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Disturbance Decoupling in Dynamic Games

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Abstract

A theory for disturbance decoupling problems has been well developed in the area of geometric control theory. The aim of the present study is to introduce disturbance decoupling problems in a dynamic game context. For this purpose, techniques from geometric control theory are applied. Necessary and sufficient conditions are derived for the solvability of the disturbance decoupling problems introduced here. Consequently, for a given game, the players can easily check if these problems are solvable or not.

Keywords

Disturbance decoupling, differential games, robust controlled invariance.

Journal of Economic Literature Classification Numbers: C67 and C73.

1 Introduction

Dynamic game theory is concerned with situations in which several agents (or "players") who have different objectives are influencing the same dynamic system. Often, the objectives are expressed through cost functions which the agents aim to minimize. The usual notion of optimum in the single-agent case is substituted in the multiple-agent situation by a notion of equilibrium. Actually several such notions exist, depending on the information that the players have about each other's decisions and about the system state; for instance, there is an essential difference between open-loop and closed-loop equilibria. See [1] for an extensive account.

In the single-player case, methods of control design are not always linked to cost functions. Common design objectives such as stability or decoupling in fact do not give rise to unique optimal controls. Design objectives of this type do not seem to have received much attention so far in the context of dynamic games. In this paper, we shall consider a dynamic game version of the well known disturbance decoupling problem [3, 5]. With the idea of closed-loop equilibria in mind, we shall consider the situation in which all players use feedback policies. Even given this, there are several solution concepts to consider depending on whether the players do or do not cooperate and depending on the information that the players have about each other's policies. Specifically, we shall discuss three cases: (i) the cooperative case; (ii) the noncooperative case without information about other players' policies; (iii) the noncooperative with information about other players' policies. In each case, we shall give easily verifiable necessary and sufficient conditions for a solution to exist.

2 Preliminaries

The system under consideration is assumed to be linear and to be described by the differential equation

$$\dot{x} = Ax + \sum_{i=1}^{N} B_i u_i + Ed, \quad x(0) = x_0,$$

where $A \in I\!R^{n\times n}$, $B_i \in I\!R^{n\times m_i}$ and $E \in I\!R^{n\times m}$ with the standing assumption that $E \neq 0$. The system is controlled by N players; player i influences the system through his control function u_i. In addition, the system is influenced by the unknown disturbance function d. The output of player i is given by a linear function z_i of the state, i.e. $z_i = H_i x$, with $H_i \in I\!R^{p_i\times n}$. In this paper only linear state feedback problems are considered, i.e. the players are able to observe the state and they choose controls of the form $u_i = F_i x$, with $F_i \in I\!R^{m_i\times n}$. With a choice of N feedback maps F_i, the output of player i can be written as

$$z_i(t) = H_i e^{\left(A+\sum_{i=1}^{N} B_i F_i\right)t} x_0 + \int_0^t T_i(t-\tau)d(\tau)d\tau,$$

where $T_i(t) := H_i e^{\left(A+\sum_{i=1}^{N} B_i F_i\right)t} E$ is the closed-loop impulse response from disturbance to the output of player i. The system is said to be disturbance decoupled for player i if $T_i(t) = 0$ for all $t \geq 0$.

Let $\mathcal{B}$ and $\mathcal{H}$ be subspaces of $I\!R^n$. The maximal $(A, \mathcal{B})$-controlled invariant contained in $\mathcal{H}$ is denoted by $\max \mathcal{V}(A, \mathcal{B}, \mathcal{H})$. The following result is well known.

Theorem 2.1 *If $N = 1$, the system is disturbance decoupled (for player 1) iff*

$$\operatorname{im} E \subset \max \mathcal{V}(A, \operatorname{im} B_1, \ker H_1).$$

In section 4.2 we use the concept of robust controlled invariance. This concept was introduced in [2]; a more detailed discussion can be found in [3], section 6.5. In the following paragraph we recall the definition and the results that are of importance for the present paper.

