The Robustness of the CAPM - A Computational Approach

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Abstract

In this paper we argue that in realistically calibrated two period general equilibrium models with incomplete markets CAPM-pricing provides a good benchmark for equilibrium prices even when agents are not mean-variance optimizers and returns are not normally distributed. We numerically approximate equilibria for a variety of different specifications for preferences, endowments and dividends and compare the equilibrium prices and portfolio-holdings to the predictions of the CAPM. While the CAPM does not hold exactly for the chosen specification, it turns out that pricing-errors are extremely small. Furthermore, two-fund separation holds approximately.

JEL codes: C61, C62, C63, C68, D52, D58, G11, G12.

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1 Introduction

The capital asset pricing model (CAPM) of Sharpe (1964) and Lintner (1965) predicts that equilibrium returns of assets are a linear function of their market β (the ratio of the covariance of the asset and the market's return over the variance of the market's return). This intuitively appealing result has long shaped the way practitioners think about average returns and risk. While the model fares poorly in explaining observed cross-sectional stock returns (see for example Fama and French (1992)) it remains one of the central building blocks in financial economics.

One of the reasons for this is that the CAPM provides a good theoretical starting point for the examination of asset prices. Geanakoplos and Shubik (1990) show that the CAPM can be viewed as a special case of the general equilibrium model with incomplete asset markets (the GEI-model). Oh (1996), Willen (1997) and others have shown that the central conclusions of the CAPM hold true under completely general dividends and endowments as long as all agents have mean-variance utility functions and common beliefs on the distribution of future states of the world.

However, without mean-variance preferences one has to make very strong assumptions on the distribution of asset pay-offs in order to derive the conclusions of the CAPM[1]. Berk (1997) shows that joint restrictions on utility functions and asset returns cannot lead to more realistic assumptions. He shows that if one assumes that agents have von-Neumann-Morgenstern utility functions, quadratic utility is necessary for the CAPM-pricing formula to hold. Since quadratic utility is an unattractive assumption, it is an important question whether CAPM-pricing provides a benchmark for the cross-section of security prices in a model with more general preferences, endowments and asset returns. Empirical contradictions of the CAPM might be explained by the fact that some agents are not mean-variance optimizers and that many securities have returns that are far from elliptical.

In this paper we show that independently of mean-variance preferences or normal returns, the CAPM pricing formula often provides a very good prediction for actual equilibrium returns. These results lead us to conjecture that the CAPM is robust with respect to variation in preferences as long as preferences, dividends and endowments in the model are realistically calibrated to annual US data.

We do not provide a theoretical justification for this claim but instead compute hundreds of examples which illustrate it (see Judd (1997) for a general discussion which favors this approach to economic theory) - we approximate equilibria numerically and compare the prices and portfolio-holdings predicted by the CAPM to the actual equilibrium prices and equilibrium portfolio-holdings.

In order to compute equilibria when asset markets are incomplete, we develop a new algorithm which differs from existing ones (see Brown, DeMarzo and Eaves (1996) and Schmedders (1998)), in that it focuses on the two-period model with a single good. This

[1] Chamberlain (1983) and Owen and Rabinovitch (1983) prove that under general preferences one has to assume that all asset returns are multivariate elliptically distributed.

limitation enables us to take advantage of the special structure of the one-good model and to operate in portfolio space instead of in the state space. Therefore the number of unknowns does not increase as the state space increases and we can approximate continuous distributions in very large discrete state-spaces.

To show that the CAPM pricing formula provides a good approximation to asset returns in realistically calibrated models, one first has to clarify what one means by 'realistically calibrated'. We follow the macroeconomic literature and we choose first and second moments of endowments and dividends to match annual US data and preferences to exhibit relative risk aversion below 10 and nonincreasing absolute risk aversion (see e.g. Mehra and Prescott (1985))[2].

More importantly one has to argue that the computed examples are not sensitive to the exact specification of the model but that they reflect some general property of asset prices. We assume that there are three agents and $32,768$ states of nature and we examine the robustness of the CAPM with respect to 600 different specifications of preferences and endowments. We first assume that endowments and dividends are log-normally distributed and consider the following 3 specifications for preferences.

- All three agents have constant absolute risk aversion (CARA) utility functions.
- All agents' utility functions exhibit constant relative risk aversion (CRRA).
- Agents' utility-functions exhibit loss aversion as in Benartzi and Thaler (1995).

For each case we randomly generate 100 economies which differ with respect to agents' (heterogeneous) degrees of risk aversion.

In the next three cases we fix preferences and vary distributions of dividends and endowments. We assume that all agents have CRRA utility functions and consider the following distributions for assets and endowments.

- Endowments and dividends are drawn from a uniform distribution. We randomly generate 100 economies with differ with respect to the support of the uniform distributions.
- Endowments and dividends are determined by two factors and an idiosyncratic shock each of which are drawn from a log-normal distribution. We randomly generate 100 economies with differ with respect to the factor-loads.
- Endowments and dividends are drawn from a log-normal distribution and there is an option on one of the stocks. We randomly generate 100 economies which differ with respect to the strike-price of the option.

[2] Note that just as in Mehra and Prescott this calibration is very unrealistic with respect to the market risk-premium. This fact might give a first indication of why CAPM-pricing does so well in our framework.

For all 600 economies under consideration we compare the computed return on individual stocks to the return predicted by the CAPM-pricing formula. We find that in all 600 cases the average mean squared pricing errors (for returns) across stocks lie below 0.04 percent. The average error across all simulations is in the order of magnitude of 0.005 percent. We also develop an R^2-measure to assess the validity of CAPM-pricing. The value of it is well above 0.9999 in all economies examined by us.

In addition to predicting asset returns, the CAPM also predicts that all agents' equilibrium portfolio-holdings will consist of the riskless bond and a mutual fund of risky assets. It is possible that CAPM-pricing is very accurate, but two-fund separation does not apply. Nevertheless, in the computed examples two-fund separation holds almost exactly in the equilibria we compute. The R^2-statistic we use to judge the presence of two-fund separation exceeds 0.99 in the vast majority of cases considered.

The paper is organized as follows. In Section 2 we give a short introduction to the model and collect several general results on the CAPM in a general equilibrium setting. Section 3 gives an example of a realistically calibrated economy with non-elliptical returns and CRRA preferences for which CAPM-pricing provides an almost perfect prediction. In Sections 4 and 5 we examine the robustness of this phenomenon. In Section 4 we vary the parameters of risk aversion for the CRRA case, we consider CARA utility functions, and we examine utility functions displaying loss aversion. In Section 5 we fix preferences to exhibit constant relative risk aversion and we examine the robustness of the CAPM with respect to dividend-distributions. In Section 6 we speculate about possible explanations and conclude the main part of the paper. In the Appendix we describe the algorithm and prove its global convergence.

2 The Two-Period Finance Economy

The finance version of the GEI-model describes an economy over two periods of time, $t = 0, 1$, with uncertainty over the state of nature resolving in period $t = 1$. We describe the model, introduce the necessary notation and discuss the CAPM. For a thorough description of the GEI-model see for example Magill and Quinzii (1996).

2.1 The Model

There are $S + 1$ states in the economy; at time $t = 0$ the economy is in state $s = 0$, at time $t = 1$ one state of nature s out of S possible states of nature realizes. In each state $s = 0, \ldots, S$, there is a single nondurable consumption good.

There are H agents, indexed by $h = 1, \ldots, H$, that participate in the economy. Agent h is characterized by initial endowments (the initial income stream) $e^h = (e_0^h, e_1^h, \ldots, e_S^h)^\top \in \mathrm{int}(X^h)$[3] and his preferences over consumption bundles (income streams available for con-

[3] $\mathrm{int}(X^h)$ denotes the interior of X^h.

sumption) $c^h = (c_0^h, c_1^h, \ldots, c_S^h)^\top \in X^h$. Here X^h is a closed subset of $\mathbb{R}^{S+1}$ that satisfies $\{x^h\} + \mathbb{R}_+^{S+1} \subset X^h$ for all $x^h \in X^h$. In most applications X^h will be equal to $\mathbb{R}_+^{S+1}$ or $\mathbb{R}^{S+1}$. To distinguish between first-period consumption and the random second period consumption, we define $\tilde{x} = (x_1, \ldots, x_S)^\top$ for any vector $x = (x_0, x_1, \ldots, x_S)^\top$. Aggregate endowments (aggregate incomes) are $e = \sum_{h=1}^H e^h$. Each agents' preferences are represented by a utility function $u^h : X^h \to \mathbb{R}$ satisfying standard assumptions; u^h is strictly quasi-concave and continuous. Moreover, the set $X^h(e^h) = \{x^h \in X^h \mid u^h(x^h) \geq u^h(e^h)\}$ is assumed to be bounded from below, a property automatically satisfied when X^h is bounded from below. In the applications of Sections $3-5$ we consider economies where all agents have separable utility functions across date-events with identical probabilities, i.e. there exist probabilities $\rho_1, \ldots, \rho_S > 0$, $\sum_{s=1}^S \rho_s = 1$, such that

$$u^h(c^h) = v_0^h(c_0^h) + \delta^h \sum_{s=1}^S \rho_s v_s^h(c_s^h),$$

where $\delta^h > 0$ is the discount factor, $X^h = \prod_{s=0}^S X_s^h$, where X_s^h is a subset of $\mathbb{R}$, and $v_s^h : X_s^h \to \mathbb{R}$ is assumed to be strictly increasing and strictly concave. In this case it follows from the properties of v_s^h that $X^h(e^h)$ is bounded from below. When the functions v_s^h are independent of s, we say that agents have von Neumann-Morgenstern utility functions.

There are J assets. Asset j pays dividends at date $t = 1$ which we denote by $d^j \in \mathbb{R}^S$. The price of asset j at time $t = 0$ is q_j. Without loss of generality we assume that the assets are in zero net supply and we collect all assets' dividends in a pay-off matrix

$$A = (d^1, ..., d^J) \in \mathbb{R}^{S \times J}.$$

At time $t = 0$ agent h chooses a portfolio-holding $\theta^h \in \mathbb{R}^J$ which uniquely defines the agents' consumption by $c_0^h = e_0^h - \theta^h \cdot q$ and $\tilde{c}^h = \tilde{e}^h + A\theta^h$. The net demand of agent h, $\tilde{c}^h - \tilde{e}^h$, belongs to the marketed subspace $\langle A \rangle = \{z \in \mathbb{R}^S \mid \exists \theta \in \mathbb{R}^J, z = A\theta\}$.

The exogenous parameters defining a finance economy $\mathcal{E} = ((X^h, u^h, e^h)_{h=1,\ldots,H}; A)$ are agents' consumption sets, utility functions and endowments, and the pay-off matrix. Without loss of generality, we assume throughout that there are no redundant assets, $\text{rank}(A) = J$. If there are redundant assets, it follows from an arbitrage argument that their price is uniquely determined by the price of the other assets. Markets are incomplete when $J < S$. We define asset prices to be arbitrage free if it is not possible to achieve a positive income stream in all states by trading in the available assets. It is well known that a price system $q \in \mathbb{R}^J$ precludes arbitrage if and only if there exists a state price vector $\pi \in \mathbb{R}_{++}^S$ such that $q = \pi^\top A$. We define Q to be the set of arbitrage free asset prices.

Definition 2.1 (Competitive Equilibrium): A competitive equilibrium for an economy $\mathcal{E}$ is a collection of portfolio-holdings $\theta^* = (\theta^{1*}, \ldots, \theta^{H*}) \in \mathbb{R}^{HJ}$ and asset prices $q^* \in \mathbb{R}^J$ that satisfy the following conditions:

(1) $\theta^{h*} \in \arg\max_{\theta^h \in \mathbb{R}^J} \; u^h(c^h) \; s.t. \; c^h = e^h + \begin{pmatrix} -q^{*\top} \\ A \end{pmatrix} \theta^h$ and $c^h \in X^h, \quad h = 1, \ldots, H;$

(2) $\sum_{h=1}^{H} \theta^{h*} = 0$.

Under an additional assumption of strictly increasing utility functions, existence of an equilibrium follows from the results of Geanakoplos and Polemarchakis (1986).

2.2 The Capital Asset Pricing Model

Sharpe (1964) and Lintner (1965) use the portfolio analysis developed by Tobin (1958) and Markowitz (1959) to examine an equilibrium model of financial markets. Under the assumption that all agents are mean-variance optimizers, and given the riskfree rate of return and the return of the market portfolio, they derive a closed-form solution for equilibrium returns of all assets, the so-called β-pricing formula. This formula relates the return of a risky asset to the return of the market portfolio by the covariance of that asset with the market. It is well known that the β-pricing formula can be derived in the finance GEI-model, see Geanakoplos and Shubik (1990). To fix notation and to give some intuition for the computational results in Sections $3-5$, we summarize and slightly generalize the findings in the literature - Geanakoplos and Shubik (1990), Magill and Quinzii (1996), Oh (1996), and Willen (1997) - to cover the case with mean-variance preferences, non-marketed endowments and a finite state space in a world with incomplete markets.

We denote by $1_n = (1, \ldots, 1)^\top \in \mathbb{R}^n$ the vector of all ones. The m-th unit vector of appropriate dimension is denoted ι_m - the dimension of ι_m is always apparent from the context. Throughout this subsection we assume that there exist objective probabilities ρ_s, $s = 1, \ldots, S$, over the possible states of nature in period 1. Moreover, asset 1 is a riskless bond, $d^1 = 1_S$. For a random variable $x \in \mathbb{R}^S$, we define its expected value $\mathrm{E}(x) = \sum_{s=1}^{S} \rho_s x_s$, for two random variables $x, y \in \mathbb{R}^S$, we define the covariance as $\mathrm{cov}(x, y) = \sum_{s=1}^{S} \rho_s x_s y_s - \mathrm{E}(x)\mathrm{E}(y)$. The variance of a random variable $x \in \mathbb{R}^S$ is given by $\mathrm{var}(x) = \mathrm{cov}(x, x)$. Finally, we define $x \cdot_\rho y = \sum_{s=1}^{S} \rho_s x_s y_s$ for vectors $x, y \in \mathbb{R}^S$.

For any competitive equilibrium (θ^*, q^*), there exists a unique state price vector in the marketed subspace $\pi_A^* \in \langle A \rangle$ such that, for all assets j, $q_j^* = \pi_A^* \cdot_\rho d^j$. Using the definitions of variance and covariance, this implies

$$q_j^* = \mathrm{E}(\pi_A^*)\mathrm{E}(d^j) + \mathrm{cov}(\pi_A^*, d^j). \tag{1}$$

We define the return of a portfolio $\theta \in \mathbb{R}^J$ with $q^* \cdot \theta \neq 0$ by $r_\theta = \frac{A\theta}{q^* \cdot \theta}$ and we denote the return of the riskless bond by $R^{\mathrm{f}} = 1/q_1^*$. With this we rewrite equation (1) as

$$q_j^* = \frac{1}{R^{\mathrm{f}}}\mathrm{E}(d^j) + \mathrm{cov}(\pi_A^*, d^j).$$

We define the pricing portfolio as the unique portfolio θ_A^* which solves $A\theta_A^* = \pi_A^*$. Notice that

$$q^* \cdot \theta_A^* = \pi_A^* \cdot_\rho A\theta_A^* = \pi_A^* \cdot_\rho \pi_A^* > 0,$$

where $\pi_A^* \neq 0$ follows from $\mathrm{E}(\pi_A^*) = q_1^* > 0$.

