered parameter set was the smallest open case.

2 Matrix tools

The following two lemmas from linear algebra (see [8]) are used in our computer search.

Lemma 2.1 *Let M be a symmetric $v \times v$ matrix with a symmetric partition*

$$M = \begin{bmatrix} M_1 & N \\ N^\top & M_2 \end{bmatrix},$$

where M_1 has order v_1 (say). Suppose M has just two distinct eigenvalues r and s $(r > s)$ with multiplicities f and $v - f$. Let $\lambda_1 \geq \ldots \geq \lambda_{v_1}$ be the eigenvalues of M_1 and let $\mu_1 \geq \ldots \geq \mu_{v-v_1}$ be the eigenvalues of M_2. Then $r \geq \lambda_i \geq s$ for $i = 1, \ldots, v_1$, and

$$\mu_i = \begin{cases} r \text{ if } 1 \leq i \leq f - v_1, \\ s \text{ if } f + 1 \leq i \leq v - v_1, \\ r + s - \lambda_{f-i+1} \text{ otherwise.} \end{cases}$$

Proof. The inequalities $r \geq \lambda_i \geq s$ and also the first two lines of the formulas for μ_i follow from eigenvalue interlacing for principal submatrices, see [8], [9], or Section 3.3 of [1]. We have $(M - rI)(M - sI) = O$. With the given block structure of M this gives $N^\top M_1 + M_2 N^\top - (r + s)N^\top = O$. Suppose that $\lambda_i \neq r, s$, let V be the corresponding eigenspace and let $\{\underline{v}_1, \ldots, \underline{v}_m\}$ be a basis for V. We claim that $B = \{N^\top \underline{v}_1, \ldots, N^\top \underline{v}_m\}$ is independent. Suppose not. Then $N^\top \underline{v} = \underline{0}$ for some $\underline{v} \in V$, $\underline{v} \neq \underline{0}$ which implies that $M \begin{bmatrix} \underline{v} \\ \underline{0} \end{bmatrix} = \lambda_i \begin{bmatrix} \underline{v} \\ \underline{0} \end{bmatrix}$, that is, λ_i is an eigenvalue of M, a contradiction. Now $N^\top M_1 + M_2 N^\top = (r+s)N^\top$ gives $M_2(N^\top \underline{v}_i) = (r + s - \lambda_i)N^\top \underline{v}_i$, thus B is an independent set of eigenvectors of M_2 for the eigenvalue $r + s - \lambda_i$. This almost proves the lemma. Only the numbers of μ_i's that are equal to r or s are not determined, but these follow from $\sum \mu_i + \sum \lambda_i = \text{trace } M_1 + \text{trace } M_2 = \text{trace } M = fr + (v - f)s$. □

Lemma 2.2 *Given a partitioned matrix*

$$M = \begin{bmatrix} M_1 & N_1 \\ N_2 & M_2 \end{bmatrix},$$

where M_1 is a square non-singular matrix of size v_1 (say). Suppose that rank $M = v_1$. *Then $M_2 = N_2 M_1^{-1} N_1$.*

Proof. The Schur complement S of M_1 equals $M_2 - N_2 M_1^{-1} N_1$, and satisfies rank $S =$ rank $M - v_1$. □

3 Structure

Consider a graph , with adjacency matrix A. Then , is a strongly regular graph with paramters $(36, 15, 6, 6)$ if and only if $A^2 = 6J + 9I$ (J denotes the all-one matrix). Suppose , is such a graph. Then , has eigenvalues 15, 3 and -3 with multiplicities 1, 15 and 20, respectively. Moreover the maximal size if a coclique of , equals six, and each point outside a maximal coclique $\mathcal{C}$ is adjacent to precisely three vertices of $\mathcal{C}$. See for example [1] p.10 or [9] p.597. Suppose that , has chromatic number six, that is, , is a POLS-6. The following proposition is clear.

