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Coalitional Provision of Pure Collective Goods*

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Abstract

I consider a *coalitional provision mechanism* in an economy with one private good and multiple non-Samuelsonian collective goods. Coalitions are assumed to have complete autonomy to determine the provision level of these collective goods. I show that there exists an optimal tax-subsidy system that guides those coalitions to a particular Pareto efficient provision scheme for the collective goods.

Second, I consider a non-cooperative framework in which the coalitions are given exogenously. In this setting I identify the conditions under which there is over- and underprovision of collective goods.

Third, I investigate overprovision in a two stage game with endogenous coalition formation. In the first stage of the game, individuals choose for which collective good to lobby. In the second stage, the formed provision coalitions determine the provision levels of the collective goods. Intuitively one expects that uniform overprovision *always* occurs in these situations. However, I show with an example that there are situations in which there is uniform underprovision, uniform overprovision, as well as mixed provision.

JEL classification codes: D61, D71, D72, H40.

Keywords: public goods; overprovision; underprovision; lobbying; implementation.

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1 Introduction

Ever since rent-seeking has been studied in relation to the provision of public goods, it has been accepted that lobbying automatically leads to overprovision; this paper shows that this conclusion is not true in general. I consider a *coalitional provision mechanism* in an economy with collective goods. In this mechanism the provision level for each collective good is assumed to be determined completely by some interest group of economic agents, implying that each group is fully effective in its lobbying for a particular collective good. Within this framework I investigate whether coalitional provision of collective goods indeed leads to overprovision of the collective goods in question. I disprove this hypothesis by identifying conditions under which exactly the opposite occurs. Moreover, these conditions are not restrictive.

Lobbying has been the subject of extensive analysis in public economics, in particular the economic theory of public policy making. Surveys of the extensive literature on this subject — theoretical as well as empirical — are given by Austen-Smith [3] and Magee, Brock and Young [15], Chapters 3–5. The literature has mainly considered lobbying activities within a political system, thus making it a subject of public choice theory and political economy.

In these models interest groups influence political parties through contributions, which implies that the political parties act as agents on behalf of these lobby groups.[1] Interest groups optimize their payoffs in economic terms, while political parties optimize payoffs in political terms, i.e., their electoral support. This creates an agency problem. The literature has concentrated on the political aspects of lobbying and the power of political parties in their relationship with the lobby groups. The public choice literature on lobbying has resulted in combining this with other theories to create a general theory of *rent-seeking*,[2] which shows that the costs of lobbying might exceed the benefits from it. Thus, this literature concludes that lobbying is subject to the "Tragedy of the Commons" and the resulting equilibrium might be socially nonoptimal. In particular, there likely is an overuse of lobbying.

The goal of my analysis is to study the consequences of the case when interest groups have complete control over the provision of collective goods. Thus, I consider an economy in which lobbying is extended to its extreme form, namely in which the provision of pure collective goods is determined completely by autonomous interest groups — or *provision coalitions*. However, I do not analyze the lobbying processes

[1] Potters [18] remarks that in principle there has to be made a distinction between influence and pressure. However, in my discussion I will not make a distinction between these two, thus following Becker [4] and [5].

[2] Here I also refer to the seminal contributions by Tullock [20] and Krueger [14].

themselves. Instead, I address two major welfare theoretic issues. The first is whether there exists a tax-subsidy system which guides the coalitional provision mechanism to attain a well specified Pareto optimal equilibrium. Second, I ask the question when an unguided decentralized decision-making mechanism based on coalitional provision of collective goods leads to over- or underprovision.

As mentioned, in my modelling I dispense with the study of the principal-agent relationship and the resulting rent-seeking approach to lobbying. Instead, the environment in which I address these questions is an abstract game theoretic model of an economy with provision coalitions. I explicitly assume that there is a central authority, which however is absent from the model; it imposes the tax schemes, but does not make any policy decisions. The provision coalitions are largely autonomous in their pursuits. Thus, I abstract from any political structure and assume that, subject to exogenously given constraints imposed by the central authority, the provision coalitions themselves make direct decisions regarding the provision of collective goods for the society as a whole. In this respect my analysis is rather close to the approach introduced by Becker [4] and [5]. In Becker the political process is viewed as a black box, represented by a *political outcome function*. On the other hand, the issues that I pursue in my analysis are very different from those addressed by Becker.

By adopting a non-cooperative game theoretic framework within which these provision coalitions operate autonomously, I am able to derive stronger results than would be the case in an alternative setting such as an extended general equilibrium model. Here I refer to the general equilibrium literature on lobbying and the provision of collective goods through voting mechanisms. Coggins, Graham-Tomasi and Roe [6] discusses the existence of an equilibrium in a partial equilibrium framework to discuss lobbying for centrally imposed prices in an open economy. Slutsky [19] and Greenberg and Shitovitz [11] consider a general equilibrium model implementing a voting mechanism for the provision of (local) public goods. These references do not achieve stronger insights than the existence of an equilibrium.

Within the framework of a game theoretic model of an economy with autonomous provision coalitions, I address three concrete welfare-theoretic questions. First, as mentioned, I discuss the existence of an *optimal* tax-subsidy system that guides the provision coalitions to select Pareto efficient provision levels of all collective goods. The main result states that such an optimal tax-subsidy system indeed exists, irrespective of the goals of the provision coalitions. In this respect the existence result can be interpreted as an implementation theorem. The construction is based on Lindahl pricing of the collective goods, and is therefore subject to the same criticism as Lindahl equilibrium. Namely, the tax-subsidy system is too complicated to be practical in achieving Pareto optimal provision of the collective goods. Furthermore, with an example it is shown that the equilibrium of such a governed economy might be unstable in the sense that the emergence of leadership among provision coalitions might be profitable for some of those coalitions.

Second, I consider a game theoretic model in which the central authority limits itself to imposing budget balance. Hence, a central authority provides exactly sufficient resources to cover the provision of the collective goods at the levels determined by the provision coalitions. Here the provision coalitions act completely autonomously and are engaged in a strategic game in which they determine the level at which the collective goods are provided. The mechanism can be interpreted to be based on a system of "self finance." I show that irrespective which one of three plausible tax systems is implemented, the resulting games are identical.

Within the setting of this autonomous provision game I identify the conditions for existence and uniqueness of Nash equilibrium. Furthermore, I determine the conditions in which there is over- or underprovision of the collective goods. In general, I conclude that provision at Pareto optimal levels is purely coincidental. In this respect the coalitional provision mechanism cannot be guided with simple means to achieve Pareto optimal allocations.