Since the return of the pricing portfolio satisfies $r_{\theta_A^*} = \frac{A\theta_A^*}{q^* \cdot \theta_A^*} = \frac{\pi_A^*}{\pi_A^* \cdot_\rho \pi_A^*}$ we can rewrite equation (1) as

$$\mathrm{E}(r_\theta) - R^{\mathrm{f}} = \frac{\mathrm{cov}(r_\theta, r_{\theta_A^*})}{\mathrm{var}(r_{\theta_A^*})}(\mathrm{E}(r_{\theta_A^*}) - R^{\mathrm{f}}). \tag{2}$$

While equation (2) relates the prices of the risky assets and looks similar to the CAPM pricing formula, this equation is rather useless if we have no further information on π_A^*. Note that so far all formulas followed simply from the absence of arbitrage. It is well known that under the assumption that at least one agent h's utility functions is differentiable and that in an equilibrium with individual consumption $(c^{h*})_{h \in H}$, agent h's utility maximization problem has an interior solution,[4] π_A^* can be characterized as

$$\pi_A^* \quad = \quad \mathrm{proj}_{\langle A \rangle} \left(\frac{\partial_{c_1^h} u^h(c^{h*})/\rho_s}{\partial_{c_0^h} u^h(c^{h*})}, \ldots, \frac{\partial_{c_S^h} u^h(c^{h*})/\rho_s}{\partial_{c_0^h} u^h(c^{h*})} \right),$$

where $\mathrm{proj}_{\langle A \rangle}$ denotes the projection on $\langle A \rangle$ under the inner product ρ.

One possibility to derive an interesting pricing formula is to assume that all preferences just depend on the mean and the variance of consumption,

$$u^h(c^h) = w^h(c_0^h, \mathrm{E}(\tilde{c}^h), \mathrm{var}(\tilde{c}^h)),$$

where w^h is strictly increasing in c_0^h and in the expected consumption and strictly decreasing in the variance of consumption.

Agent h's first period endowments can be decomposed into a marketed part and a non-marketed part, where the latter part lies orthogonal to the marketed subspace under the inner product ρ. We write

$$\tilde{e}^h = \tilde{e}_{\mathrm{M}}^h + \tilde{e}_{\perp}^h$$

and have by definition $\tilde{e}_{\perp}^h \cdot_\rho z = 0$ for all $z \in \langle A \rangle$. This decomposition is uniquely determined. We define the marketed endowments $\tilde{e}_{\mathrm{M}} = \sum_{h=1}^H \tilde{e}_{\mathrm{M}}^h$ and the market portfolio θ_{M} as the unique portfolio satisfying

$$A\theta_{\mathrm{M}} = \tilde{e}_{\mathrm{M}}.$$

Note that it may happen that $q^* \cdot \theta_{\mathrm{M}} = 0$, even when $\tilde{e} \gg 0$.[5]

To simplify matters, we first assume $q^* \cdot \theta_{\mathrm{M}} \neq 0$ and then argue that this assumption is not necessary. Given a competitive equilibrium (θ^*, q^*), we define β_θ for a portfolio $\theta \in \mathrm{I\!R}^J$ by

$$\beta_\theta = \frac{\mathrm{cov}(r_\theta, r_{\theta_{\mathrm{M}}})}{\mathrm{var}(r_{\theta_{\mathrm{M}}})}.$$

[4]The assumption of smooth preferences we make in the appendix ensures that we always have interior solutions.

[5]For a vector $x \in \mathrm{I\!R}^m$ we use the notation $x \geq 0$ if $x \in \mathrm{I\!R}_+^m$, $x > 0$ if $x \in \mathrm{I\!R}_+^m \setminus \{0\}$, and $x \gg 0$ if $x \in \mathrm{I\!R}_{++}^m$.

Then we have the following result.

THEOREM 2.3: *Under the assumptions that all agents maximize mean-variance utility functions with objective probabilities ρ, $\text{var}(\tilde{e}_{\text{M}}) > 0$, and there is a riskless bond, each equilibrium (θ^*, q^*) of $\mathcal{E}$ with equilibrium consumption $(c^{1*}, \ldots, c^{H*})$ has the following properties.*

1. *The CAPM-pricing formula holds; when $q^* \cdot \theta_{\text{M}} \neq 0$, then for each $\theta \in \mathbb{R}^J$,*
$$E(r_\theta) - R^{\text{f}} = \beta_\theta(E(r_{\theta_{\text{M}}}) - R^{\text{f}}). \tag{3}$$
2. *Two-fund separation holds; for each agent h there exists $(\alpha_1^h, \alpha_2^h) \in \mathbb{R} \times \mathbb{R}_+$, where $\sum_{h=1}^H \alpha_1^h = 0$ and $\sum_{h=1}^H \alpha_2^h = 1$, such that*
$$\tilde{c}^{h*} - \tilde{e}_\perp^h = \alpha_1^h 1_S + \alpha_2^h A\theta_{\text{M}}.$$
3. *The pricing vector satisfies $\pi_A^* = \alpha_1 1_S - \alpha_2 \tilde{e}_{\text{M}}$, with $\alpha_1 > \alpha_2 \text{E}(\tilde{e})$ and α_2 strictly positive.*

PROOF. We first show that a pseudo two-fund separation holds in the sense that the agents' consumption bundles can be written as $\tilde{c}^{h*} = \tilde{e}_\perp^h + \tilde{\alpha}_1^h 1_S + \tilde{\alpha}_2^h \pi_A^*$ for some $\tilde{\alpha}_1^h, \tilde{\alpha}_2^h \in \mathbb{R}$. Define
$$\tilde{c}^h = \tilde{e}_\perp^h + \text{proj}_{\langle 1_S, \pi_A^* \rangle}(\tilde{c}^{h*}).$$
Suppose pseudo two-fund separation does not hold, so $\tilde{c}^h \neq \tilde{c}^{h*}$. Since $\pi_A^* \cdot_\rho (\tilde{c}^{h*} - \tilde{c}^h) = 0$, it follows that the portfolios needed to consume $\tilde{c}^{h*}$ and $\tilde{c}^h$ are as expensive at date 0. Moreover, $\tilde{e}_\perp^h \cdot_\rho (\tilde{c}^{h*} - \tilde{c}^h) = 0$ and $1_S \cdot_\rho (\tilde{c}^{h*} - \tilde{c}^h) = 0$, so it follows that $\text{E}(\tilde{c}^{h*} - \tilde{c}^h) = 0$ and $\text{cov}(\tilde{c}^{h*} - \tilde{c}^h, \tilde{c}^h) = 0$. Therefore, $\text{E}(\tilde{c}^{h*}) = \text{E}(\tilde{c}^h)$ and $\text{var}(\tilde{c}^{h*}) > \text{var}(\tilde{c}^h)$, giving a contradiction to the optimality of $\tilde{c}^{h*}$ at prices q^*. We obtain pseudo two-fund separation.

Since in equilibrium $\tilde{e}_{\text{M}} = \sum_{h=1}^H (\tilde{c}^{h*} - \tilde{e}_\perp^h)$, the two-fund separation property implies that $\tilde{e}_{\text{M}} \in \langle 1_S, \pi_A^* \rangle$. The assumption $\text{var}(\tilde{e}_{\text{M}}) > 0$ implies that $\tilde{e}_{\text{M}}$ is not collinear to 1_S and it holds that $\pi_A^* = \alpha_1 1_S - \alpha_2 \tilde{e}_{\text{M}}$ for some numbers α_1, α_2. Two-fund separation follows immediately, $\tilde{c}^{h*} - \tilde{e}_\perp^h = \alpha_1^h 1_S + \alpha_2^h A\theta_{\text{M}}$ for some numbers α_1^h, α_2^h.

Since $\tilde{e}_{\text{M}} = \sum_{h=1}^H \alpha_1^h 1_S + \sum_{h=1}^H \alpha_2^h \tilde{e}_{\text{M}}$ and $\text{var}(\tilde{e}_{\text{M}}) > 0$, we have $\sum_{h=1}^H \alpha_1^h = 0$ and $\sum_{h=1}^H \alpha_2^h = 1$. Consider a consumption bundle c_-^h that results from using the income that is invested in the market portfolio to buy the riskless bond, so $\tilde{c}_-^h = \tilde{e}_\perp^h + \alpha_1^h 1_S + \alpha_2^h (\pi_A^* \cdot_\rho A\theta_{\text{M}}/q_1^*) 1_S$. The portfolios needed to consume $\tilde{c}^{h*}$ and $\tilde{c}_-^h$ are as expensive since $\pi_A^* \cdot_\rho (\tilde{c}^{h*} - \tilde{c}_-^h) = 0$. Since $\text{var}(\tilde{c}_-^h) \leq \text{var}(\tilde{c}^{h*})$ and $u^h(c_-^h) \leq u^h(c^{h*})$, it holds that $\text{E}(\tilde{c}_-^h) - \text{E}(\tilde{c}^{h*}) = -(\alpha_2^h \alpha_2 / q_1^*) \text{var}(\tilde{e}_{\text{M}}) \leq 0$, where we use that $\pi_A^* \cdot_\rho A\theta_{\text{M}} = q_1^* \text{E}(\tilde{e}_{\text{M}}) - \alpha_2 \text{var}(\tilde{e}_{\text{M}})$. The preceding inequalities are strict inequalities when $\alpha_2^h > 0$, which is the case for at least one agent. Then it follows that $\alpha_2 > 0$ and $\alpha_2^h \geq 0$, $h = 1, \ldots, H$. Since $0 < q_1^* = \text{E}(\pi_A^*) = \alpha_1 - \alpha_2 \text{E}(\tilde{e}_{\text{M}})$, and $1_S \cdot_\rho \tilde{e}_\perp = 0$, so $\text{E}(\tilde{e}_{\text{M}}) = \text{E}(\tilde{e})$, it holds that $\alpha_1 > \alpha_2 \text{E}(\tilde{e})$.

The CAPM pricing formula is obtained by substituting $\theta^*_A = \alpha_1 \iota_1 - \alpha_2 \theta_{\mathrm{M}}$ in equation (2). From

$$\frac{\mathrm{cov}(r_\theta, r_{\theta^*_A})}{\mathrm{var}(r_{\theta^*_A})} = -\frac{q^* \cdot \theta^*_A}{\alpha_2 q^* \cdot \theta_{\mathrm{M}}} \frac{\mathrm{cov}(r_\theta, r_{\theta_{\mathrm{M}}})}{\mathrm{var}(r_{\theta_{\mathrm{M}}})}$$

and

$$\mathrm{E}(r_{\theta^*_A}) - R^{\mathrm{f}} = \frac{\alpha_1 - \alpha_2 \mathrm{E}(A\theta_{\mathrm{M}})}{q^* \cdot \theta^*_A} - R^{\mathrm{f}},$$

it follows that

$$\mathrm{E}(r_\theta) - R^{\mathrm{f}} = \beta_\theta \left(\frac{\alpha_1 - \alpha_2 \mathrm{E}(A\theta_{\mathrm{M}})}{-\alpha_2 q^* \cdot \theta_{\mathrm{M}}} + \frac{R^{\mathrm{f}}(\alpha_1 q^*_1 - \alpha_2 q^* \cdot \theta_{\mathrm{M}})}{\alpha_2 q^* \cdot \theta_{\mathrm{M}}} \right) = \beta_\theta (E(r_{\theta_{\mathrm{M}}}) - R^{\mathrm{f}}).$$

Q.E.D.

We assume in the theorem that $\mathrm{var}(\tilde{e}_{\mathrm{M}}) > 0$. The theorem also holds true for the degenerate case where $\tilde{e}_{\mathrm{M}}$ is collinear to 1_S, but since the proof of this simple fact is rather tedious it is omitted.

Note that for the case where the endowments are spanned, i.e. where $e^h_\perp = 0$ for all h, the pricing formula reduces to the standard CAPM-formula (see Magill and Quinzii (1996)).

It might be sensible to define the market portfolio somewhat differently as a portfolio of risky assets only. This clarifies the concept of two-fund separation, since then one fund consists of risky assets only. In this case define $\widehat{\theta}_{\mathrm{M}} = (0, \theta_{\mathrm{M},2}, \ldots, \theta_{\mathrm{M},J})$. If we define $\widehat{\beta}_\theta = \mathrm{cov}(r_\theta, r_{\widehat{\theta}_{\mathrm{M}}})/\mathrm{var}(r_{\widehat{\theta}_{\mathrm{M}}})$ it turns out that the pricing formula still holds. After some substitutions, one obtains

$$\mathrm{E}(r_\theta) - R^{\mathrm{f}} = \widehat{\beta}_\theta(\mathrm{E}(r_{\widehat{\theta}_{\mathrm{M}}}) - R^{\mathrm{f}}).$$

Even more generally, define the market portfolio $\widetilde{\theta}_{\mathrm{M}}$ as an arbitrary combination of a portfolio consisting of the riskless asset only and the portfolio $\widehat{\theta}_{\mathrm{M}}$, so

$$\widetilde{\theta}_{\mathrm{M}} = \gamma_1 \iota_1 + \gamma_2 \widehat{\theta}_{\mathrm{M}},$$

where $\gamma_2 \neq 0$. Then it holds that $\theta^*_A = \tilde{\alpha}_1 \iota_1 - \tilde{\alpha}_2 \widetilde{\theta}_{\mathrm{M}}$, where $\tilde{\alpha}_2 \neq 0$. If we define $\widetilde{\beta}_\theta = \mathrm{cov}(r_\theta, r_{\widetilde{\theta}_{\mathrm{M}}})/\mathrm{var}(r_{\widetilde{\theta}_{\mathrm{M}}})$, then

$$\mathrm{E}(r_\theta) - R^{\mathrm{f}} = \widetilde{\beta}_\theta(\mathrm{E}(r_{\widetilde{\theta}_{\mathrm{M}}}) - R^{\mathrm{f}}).$$

The proof is identical to the one of Theorem 2.3, when α_1, α_2, and θ_{M} are substituted by $\tilde{\alpha}_1$, $\tilde{\alpha}_2$, and $\widetilde{\theta}_{\mathrm{M}}$. This result also offers a way out when $q^* \cdot \theta_{\mathrm{M}} = 0$. One may simply use $\widetilde{\theta}_{\mathrm{M}} = \theta_{\mathrm{M}} + \varepsilon \iota_1$ with $\varepsilon > 0$ to derive the pricing formula. Indeed, $q^* \cdot \widetilde{\theta}_{\mathrm{M}} = \varepsilon q^*_1 > 0$.