Proposition 3.1 *Without loss of generality A admits the following block structure:*

$$A = \begin{bmatrix} 0 & A_{1,2} & A_{1,3} & A_{1,4} & A_{1,5} & A_{1,6} \\ A_{2,1} & 0 & A_{2,3} & A_{2,4} & A_{2,5} & A_{2,6} \\ A_{3,1} & A_{3,2} & 0 & A_{3,4} & A_{3,5} & A_{3,6} \\ A_{4,1} & A_{4,2} & A_{4,3} & 0 & A_{4,5} & A_{4,6} \\ A_{5,1} & A_{5,2} & A_{5,3} & A_{5,4} & 0 & A_{5,6} \\ A_{6,1} & A_{6,2} & A_{6,3} & A_{6,4} & A_{6,5} & 0 \end{bmatrix},$$

where $A_{i,j} = A_{j,i}^{\top}$ and $A_{i,j}$ $(i \neq j)$ is a 6×6 matrix with 3 ones and 3 zeros in each row and column.

Consider the following two operations on , . *Block complementation* is the replacement of each off-diagonal block $A_{i,j}$ by its complement $J - A_{i,j}$. *Block switching* with respect to a colour class k is the replacement of the blocks $A_{i,k}$ and $A_{k,i}$ $(i = 1, \ldots, 6,\ i \neq k)$ by their complements. Thus block switching is just Seidel switching with respect to a colour class (see [5] or [4]). It is straightforward that block complementation and block switching leave the equation $A^2 = 6J + 9I$ valid and therefore , remains a POLS-6 under these operations. There are 32 possible ways to apply one or more block switchings. Together with block complementation one can obtain 64 POLS-6's from a given one. Of course, some of these may be isomorphic.

Proposition 3.2 *Up to the ordering of rows and colums there are just seven candidates $C_1, \ldots, C_7$ for the block matrices $A_{i,j}$, being:*

$$C_1 = \begin{bmatrix} 1&1&1&0&0&0\\ 1&1&1&0&0&0\\ 1&1&1&0&0&0\\ 0&0&0&1&1&1\\ 0&0&0&1&1&1\\ 0&0&0&1&1&1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1&1&1&0&0&0\\ 1&1&1&0&0&0\\ 1&1&0&1&0&0\\ 0&0&1&0&1&1\\ 0&0&0&1&1&1\\ 0&0&0&1&1&1 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 1&1&1&0&0&0\\ 1&1&1&0&0&0\\ 1&0&0&1&1&0\\ 0&1&0&1&0&1\\ 0&0&1&0&1&1\\ 0&0&0&1&1&1 \end{bmatrix},$$

$$C_5 = \begin{bmatrix} 1&1&1&0&0&0\\ 1&1&0&1&0&0\\ 1&0&1&0&1&0\\ 0&1&0&1&0&1\\ 0&0&1&0&1&1\\ 0&0&0&1&1&1 \end{bmatrix}, \quad C_6 = \begin{bmatrix} 1&1&1&0&0&0\\ 1&1&0&1&0&0\\ 1&0&1&0&1&0\\ 0&1&0&0&1&1\\ 0&0&1&1&0&1\\ 0&0&0&1&1&1 \end{bmatrix}, \quad C_7 = \begin{bmatrix} 1&1&1&0&0&0\\ 1&1&0&1&0&0\\ 1&0&0&0&1&1\\ 0&1&0&0&1&1\\ 0&0&1&1&1&0\\ 0&0&1&1&0&1 \end{bmatrix},$$

and $C_4 = C_3^\top$.

Proof. First note that C_1, C_2, C_5, C_6 and C_7 are equal to their transpose. In [2] all connected cubic graphs on 12 vertices are generated. Among them there are five bipartite ones. These correspond to candidate blockmatrices C_2 to C_7, where C_3 and C_4 belong to the same graph. The only disconnected bipartite cubic graph is the disjoint union of two $K_{3,3}$'s, which corresponds to C_1.