Finally, I address provision with endogenous formation of provision coalitions in a two stage game. In the first stage the provision coalitions are formed. Here, agents are assumed to become members of one and only one provision coalition. Implicitly this supposes the existence of a significant membership cost. In the second stage the provision coalitions formed engage in the coalitional provision mechanism discussed in the preceding paragraphs. A simple example satisfying all standard regularity conditions shows that there are no clear conclusions to be drawn. I identify two sets of parameter values. One set of parameter values identifies situations with equilibria in which there is possibly uniform overprovision or uniform underprovision of the collective goods. A second set of parameter values identifies situations in which there are equilibria with mixed provision, i.e., in which some collective goods are overprovided, while others are underprovided.

2 The foundations

I consider an economy with a finite set of agents represented by $A = \{1, \ldots, n\}$. There is one private good in the economy, usually denoted by x, and all agents are endowed with the private good only. This endowment is assumed to be strictly positive and given by $w : A \to \mathbb{R}_{++}$. Furthermore, I introduce $\overline{w} := \sum_{a \in A} w_a > 0$.

There is a set of *collective goods* given by $\mathcal{P} := \{1, \ldots, \mathbf{P}\}$, where each collective good $p \in \mathcal{P}$ can be delivered at different *provision levels* indicated by $q_p \in I \equiv [0,1]$.[3] In this representation $q_p = 0$ indicates the non-provision of collective good p. The level $q_p = 1$ indicates a "maximal" provision level. In many applications this provision level is purely artificial. I define $\mathcal{P}_{-p} := \mathcal{P} \setminus \{p\} = \{1, \ldots, p-1, p+1, \ldots, \mathbf{P}\}$ as the set of collective goods except for collective good $p \in \mathcal{P}$.

Each collective good $p \in \mathcal{P}$ is provided through a production technology using the single private good as an input. This production technology is described by the cost function $c_p : I \to \mathbb{R}_+$, where $c_p(q_p) \geqq 0$ denotes the cost of providing $p \in \mathcal{P}$ at level q_p. It is assumed that c_p is a twice continuously differentiable, increasing convex function with $c_p(0) = 0$, $c_p' > 0$, and $c_p'' \geqq 0$. In the following we might discuss particular examples in which $\sum_{p \in \mathcal{P}} c_p(1) > \overline{w}$, although this is not a requirement in the proof of some of the main results.

To complete the model we introduce for each agent $a \in A$ a quasi-linear utility function $U_a : \mathbb{R} \times I^{\mathcal{P}} \to \mathbb{R}$ with

$$U_a(x, q) = x + v_a(q)$$

where $q = (q_1, \ldots, q_{\mathbf{P}})$ is the vector of provision levels of the collective goods and $v_a : I^{\mathcal{P}} \to \mathbb{R}$ is an arbitrary $\mathcal{C}^2$ function representing a preference relation over the collective goods. It is clear from this formulation that the private good acts as an enumerator of the utility function and that agents are allowed to have short positions in the private good.

I do not assume the utility functions v_a, $a \in A$, to be monotonically increasing with respect to the collective goods. In that respect these goods in principle retain a non-Samuelsonian character. Also, there might be spillover effects between the collective goods; while marginal utilities for some agents are positive, the marginal utilities for some other agents with respect to the same collective goods might be negative. In particular I explicitly allow these spillover effects to be negative as well as positive. For a more elaborate discussion I also refer to Diamantaras, Gilles and Scotchmer [8] and Gilles and Hahn [10].

Definition 2.1 *Let $a \in A$. The utility function v_a is* ***single-peaked for*** $p \in \mathcal{P}$ *if for every $q_{-p} \in I^{\mathcal{P}_{-p}}$ there exists a "peak" $q_p^*(q_{-p}) \in I$ with the function $v_a(q_{-p}, \cdot)$ mono-*

[3] The main reasons for imposing a compact comodity space is technical. This assumption can be relaxed to be the non-negative real line as the commodity space. In particular I will apply the latter in some examples to make the computational operations more tractable.

tonically strictly increasing on the interval $\left[0, q_p^*\left(q_{-p}\right)\right]$ *and monotonically strictly decreasing on the interval* $\left[q_p^*\left(q_{-p}\right), 1\right]$.
The utility function v_a *is* ***single-peaked*** *if it is single-peaked for every* $p \in \mathcal{P}$.

The utility function v_a is *monotone* for collective good $p \in \mathcal{P}$ if it is single-peaked for $p \in \mathcal{P}$ and every $q_{-p} \in I^{\mathcal{P}_{-p}}$ the peak is given by $q_p^*\left(q_{-p}\right) = 1$. Note that this is equivalent to requiring that $\frac{\partial v_a}{\partial q_p}(q) > 0$ for every $q \in I^{\mathcal{P}}$, i.e., collective good $p \in \mathcal{P}$ is uniformly desired by all members of the society.

I make the following assumption.

Axiom 2.2 *Every agent* $a \in A$ *has a single-peaked utility function* v_a *such that for every* $p \in \mathcal{P}$*, if* $q_p^*\left(q_{-p}\right) = 1$ *for some* $q_{-p} \in I^{\mathcal{P}_{-p}}$*, then* $\lim_{r \uparrow 1} \frac{\partial v_a}{\partial q_p}\left(q_{-p}, r\right) = 0$.

Axiom 2.2 does not exclude that preferences could be monotone. Furthermore, I remark that the second part of Axiom 2.2 is a technicality and might be indicated as *Asymptotic Satiation in Collective Goods*. Conley [7] used a similar property for an economy with Samuelsonian public goods.

Definition 2.3 *A pair* (f, q) *is an* ***allocation*** *if* $f : A \to \mathbb{R}$ *and* $q \in I^{\mathcal{P}}$. *An allocation* (f, q) *is called* ***feasible*** *if*

$$\sum_{a \in A} f(a) + \sum_{p \in \mathcal{P}} c_p\left(q_p\right) = \overline{w} \tag{1}$$

An allocation (f, q) *is* ***Pareto optimal*** *if it is feasible and there is no alternative feasible allocation* (g, r) *such that* $f(a) + v_a(q) \leqq g(a) + v_a(r)$ *for every agent* $a \in A$ *and* $f(b) + v_b(q) < g(b) + v_b(r)$ *for some* $b \in A$.

Note that short positions in the private good are allowed and taken into account in the definition of an allocation. From the model formulation it can easily be concluded that there exist Pareto optimal allocations in this economy.[4]

The discussion of the foundations is completed with a description of the elements of the *coalitional provision mechanism*. This mechanism is based on the hypothesis that each collective good is provided through intermediation of an exogenously specified group of economic agents. Such an interest group for a particular collective good is called a *provision coalition* for that collective good.