The version of two-fund separation we consider in Theorem 2.3 is slightly more general than the usual one, where it is assumed that the initial income stream e^h of every agent is marketed. As a consequence one obtains the formula

$$\tilde{c}^{h*} = \alpha_1^h 1_S + \alpha_2^h \tilde{e}$$

when endowments are marketed. In the more general case considered in Theorem 2.3, the final income stream consumed by each agent consists not only of the returns of a linear combination of the riskless bond and the market portfolio, but also of the undiversifiable non-marketed individual part of the initial income stream, $\tilde{e}^h_\perp$.

Finally, note that the concept of marketed endowments is not needed to define the pricing vector. Since $\tilde{e}_\perp$ is orthogonal to $\langle A\rangle$, the pricing vector can also be defined by $\tilde{\pi}^*_A = \alpha_1 1_S - \alpha_2 \tilde{e}$. Of course it no longer holds that $\tilde{\pi}^*_A \in \langle A\rangle$. Moreover, income streams not in $\langle A\rangle$ are typically priced differently by $\tilde{\pi}^*_A$ than by π^*_A.

As we have discussed in the introduction, Theorem 2.3 can only be obtained when one is willing to make very restrictive assumptions. As Magill and Quinzii (1996) put it when commenting on representative agent models and the CAPM: "As we indicated above these models are interesting since they lead to clearcut results which have strong intuitive appeal. However the restrictive nature of the hypothesis made could cast doubt on the generality of the results." The important question we want to address is how much actual equilibrium prices and actual portfolio-holdings in a general setting will differ from the predictions of the CAPM.

3 The CAPM Without Mean-Variance Preferences

The assumption that all agents maximize a quadratic utility function is unattractive because it implies increasing absolute risk aversion. A more realistic assumption, and one commonly made in macroeconomics and finance, is that agents' preferences exhibit constant relative risk aversion. It is clear, however, that with these preferences agents' will care about higher moments and that therefore a mean-variance analysis is not valid. The following example shows that a mean-variance utility function does not even serve as a good approximation of a constant relative risk aversion utility function.

Example 3.1: Consider an agent with utility function $u^h(c^h) = \sum_{s=1}^3 \rho_s v^h(c^h_s)$, where $\rho_s = 1/3$, $s = 1, 2, 3$, and $v^h(c^h_s) = -1/3(c^h_s)^{-3}$, which corresponds to a utility function with constant relative risk aversion equal to 4. For simplicity we assume that the household has no income at $t = 0$ and does not derive utility from consumption in that period. Consider the consumption of two income streams, $(0.8, 0.8, 1.4)$ and $(0.6, 1.2, 1.2)$, that have the same mean and variance. Any mean-variance utility function should therefore consider both income streams as being equally good. When an agent has a constant relative risk aversion utility function, the second income stream is less preferred, as the income in the first state is 40% lower than average income, whereas the income at the bad states of the first income stream are only 20% below average income, $u^h(0.8, 0.8, 1.4) = -0.475$ and $u^h(0.6, 1.2, 1.2) = -0.643$. Even if for the second income profile, income is increased by 10% in every state, we get $u^h(0.66, 1.32, 1.32) = -0.483$, so it would still be inferior to the income stream $(0.8, 0.8, 1.4)$. This phenomenon becomes even more severe when two income streams with the same, higher variance are compared or when a more risk averse agent is considered.

A standard way to calibrate equilibrium models under uncertainty is to assume that there are several uncorrelated shocks and to choose the magnitude of the shocks to match aggregate first and second moments. From now on we examine an economy with three heterogeneous agents, representing classes of agents with low, medium and high incomes.

Each agent is endowed with an initial portfolio $(0, \theta^h_-)$ of the riskless bond and the available stocks,[6] with current income, representing current labor income plus dividends from θ^h_-, $e^1_0 = 2/3$, $e^2_0 = 1$, and $e^3_0 = 4/3$, and with stochastic future labor income given by some $l^h \in \mathbb{R}^S_{++}$. We are back in the framework of Section 2 by setting $e^1_0 = 2/3$, $e^2_0 = 1$ and $e^3_0 = 4/3$, and $\tilde{e}^h = l^h + \sum_{j=2}^J \theta^h_{-j} d^j$ for $h = 1, ..., H$. For each household h, the labor incomes l^h_s are generated by S independent draws from some given distribution. In this way we can obtain a discrete approximation of any continuous distribution.

The first agent has no capital income, $\theta^1_- = 0$. For the other agents we have $\theta^2_- = 1/3 \cdot 1_{J-1}$ and $\theta^3_- = 2/3 \cdot 1_{J-1}$. In most applications agents have heterogeneous von Neumann-Morgenstern utility functions with identical uniform probabilities over states and identical discount factors $\delta^h = 0.95$.

The assets available are given by a riskless bond and seven stocks. In most examples the dividends of asset j depend on a single common factor $f \in \mathbb{R}^S$ as well as on an idiosyncratic shock $\varepsilon^j \in \mathbb{R}^S$. We denote asset j's load in the factor by c_j, varying from 0.25 to 1.75 in steps of 0.25. The examples are calibrated to yearly US data. The expected growth rate of aggregate consumption equals two percent and the standard deviation of both the factor and the idiosyncratic shock determining the dividends are about 0.13 - giving an overall standard deviation of the stock market of about 0.17. The standard deviation of labor income is chosen to be around 0.10 and labor income constitutes around 2/3 of total income. The eleven random variables in the model are therefore $((l^h)_{h=1,...,H}, f, (\varepsilon^j)_{j=2,...,J})$.

As a first example we analyze the case where the realization of each random variable is either high or low with equal probabilities, and all random variables are independent. The minimal state space to achieve this consists of $2^{11} = 2{,}048$ states. More specifically we have that

$$\begin{aligned} l^h_s &\in \{2/3 \cdot (1.02 - 0.1), 2/3 \cdot (1.02 + 0.1)\}, \\ f_s &\in \{-0.13, 0.13\}, \\ \varepsilon^j_s &\in \{-0.13, 0.13\}. \end{aligned}$$

[6] Note that contrary to the model described in Section 2, we assume now that stocks are in unit net supply.

Dividends of asset j are then determined by

$$d_s^j \;=\; 1/7 \cdot (1.02 + \sqrt{c_j} f_s + \varepsilon_s^j).$$

We assume that all agents have constant relative risk aversion utility functions of the form

$$v^h(c_s^h) = \frac{(c_s^h)^{1-\gamma^h}}{1-\gamma^h}, \quad c_s^h > 0,$$

where γ^h is the coefficient of relative risk aversion. We choose $\gamma^1 = 6$, $\gamma^2 = 4$ and $\gamma^3 = 2$.

This specification completes the description of the economy $\mathcal{E}$. We compute the equilibrium prices and portfolio-holdings and compare them to the predictions of the CAPM in Figure 1. To do those computations, we could in principle use the homotopy algorithms as reported in Brown, DeMarzo and Eaves (1996) or Schmedders (1998), which can solve for an equilibrium in the general multiple commodities GEI-model. The problem is that for both algorithms the number of equations to be solved is a multiple of the number of states, whereas the number of states is 1,024 for the current economy and 32,768 for the other economies considered in this paper. This makes both algorithms unsuitable for our purposes. In the appendix we develop an algorithm that is tailored to the finance GEI-model with one good per state, and that is independent of the number of states. Instead, the number of equations to be solved is related to the number of assets, which is 8 for most economies analyzed in this paper. The specifics of the algorithm, as well as a proof of its global convergence, are treated in detail in the appendix.

The solid line in Figure 1 is the security market line, i.e. the CAPM relationship between a portfolio's β and its risk premium. The actual equilibrium expected returns of the seven securities are depicted by + and lie all almost exactly on the security market line. The CAPM turns out to be an extraordinarily good predictor for the actual equilibrium returns of assets in this example. This is surprising as preferences are far from mean-variance, and asset returns are far from being normally distributed.

Although the graph of Figure 1 looks very convincing, it is clear that we need more objective measures to quantify the deviation of equilibrium prices and portfolio-holdings from the CAPM predictions. Note that we need to check both the robustness of two-fund separation and the robustness of the pricing-formula. With general preferences CAPM-pricing is neither necessary nor sufficient for two-fund separation. It is easy to see that two-fund separation does not imply CAPM-pricing. Consider a model with complete markets where all agents have identical constant absolute risk aversion preferences. It is well known that two-fund separation holds since there exists a linear sharing rule, see also Cass and Stiglitz (1970). However, in general it holds that

$$\begin{aligned}
\pi_A^* \;&=\; \mathrm{proj}_{\langle A\rangle}\left(\frac{\delta^h \partial_{c_1^h} v_1^h(e_1)}{\partial_{c_0^h} v_0^h(e_0)}, \ldots, \frac{\delta^h \partial_{c_S^h} v_S^h(e_S)}{\partial_{c_0^h} v_0^h(e_0)}\right) \\
&=\; \left(\frac{\delta^h \partial_{c_1^h} v_1^h(e_1)}{\partial_{c_0^h} v_0^h(e_0)}, \ldots, \frac{\delta^h \partial_{c_S^h} v_S^h(e_S)}{\partial_{c_0^h} v_0^h(e_0)}\right) \notin \langle 1_S, \tilde{e}\rangle.
\end{aligned}$$

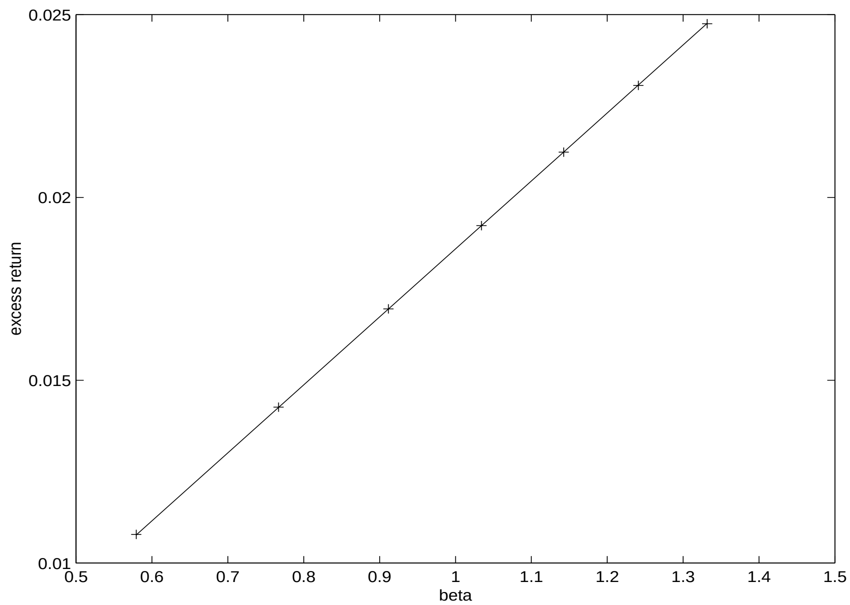

Figure 1: Security market line with high-low returns.

Conversely, suppose prices of assets consistent with CAPM-pricing are given, and let asset markets be complete for simplicity. It is easy to choose individual consumption bundles which do not belong to $\langle 1_S, \widetilde{e}\rangle$ and utility functions for which the individual consumption bundles are optimal at the asset prices chosen.

The most straightforward approach to measure the accuracy of CAPM-pricing is to take the Mean Squared Error (MSE), which is defined by

$$\mathrm{MSE} = \sqrt{\frac{1}{J-1}\sum_{j=2}^{J}(r_j^* - \widehat{r}_j)^2},$$

where r_j^* denotes the equilibrium expected return of asset j and $\widehat{r}_j$ the prediction by the CAPM.

A different approach consists of the following. By the arguments used in the proof of Theorem 2.3 it is obvious that $\pi_A^* \in \langle 1_S, \widetilde{e}_{\mathrm{M}}\rangle$ is sufficient for CAPM-pricing. That this is necessary as well follows from the observation that otherwise π_A^* is equal to the sum of its projection on $\langle 1_S, \widetilde{e}_{\mathrm{M}}\rangle$ plus a non-zero orthogonal part in $\langle A\rangle$ under the inner product ρ. When CAPM-pricing is valid, the orthogonal part should have zero price, which is obviously not the case when priced by π_A^*. Therefore, an interesting alternative to MSE is to take the OLS R^2 of the regression with

$$\mathrm{proj}_{\langle A\rangle}\left(\frac{\partial_{c_1^h} u^h(c^{h*})/\rho_1}{\partial_{c_0^h} u^h(c^{h*})}, \ldots, \frac{\partial_{c_S^h} u^h(c^{h*})/\rho_s}{\partial_{c_0^h} u^h(c^{h*})}\right)$$

as regressand and 1_S and $\widetilde{e}_{\mathrm{M}}$ as regressors. Notice that this measure is independent of h. We call it Pricing R^2.

To measure how well two-fund separation holds for agent h, we take the OLS R^2 of the regression with $(\theta_j^{h*})_{j=2,\ldots,J}$ as regressand and $\widehat{\theta}_{\mathrm{M}}$, the risky part of the market portfolio, as regressor.

The following table confirms that the CAPM provides an outstanding prediction for the economy under consideration.

R^{f}	1.0633
Equity Premium	0.0185
MSE	0.0000530
Pricing R^2	0.99999998
Two-fund R^2 $h=1$	0.9999988
Two-fund R^2 $h=2$	0.9999994
Two-fund R^2 $h=3$	0.9999998

Table 1: The CAPM for CRRA preferences and two-point distributions.

Although the high-low specifications for the random variables are two-point approximations to normal random variables, the well-known fact that the CAPM holds with normally distributed returns does not imply anything about the validity of the CAPM in this framework. It is easy to see that two-point approximations to normal random variables do not satisfy the properties of elliptical distributions. The following trivial example shows that while each dividends distribution is characterized by its mean and variance it is not true that a linear combination of these random variables is also fully characterized by its mean and variance.

EXAMPLE 3.2: Consider a model with 4 states where all probabilities are equal. Let

$$d^1 = \begin{pmatrix} -0.5 \\ -0.5 \\ 0.5 \\ 0.5 \end{pmatrix}, \qquad d^2 = \begin{pmatrix} 0.5 \\ -0.5 \\ 0.5 \\ -0.5 \end{pmatrix}.$$

Both d^1 and d^2 are discrete distributions such that with probability 1/2 the realization is -0.5 and with probability 1/2 the realization is 0.5. However, both $d^1 + d^2$ and $\sqrt{2}d^1$ have expectation zero and variance 1/2, but correspond to different distributions and different utilities when the utility function is not mean-variance.