It is also a straightforward task to generate the seven matrices (even without a computer). □

Proposition 3.3 *Candidate C_1 cannot occur.*

Proof. Suppose A has a submatrix $A_1 = \begin{bmatrix} 0 & C_1 \\ C_1 & 0 \end{bmatrix}$. Apply Lemma 2.1 with $M = A - \frac{1}{2}J$ and $M_1 = A_1 - \frac{1}{2}J$. Then $r = -s = 3$, $f = 15$, $\lambda_1 = \ldots = \lambda_{10} = 0$, $\lambda_{11} = \lambda_{12} = -3$ and hence $\mu_1 = \ldots = \mu_5 = 3$, $\mu_6 = \ldots = \mu_{15} = 0$ and $\mu_{16} = \ldots = \mu_{24} = -3$. Therefore $A_2 = M_2 + \frac{1}{2}J$ is the adjacency matrix of a regular graph of degree 9 with just four distinct eigenvalues. Such a graph is walk regular (see Van Dam [6]) which implies that A_2^k has a constant diagonal for every positive integer k. The entry $(A_2^3)_{1,1}$ counts the number of closed walks of length three at point 1, so should be an even integer, but we find $(A_2^3)_{1,1} = \frac{1}{24}\text{trace } A_2^3 = \frac{1}{24}(9^3 + \mu_1^3 + \ldots + \mu_{23}^3) = 27$, a contradiction. □

Also the next result follows from Lemma 2.1 applied to $A - \frac{1}{2}J$, but now with a different partition.

Proposition 3.4 *Let $,_1$ be a subgraph of , induced by three colour classes and let $,_2$ be the subgraph induced by the remaining three colour classes. Let $\lambda_1 \geq \ldots \geq \lambda_{18}$*

and $\mu_1 \geq \ldots \geq \mu_{18}$ *be the eigenvalues* $,_1$ *and* $,_2$*, respectively. Then* $\lambda_1 = \mu_1 = 6$*,* $\lambda_{17} = \mu_{17} = \lambda_{18} = \mu_{18} = -3$ *and* $3 \geq \lambda_i = -\mu_{18-i} \geq -3$ *for* $i = 2, \ldots, 16$*.*

Proposition 3.5 *With the notation of the previous proposition, if* $,_1'$ *is obtained from* $,_1$ *by block switching, then* $,_1'$ *has the same spectrum as* $,_1$*. If* $,_1''$ *is obtained from* $,_1$ *by block complementation, then* $,_1''$ *has the same spectrum as* $,_2$*.*

Proof. Let $H = 2A_1 - J$, where A_1 is the adjacency matrix of $,_1$. In H, block switching corresponds to multiplications of some rows and the corresponding columns with -1, an operations which leaves the spectrum invariant. Moreover, from the structure of A it follows that $HJ = 6J$ before and after switching. This implies that H and J have a common set of eigenvectors and hence the spectrum of $\frac{1}{2}(H + J) = A_1$ is also invariant under block switching.

Next define $K = J_{18} - \mathrm{diag}(J_6, J_6, J_6)$ (indices indicate the size; so K is the adjacency matrix of the complete tripartite graph $K_{6,6,6}$). The block structure of A_1 and K implies that A_1 and K have three common eigenvectors, whose entries are constant over each block. The corresponding eigenvalues being 6, -3 and -3 for A_1, and 12, -6 and -6 for K. All other eigenvectors of A_1 are in the kernel of K. This implies that $K - A_1$, which is the adjacency matrix of $,_1''$ has eigenvalues 6, -3 (twice) and $-\lambda_i$ for $i = 2, \ldots, 16$. □

4 The search

We saw that there are just five candidates for the subgraph of , induced by two colour classes; the five connected cubic bipartite graphs on 12 vertices. presented by the block matrices C_2 to C_7 (where C_3 and C_4 give the same graph).

The next step is to determine the possible candidates for $,_1$, the subgraph induced by three colour classes, by taking three candidate blocks and combine them in all possible ways. In doing so we used the eigenvalue inequalities of Proposition 3.4 (the fact that $\lambda_1 = 6$ and $\lambda_{17} = \lambda_{18} = -3$ follows from the block structure and therefore excludes no condidate, but the inequalities do). Furthermore we ruled out isomorphic candidates assuming that the binary numbers represented by the rows are as small as possible. This lead to 705838 candidates for $,_1$. In the subsequent step we reduced the number of candidates to 179126 by taking just one representative among the ones that are equivalent under block switching. The remaining candidates were tested on a possible extension to a POLS-6 by a method described in [3] based on Lemma 2.2. Suppose $\lambda_{16} \neq -3$. Then $M = A + 3I$ has rank 16, and $M_1' = A_1 + 3I$ has rank 16 too, and therefore M_1' has a non-singular principal submatrix M_1 of size 16. Now apply Lemma 2.2. It follows that a candidate row of $N_2 = N_1^\top$ is a $\{0,1\}$-vector $\underline{v}$