[4]This is immediate from the continuity of the utility functions and the compactness of the collective good space $I^{\mathcal{P}}$.

Formally, this is represented by a *weight system* $\Lambda = \{\lambda_a^p \mid a \in A \text{ and } p \in \mathcal{P}\}$, where for every agent $a \in A$ the weight $\lambda_a^p \geqq 0$ reflects the "power" that this agent has in the processes leading to the determination of the provision level for collective good $p \in \mathcal{P}$. Implicitly this introduces the *provision coalition structure* $\mathcal{L}_\Lambda = \{E_p \mid p \in \mathcal{P}\}$ with for every $p \in \mathcal{P}$ the set

$$E_p := \{a \in A \mid \lambda_a^p > 0\} \tag{2}$$

denoting the provision coalition for collective good p. It might be clear that through (2) the weight system completely determines the provision coalition structure. I emphasize that an agent can be member of multiple provision coalitions or of none, i.e., the provision coalition structure $\mathcal{L}_\Lambda$ is not necessarily a partition of A. Thus, $\mathcal{L}_\Lambda$ is not a "coalition structure" in the sense of Aumann and Drèze [2]; it is better to categorize a provision coalition $E \in \mathcal{L}_\Lambda$ as a "club" in the sense of Ellickson, Grodal, Scotchmer and Zame [9].

Provision coalition E_p should be considered to be a group of agents with enough political clout to completely determine the political decision-making processes with regard to the provision of collective good p at a certain provision level. Thus, the goal of coalition E_p is to promote the advancement of collective good p.[5] However, the introduction of a provision coalition through a weight system described above implies rather strong a hypothesis, namely that all political decision-making processes with regard to a collective good are explicitly made *within* that provision coalition. Thus, provision coalitions are essentially autonomous decision-making institutions, possibly only constrained by exogenously given tax-subsidy systems. Obviously this is an abstraction of real practice. For a precedent in the literature for such a construction I refer to Becker [4] and [5].

In the subsequent sections of this paper I consider different formulations of decision-making with regard to the provision of the collective goods within the developed setting. Throughout the following two sections the weight system Λ, and therefore the corresponding provision coalition structure $\mathcal{L}_\Lambda$, is given exogenously. In the final section of this paper I consider the endogenous formation of provision coalitions. In that model the weight system is endogenously generated through adoption of certain strategies by the agents in the economy.

[5]It might be clear that in practice interest groups such as the NRA, Amesty International, and Greenpeace act as lobby groups. Indeed these groups are able to influence decisions regarding the provision of a certain public good, but normally these coalitions will not be able to determine its provision *completely*. Hence, these groups are *not* provision coalitions in the sense of this paper. For further discussion I also refer to Potters [18].

3 Implementation of Pareto optimal provision

In this section I consider the optimal provision of collective goods through provision coalitions guided by a government imposed system of taxes and subsidies. A central authority provides a basic budget for the provision of each collective good, and the provision coalitions play a Nash equilibrium in response.

Definition 3.1 *A* ***complete tax-subsidy system*** *is a pair* (B, t)*, where* $B : \mathcal{P} \to \mathbb{R}$ *assigns to each collective good* $p \in \mathcal{P}$ *a (positive or negative)* ***budget*** B_p *and* $t : \mathcal{P} \times A \to \mathbb{R}$ *assigns to each agent* $a \in A$ *a personalized* ***admission price*** $t_p(a)$ *to each collective good* $p \in \mathcal{P}$ *provided.*

Let (B, t) be some complete tax-subsidy system. For collective good $p \in \mathcal{P}$ the admission price function is given by $t_p : A \to \mathbb{R}$. This admission price might be either a tax ($t_p(a) \geqq 0$) or a subsidy ($t_p(a) < 0$). With regard to an individual economic agent $a \in A$ this implies that under (B, t) he contributes $t_p(a)\, q_p \in \mathbb{R}$ towards the provision of collective good $p \in \mathcal{P}$, when that collective good is provided at level $q_p \in I$. The budget B_p for collective good $p \in \mathcal{P}$ is a lump-sum that could be positive or negative. If $B_p > 0$, the central authority provides a subsidy in support of collective good p. However, if $B_p < 0$, the provision of collective good p is subject to a *fixed cost*.

For collective good $p \in \mathcal{P}$ the available resources for its provision at level $r \in I$ are given by $\sum_{a \in A} t_p(a)\, r + B_p$. It is assumed that provision coalition E_p has complete control over those (variable) resources and fully determines the provision level.

Given the available (variable) resources as well as the provision levels of all other collective goods, a provision coalition tries to optimize the aggregate net utility of its constituents. Here, for each constituent $a \in E_p$, provision coalition E_p takes account of his net utility at provision level r, being $v_a(q_{-p}, r) - t_p(a)\, r$, where $q_{-p} \in I^{\mathcal{P}_{-p}}$ is the vector of the provision levels chosen by the other provision coalitions $E_{p'}$, $p' \neq p$. This implies that all provision coalitions are assumed to be engaged in a strategic game in which the tax-subsidy system (B, t) imposed by the central authority is taken as given.

Definition 3.2 *Let* $\Lambda = \{\lambda_a^p \mid a \in A \text{ and } p \in \mathcal{P}\}$ *be a weight system and let* $\mathcal{L}_\Lambda = \{E_p \mid p \in \mathcal{P}\}$ *be the corresponding provision coalition structure such that* $E_p \neq \emptyset$*,* $p \in \mathcal{P}$*. For any complete tax-subsidy system* (B, t) *the* ***provision game*** *is defined as the normal form game* $\Gamma(B, t) = \langle \mathcal{L}_\Lambda, \Sigma, \{V_p \mid p \in \mathcal{P}\} \rangle$*, where*

- $\mathcal{L}_\Lambda = \{E_p \mid p \in \mathcal{P}\}$ *is the set of players;*

- $\Sigma = \Sigma_1 \times \cdots \times \Sigma_{\mathbf{P}}$ *is the set of strategy tuples with for every provision coalition* E_p *its strategy set given by the feasible provision levels for collective good* $p \in \mathcal{P}$*, i.e.,*

$$\Sigma_p = \left\{ r \in I \;\middle|\; c_p(r) \leqq \sum_{a \in A} t_p(a)\, r + B_p \right\}, \text{ and} \tag{3}$$