One should not expect the CAPM to hold in this model even though the distributions provide a (very crude) approximation to normal distributions.

Also the fact that two-fund separation holds so well, comes as a surprise. Since the households we are dealing with have different parameters of relative risk aversion, there is no reason to expect that two fund separation obtains, see Cass and Stiglitz (1970) and Detemple and Gottardi (1998).

4 Robustness in Preferences

In order to show that the predictions of the CAPM are a good approximation for equilibria in a wide variety of economic settings we compute 600 examples. To avoid the suspicion that high-low shocks are close substitutes to normal shocks, which might explain the results, we use log-normally distributed shocks throughout this section. We assume that there are $S = 32,768$ states of nature. Using a large number of states guarantees that our final samples are good approximations of continuous distributions. By taking a large number of states we rule out finite sample effects on the prices of assets. When we replicate the experiment and generate economies out of a newly drawn sample, the equilibrium will be almost the same if the number of states is sufficiently large.

The assumption that all random variables are log-normally distributed means that l_s^h, f_s, and ε_s^j are drawn independently from a log-normal distribution. The log-normal distribution with mean μ and variance σ^2 is denoted by $\mathrm{LN}(\mu, \sigma^2)$. Since we are considering finite samples, the drawing will be of (some) influence on the equilibrium we compute. As before asset 1 is the riskless bond. For $j \geq 2$, we define asset j's dividend to be

$$d_s^j = 1/7 \cdot 1.02 \cdot f_s^j \cdot \varepsilon_s^j$$

and we choose

$$\begin{aligned} l_s^h &\sim \text{LN}(2/3 \cdot 1.02, (2/3)^2 \cdot 0.01), \\ f_s^j &\sim \text{LN}(1, c_j \cdot 0.0161), \\ \varepsilon_s^j &\sim \text{LN}(1, 0.0161). \end{aligned}$$

The actual $(f_s^j)_{j=2}^J$ are all based on a single realization of a normal random variable $\widehat{f}_s$. For each asset j, we linearly transform the realization of this random variable in such a way that after taking the exponent a log-normally distributed random variable with mean 1 and variance $c_j \cdot 0.0161$ results. The construction of the random variables implies that all dividends themselves are log-normally distributed. To get a similar variance of the entire stock market as before the variance of the factors and the idiosyncratic shock have to be chosen to be 0.0161 instead of 0.0169. Notice that the factor realization does not enter linearly in the formula for the asset's dividends, an assumption that is made in most models describing factor economies. This is an additional advantage as it puts the CAPM only more seriously to the robustness test. Finally, it follows from the work of Feldstein (1969) that log-normal distributions do not belong to the elliptic class, and would not admit of two-fund separation.

We consider three different families of utility functions and compute one hundred randomly generated examples within each class. For each class we report histograms of the MSE, the Pricing R^2, and the Two-fund R^2 of agents 2 and 3. By market clearing, the portfolio-holdings of agent 1 are fully dependent on those of agents 2 and 3. If two-fund separation holds exactly for agents 2 and 3 it will hold exactly for agent 1 as well. Therefore, we safe space and do not report the Two-fund R^2 of agent 1. In all histograms the scaling is taken identically, so that results for different models can be compared easily.

We first assume that all agents' utility functions exhibit constant absolute risk aversion, i.e.

$$v^h(c_s^h) = -\exp(\alpha^h c_s^h), \quad c_s^h \in \mathbb{R},$$

where α^h is the coefficient of absolute risk aversion. We then move on and examine an economy where all agents' utility functions exhibit constant relative risk aversion, i.e.

$$\begin{aligned} v^h(c_s^h) &= \frac{(c_s^h)^{1-\gamma^h}}{1-\gamma^h}, \quad c_s^h > 0, \quad \gamma^h \neq 1, \\ v^h(c_s^h) &= \log(c_s^h), \quad c_s^h > 0, \quad \gamma^h = 1, \end{aligned}$$

where γ^h is the coefficient of relative risk aversion.

The rationale for examining both constant absolute risk aversion and constant relative risk aversion is as follows. Kenneth Arrow has repeatedly argued that it is realistic to assume increasing relative risk aversion and non-increasing absolute risk aversion. By covering the two extreme cases of constant absolute and constant relative risk aversion we want to argue that the CAPM provides a good approximation for pricing for all specifications which satisfy Arrow's criteria.

We end this section by assuming that agents have utility functions with loss aversion, for the specifics see Subsection 4.3. The analysis of that case shows that the assumption of expected utility maximization is not essential for the CAPM to be a valuable tool.

4.1 Random CARA

We randomly generate 100 examples of economies where all agents have constant absolute risk aversion. For each example we draw the coefficient of risk aversion α^h, $h = 1, 2, 3$, from a uniform distribution on the interval $[0.5, 10]$. Comparisons between the computed equilibria and the CAPM predictions are depicted in the histograms of Figures 2a-d.[7]

Obviously the CAPM predicts extremely well. The mean squared error always lies below 0.04 percent. In most cases it is around $0.5 \cdot 10^{-4}$. The Pricing R^2 exceeds 0.9999 in all examples. The Two-fund R^2 exceeds 0.99 in most cases. Compared to the single example examined in Section 3, the results are slightly worse on average. Figure 3 clarifies that this can be entirely explained by higher values for the average rate of risk aversion present in the economy. The MSE increases with average risk aversion in the economy, as measured by the harmonic mean of the α^h's (it is well known that the harmonic mean is the right measure for average risk aversion in an economy where all agents have constant absolute risk aversion).

Although the CAPM remains an excellent predictor for all cases examined so far, Figure 3 indicates that the CAPM is a better tool in environments with lower average risk aversion. In the light of this result one might be tempted to draw a parallel between our results and the observation of Mehra and Prescott (1985) that realistic values of risk-aversion do not produce a realistic equilibrium risk-premium. If the equilibrium returns of risky assets do not change significantly with small variations of agents' coefficient of relative risk aversion it can be expected that the cross-section remains almost unchanged and that the CAPM (which predicts excess returns independently of preferences) provides a good prediction for a variety of attitudes towards risk. Note, however, that this can only explain one side of the phenomena - the question remains why the cross-section of returns can be described by the assets' β's.

4.2 Random CRRA

We now assume that all agents have constant relative risk aversion and we draw γ^h, $h = 1, 2, 3$, from a uniform distribution on the interval $[0.5, 10]$. With mean household income equal to 1, the degree of risk-aversion in the economy is similar to the CRRA-case examined

[7] The Pricing R^2 is multiplied by 100 to avoid round-off to 1.000 by our software.

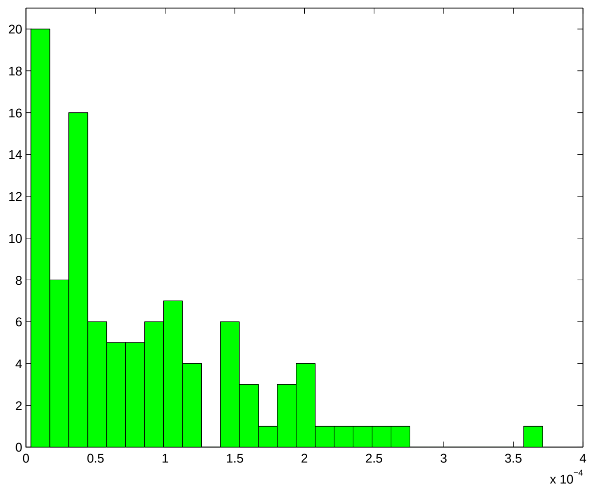

FIGURE 2A: CARA: MSE.

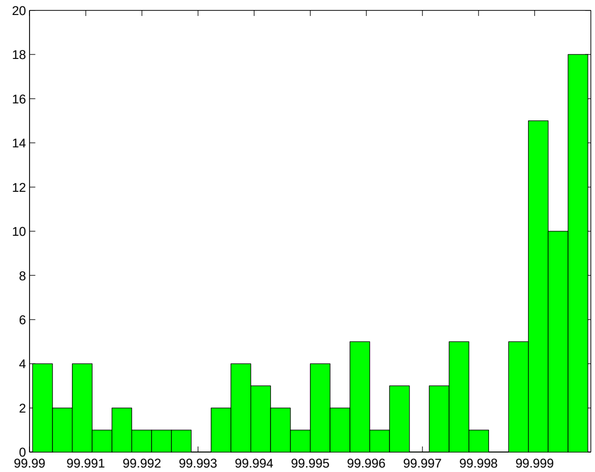

FIGURE 2B: CARA: 100· Pricing R^2.

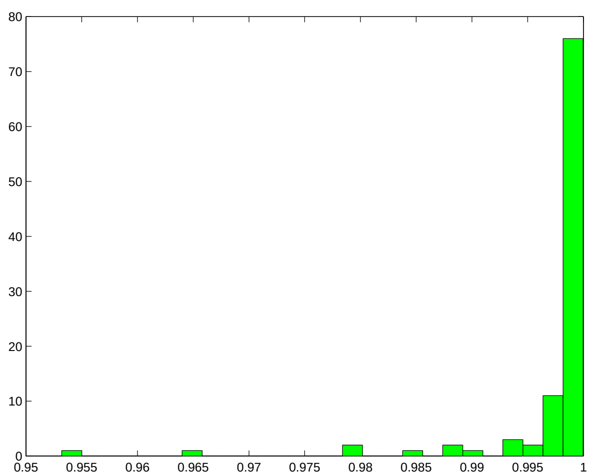

FIGURE 2C: CARA: Two-fund separation agent 2.

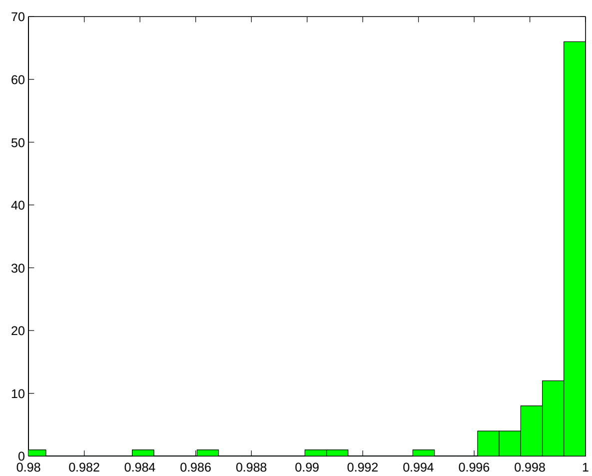

FIGURE 2D: CARA: Two-fund separation agent 3.

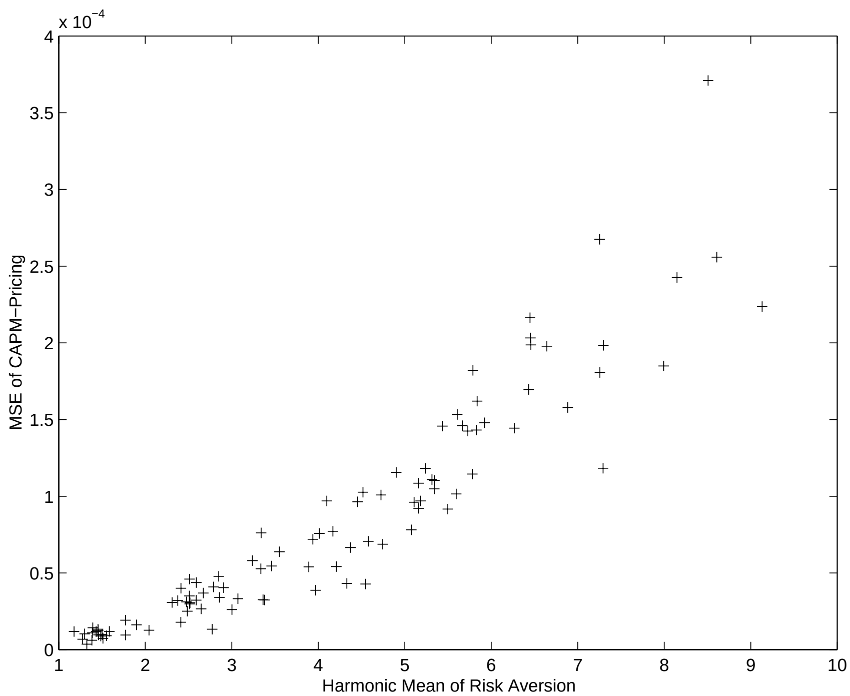

FIGURE 3: MSE against Risk-Aversion for CARA preferences.

in Subsection 4.1. As before we compute 100 examples - Figures 4a-d report the analogues of Figures 2a-d for the CRRA case.

Figure 4 shows that the CAPM is an excellent predictor for the class of CRRA utility functions, both in terms of pricing and in terms of two-fund separation. In most cases MSE is around $1 \cdot 10^{-4}$. The worst Pricing R^2 found is 0.99995 and the worst Two-fund R^2 is 0.95.

The high values of the Pricing R^2 provides very useful information for the pricing of assets. Recall that the price of asset j is given by $\pi_A^* \cdot_\rho d^j = \pi_A^* \cdot d^j/S$. Any vector that is highly correlated with π_A^* should lead to a similar price for asset j. In particular, when the Pricing R^2 is close to one, the CAPM is bound to give almost exact equilibrium prices and the use of the CAPM leads to a low MSE.

4.3 Loss Aversion

To demonstrate that our results do not depend on state independent utility, we analyze a class of utility functions that are state dependent and that are characterized by loss aversion. Such utility functions get support from empirical work on the decision making of agents. They are also claimed to be helpful in explaining the equity premium puzzle of Mehra and Prescott (1985), see Benartzi and Thaler (1995).

We cannot use exactly the same utility functions as Benartzi and Thaler, as these are not everywhere quasi-concave, and as a consequence a competitive equilibrium may not exist. The important characteristic of loss-aversion is not so much the existence of non-concavities, but a sharp decrease in utility when loosing income compared to the status quo and only a mild increase in utility when gaining income. This is usually modeled by a utility function that has a kink at the status quo.