Let A and $\mathcal{B}$ depend on a parameter $q \in Q \subset I\!R^r$. A subspace $\mathcal{W}$ is called a robust $(A(q), \mathcal{B}(q))$-controlled invariant relative to Q if $A(q)\mathcal{W} \subset \mathcal{W} + \mathcal{B}(q)$ for all $q \in Q$. The set of all robust controlled invariants contained in a given subspace is closed under addition. The maximal robust $(A(q), \mathcal{B}(q))$-controlled invariant contained in $\mathcal{H}$ is denoted by $\max \mathcal{V}_R(A(q), \mathcal{B}(q), \mathcal{H})$. For the computation of this space the following algorithm, first presented in [4], is available. In this algorithm we use the notation $f^{\leftarrow}(\mathcal{Y}_0) \subset \mathcal{X}$ for the inverse image of $\mathcal{Y}_0$ under f, where f is a map from the set $\mathcal{X}$ to the set $\mathcal{Y}$ and $\mathcal{Y}_0 \subset \mathcal{Y}$.

Algorithm 2.2 The subspace $\max \mathcal{V}_R(A(q), \mathcal{B}(q), \mathcal{H})$ coincides with the last term of the sequence

$$\mathcal{Z}_0 := \mathcal{H},$$
$$\mathcal{Z}_i := \mathcal{H} \cap \left(\bigcap_{q \in Q} A(q)^{\leftarrow}(\mathcal{Z}_{i-1} + \mathcal{B}(q)) \right), \quad i = 1, \cdots, k,$$

where the value of $k \leq n$ is determined by the condition $\mathcal{Z}_k = \mathcal{Z}_{k-1}$.

We finally present the hyper-robust disturbance localization problem (section 6.5.1 of [3]). Consider the case $N = 1$ and let A, B_1 depend on the parameter q. Then, the hyper-robust disturbance localization problem is to find a feedback map $F_1(q)$ such that the system is disturbance decoupled (for player 1) for all $q \in Q$. For this problem the following result was established [3, Thm. 6.5-1].

Theorem 2.3 *The hyper-robust disturbance localization problem is solvable iff*

$$\operatorname{im} E \subset \max \mathcal{V}_R(A(q), \operatorname{im} B_1(q), \ker H_1).$$

3 Cooperative Disturbance Decoupling

If the players cooperate it is of common interest to decouple outputs from the disturbance. In the following definition we define the problem of decoupling all the outputs from the disturbance using all the inputs.

Definition 3.1 The *Cooperative Disturbance Decoupling Problem* (CDDP) by state feedback is to find feedback maps F_i such that the system is disturbance decoupled for all the players.

In this problem there is neither a distinction between the controls nor between the outputs. Instead of considering a system with N controls and N outputs, we can equivalently consider a system with one control and one output in the following way:

$$\dot{x} = Ax + \begin{bmatrix} B_1 & \cdots & B_N \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} + Ed, \quad x(0) = x_0,$$

$$\begin{bmatrix} z_1 \\ \vdots \\ z_N \end{bmatrix} = \begin{bmatrix} H_1 \\ \vdots \\ H_N \end{bmatrix} x.$$

The standard results from geometric control theory can be applied to this system. If we apply theorem 2.1 we get the following result.

Theorem 3.2 *The CDDP by state feedback is solvable iff*

$$\operatorname{im} E \subset \max \mathcal{V} \left(A, \sum_{i=1}^{N} \operatorname{im} B_i, \bigcap_{i=1}^{N} \ker H_i \right).$$

4 Noncooperative Disturbance Decoupling

We now turn to the noncooperative setting. Different problems can be posed depending on whether or not players choose their feedback policies in a certain order and are aware of each other's decisions. In this section we shall fix one player (denoted by subscript *d*) and consider the situations in which this player is or is not aware of the other players' chosen feedback policies. For ease of notation we take $N = 2$; the second player will be denoted by subscript *o*.