- *for every* $p \in \mathcal{P}$*,* $V_p : \Sigma \to \mathbb{R}$ *is the weighted sum of the net utilities of the constituents in* E_p*, i.e.,*

$$V_p(q) = \sum_{a \in E_p} \lambda_a^p \left[v_a(q) - t_p(a)\, q_p \right]. \tag{4}$$

It is assumed that the provision coalitions $\{E_p \mid p \in \mathcal{P}\}$ act *Nash rational* in the sense that given the tax-subsidy system (B, t) each provision coalition E_p $(p \in \mathcal{P})$ selects a best response to the provision levels of all other collective goods $p' \neq p$. For each $p \in \mathcal{P}$ this implies that given the vector of provision levels by the other provision coalitions q_{-p} provision coalition E_p solves the following maximization problem:

$$\max_r V_p(q_{-p}, r) \text{ such that } c_p(r) \leqq \sum_{a \in A} t_p(a)\, r + B_p \tag{5}$$

If all provision coalitions act Nash rational there results a Nash equilibrium in the provision game $\Gamma(B, t)$. I remark that imposing Nash rationality treats all provision coalitions as perfectly competitive, since they take the provision levels of the other collective goods as given in their decision-making processes to determine the provision level of "their" collective good. It might be clear that this is a demanding assumption. (See Example 3.4 below.)

The main result states the existence of a complete tax-subsidy system at which Nash rational responses are optimal in the sense that the provision levels selected are Pareto optimal.

Theorem 3.3 *Let* (f, q) *be a Pareto optimal allocation. If Axiom 2.2 is satisfied, there exists a complete tax-subsidy system* (B, t) *such that* q *is a Nash equilibrium in the provision game* $\Gamma(B, t)$*.*

Proof. Let (f, q) be a Pareto optimal allocation. Given the structure of the model, all conditions of Theorem 3.3(a) in Gilles and Hahn [10] are satisfied. Hence, there exist $t : \mathcal{P} \times A \to \mathbb{R}$ and $T : A \to \mathbb{R}^{\mathcal{P}}$ such that the following conditions are satisfied:

A For every $p \in \mathcal{P}$:

$$\sum_{a \in A} t_p(a)\, q_p + \sum_{a \in A} T_p(a) = c_p(q_p). \tag{6}$$

B For every $p \in \mathcal{P}$ and for any $r \in I$:

$$\sum_{a \in A} t_p(a)\; q_p - c_p(q_p) \leqq \sum_{a \in A} t_p(a)\; r - c_p(r). \tag{7}$$

C For every agent $a \in A$, $(f(a), q)$ maximizes the utility function U_a over the set

$$\left\{ (x, r) \in \mathbb{R}_+ \times I^{\mathcal{P}} \;\middle|\; x + t(a) \cdot r + \sum_{p \in \mathcal{P}} T_p(a) \leqq w_a \right\}, \tag{8}$$

where $t(a) := (t_1(a), \ldots, t_{\mathbf{P}}(a))$.

Define the function t as above and let B be given by $B_p := \sum_{a \in A} T_p(a)$ for every collective good $p \in \mathcal{P}$.

I now show that $q \in I^{\mathcal{P}}$ is indeed a Nash equilibrium of $\Gamma(B, t)$. Let $p \in \mathcal{P}$. Given q_{-p}, according to property **C** above for any agent $a \in E_p$ $(f(a), q_p)$ maximizes $x + v_a(q_{-p}, r)$ over

$$\mathfrak{A}(q_{-p}) := \left\{ (x, r) \in \mathbb{R}_+ \times I^{\mathcal{P}} \;\middle|\; x + t_p(a)\; r + t_{-p}(a) \cdot q_{-p} + \sum_{p \in \mathcal{P}} T_p(a) \leqq w_a \right\}.$$

Single-peakedness of v_a for $p \in \mathcal{P}$— by Axiom 2.2 — now implies that either $q_p = q_p^*(q_{-p})$, or $q_p < q_p^*(q_{-p})$. The first case is attained when $q_p^*(q_{-p})$ is feasible with respect to the given feasible set $\mathfrak{A}(q_{-p})$, while the second case is attained when $q_p^*(q_{-p})$ is not feasible with respect to the given feasible set $\mathfrak{A}(q_{-p})$. This in turn implies that q_p maximizes $v_a(q_{-p}, r) - t_p(a)\; r$ over r. Together with properties **A** and **B** and the fact that $\lambda_a^p > 0$ for every constituent $a \in E_p$, this immediately implies that q_p solves

$$\max_r \sum_{a \in E_p} \lambda_a^p \left[v_a(q_{-p}, r) - t_p(a)\; r \right] \text{ subject to } c_p(r) \leqq \sum_{a \in A} t_p(a)\; r + B_p.$$

This completes the proof of the assertion. ■

In Theorem 3.3 all provision coalitions act perfectly competitive. However, if provision coalitions fail to compete in this fashion the provision levels might be nonoptimal. This is illustrated by the following example in which there emerge leadership and follower roles for the provision coalitions under consideration.

Example 3.4 Consider an economy consisting of two agents, $A = \{a, b\}$, two collective commodities $\mathcal{P} = \{1, 2\}$, and two provision coalitions $E_1 = \{a\}$ and $E_2 = \{b\}$. I only develop this economy partially, omitting the endowments. Regarding the preferences I assume that

$$\begin{aligned} v_a(q_1,q_2) &= q_1 q_2 \\ v_b(q_1,q_2) &= q_1 q_2^2 \end{aligned}$$

and with respect to the cost functions I introduce

$$\begin{aligned} c_1(r) &= \alpha r \\ c_2(r) &= \beta r + \gamma \end{aligned}$$

where $0 < \alpha < 2$, $0 < 2\beta < 3\alpha$, and $\gamma \in \mathbb{R}$.

From the specification and the proofs given in Gilles and Hahn [10], page 386, as well as the proof of Theorem 3.3 given above, it is clear that the complete tax-subsidy system (B,t) given by $t_1(a) = \frac{\alpha}{2}$, $t_1(b) = \frac{\alpha}{2}$, $t_2(a) = \frac{\beta}{3}$, $t_2(b) = \frac{2\beta}{3}$, $B_1 = 0$, and $B_2 = \gamma$ is an optimal tax-subsidy system supporting a Pareto optimal allocation of collective goods. Indeed, the strategic game $\Gamma(B,t)$ has a Nash equilibrium given by $\widehat{q}_1 = \frac{\alpha}{2}$ and $\widehat{q}_2 = \frac{2\beta}{3\alpha}$. In this equilibrium the net utilities for the two agents are given by $\widehat{U}_a = \frac{\beta}{3} - \frac{\alpha^2}{4} - \frac{2\beta^2}{9\alpha}$ and $\widehat{U}_b = -\frac{\alpha^2}{4} - \frac{2\beta^2}{9\alpha}$.