We generate a utility function with loss aversion as follows. We identify the status quo of an agent h in state $s \geq 1$ with e_s^h. Then loss aversion applies to making good or bad investment decisions on the stock market. Consistent with Benartzi and Thaler (1995), we want a Bernoulli function v_s^h such that $\lim_{c_s^h \uparrow e_s^h} \partial v_s^h(c_s^h) = 2 \lim_{c_s^h \downarrow e_s^h} \partial v_s^h(c_s^h)$. For each h, we choose parameters γ_1^h and γ_2^h. When $c_s^h \geq e_s^h$, then v_s^h coincides with a CRRA utility function with parameter of relative risk aversion γ_1^h. When $c_s^h \leq e_s^h$, then v_s^h coincides with a CRRA utility function with parameter of relative risk aversion γ_2^h, plus a term linear in c_s^h to get $\lim_{c_s^h \uparrow e_s^h} \partial v_s^h(c_s^h) = 2 \lim_{c_s^h \downarrow e_s^h} \partial v_s^h(c_s^h)$, plus a constant k_s to make v_s^h continuous. More precisely, we assume that $u^h(c^h) = v_0^h(c_0^h) + \delta^h \sum_{s=1}^S \rho_s v_s^h(c_s^h)$, where

$$\begin{aligned} v_0^h(c_0^h) &= (c_0^h)^{1-\gamma_1^h}/(1-\gamma_1^h), \\ v_s^h(c_s^h) &= (c_s^h)^{1-\gamma_2^h}/(1-\gamma_2^h) + (\frac{\gamma_2}{\gamma_1}(e_s^h)^{-\gamma_1^h} - (e_s^h)^{-\gamma_2^h})c_s^h + k_s, \quad c_s^h \leq e_s^h, \\ v_s^h(c_s^h) &= (c_s^h)^{1-\gamma_1^h}/(1-\gamma_1^h), \quad c_s^h \geq e_s^h, \end{aligned}$$

with

$$k_s = \frac{\gamma_2}{1-\gamma_2}(e_s^h)^{1-\gamma_2^h} + \frac{\gamma_1^h\gamma_2^h + \gamma_1^h - \gamma_2^h}{\gamma_1^h(1-\gamma_1^h)}(e_s^h)^{1-\gamma_1^h}.$$

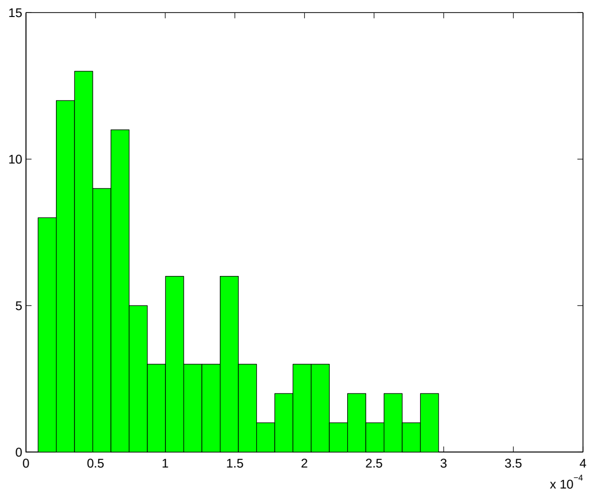

Figure 4a: CRRA: MSE.

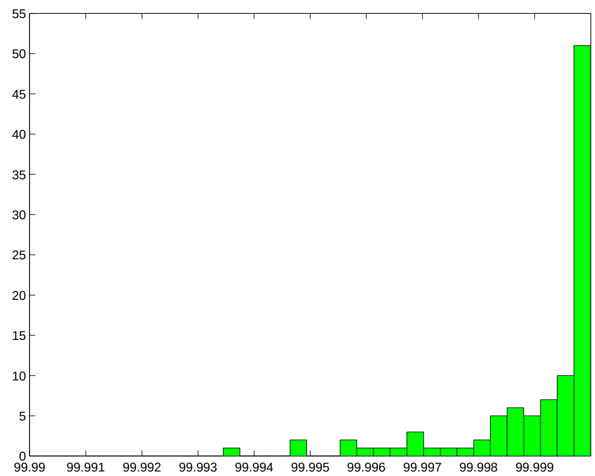

Figure 4b: CRRA: 100· Pricing R^2.

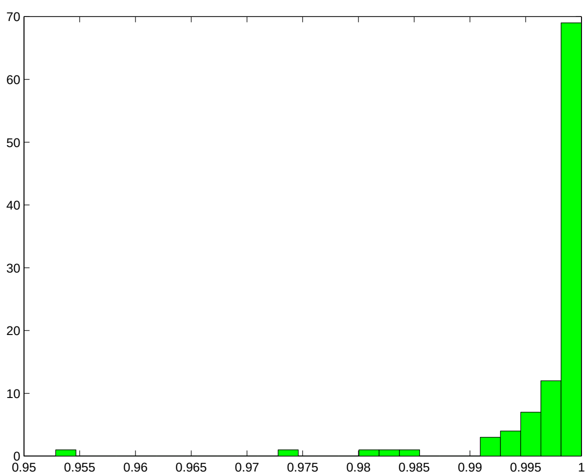

Figure 4c: CRRA: Two-fund separation agent 2.

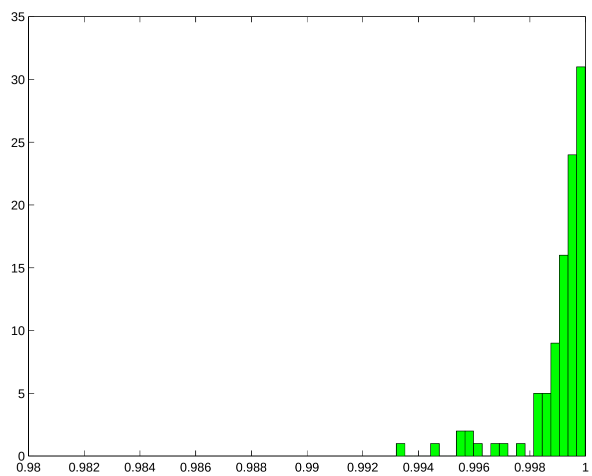

Figure 4d: CRRA: Two-fund separation agent 3.

The Bernoulli function v_s^h is continuous and is continuously differentiable except at e_s^h where it has a kink. It can be shown that the coefficient of relative risk aversion varies continuously in c_s^h and is given by $(\gamma_1^h \gamma_2^h (c_s^h)^{-\gamma_2^h})/(\gamma_2^h (e_s^h)^{-\gamma_1^h} - \gamma_1^h (e_s^h)^{-\gamma_2^h} + \gamma_1^h (c_s^h)^{-\gamma_2^h})$ if $c_s^h \leq e_s^h$, so it approaches γ_2^h as $c_s^h \to 0$. The coefficient of relative risk aversion is given by γ_1^h if $c_s^h \geq e_s^h$.

Since v_s^h is not differentiable at e_s^h it does not satisfy the assumptions under which the algorithm has been shown to be convergent. We have to smooth out the kinks of the utility function. We can do this by taking any e_s^{h-}, e_s^{h+} such that $e_s^{h-} < e_s^h < e_s^{h+}$ and defining

$$\partial v_s^h(c_s^h) = \frac{e_s^{h+} - c_s^h}{e_s^{h+} - e_s^{h-}} \partial v_s^h(e_s^{h-}) + \frac{c_s^h - e_s^{h-}}{e_s^{h+} - e_s^{h-}} \partial v_s^h(e_s^{h+}).$$

In principle, the parameter k_s has to be adjusted to make v_s^h continuous. Since our algorithm works entirely with first order conditions, this is of no concern to us. In the numerical experiments we took $e_s^{h-} = 0.95 e_s^h$ and $e_s^{h+} = 1.05 e_s^h$. For each example we take $\gamma_1^h = \gamma_2^h/2$ and we draw γ_2^h, $h = 1, 2, 3$, from a uniform distribution on the interval $[1, 6]$. In this way one hundred economies are randomly generated. The outcomes of our computations are presented in Figures 5a-d.

It turns out that the CAPM is an extraordinarily good predictor for the case with loss aversion. The results seem to be even better than for the CARA and CRRA cases examined before. In most cases, MSE is below $1 \cdot 10^{-4}$, Pricing R^2 exceeds 0.99999, and the Two-fund R^2 exceeds 0.98. If we take into account that the examples with loss aversion are such that the degree of risk aversion is lower on average than before, the Pricing R^2 is comparable to the one found for CRRA and CARA preferences.

5 Robustness in Return Processes

We now fix agents' preferences to exhibit constant relative risk aversion and choose $\gamma^1 = 6$, $\gamma^2 = 4$, and $\gamma^3 = 2$. We test the robustness of our results to variations in the distributions of endowments and assets. We consider three different families of return processes and compute 100 randomly generated examples within each class. We show the histograms of MSE, Pricing R^2, and Two-fund R^2 of agents 2 and 3.

5.1 Uniform Returns

In order to verify whether our results depend on the assumption of log-normal shocks, we now assume that all shocks are uniformly distributed. We also allow for some variation in the ratio of labor income to total income, in the variance of the factor and in the variance of the idiosyncratic shocks.

We start each example by randomly generating parameters a_1, a_2, a_3 and a_4, where

$$a_1 \sim \mathrm{U}(1.02 \cdot 0.5, 1.02 \cdot 0.9),$$

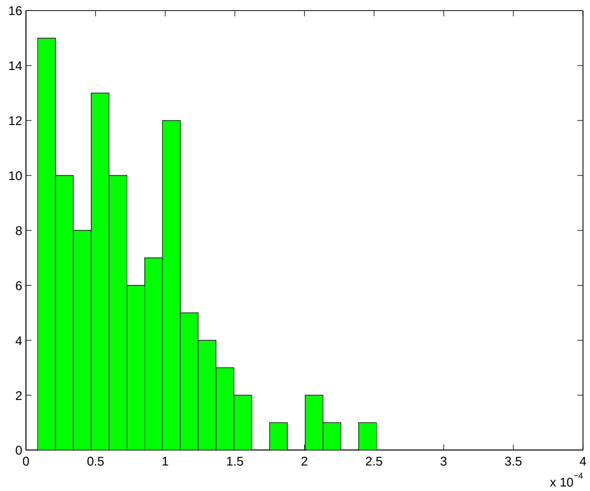

Figure 5a: LA: MSE.

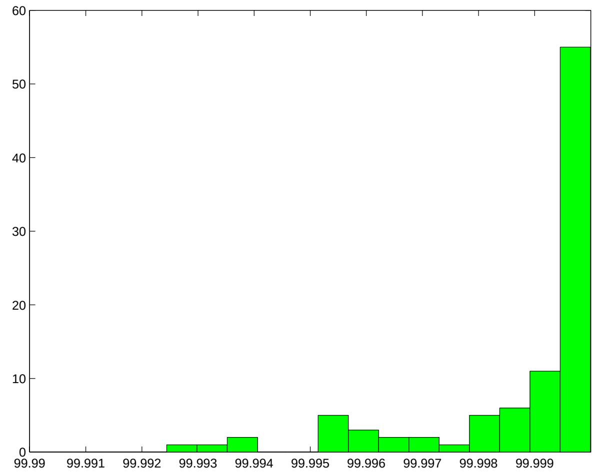

Figure 5b: LA: 100· Pricing R^2.

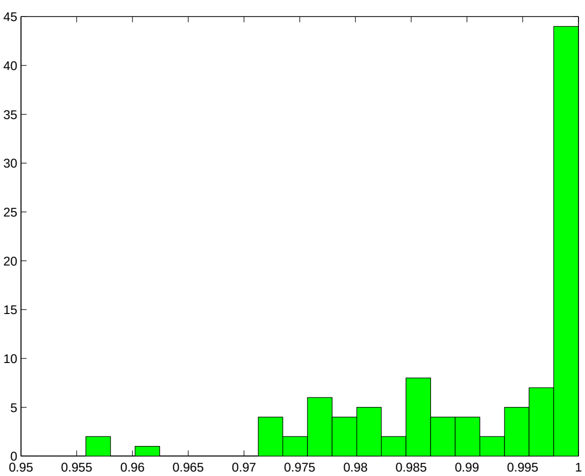

Figure 5c: LA: Two-fund separation agent 2.

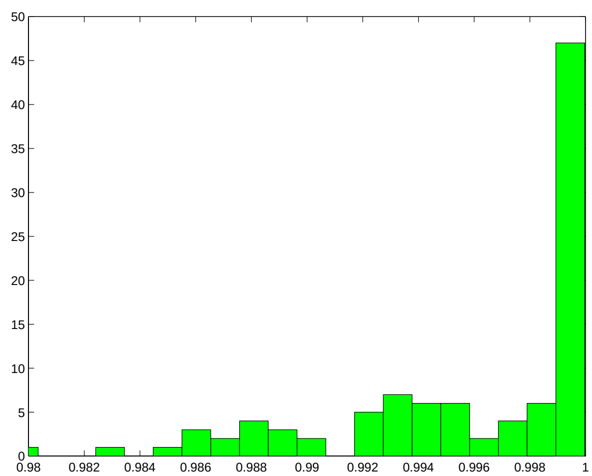

Figure 5d: LA: Two-fund separation agent 3.

$$
\begin{aligned}
a_2 &\sim \mathrm{U}(1.02 \cdot 1.1, 1.02 \cdot 1.5), \\
a_3 &\sim \mathrm{U}(-0.5, -0.1), \\
a_4 &\sim \mathrm{U}(0.1, 0.5).
\end{aligned}
$$

Given a realization for $a_1, \ldots, a_4$, we continue the construction of the economy by taking independent drawings for l_s^h, f_s and ε_s^j, where

$$
\begin{aligned}
l_s^h &\sim \mathrm{U}(2/3 \cdot 0.8, 2/3 \cdot 1.24), \\
f_s &\sim \mathrm{U}((a_1 - a_2)/2, (a_2 - a_1)/2), \\
\varepsilon_s^j &\sim \mathrm{U}(a_3, a_4).
\end{aligned}
$$

Finally, dividends are determined by

$$
d_s^j = 1/7 \cdot (\frac{a_1 + a_2}{2} + \sqrt{c_j} f_s + \varepsilon_s^j).
$$

Given the realizations for the parameters a_1 and a_2, $1/7 \cdot (a_1 + a_2)/2$ equals expected dividends from asset j. The realization of the factor belongs to the interval $[(a_1 - a_2)/2, (a_2 - a_1)/2]$ and the realizations of the idiosyncratic shocks to the interval $[a_3, a_4]$. The expected labor income and the variance of labor income are as before.

Figures 6a-d show that the ability of the CAPM to predict portfolio-holdings and excess returns is robust to the exact specification of the distribution of shocks. The results are very close to the ones obtained for the base case with log-normal shocks examined in Section 3, where the average degree of risk aversion in the economy is similar.

5.2 More Factors

One might wonder whether our results are not simply due to the fact that we have all risky assets being influenced by a single common factor. In fact, it is possible to derive the CAPM as a special case of the APT where there is only one factor, see for instance Connor (1984). However, such a derivation requires an uncountable number (or at least very large number) of assets to diversify the idiosyncratic shocks away. The influence of idiosyncratic shocks is quite substantial in our economies with only seven risky assets. Moreover, usually factors enter linearly in the definition of an asset's pay-off, which is not always the case in our economies. It seems therefore not likely that our results are due to the single factor set-up.

Other elements of the set-up we used so far are that factor loads are distributed very symmetrically and balanced, and that the importance of idiosyncratic shocks is the same for all assets. We drop these assumptions in the economies of this section. Finally, we consider a wider range for the variance of the entire stock market.