4.1 Strong Noncooperative Disturbance Decoupling

Here we consider the situation without information about other players' choices of feedback policies. The corresponding problem can be formally defined as follows.

Definition 4.1 The *Strong Noncooperative Disturbance Decoupling Problem* (SNDDP) by state feedback is to find a feedback map F_d such that the system is disturbance decoupled for player *d* for all feedback maps F_o.

Assume that F_d solves the SNDDP by state feedback. We then have that $T_d(t) = 0$ for all F_o, or, equivalently

$$H_d(A + B_dF_d + B_oF_o)^k E = 0, \text{ for all } F_o \text{ and for all } k \geq 0. \tag{1}$$

By taking $F_o = 0$, it follows that

$$\operatorname{im} E \subset \max \mathcal{V}(A, \operatorname{im} B_d, \ker H_d). \tag{2}$$

Obviously, this condition (which is stated completely in terms of the system matrices) is a necessary condition for solvability of the SNDDP by state feedback. In the following lemma we prove another consequence of (1).

Lemma 4.2 *If there exists an F_d for which* (1) *holds, we have*

$$H_d(A + B_dF_d)^l B_o = 0, \text{ for all } l \geq 0. \tag{3}$$

Proof We prove this lemma by induction. Setting $k = 1$ in (1), we find $H_d(A+B_dF_d+B_oF_o)E = 0$ for all F_o. Taking $F_o = 0$ yields $H_d(A + B_dF_d)E = 0$ so that we have $H_dB_oF_oE = 0$ for all F_o. Since $E \neq 0$ this gives us $H_dB_o = 0$. Now, fix an $l_0 \geq 0$ and assume that (3) holds for all $l = 0, \cdots, l_0$. Take $k = l_0 + 2$ and $F_o = 0$ in (1) to get $H_d(A + B_dF_d)^{l_0+2}E = 0$. Use this and the formula (which can also be seen by induction)

$$\begin{aligned}(A + B_dF_d + B_oF_o)^{l_0+2} =& (A + B_dF_d)^{l_0+2} + \\ &+ \sum_{j=0}^{l_0+1} (A + B_dF_d)^j B_oF_o(A + B_dF_d + B_oF_o)^{l_0+1-j}\end{aligned}$$

to see that (1) (with $k = l_0 + 2$) implies

$$\sum_{j=0}^{l_0+1} H_d(A + B_dF_d)^j B_oF_o(A + B_dF_d + B_oF_o)^{l_0+1-j}E = 0, \text{ for all } F_o.$$

With the induction assumption, this reduces to $H_d(A + B_dF_d)^{l_0+1}B_oF_oE = 0$ for all F_o. Since $E \neq 0$ this gives us $H_d(A + B_dF_d)^{l_0+1}B_o = 0$, which is the desired result.

□

So the feedback maaping F_d that solves the SNDDP by state feedback also satisfies (3). As a consequence the system matrices must satisfy

$$\operatorname{im} B_o \subset \max \mathcal{V}(A, \operatorname{im} B_d, \ker H_d).$$

Again we find a necessary condition for the solvability of the SNDDP by state feedback completely in terms of the system matrices. In the following theorem it is stated that (2) and (3) are also sufficient.

Theorem 4.3 *The SNDDP by state feedback is solvable iff*

$$\operatorname{im} \begin{bmatrix} B_o & E \end{bmatrix} \subset \max \mathcal{V}(A, \operatorname{im} B_d, \ker H_d). \tag{4}$$

Proof The necessity was already proven. For the sufficiency, it suffices to consider the control input of player *o* as an extra distrubance. Indeed, assume that (4) holds and write $\mathcal{W} := \max \mathcal{V}(A, \operatorname{im} B_d, \ker H_d)$. Since $\mathcal{W}$ is an $(A, \operatorname{im} B_d)$-controlled invariant, there exists an F_d such that $\mathcal{W}$ is an $A + B_d F_d$-invariant, so that

$$(A + B_d F_d + B_o F_o)\mathcal{W} \subset \mathcal{W} + \operatorname{im} B_o \subset \mathcal{W}.$$

Together with (4) shows that the closed loop system is disturbance decoupled for player *d*. □

Formula (4) provides a compact necessary and sufficient condition for the existence of a feedback control for the decoupling player that achieves disturbance decoupling for him whatever the other player will do. In fact, if condition (4) holds, the corresponding feedback control of the decoupling player achieves disturbance decoupling for any control of the other player.