Next consider the situation in which provision coalition E_1 is a Von Stackelberg leader and provision coalition E_2 is a Von Stackelberg follower for the given tax-subsidy system. In that case $E_2 = \{b\}$ solves

$$\max_{q_2} \; q_1 q_2^2 - \frac{\alpha}{2} q_1 - \frac{2\beta}{3} q_2, \tag{9}$$

resulting in the best response function

$$\widetilde{q}_2(q_1) = \frac{\beta}{3q_1}.$$

I emphasize that the constraints in this optimization problem imposed by tax-subsidy system (B,t) are always satisfied. Now provision coalition $E_1 = \{a\}$ takes the best response function of E_2 into account and solves

$$\max_{q_1} \; q_1 \widetilde{q}_2(q_1) - \frac{\alpha}{2} q_1 - \frac{\beta}{3} \widetilde{q}_2(q_1), \tag{10}$$

resulting in the equilibrium described by $\widetilde{q}_1 = \frac{\beta}{3\alpha}\sqrt{2\alpha}$ and $\widetilde{q}_2 = \sqrt{\frac{\alpha}{2}}$. In the equilibrium the net utilities are given by $\widetilde{U}_a = \frac{\beta}{3}\left(1 - \sqrt{2\alpha}\right)$ and $\widetilde{U}_b = -\frac{\beta}{3}\sqrt{2\alpha}$. It is obvious that for the given parameter values $\widetilde{U}_a > \widehat{U}_a$. ♦

The example leads to some interesting insights with regard to the stability of the coalitional provision mechanism. Indeed it shows that there are certain situations in which the tax-subsidy system guides the economy to an equilibrium that is unstable. In particular, being a Von Stackelberg leader might be profitable and that consequently provision coalitions may engage in a race to commit first to a certain provision level.

4 Autonomous coalitional provision

In this section there is given a *normalized* weight system $\Lambda = \{\lambda_a^p \mid a \in A \text{ and } p \in \mathcal{P}\}$ with $\sum_{a \in A} \lambda_a^p = 1$ for every collective good $p \in \mathcal{P}$. Furthermore, $\mathcal{L}_\Lambda = \{E_p \mid p \in \mathcal{P}\}$ is defined as the corresponding provision coalition structure. Note that, for every $p \in \mathcal{P}$, $E_p \equiv \{a \in A \mid \lambda_a^p > 0\} \neq \emptyset$.

I consider a model in which the provision coalitions are completely *autonomous* in their determination of the level at which the collective good is provided. Hence, they are not constrained by an imposed system of taxes and subsidies. Each provision coalition $E \in \mathcal{L}_\Lambda$ is assumed to maximize the weighted sum of the net utilities of its constituents. A central authority is assumed to impose a certain tax system to finance the collective good at the level determined by the provision coalition in question. This central authority is assumed to be absent from the model. I assume that the central authority levies a tax such that the budget is exactly balanced, i.e., all tax rates are determined by the budget balance condition.

I consider three different tax systems that might be imposed by the central authority. The common feature of these taxation schemes is that taxation is uniform, i.e., for each collective good *all* agents in the economy have to pay the specified tax to support its provision. This implies the exclusion of discrimination between constituents and non-constituents of a provision coalition. In this respect the central authority is assumed to be impartial.

Admission tax The central authority levies a uniform admission tax t_p to finance collective good $p \in \mathcal{P}$. All citizens pay this tax. Using the imposed budget balance this implies that given the provision levels of the other collective goods $q_{-p} \in I^{\mathcal{P}_{-p}}$ provision coalition $E_p \in \mathcal{L}_\Lambda$ for $p \in \mathcal{P}$ solves the following optimization problem

$$\max_{r, t_p} \sum_{a \in E_p} \lambda_a^p \left[v_a \left(q_{-p}, r \right) - t_p \, r \right] \text{ subject to } c_p \left(r \right) \leqq n \, t_p \, r. \tag{11}$$

Note that the term between the brackets in optimization problem (11) is the net utility of constituent $a \in E_p$.

Poll tax The central authority levies an egalitarian lump-sum tax T_p to finance collective good $p \in \mathcal{P}$. Such a lump-sum tax system is also known as a poll tax system.[6] Each citizen pays this tax. Again, given the imposed budget balance and the provision levels of the other collective goods $q_{-p} \in I^{\mathcal{P}_{-p}}$ provision coalition E_p solves the following optimization problem

$$\max_{r,T_p} \sum_{a \in E_p} \lambda_a^p \left[v_a \left(q_{-p}, r \right) - T_p \right] \text{ subject to } c_p \left(r \right) \leqq n\, T_p. \tag{12}$$

Proportional income tax The central authority levies a proportional income tax rate τ_p to finance collective good $p \in \mathcal{P}$. Each citizen pays this income tax. Thus, given the imposed budget balance and the provision levels of the other collective goods $q_{-p} \in I^{\mathcal{P}_{-p}}$ provision coalition E_p solves the following optimization problem

$$\max_{r,\tau_p} \sum_{a \in E_p} \lambda_a^p \left[v_a \left(q_{-p}, r \right) - \tau_p\, w_a \right] \text{ subject to } c_p \left(r \right) \leqq \tau_p\, \overline{w}, \tag{13}$$

where $\overline{w} \equiv \sum_{a \in A} w_a$.

I am now in the position to introduce a non-cooperative game Γ_Λ in which the provision coalitions are the players and the strategies selected are the provision levels. However, before this game can be defined properly, I introduce some preliminary concepts.

Axiom 4.1 *The central authority finances the provision of collective good $p \in \mathcal{P}$ by using a tax system that is based either on an admission tax, or on poll tax, or on an income tax, and selects the appropriate tax rate to establish budget balance.*
If collective good p is financed through a proportional income tax, it is additionally assumed that $\sum_{a \in E_p} \lambda_a^p\, w_a = \frac{1}{n} \sum_{a \in A} w_a$.

Axiom 4.1 separates the financing of a collective through a proportional income tax system from financing by using the two other tax systems considered. The additional requirement on the weight system is just a technicality; it allows to treat all tax systems equivalently. In particular, Axiom 4.1 implies that all provision coalitions have exactly the same objective function.

[6] Here I also refer to Konishi, Le Breton and Weber [12], who consider similar tax systems for a game theoretic model of an economy with local public goods.