In this subsection we generate a number of economies where risky assets depend on two factors, f and $\hat{f}$, and factor loads for each one of the assets are randomly drawn. On top of this, also the importance of the idiosyncratic shock is randomly determined.

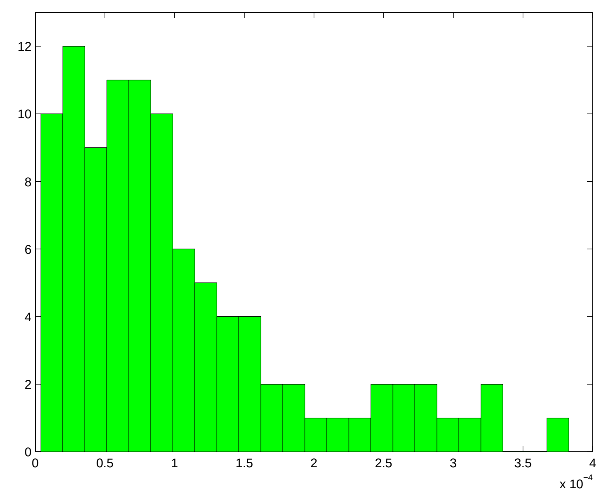

Figure 6a: Uniform: MSE.

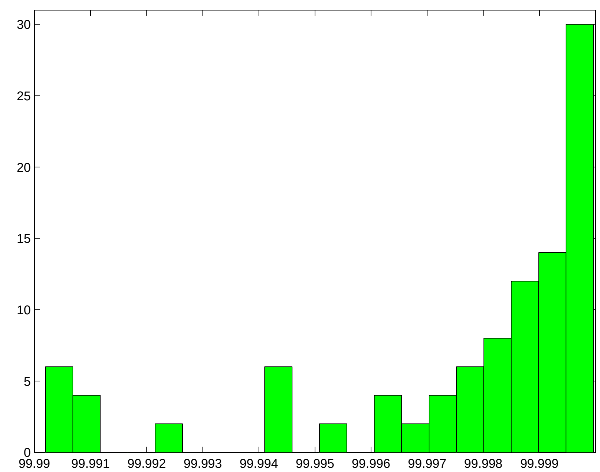

Figure 6b: Uniform: 100· Pricing R^2.

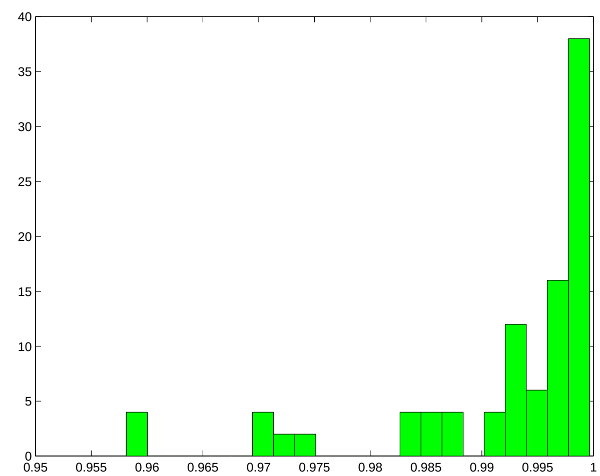

Figure 6c: Uniform: Two-fund separation agent 2.

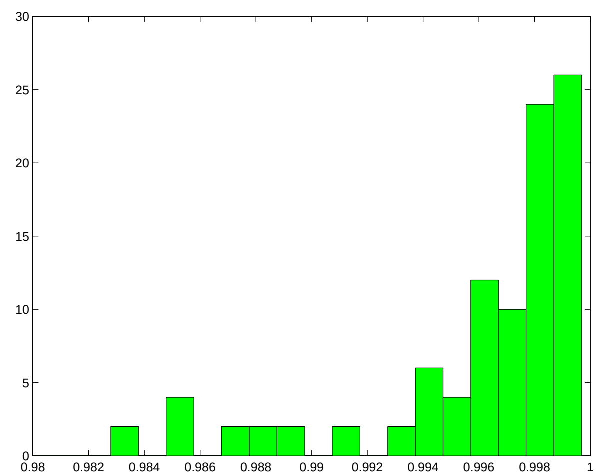

Figure 6d: Uniform: Two-fund separation agent 3.

We start each example by randomly generating, for each asset $j = 2, \ldots, J$, parameters c_j, $\widehat{c}_j$, and i_j. These parameters represent the load in factor 1, the load in factor 2 and the importance of the idiosyncratic shock. More specifically it holds that

$$\begin{aligned} c_j &\sim \mathrm{U}(0,2), \\ \widehat{c}_j &\sim \mathrm{U}(0,2), \\ i_j &\sim \mathrm{U}(0,4). \end{aligned}$$

Labor income, the two factors and assets' idiosyncratic shocks are independently log-normally distributed, so l_s^h, f_s, $\widehat{f}_s$, and ε_s^j are drawn from a log-normal distribution,

$$\begin{aligned} l_s^h &\sim \mathrm{LN}(2/3 \cdot 1.02, (2/3)^2 \cdot 0.01), \\ f_s^j &\sim \mathrm{LN}(1, c_j \cdot 0.0161), \\ \widehat{f}_s^j &\sim \mathrm{LN}(1, \widehat{c}_j \cdot 0.0161), \\ \varepsilon_s^j &\sim \mathrm{LN}(1, i_j \cdot 0.0161). \end{aligned}$$

Finally, dividends are determined by

$$d_s^j = 1/7 \cdot 1.02 \cdot f_s^j \cdot \widehat{f}_s^j \cdot \varepsilon_s^j.$$

The way to generate f_s^j, $j = 2, \ldots, J$, from a single realization of a normally distributed random variable is the same as in Section 4. The same applies to the other factor.

From Figure 7 we may conclude that the one factor framework is certainly not the driving force that makes the CAPM work. Also in the two factor set-up, for a variety of factor loads, with assets that are different in the importance of the idiosyncratic shocks, the CAPM turns out to be an excellent model.

5.3 Options

Since markets are incomplete the introduction of an option on one of the assets will generally change all equilibrium prices (see Detemple and Selden (1991)). Therefore one might expect that the introduction of an option worsens CAPM-pricing considerably. Furthermore, given the robustness of the CAPM in the earlier examples, it is interesting to see if it is possible to give an equilibrium pricing formula for options in incomplete markets via the CAPM.

Another reason to introduce an option is that this is an asset with the capacity to seriously alter the higher order moments of an asset portfolio. One possible explanation for our results obtained so far is that asset markets are very incomplete, which makes it difficult for households to change the higher order moments of the returns of their portfolios. Although households care for higher order moments, the mix of marketed assets makes it difficult to affect the higher order moments. With the introduction of an option this clearly changes. Agents have then a possibility to limit downwards risk, which is exactly the kind of

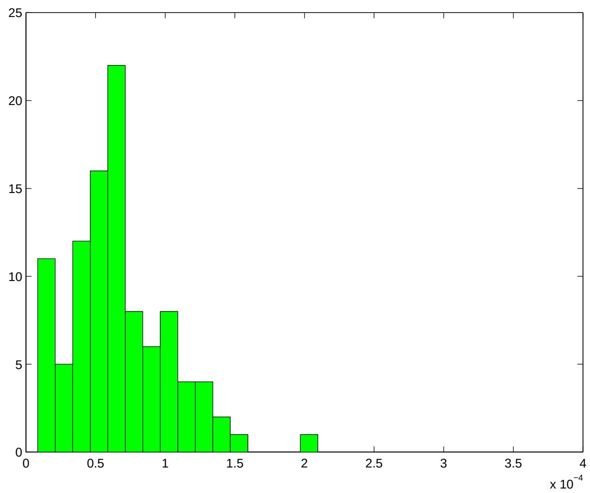

Figure 7a: Two-factor: MSE.

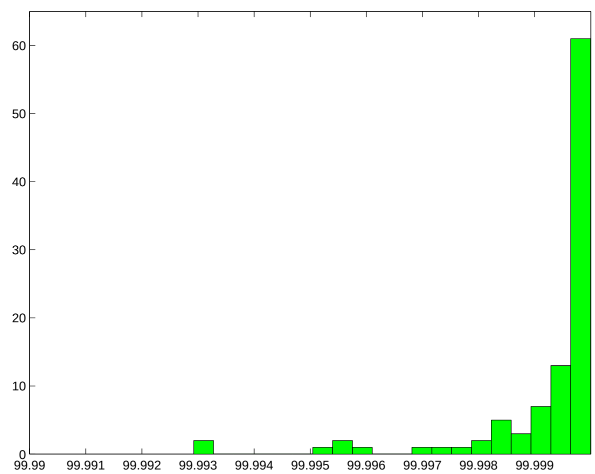

Figure 7b: Two-factor: 100· Pricing R^2.

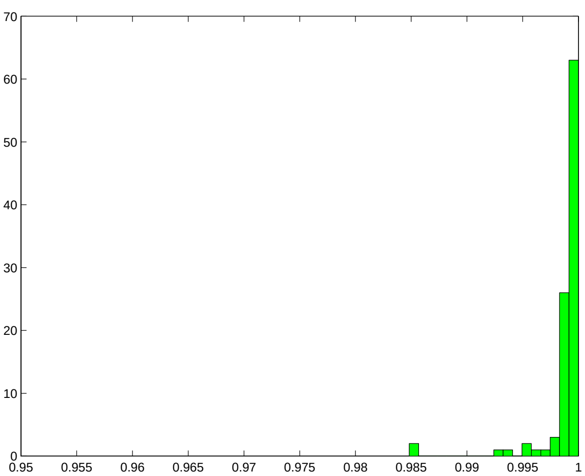

Figure 7c: Two-factor: Two-fund separation agent 2.

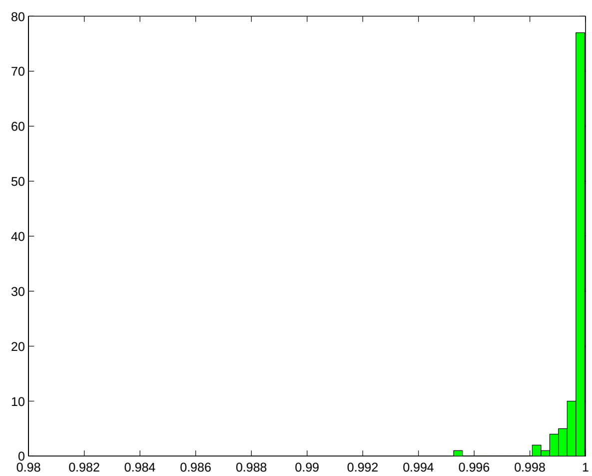

Figure 7d: Two-factor: Two-fund separation agent 3.

risk agents with CRRA utility functions are concerned about, but mean-variance optimizers are not.

In order to investigate this issue more closely we introduce a call option on the most risky asset. Specifically we have a 9-th security which pays $\max(d_s^j - X, 0)$ in state s, with X the strike price of the call option.

Suppose we consider the uniquely determined equilibrium pricing vector π_A^* of the economy without the option, and we use this pricing vector to price the option. Given the reasoning of the previous paragraph, at those prices one would expect the call option (in combination with the bond) to be more attractive to the agents than the stock, exactly because of the higher order moments. So the equilibrium price of the call option should be higher than the one computed by CAPM-pricing, in order to make that asset less appealing. As a consequence, the expected equilibrium return of the call option should be less than the one predicted by the CAPM.

To examine different options, we draw X out of the uniform distribution for each example. To avoid options that are either too far in or too far out of the money we determine in each example the minimal dividend paid out by asset 8, $\underline{d}^8 = \min_{s=1,\dots,S} d_s^8$, and the maximal dividend paid out, $\overline{d}^8 = \max_{s=1,\dots,S} d_s^8$. We then draw X out of a uniform distribution on $[0.5 \cdot (1.02 + \underline{d}^8), 0.5 \cdot (1.02 + \overline{d}^8)]$. Note that 1.02 is the expected dividend of asset 8. The strike price is always between the average of the minimal dividend and the expected dividend, and the average of the expected dividend and the maximal dividend. The results are given in Figures 8a-d.

The MSE in Figure 8 refers to the MSE of the pricing of the stocks only. The option is analyzed in detail in Figure 9. It turns out that the MSE, and the Two-fund R^2 are comparable to the ones given before. The Pricing R^2 is somewhat less good than before, but is still excellent. Surprisingly, we have found no systematic effect of the introduction of the option on the price of asset 8. In some examples the introduction of an option raised the price above the CAPM-prediction, in others it has been lower.

Figure 9 analyzes the pricing of the option by the CAPM. According to the CAPM, a call option is a very risky asset. It has zero pay-offs in bad states of nature, and very high in good states of nature. The covariance of a call option with the market portfolio is very high, which is also clear from Figure 9, where it is shown that the option's β varied from 5 to 35 in the economies generated. Notice that, as we expected, there is indeed an over-prediction of the expected return of an option by the CAPM. In all economies generated, the CAPM underpriced the call option. The misprediction was relatively small when the option's β is low, say below 10, but may get quite severe for call options with a very high strike price, which are the ones with a high β. Notice, however, that a higher β of an option also corresponds to a higher excess return, which makes the relative misprediction less bad. Still, the over-prediction of call option returns is more than linearly increasing in an option's β, whereas the excess return itself is still roughly linear.

It is surprising that the Pricing R^2 and the MSEs of stocks remained so good in all economies, even when the option was sometimes seriously under-priced by the CAPM.

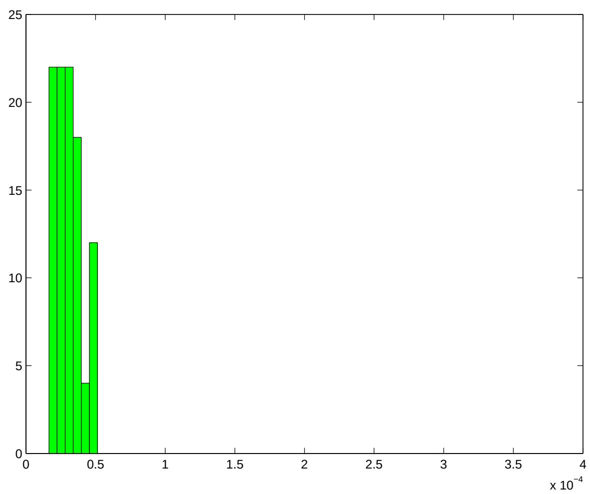

FIGURE 8A: Option: MSE.