4.2 Weak Noncooperative Disturbance Decoupling

In this subsection we consider the case in which information about other players' feedback policies is available. The corresponding problem can be formally defined as follows.

Definition 4.4 The *Weak Noncooperative Disturbance Decoupling Problem* (WNDDP) by state feedback is to find for any given feedback map F_o a feedback map F_d such that the system is disturbance decoupled for player *d*.

If the WNDDP by state feedback is solvable, the decoupling player is able to respond to any control of the other player in such a way that he can decouple his output from the disturbance. In particular, if the other player chooses $F_o = 0$, the decoupling player can decouple iff

$$\operatorname{im} E \subset \max \mathcal{V}(A, \operatorname{im} B_d, \ker H_d). \tag{5}$$

Hence condition (2) is also a necessary condition for the solvability of the WNDDP by state feedback.

For a given F_o the system equations are

$$\begin{aligned} \dot{x} &= (A + B_o F_o)x + B_d u_d + Ed, \\ z_d &= H_d x. \end{aligned}$$

Although F_o is not manipulable for the decoupling player, it is accessible for his decoupling purposes. We recognize this problem as a particular case of the hyper-robust disturbance localization problem (see the end of section 2). According to Theorem 2.3 the problem for the decoupling player is solvable iff

$$\operatorname{im} E \subset \max \mathcal{V}_R(A + B_o\cdot, \operatorname{im} B_d, \ker H_d), \tag{6}$$

i.e. the maximal robust $(A+B_o\cdot, \mathrm{im}\, B_d)$-controlled invariant relative to $I\!R^{m_o\times n}$ contained in $\ker H_d$ (we write a dot instead of an F_o in order to indicate that the robustness needs to be interpreted with respect to this parameter). According to Algorithm 2.2 this space coincides with the last term of the sequence

$$\mathcal{Z}_0 := \ker H_d,$$
$$\mathcal{Z}_i := \ker H_d \cap \left(\bigcap_{F_o}(A+B_oF_o)^{\leftarrow}(\mathcal{Z}_{i-1}+\mathrm{im}\, B_d)\right), \quad i=1,\cdots,k.$$

Now, assume that the WNDDP is solvable. Then, since $E\neq 0$ and according to (6), there exists an $x_0\neq 0$ in the space $(A+B_oF_o)^{\leftarrow}(\mathcal{Z}_i+\mathrm{im}\, B_d)$ for all F_o and for all $i=0,1,\cdots$. Equivalently, $(A+B_oF_o)x_0\in\mathcal{Z}_i+\mathrm{im}\, B_d$ for all F_o and for all $i=0,1,\cdots$. Because this holds for all F_o and because $x_0\neq 0$, we have $Ax_0+b_o\in\mathcal{Z}_i+\mathrm{im}\, B_d$ for all $b_o\in\mathrm{im}\, B_o$ and for all $i=0,1,2,\cdots$. But this implies that

$$\mathrm{im}\, B_o\subset\mathcal{Z}_i+\mathrm{im}\, B_d, \quad i=0,1,\cdots. \tag{7}$$

We have now shown that solvability of the WNDDP by state feedback implies (5) and (7). The latter condition is however not yet in terms of the system matrices. We will remedy this after the following lemma.

Lemma 4.5 *Consider the sequence $\mathcal{V}_i$, defined by*

$$\mathcal{V}_0 := \ker H_d,$$
$$\mathcal{V}_i := \ker H_d\cap A^{\leftarrow}(\mathcal{V}_{i-1}+\mathrm{im}\, B_d), \quad i=1,\cdots,k.$$

Then we have the inclusion $\mathcal{Z}_i\subset\mathcal{V}_i$ for all i.