Lemma 4.2 *If Axiom 4.1 holds, irrespective of the tax system for the financing of collective good* $p \in \mathcal{P}$ *the provision coalition* $E_p \in \mathcal{L}_\Lambda$ *applies the objective function given by* $V_p : I^{\mathcal{P}} \to \mathbb{R}$ *with*

$$V_p(q) = \sum_{a \in E_p} \lambda_a^p \, v_a(q) - \frac{c_p(q_p)}{n}, \; q \in I^{\mathcal{P}}. \tag{14}$$

Proof. Let $p \in \mathcal{P}$. For each of the three tax systems introduced I consider the formulation of the objective function for provision coalition $E_p \in \mathcal{L}_\Lambda$. First, note that the balanced financing of the collective good required in Axiom 4.1 implies that all tax rates are determined endogenously by the budget balance equations.

Admission taxation: It is immediate from (11) and budget balance that the central authority imposes $t_p = \frac{c_p(r)}{n\,r}$ and thus (11) becomes a simple maximization problem given by

$$\max_r \sum_{a \in E_p} \lambda_a^p \left[v_a(q_{-p}, r) - \frac{c_p(r)}{n} \right]$$

Since by assumption $\sum_{a \in E_p} \lambda_a^p = 1$ it follows immediately that this objective function indeed translates into the one given in equation (14).

Equal taxation: Again, from (12) it immediately follows that the central authority selects $T_p = \frac{c_p(r)}{n}$ to achieve budget balance and thus the result follows immediately.

Income taxation: It is immediate from (13) that the central authority imposes the income tax rate $\tau_p = \frac{c_p(r)}{\overline{w}}$ and thus (13) becomes

$$\max_r \sum_{a \in E_p} \lambda_a^p \left[v_a(q_{-p}, r) - \frac{w_a}{\overline{w}} \, c_p(r) \right]$$

Recall that by Axiom 4.1 $\sum_{a \in E_p} \lambda_a^p = 1$ and $n \sum_{a \in E_p} \lambda_a^p \, w_a = \overline{w}$. Now, the above can be rewritten as (14). ∎

Using Lemma 4.2 I now introduce a normal form game that describes the provision of collective goods through autonomous provision coalitions.

Definition 4.3 *Let* Λ *be a normalized weight system and let Axiom 4.1 be satisfied. The* ***autonomous provision game*** *is defined to be the normal form game*

$$\Gamma_\Lambda := \left\langle \mathcal{L}_\Lambda, I^{\mathcal{P}}, \{ V_p \mid p \in \mathcal{P} \} \right\rangle,$$

where

- $\mathcal{L}_\Lambda = \{ E_p \mid p \in \mathcal{P} \}$ *is the set of players;*

- *$I^{\mathcal{P}}$ is the set of strategy tuples with for every provision coalition E_p its strategy set given by $I \equiv [0,1]$, the set of all potential provision levels for collective good $p \in \mathcal{P}$, and*
- *for every $p \in \mathcal{P}$, $V_p : I^{\mathcal{P}} \to \mathbb{R}$ is the objective function given in Lemma 4.2, i.e.,*

$$V_p(q) = \sum_{a \in E_p} \lambda_a^p v_a(q) - \frac{c_p(q_p)}{n}, q \in I^{\mathcal{P}}.$$

For the autonomous provision game I investigate the existence and uniqueness of Nash equilibrium. Furthermore, I address whether there is over- or underprovision in Nash equilibrium when compared to Pareto optimal provision levels.

4.1 Nash equilibria: Existence and uniqueness

The first question to be addressed with regard to the autonomous provision game Γ_Λ is the existence of Nash equilibria.

Theorem 4.4 (Existence) *Suppose Axioms 2.2 and 4.1 are satisfied. Then the first order conditions for a Nash equilibrium $\widehat{q} \in I^{\mathcal{P}}$ in the autonomous provision game Γ_Λ are given by*

$$\sum_{a \in E_p} \lambda_a^p \frac{\partial v_a}{\partial q_p}(\widehat{q}) = \frac{1}{n} c_p'(\widehat{q}_p), \quad p \in \mathcal{P}. \tag{15}$$

Furthermore, suppose that for every $a \in A$ and $p \in \mathcal{P}$

$$\frac{\partial^2 v_a}{\partial q_p^2}(q) < 0, \quad q \in I^{\mathcal{P}}, \tag{16}$$

then there exists a Nash equilibrium in pure strategies in the autonomous provision game Γ_Λ.

Proof. The first order condition (15) follows immediately from the definition of the autonomous provision game Γ_Λ.

Next I investigate the existence of Nash equilibrium by showing that there exists a solution to the first order conditions (15). Namely, the additional assumption that $\frac{\partial^2 v_a}{\partial q_p^2} < 0$ for every agent $a \in A$ and each collective good $p \in \mathcal{P}$ justifies relying on the first order conditions only.
First, I will show that for each provision coalition the best response correspondence is a well-defined continuously differentiable function.

Uniqueness of the best response.

From the conditions on v_a, $a \in A$, and c_p it follows that $\sum_{a \in E_p} \lambda_a^p \frac{\partial v_a}{\partial q_p}$ is a strictly decreasing function and c_p' is a decreasing function. Thus, for $0 < q_p < 1$

$$\frac{\partial}{\partial q_p}\left[\sum_{a \in E_p} \lambda_a^p \frac{\partial v_a}{\partial q_p}(q) - \frac{1}{n} c_p'(q_p)\right] = \sum_{a \in E_p} \lambda_a^p \frac{\partial^2 v_a}{\partial q_p^2}(q) - \frac{1}{n} c_p''(q_p) < 0 \tag{17}$$

I consider two cases:

Case A $\frac{1}{n} \lim_{r \downarrow 0} c_p'(r) > \lim_{r \downarrow 0} \sum_{a \in E_p} \lambda_a^p \frac{\partial v_a}{\partial q_p}(q_{-p}, r)$.
In this case $q_p = 0$ is the unique best response to q_{-p} as also follows from the second order derivatives of c_p and v_a, $a \in A$.

Case B $\frac{1}{n} \lim_{r \downarrow 0} c_p'(r) \leqq \lim_{r \downarrow 0} \sum_{a \in E_p} \lambda_a^p \frac{\partial v_a}{\partial q_p}(q_{-p}, r)$.
Now if $\lim_{r \uparrow 1} \frac{\partial v_a}{\partial q_p}(q_{-p}, r) < 0$ for some $a \in E_p$, it follows together with Axiom 2.2 that $\lim_{r \uparrow 1} \sum_{a \in E_p} \frac{\partial v_a}{\partial q_p}(q_{-p}, r) < 0$. Hence, with the intermediate value theorem (Theorem 5.11 in Apostol [1]) there exists a solution to the first order condition given as (15) for p. By (17) it follows that this solution is unique and indeed is a best response to q_{-p}.
Now suppose that $\lim_{r \uparrow 1} \frac{\partial v_a}{\partial q_p}(q_{-p}, r) = 0$ for all $a \in E_p$. In this case there is either a solution to the first order condition described in the previous paragraph, implying the existence of a unique best response to q_{-p}, or $q_p = 1$ is a best response to q_{-p}.