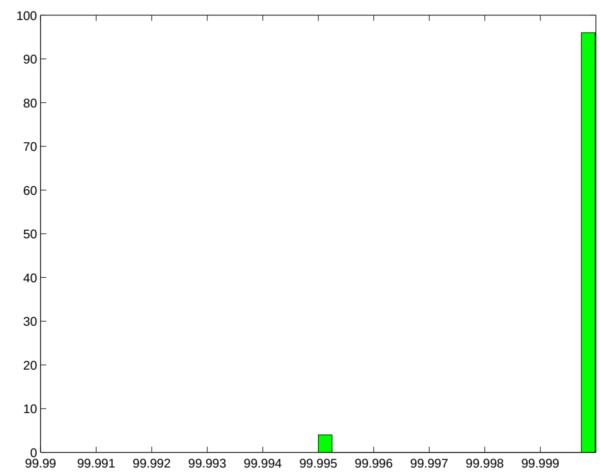

FIGURE 8B: Option: 100· Pricing R^2.

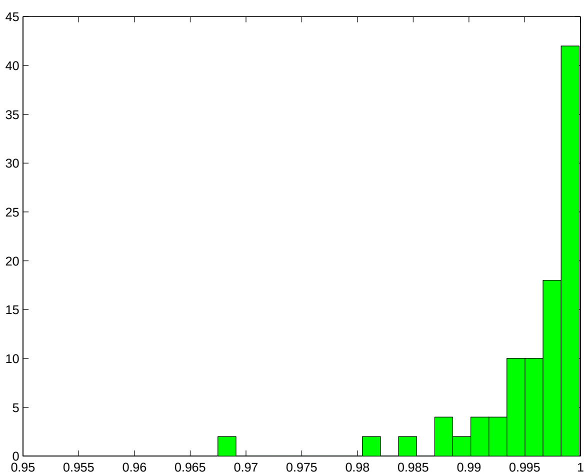

FIGURE 8C: Option: Two-fund separation agent 2.

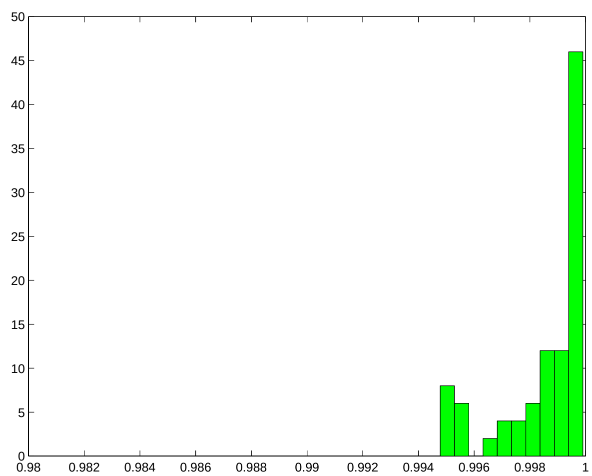

FIGURE 8D: Option: Two-fund separation agent 3.

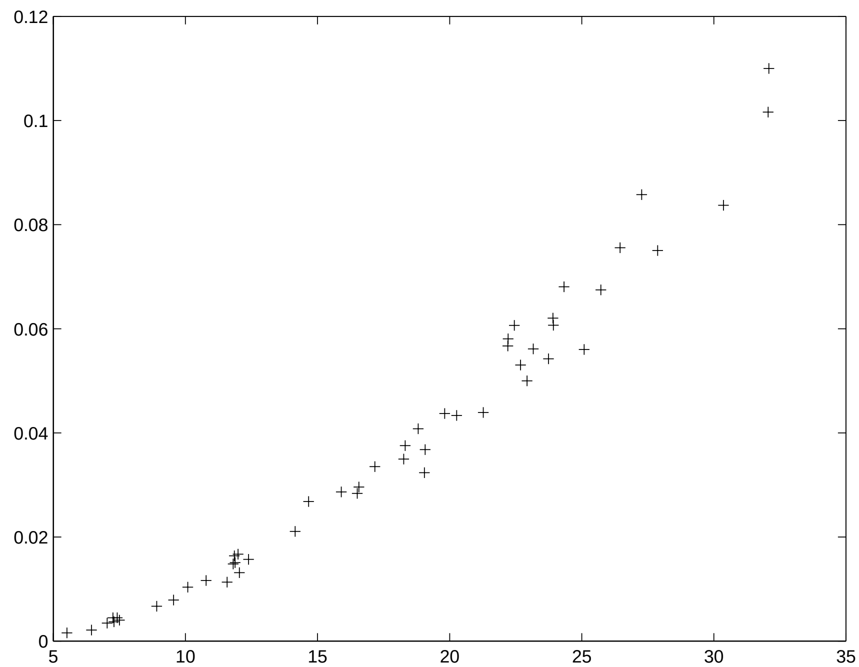

FIGURE 9: Option: over-prediction of return against option's β.

In fact, it may even be perceived as an inconsistency that the Pricing R^2 is virtually exactly correct, and the option is seriously mispriced. Indeed, when CAPM-pricing is highly correlated with π_A^*, almost all assets are priced very well. The only exceptions are those like options with a very high strike price. Such an asset pays off in a few (less than 10) states of the 32,768 only. A high correlation with π_A^* is not inconsistent with a fairly different state price in a negligible fraction of states only.

6 Interpretation and Conclusion

In this paper we develop a new algorithm to compute equilibria in the finance version of the GEI-model that is tailored to situations with a large state space. We use the algorithm to examine the cross-section of stock returns with general preferences and general specifications of dividends, endowments and prices. We find that the CAPM provides an excellent approximation to equilibrium excess returns and portfolio-holdings for a wide variety of preferences, dividends and endowments.

This result certainly does not confirm our intuition. In the CAPM-pricing formula, excess returns only depend on the first two moments of dividends and endowments, while under general preferences agents care about all moments of the relevant distributions. While the preferences we consider are in no way similar to mean-variance preferences, nor are the endowments or dividends generated by elliptical distributions, the resulting equilibrium portfolio-holdings and excess returns do not reflect this fact at all.

In this paper we have generated over 600 economies, and the worst case has been a Pricing R^2 of 0.9999 and a Two-fund R^2 of 0.95. Our current computational experience suggests that the CAPM is an excellent tool to price assets in realistically calibrated economies with incomplete markets. The results seem to be robust for a wide variety of cases. The form of the utility functions, the distribution of the shocks, the number of factors, and the introduction of options do not affect our results. We have heterogeneous households, but their number, three, is too small to expect that nice structural properties that tend to come out of aggregation in a heterogeneous economy, see Hildenbrand (1983, 1994) and Grandmont (1992), can explain our findings. An explanation invoking the central limit theorem does not apply either, as the total marketed risk in our economy is the sum of only seven stocks that are for instance lognormally distributed. More importantly, each possible linear combination of individual assets has to be priced correctly, in particular each, for instance lognormally distributed, stock. The excellent performance of the CAPM in all the economies generated therefore seems to be a puzzle.

Appendix: The Algorithm

In order to investigate the cross-sectional distribution of asset returns in more general models we have to compute equilibria. In this section we develop a globally convergent algorithm to compute equilibria for the model introduced in Section 2. The presentation of the algorithm, and the convergence proof, is simplified by restricting attention to an economy without first period consumption. From the arguments given in Geanakoplos and Polemarchakis (1986) it follows that this is without loss of generality. Indeed, given the pay-off matrix A of the previous section, if we define the matrix $\overline{A} \in \mathbb{R}^{(S+1)\times(J+1)}$ by $\overline{A}_{00} = 1$, $\overline{A}_{0j} = 0$, $j = 1,\ldots,J$, $\overline{A}_{s0} = 0$, $s = 1,\ldots,S$, and $\overline{A}_{sj} = A_{sj}$, $s = 1,\ldots,S$, $j = 1,\ldots,J$, then state 0 can be identified with the first period, and purchasing one unit of asset 0 corresponds to having one more unit of first period consumption. In this section, the index of assets runs from 0 to J.

We strengthen the assumptions made so far to Assumption A below, which states the standard assumptions that are invoked when twice differentiability of demand is required.

Assumption A1

1. $X^h = \mathbb{R}^{S+1}_{++}$.
2. u^h is three times continuously differentiable, $\partial u^h(x^h) \in \mathbb{R}^{S+1}_{++}$ for all $x^h \in X^h$ (strong monotonicity), $y^\top \partial^2 u^h(x^h) y < 0$ for all $y \neq 0$ such that $\partial u^h(x^h) y = 0$, for all $x^h \in X^h$ (negative Gaussian curvature), and $\{x^h \in \mathbb{R}^{S+1}_{++} \mid u^h(x^h) \geq u^h(\overline{x}^h)\}$ is closed in $\mathbb{R}^{S+1}$ for all $\overline{x}^h \in X^h$ (boundary condition).
3. $e^h \in X^h$.
4. $\mathrm{rank}(\overline{A}) = J+1$ and there is $\overline{\theta} \in \mathbb{R}^{J+1}$ such that $\overline{A\theta} > 0$.

A1(4) is actually weaker than the assumptions of Section 2, as it is clearly satisfied if there is first period consumption. All results of this section remain true, at the cost of slightly more complicated proofs, when the assumption $X^h = \mathbb{R}^{S+1}_{++}$ is replaced by the weaker assumption on consumption sets of Section 2, and the boundary condition in A1(2) is modified accordingly.

Without loss of generality it can be assumed that asset 0 pays off a non-negative amount in each state. Indeed, we can replace the original asset structure by taking $\overline{A\theta}$ as asset 0 and deleting an asset j for which $\overline{\theta}_j \neq 0$. Then it holds that $q_0 > 0$ for all $q \in Q$.

The following properties are useful when showing convergence of the algorithm. The function $G : Q \to \mathbb{R}^{J+1}$ denotes the total demand function for assets.

Lemma A.1: *If the economy $\mathcal{E}$ satisfies Assumption A1, then the following properties hold.*

1. *The function $G : Q \to \mathbb{R}^{J+1}$ is twice continuously differentiable.*
2. *For all $q \in Q$, $q \cdot G(q) = 0$.*

3. If $(q^n)_{n\in\mathbb{N}}$ *is a sequence in* Q, $q^n \to \overline{q} \in \partial Q$,[8] $\overline{q} \neq 0$, *then for all* $\widehat{q} \in Q$, $\widehat{q}\cdot G(q^n) \to \infty$.

PROOF. See Hens (1991). Q.E.D.

Let $g^0 : Q \to \mathbb{R}^{J+1}$ be the excess demand function for assets of some artificial agent having a utility function and initial endowments satisfying Assumption A1. We will discuss a sensible choice for this agent later on. Since Lemma A.1 also applies to an economy consisting of just one agent, we obtain the properties of Lemma A.1 for g^0.

The function g^0 with component zero deleted is denoted by $\widehat{g}^0$; G with component zero deleted is denoted by $\widehat{G}$. We normalize prices by taking $\sum_{j=0}^{J}(q_j)^2 = 1$ and we propose to use the homotopy $\mathcal{H} : [0,1] \times Q \to \mathbb{R}^{J+1}$ defined by

$$\mathcal{H}(t,q) = \begin{cases} \sum_{j=0}^{J}(q_j)^2 - 1 \\ t\widehat{G}(q) + (1-t)\widehat{g}^0(q) \end{cases}$$

to compute equilibria in the finance GEI economy.

6.1 Generic Convergence

We are looking for solutions to $\mathcal{H}(t,q) = 0$. If $\mathcal{H}(\overline{t},\overline{q}) = 0$, then $\overline{q}_0$ is positive and Lemma A.1.2 implies $\overline{t}G_0(\overline{q}) + (1-\overline{t})g_0^0(\overline{q}) = 0$, so $\overline{t}G(\overline{q}) + (1-\overline{t})g^0(\overline{q}) = 0$. In particular, if $\overline{t} = 1$, it follows that $\overline{q}$ is a competitive equilibrium price system.

A homotopy is in general constructed in such a way that there is a unique solution to $\mathcal{H}(0,q) = 0$, solutions to $\mathcal{H}(1,q) = 0$ are solutions to the problem of interest, and the unique solution to $\mathcal{H}(0,q) = 0$ is linked by a path to one solution to $\mathcal{H}(1,q) = 0$. By following this path, which is feasible by several numerical techniques, a solution to the problem of interest is found. When the unique solution to $\mathcal{H}(1,q) = 0$ is indeed linked by a path to a solution to $\mathcal{H}(1,q) = 0$, then the homotopy is said to converge.

Theorem A.2 analyses the structure of the solutions to our homotopy.

THEOREM A.2: *Let* $\mathcal{E}$ *be an economy satisfying Assumption A1. Then, for an open set of initial endowments with full Lebesgue measure,*

- $\mathcal{H}^{-1}(\{0\})$ *is a compact* C^2 *1-dimensional manifold with boundary, with boundary given by* $\mathcal{H}^{-1}(\{0\}) \cap (\{0,1\} \times Q)$.
- *there is an odd number of solutions in* $\mathcal{H}^{-1}(\{0\}) \cap (\{1\} \times Q)$, *i.e. there is an odd number of competitive equilibria.*

For any choice of initial endowments,

- *there is one solution in* $\mathcal{H}^{-1}(\{0\}) \cap (\{0\} \times Q)$,

[8] ∂Q represents the boundary of Q.

- *there is no sequence $(t^n, q^n)_{n\in\mathbb{N}}$ in $\mathcal{H}^{-1}(\{0\})$ converging to $(t,q) \in [0,1] \times \partial Q$.*

PROOF. The only solution in $\mathcal{H}^{-1}(\{0\}) \cap (\{0\} \times Q)$ is obviously given by

$$(0, q^0) = (0, \partial u^0(e^0)\overline{A}/\|\partial u^0(e^0)\overline{A}\|_2).$$

Suppose $(t^n, q^n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{H}^{-1}(\{0\})$ converging to $(t,q) \in [0,1] \times \partial Q$. Then, for $\overline{q} \in Q$,

$$0 = \overline{q} \cdot (t^n G(q^n) + (1-t^n) g^0(q^n)),$$

but

$$\overline{q} \cdot (t^n G(q^n) + (1-t^n) g^0(q^n)) \to \infty$$

by Lemma A.1.3, a contradiction. Solutions to the homotopy equations stay away from $[0,1] \times \partial Q$. Now it also follows that $\mathcal{H}^{-1}(\{0\})$ is compact.

The proof is completed by showing that $\partial_q \mathcal{H}(0,q)$, and, generic in initial endowments, $\partial_q \mathcal{H}(1,q)$, and $\partial_{t,q}\mathcal{H}(t,q)$ have full rank for points in $\mathcal{H}^{-1}(\{0\})$.