Remark The last term of the sequence $\mathcal{V}_i$ equals the maximal $(A,\mathrm{im}\, B_d)$-controlled invariant contained in $\ker H_d$.

Proof We prove the lemma by induction. Clearly, the statement holds for $i=0$. Assume that $\mathcal{Z}_i\subset\mathcal{V}_i$ for a certain i. According to the definition of the $\mathcal{Z}_i$'s we have $\mathcal{Z}_{i+1}\subset\ker H_d$ and $\mathcal{Z}_{i+1}\subset(A+B_oF_o)^{\leftarrow}(\mathcal{Z}_i+\mathrm{im}\, B_d)$ for all F_o, so in particular for $F_o=0$. This results in

$$A\mathcal{Z}_{i+1}\subset\mathcal{Z}_i+\mathrm{im}\, B_d\subset\mathcal{V}_i+\mathrm{im}\, B_d,$$

and hence, $\mathcal{Z}_{i+1}\subset A^{\leftarrow}(\mathcal{V}_i+\mathrm{im}\, B_d)\cap\ker H_d=\mathcal{V}_{i+1}$, which completes the proof. □

Applying this lemma to (7) gives us $\mathrm{im}\, B_o\subset\mathcal{S}_i+\mathrm{im}\, B_d$ for all i and by taking the last term in this sequence (as noted in the preceding remark) we arrive at a necessary condition in terms of the system matrices. In the following theorem we claim that this condition together with condition (5) is actually also sufficient for solvability of WNDDP.

Theorem 4.6 *The WNDDP by state feedback is solvable iff*

$$\operatorname{im} E \subset \max \mathcal{V}(A, \operatorname{im} B_d, \ker H_d), \tag{8}$$

$$\operatorname{im} B_o \subset \max \mathcal{V}(A, \operatorname{im} B_d, \ker H_d) + \operatorname{im} B_d. \tag{9}$$

Proof The necessity was already proven. The sufficiency can be concluded by considering the input of player *o* as an accessible disturbance (cf. [5, Exc. 4.10], [3, p. 225]). Specifically, let $F_o \in \mathbb{R}^{m_o \times n}$ and write $\mathcal{W} := \max \mathcal{V}(A, \operatorname{im} B_d, \ker H_d)$. According to (9), we have $\operatorname{im} E \subset \mathcal{W} \subset \ker H_d$. Furthermore, since $\mathcal{W}$ is an $(A, \operatorname{im} B_d)$-controlled invariant and because of (9), we also have $(A + B_o F_o)\mathcal{W} \subset \mathcal{W} + \operatorname{im} B_d$, so that $\mathcal{W}$ is also an $(A + B_o F_o, \operatorname{im} B_d)$-controlled invariant; hence there exists a mapping F_d such that $\mathcal{W}$ is an $(A + B_o F_o + B_d F_d)$-invariant. □

Formula (8) and (9) provide compact necessary and sufficient conditions for the decoupling player to respond to the action of the other player always in such a way that his output is decoupled from the disturbance.

5 Concluding Remarks

The aim of the paper was to define some disturbance decoupling problems in the area of differential games. Necessary and sufficient conditions for the solvability of these problems in terms of the system data were derived; consequently, one can easily verify whether the considered problems are solvable for a given system.

The idea of considering games in which the players follow certain control objectives that are not necessarily given by cost functions can be carried further in many ways. In the context of decoupling, extensions can for instance be made in the direction of incorporating stability and to the case where the players are not able to completely observe the state and/or each other's actions. The notion of equilibrium could be incorporated to a greater extent by considering games in which the players know that the other players will look for a decoupling feedback but are not informed about each other's decisions. The two noncooperative problems considered in this paper provide necessary and sufficient conditions respectively for a decoupling solution to exist in this situation.

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