Cases **A** and **B** shown above imply that for each provision coalition E_p there is indeed a unique best response to q_{-p}. From (17) it follows from the implicit function theorem (Theorem 13.7 in Apostol [1]) that there exists a continuous best response function φ_p such that for situations covered by Case **B** above it holds that

$$\sum_{a \in E_p} \lambda_a^p \frac{\partial v_a}{\partial q_p}\left(q_{-p}, \varphi_p(q_{-p})\right) = \frac{1}{n} c_p'\left(\varphi_p(q_{-p})\right).$$

From regularity conditions imposed it should also be clear that in the situations covered by Case **A** the best response function φ_p is continuous.

Existence of Nash equilibrium.
The collection of best response functions $\{\varphi_p \mid p \in \mathcal{P}\}$ defines a continuous function $\varphi : I^{\mathcal{P}} \to I^{\mathcal{P}}$. Brouwer's fixed point theorem can be applied to show the existence of a fixed point $\widehat{q} \in I^{\mathcal{P}}$ with $\varphi_p(\widehat{q}_{-p}) = \widehat{q}_p$, thus implying the existence of a Nash equilibrium in pure strategies in Γ_Λ. ■

I remark that in Theorem 4.4 the condition on the utility functions cannot be weakened straightforwardly. In particular, the fixed point argument employed in the proof of Theorem 4.4 cannot be replaced by an argument based on Kakutani's fixed point theorem for correspondences. Namely, the best response correspondences are not guaranteed to have convex values. A counterexample can easily be constructed in which multiple, strictly separated best responses for each provision coalition can be identified.[7]

Next we turn to the question when there exists only one Nash equilibrium in the autonomous provision game Γ_Λ. This is an issue that has to be resolved in order to address the endogenous formation of provision coalitions in a two-stage game in which the second stage is the provision game among the formed provision coalitions. This is addressed in the next theorem.

Theorem 4.5 (Uniqueness) *Let Axiom 2.2 be satisfied and let* $c_p'' > 0$ *for every* $p \in \mathcal{P}$. *Furthermore, let for every agent* $a \in A$ *and every* $q \in I^{\mathcal{P}}$ *the modified Hessian* $H_p^\Lambda(q)$ *be given by*

$$H_a^\Lambda(q) := \left[\lambda_a^p \frac{\partial^2 v_a}{\partial q_p \partial q_{p'}}(q) \right]_{p,p' \in \mathcal{P}}.$$

Assume that for the given weight system Λ, *(15) has an internal solution, say* $\widehat{q}$. *If for every agent* $a \in A$ *and every* $q \in I^{\mathcal{P}}$ *the modified Hessian* $H_a^\Lambda(q)$ *is negative semidefinite, then* $\widehat{q}$ *is the only internal Nash equilibrium of the autonomous provision game* Γ_Λ.

Proof. I first remark that the system of equations (15) for the internal solution $\widehat{q} \in (0,1)^{\mathcal{P}}$ can be rewritten as a zero point condition $f(\widehat{q}) = 0$ for a continuously differentiable function $f : (0,1)^{\mathcal{P}} \to \mathbb{R}^{\mathcal{P}}$, where the function f is defined by (15). As Mas-Colell [16] states on page 216, it follows from the Index Theorem (Theorem 5.5.3, [16]) that if for this function f the matrix of partial derivatives Df is negative definite, the zero point indicated is unique.

Here the matrix of partial derivatives of the introduced function f is given by

$$Df(q) = \left[\sum_{a \in E_p} \lambda_a^p \frac{\partial^2 v_a}{\partial q_p \partial q_{p'}}(q) \right]_{p,p' \in \mathcal{P}} - \frac{1}{n} \operatorname{diag} \left[c_p''(q_p) \right]_{p \in \mathcal{P}}.$$

This implies that for every vector $z \neq 0$ we can write

$$z \cdot Df(q) \cdot z = \sum_{p \in \mathcal{P}} \left[z_p \sum_{a \in E_p} \lambda_a^p \sum_{p' \in \mathcal{P}} \frac{\partial^2 v_a}{\partial q_p \partial q_{p'}}(q)\, z_{p'} - \frac{1}{n} z_p^2 \, c_p''(q_p) \right]$$

[7] Such a counter example can be based on a higher order polynomial utility function that still satisfies Axiom 2.2.

$$= \quad z \cdot H_a^\Lambda(q) \cdot z - \frac{1}{n} \sum_{p \in \mathcal{P}} z_p^2 \, c_p''(q_p)$$

Here we used the definition of the modified Hessian. Since by assumption $z \cdot H_a^\Lambda(q) \cdot z \leqq 0$, $c_p''(q_p) > 0$ $(p \in \mathcal{P})$, and $z \neq 0$, I conclude that $z \cdot Df(q) \cdot z < 0$. Thus, Df is indeed negative definite, thus implying the uniqueness of the solution to the system of equations given by (15). ■

The condition for uniqueness formulated in Theorem 4.5 can be interpreted as a concavity condition on the utility structure and the production technology. For a special class of utility functions this condition can be simplified significantly.

Definition 4.6 *Let $a \in A$. The utility function v_a is called* ***additively separable*** *if for every $q \in I^{\mathcal{P}}$ it can be represented as $v_a(q) = \sum_{p \in \mathcal{P}} v_a^p(q_p)$.*

From the proofs of Theorem 4.4 it follows immediately that if all utility functions are additively separable, the best response function constructed for $p \in \mathcal{P}$ only depends on one variable, namely q_p. Together with the proof of Theorem 4.5 uniqueness of Nash equilibrium follows immediately. A proof of the following corollary is therefore omitted.