which satisfies $\underline{v}^\top M_1^{-1}\underline{v} = 3$. It also follows that no two distinct rows $\underline{v}_i$ and $\underline{v}_j$ of N_2 can be equal, indeed $\underline{v}_i = \underline{v}_j$ implies $3 = \underline{v}_i^\top M_1^{-1}\underline{v}_i = \underline{v}_i^\top M_1^{-1}\underline{v}_j = (M_2)_{i,j} = 0$ or 1, a contradiction. In addition, we have information on the distribution of zeros and ones in $\underline{v}$ and we know that $\underline{v}^\top M_1^{-1}\underline{v}_1$ and $\underline{v}^\top M_1^{-1}\underline{v}_2$ are equal to 0 or 1, when $\underline{v}_1$ and $\underline{v}_2$ are the first two rows of N_2 (and $\underline{v} \neq \underline{v}_1$, $\underline{v}_2$). We computed all vectors $\underline{v}$ with these properties. In some cases there were fewer than 20 which is not enough to make the extension. In case of enough candidate vectors, we define a graph Δ on the $\underline{v}$'s where two vectors $\underline{v}_i$ and $\underline{v}_j$ are adjacent whenever $\underline{v}_i^\top M_1^{-1}\underline{v}_j = 0$, or 1. Inside Δ we searched for cliques of size 20, which would give feasible sets of candidate vectors. In fact, we also used the information we have on the distrubution of zeros and ones in M_2. In all cases there were no such cliques. This ruled out all candidates for , $_1$ with $\lambda_{16} \neq -3$. In doing so, we used the fact that if a candidate , $_1$ is ruled out, then so is its block complement. This also took care of the case $\lambda_{16} = -3$, $\lambda_2 \neq 3$, since then the block complement has $\lambda_{16} \neq -3$. This left us with only 316 candidates for , $_1$ satisfying $\lambda_2 = -\lambda_{16} = 3$. The last step is to eliminate these candidates. In doing so we used Proposition 3.4, which gives the spectrum of , $_2$. For each candidate for , $_1$, there were a few (mosly just one, being the block complement) candidates for , $_2$ and we tried to complete the matrix in a staightforward manner. This reduced the number of candidates for , $_1$ to zero, showing that no POLS-6 exits.

5 History

There are 32548 strongly regular graphs known with parameters $(36, 15, 6, 6)$. This includes the twelve Latin square graphs of order six. Starting from these twelve and the 80 Steiner triple system on 15 points, Bussemaker, Mathon and Seidel [4] found, by use of Seidel switching, 16428 strongly regular graphs with the above parameters (they claimed to have found 16448, but see [13]). Later Spence [13] found 16120 more of these graphs. It has been verified by computer that none of the known ones admits a Hoffman colouring.

The search for POLS-6 as described in this report started in 1997 with the first two authors. Frans Bussemaker, was responsible for the computer programming. The major work was the generation of the candidates for , $_1$ and the subsequent reduction to just 316 cases. This part was finished in the beginning of 1998. Then Frans stopped working because of a severe illness, which led to his tragic death in December 1998. Then the third author finished the last part of the search. Surprisingly, this took only a few minutes of computing time, while the first part took many days (making use of the computers of colleagues in the weekend). In addition the third author performed an independent check for the first part. The search was completed in February 1999.

In the meantime Brendan McKay and the third author [12] have completed an exhaustive computer search for strongly regular graphs with parameters $(36,15,6,6)$ which shows that the 32548 indicated in [13] constitute the complete set. Thus, in view of the fact that it has been verified that none of these graphs admits a Hoffman colouring, we have one more independent check for our claim.

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