It holds that $g^0(q) = \theta$ if and only if there is $\lambda \neq 0$ such that

$$\begin{aligned} &\partial u^0(e^0 + \overline{A}\theta)\overline{A} - \lambda q^\top = 0, \\ &q \cdot \theta = 0. \end{aligned}$$

By the inverse function theorem it holds that

$$\begin{pmatrix} \partial_q g^0(q) \\ \partial_q \lambda^0(q) \end{pmatrix} = \begin{bmatrix} \overline{A}^\top \partial^2 u^0(e^0)\overline{A} & -q \\ q^\top & 0 \end{bmatrix}^{-1} \begin{bmatrix} -\lambda I \\ \theta^\top \end{bmatrix},$$

where I denotes the $(J+1)$-dimensional unit matrix. The first matrix on the right-hand side is indeed invertible. Suppose not, then there is $(y,z) \in (\mathbb{R}^{J+1} \times \mathbb{R}) \setminus \{0\}$ such that

$$\begin{bmatrix} \overline{A}^\top \partial^2 u^0(e^0)\overline{A} & -q \\ q^\top & 0 \end{bmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = 0.$$

It follows that $y \neq 0$, since otherwise $y = 0$ and $\overline{A}^\top \partial^2 u^0(e^0)\overline{A}y - qz = 0$ implies $z = 0$, contradicting $(y,z) \neq 0$. Since $\overline{A}$ has full column rank, $\overline{A}y \neq 0$. Moreover, $\partial u^0(e^0)\overline{A}y = \lambda q \cdot y = 0$, so the non-zero Gaussian curvature of u^0 implies

$$0 = y^\top \overline{A}^\top \partial^2 u^0(e^0)\overline{A}y + y^\top qz = y^\top \overline{A}^\top \partial^2 u^0(e^0)\overline{A}y \neq 0,$$

a contradiction.

Consider $(0, q^0) \in \mathcal{H}^{-1}(\{0\}) \cap (\{0\} \times Q)$. Since $\lambda^0(q^0) \neq 0$, $(\partial_q g^0(q^0), \partial_q \lambda^0(q^0))$ has rank $J+1$, so $\partial_q g^0(q^0)$ has at least rank J. It follows that $\partial_q \widehat{g}^0(q^0)$ has rank J, since $q \cdot g^0(q) = 0$

for $q \in Q$ and $g^0(q^0) = 0$ imply $\partial_q g^0_0(q^0) = -\sum_{j=1}^J (q_j/q_0)\partial_q g^0_j(q^0)$. Since homogeneity of degree zero of g^0 in asset prices implies $\partial_q \hat{g}^0(q^0)q^0 = 0$, it follows that

$$\partial_q \mathcal{H}(0, q^0) = \begin{bmatrix} -2q^0_0 & \cdots & -2q^0_J \\ & \partial_q \hat{g}^0(q^0) & \end{bmatrix}$$

has full rank, $J+1$.

We define $\overline{\mathcal{H}} : [0,1] \times Q \times \mathbb{R}^{S+1}_{++} \to \mathbb{R}^{J+1}$ by

$$\overline{\mathcal{H}}(t, q, e^1) = \begin{cases} \sum_{j=0}^J (q_j)^2 - 1, \\ t\hat{G}(q, e^1) + (1-t)\hat{g}^0(q), \end{cases}$$

where $\hat{G}(q, e^1) = \hat{g}^1(q, e^1) + \sum_{h=2}^H \hat{g}^h(q)$. We show next that $\overline{\mathcal{H}} : (0,1) \times Q \times \mathbb{R}^{S+1}_{++} \to \mathbb{R}^{J+1}$ is transversal to zero, or equivalently, that $\partial_{t,q,e^1}\overline{\mathcal{H}}(\bar{t}, \bar{q}, \bar{e}^1)$ has full row rank whenever $\overline{\mathcal{H}}(\bar{t}, \bar{q}, \bar{e}^1) = 0$.

For $j' = 1, \ldots, J$, define the asset portfolio $\bar{\theta}^{j'}$ by $\bar{\theta}^{j'}_0 = -\bar{q}_{j'}$, $\bar{\theta}^{j'}_{j'} = \bar{q}_0$, and $\bar{\theta}^{j'}_j = 0$, $j \neq 0$, $j \neq j'$. Then changing the initial endowment of agent 1 to $\bar{e}^1 + \alpha \overline{A}\bar{\theta}^{j'}$ with α sufficiently small, changes his asset demand to $g^1(\bar{q}, \bar{e}^1) - \alpha\bar{\theta}^{j'}$. Since the vectors $\bar{\theta}^{j'}$, $j' = 1, \ldots, J$, are independent, even with component 0 deleted, it follows that $\partial_{e^1}\hat{G}(\bar{q}, \bar{e}^1)$ has rank J.

Homogeneity of degree zero of $\hat{G}$ in asset prices implies $\partial_q \hat{G}(\bar{q}, \bar{e}^1)\bar{q} = 0$. It follows that $\partial_{t,q,e^1}\overline{\mathcal{H}}(\bar{t}, \bar{q}, \bar{e}^1)$ has rank $J+1$. By the transversal density theorem, see Mas-Colell (1985), I.2.2, page 45, the set of economies for which $\partial_{t,q}\mathcal{H}(\bar{t}, \bar{q})$ has full rank for all points in $\mathcal{H}^{-1}(\{0\})$ has full Lebesgue measure.

Exactly the same argument shows that for a set of initial endowments with full Lebesgue measure $\partial_q \mathcal{H}(1, \bar{q})$ has full rank for points in $\mathcal{H}^{-1}(\{0\}) \cap (\{1\} \times Q)$.

The transversality proofs given, show that for a set of initial endowments with full Lebesgue measure $\mathcal{H}^{-1}(\{0\})$ is a C^2 1-dimensional manifold with boundary, where the boundary is given by $\mathcal{H}^{-1}(\{0\}) \cap (\{0,1\} \times Q)$.

Using Lemma A.1.3, it follows by a standard argument that the set of initial endowments for which transversality holds can be taken open and of full Lebesgue measure.

Concluding, for an open set of initial endowments with full Lebesgue measure, $\mathcal{H}^{-1}(\{0\})$ is a compact C^2 1-dimensional manifold with boundary, therefore a finite collection of arcs and loops.[9] Each arc has two boundary points. Since all boundary points belong to $\{0,1\} \times Q$, and there is exactly one boundary point in $\{0\} \times Q$, it follows that for an open set of initial endowments with full Lebesgue measure, there is an odd number of solutions in $\mathcal{H}^{-1}(\{0\}) \cap (\{1\} \times Q)$. Q.E.D.

Since $\mathcal{H}$ is a system of $J+1$ independent continuous equations in $J+2$ variables, it is not surprising that $\mathcal{H}^{-1}(\{0\})$ is generically a compact 1-dimensional manifold with boundary, i.e. a finite collection of arcs and loops. There is a unique solution to $\mathcal{H}(0, q) = 0$, obtained by taking q equal to $\partial u^0(e^0)\overline{A}$. The boundary behavior of G guarantees that there is no

[9] An arc is a set homeomorphic to the unit interval and a loop a set homeomorphic to the unit circle.

sequence $(t^n, q^n)_{n\in\mathbb{N}}$ in $\mathcal{H}^{-1}(\{0\})$ converging to $(t,q) \in [0,1] \times \partial Q$. Therefore the unique solution to $\mathcal{H}(0,q) = 0$ is generically part of a path in $\mathcal{H}^{-1}(\{0\})$ that does not run off to the boundary, but reaches $t = 1$. The unique solution to $\mathcal{H}(0,q) = 0$ is thereby connected to exactly one point $(1, q^*) \in \mathcal{H}^{-1}(\{0\})$, a competitive equilibrium for $\mathcal{E}$, and the homotopy converges. A more detailed description of homotopy like methods, as well as similar generic convergence results, can be found in Brown, DeMarzo and Eaves (1996), Herings (1997), or Schmedders (1998). Notice that there is no need to compute the set Q explicitly. Our homotopy is constructed in such a way that its projection on the set Q stays away from ∂Q.

Corollary A.3: *Let $\mathcal{E}$ be an economy satisfying Assumption A1. Then, for an open set of initial endowments with full Lebesgue measure, the homotopy $\mathcal{H}$ converges to a competitive equilibrium.*

If there are multiple equilibria, then in addition to the arc connecting q^0 and a competitive equilibrium q^*, there is a finite number of arcs, each one having two more competitive equilibria as its end points. This gives a constructive proof of the fact that there is an odd number of competitive equilibria. In fact, using the properties of a homotopy, we can get an index theorem for our economy, a result already obtained by Hens (1991), and, for certain classes of economies with more than one good per state, by Schmedders (1998).

The computation of the demand for assets as a function of prices is not necessarily an easy problem. It is notoriously hard when the asset market is incomplete. The theoretical homotopy $\mathcal{H}$ will therefore be replaced by the diffeomorphic implementable homotopy $\mathcal{H}^* : [0,1] \times Q \times \mathbb{R}^{(H+1)(J+1)} \times \mathbb{R}^{H+1} \to \mathbb{R}^{1+J+(H+1)(J+1)}$,

$$\mathcal{H}^*(t,q,\theta,\lambda) = \begin{cases} \sum_{j=0}^{J} (q_j)^2 - 1, \\ \sum_{h=0}^{H} \theta_j^h, \quad j = 1, \ldots, J, \\ \partial u^h(e^h + \overline{A}\theta^h)\overline{A} - \lambda^h q^\top, \quad h = 0, \ldots H, \\ q \cdot \theta^h, \quad h = 0, \ldots, H. \end{cases}$$

We have replaced the demand functions of the agents by their first order conditions, an approach proposed in Garcia and Zangwill (1981).

Theorem A.4: *Let $\mathcal{E}$ be an economy satisfying Assumption A1. Then ${\mathcal{H}^*}^{-1}(\{0\})$ is C^2 diffeomorphic to $\mathcal{H}^{-1}(\{0\})$.*

Proof. It holds that $(\overline{t}, \overline{q}, \overline{\theta}, \overline{\lambda}) \in {\mathcal{H}^*}^{-1}(\{0\})$ if and only if $(\overline{t}, \overline{q}) \in \mathcal{H}^{-1}(\{0\})$, $\overline{\theta}^h = g^h(\overline{q})$, $h = 0, \ldots, H$, and $\overline{\lambda}^h = \partial u^h(e^h + \overline{A} g^h(\overline{q}))\overline{A}$, $h = 0, \ldots, H$. The claim follows since g^h and ∂u^h are twice continuously differentiable functions. Q.E.D.

Since ${\mathcal{H}^*}^{-1}(\{0\})$ is diffeomorphic to $\mathcal{H}^{-1}(\{0\})$, the results of Theorem A.2 carry over to ${\mathcal{H}^*}^{-1}(\{0\})$.

Corollary A.5: *Let $\mathcal{E}$ be an economy satisfying Assumption A1. Then, for an open set of initial endowments with full Lebesgue measure, the homotopy $\mathcal{H}^*$ converges to a competitive equilibrium.*

The speed of homotopy algorithms depends mainly on two factors, the number of equations and the arc length of the homotopy path. A quick comparison shows the great benefits of developing a special purpose homotopy tailored to the finance GEI-model. The homotopy algorithms as reported in Brown, DeMarzo and Eaves (1996) and Schmedders (1998) are designed to deal with the general GEI-model with multiple commodities per state, but can be applied to finance economies.

The homotopy proposed by Brown, DeMarzo and Eaves (1996) needs closed form solutions for excess demand functions and should therefore be compared with our homotopy $\mathcal{H}$. Applied to two-period finance economies, their algorithm has $2S+1$ equations, whereas ours only has $J+1$. The algorithm of Schmedders (1998) does not require closed-form solutions for excess demand functions, and also uses the first order conditions. The number of equations of his algorithm amounts to $2(H+1)(S+1)+HJ+1$, whereas the number of equations in our algorithm $\mathcal{H}^*$ equals $(H+2)(J+1)$.

In both cases, we roughly need a fraction $J/2S$ only of the equations of alternative algorithms. This is especially favorable when S is high, which is exactly the case for the kind of applications in this paper. In fact, in most applications we have $J=8$ and $S=32,768$. The high number of states is used to get a good discrete approximation of a continuously distributed multivariate random variable. On top of the great number of equations saved, our method also has the flexibility of choosing the initial price system as desired, contrary to the homotopies of Brown, DeMarzo and Eaves (1996) or Schmedders (1998). Since it is not too hard to make a reasonable guess for an equilibrium price system using the CAPM or the method of the next subsection, our algorithm will generally substantially reduce the arc length of the homotopy path.

Implementation

We implemented the algorithm using HOMPACK - a suite of FORTRAN 77 subroutines designed to solve systems of non-linear homotopy equations with path-following methods. See Watson (1979) and Watson, Billups and Morgan (1987) for details on HOMPACK.

We now turn to the determination of the starting point and the specification of the artificial agent's demand function.

The demand function $g^0(q)$ should be chosen such that an a priori selected starting point $q^0 \in Q$ with $\sum_{j=0}^{J}(q_j^0)^2 = 1$ is the unique solution to $g^0(q)=0$ and $\sum_{j=0}^{J}(q_j)^2 - 1 = 0$.

We take a Cobb-Douglas utility function for the artificial agent,

$$u^0(c^0) = \sum_{s=0}^{S} \rho_s \gamma_s \ln(c_s^0), \quad c^0 \in \mathbb{R}_{++}^{S+1}.$$

Let $\pi^0 \in \mathbb{R}^{S+1}_{++}$ be any state price vector such that ${\pi^0}^\top \overline{A} = q^0$. If the artificial agent is defined by

$$
\begin{aligned}
e^0_s &= 1, \quad s = 0, \ldots, S, \\
\gamma_s &= \pi^0_s / \rho_s, \quad s = 0, \ldots, S,
\end{aligned}
$$

then the unique solution in Q to $g^0(q) = 0$ and $\sum_{j=0}^{J} (q^0_j)^2 - 1 = 0$ is indeed given by q^0.

In applications, there is usually no need to solve for $\pi^\top \overline{A} = q^0$.[10] Instead, we take π^0 equal to the weighted average over all agents of $\partial u^h(e^h)$, with weight for agent h equal to $1/\lambda^h$, where λ^h denotes the marginal utility of first period consumption at the initial endowment e^h. Next we take q^0 equal to ${\pi^0}^\top \overline{A} / \| {\pi^0}^\top \overline{A} \|_2$.

The use of a Cobb-Douglas utility function yields a simple way to get any a priori specified asset price system as the unique zero point. Another advantage of using a Cobb-Douglas utility function is that numerical experience shows that convergence takes place faster for low values of relative risk aversion.

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[10] If $q^0 \in Q$ with $\sum_{j=0}^{J} (q^0_j)^2 = 1$ is given and there is a need to solve for $\pi^\top \overline{A} = q^0$, an easy way to achieve this for an economy with first-period consumption is to solve the following linear program

$$
\min \sum_{s=0}^{S} \pi_s \quad \text{s.t.} \quad \overline{A}^\top \pi - \pi_0 q^0 / q^0_0 = 0, \qquad \pi - 1_{S+1} \geq 0,
$$

and divide the solution found, say π^0, by $\| {\pi^0}^\top \overline{A} \|_2$.

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