Corollary 4.7 *Assume that Axioms 2.2 and 4.1 are satisfied and let for every agent $a \in A$ the utility function v_a be additively separable. Furthermore, suppose that for every $a \in A$ and $p \in \mathcal{P}$ $\frac{\partial^2 v_a^p}{\partial q_p^2}(q) < 0$, $q \in I^{\mathcal{P}}$, then there exists a unique Nash equilibrium in pure strategies in the autonomous provision game Γ_Λ.*

4.2 Over- and underprovision of collective goods

The next issue to be addressed is under which conditions there is over- or underprovision of the collective goods when provided autonomously by the provision coalitions. The benchmark in this discussion is the notion of a Pareto optimal allocation introduced in Definition 2.3. Pareto optimal allocations are also identified by solving the following maximization problem:

$$\max_{(f,q)} \sum_{a \in A} \mu_a \left[f(a) + v_a(q) \right] \quad \text{subject to} \quad \sum_{a \in A} f(a) + \sum_{p \in \mathcal{P}} c_p(q_p) = \overline{w} \tag{18}$$

for arbitrary weights $\mu \in \mathbb{R}^A$. From (18) we find the first order condition for a Pareto optimal provision scheme $q^* \in I^{\mathcal{P}}$ to be the standard Samuelson conditions,

$$\sum_{a \in A} \frac{\partial v_a}{\partial q_p}(q^*) = c_p'\left(q_p^*\right), \quad p \in \mathcal{P}. \tag{19}$$

Also I introduce for every agent $a \in A$ and every $q \in I^{\mathcal{P}}$ the *Hessian* of second order partial derivatives of the utility function v_a by the $\mathbf{P} \times \mathbf{P}$-matrix

$$H_a(q) := \left[\frac{\partial^2 v_a}{\partial q_p \partial q_{p'}}(q) \right]_{p,p' \in \mathcal{P}}. \tag{20}$$

In order to make a careful comparison of Pareto optimal levels of provision with the provision levels resulting in Nash equilibrium in the game Γ_Λ I have to make the following assumption which extends Axiom 2.2 with a condition on the second order partial derivatives of the utility function. This additional requirement simply is strict concavity of the utility functions for the collective goods.

Axiom 4.8 *Axioms 2.2 and 4.1 are satisfied with the additional provision that for every $a \in A$ and every $q \in I^{\mathcal{P}}$ the Hessian $H_a(q)$ is negative definite.*

The over- and underprovision of collective goods through autonomous provision is addressed in the following two propositions. The first proposition confirms the intuitive property that under certain regularity conditions overprovision occurs when for all provision coalitions the marginal utility is above the overall average of the marginal utilities of all agents in the economy. The second proposition confirms the reverse of this intuitive property.

Proposition 4.9 (Overprovision) *Let Axiom 4.8 be satisfied. Suppose that there exists an internal solution to the Pareto optimality problem (18), indicated by $q^* \in I^{\mathcal{P}}$. Furthermore, suppose that there exists a Nash equilibrium in the autonomous provision game Γ_Λ, indicated by $\widehat{q} \in I^{\mathcal{P}}$, which is an internal solution to the system of equations (15).*

(a) *If for every $p \in \mathcal{P}$*

$$\sum_{a \in E_p} \lambda_a^p \frac{\partial v_a}{\partial q_p}(q^*) > \frac{1}{n} \sum_{a \in A} \frac{\partial v_a}{\partial q_p}(q^*), \tag{21}$$

or for every $p \in \mathcal{P}$

$$\sum_{a \in E_p} \lambda_a^p \frac{\partial v_a}{\partial q_p}(\widehat{q}) > \frac{1}{n} \sum_{a \in A} \frac{\partial v_a}{\partial q_p}(\widehat{q}), \tag{22}$$

then $\widehat{q}_{p'} > q^*_{p'}$ for some $p' \in \mathcal{P}$.

(b) *Let $p \in \mathcal{P}$. Suppose that property (21) or property (22) holds for $p \in \mathcal{P}$. If, additionally, for every $a \in A$ the utility function v_a is separable with respect to the collective good $p \in \mathcal{P}$ in the sense that for all $p' \in \mathcal{P}_{-p}$ $\frac{\partial^2 v_a}{\partial q_p \partial q_{p'}}(q) = 0$, $q \in I^{\mathcal{P}}$, then $\widehat{q}_p > q^*_p$.*

Proof. Let $\widehat{q}$ and q^* be as asserted in the statement of the theorem.

Proof of (a)

For every $a \in A$ I introduce $\nabla v_a(q) := \left(\frac{\partial v_a}{\partial q_1}(q), \ldots, \frac{\partial v_a}{\partial q_{\mathbf{P}}}(q)\right)$ as the gradient of the utility function v_a at $q \in I^{\mathcal{P}}$. Since the Hessian $H_a(q)$ is negative definite for every q it follows from Proposition 5.7.3 (iii) in Mas-Colell [16], page 217, that ∇v_a is a monotone function in the sense that for all $q \neq \widetilde{q}$

$$(q - \widetilde{q}) \cdot (\nabla v_a(q) - \nabla v_a(\widetilde{q})) < 0.$$

I show assertion (a) by contradiction. Suppose to the contrary that $\widehat{q} \leqq q^*$, i.e., $\widehat{q}_p \leqq q_p^*$ for all $p \in \mathcal{P}$. Then it immediately follows that $c_p'(\widehat{q}_p) \leqq c_p'(q_p^*)$ for each $p \in \mathcal{P}$. Next we show that $\sum_{a\in A} \frac{\partial v_a}{\partial q_p}(q^*) \leqq \sum_{a\in A} \frac{\partial v_a}{\partial q_p}(\widehat{q})$ for some $p \in \mathcal{P}$.

Indeed if $\widehat{q} = q^*$, it is trivially the case that $\frac{\partial v_a}{\partial q_p}(q^*) = \frac{\partial v_a}{\partial q_p}(\widehat{q})$ for every $a \in A$ and every $p \in \mathcal{P}$. This implies the desired conclusion that $\sum_{a\in A} \frac{\partial v_a}{\partial q_p}(q^*) \leqq \sum_{a\in A} \frac{\partial v_a}{\partial q_p}(\widehat{q})$.

Hence, suppose that $\widehat{q} \neq q^*$, implying $\widehat{q} < q^*$. Now since the function ∇v_a is monotone it follows immediately that the function $\sum_{a\in A} \nabla v_a$ is monotone as well. Hence, $(q^* - \widehat{q}) \cdot \left(\sum_{a\in A} \nabla v_a(q^*) - \sum_{a\in A} \nabla v_a(\widehat{q})\right) < 0$. Since $q^* - \widehat{q} > 0$ it has to hold that $\sum_{a\in A} \frac{\partial v_a}{\partial q_p}(q^*) \leqq \sum_{a\in A} \frac{\partial v_a}{\partial q_p}(\widehat{q})$ for some $p \in \mathcal{P}$.

Hence, the derived property for $p \in \mathcal{P}$ now implies that

$$\begin{aligned} \frac{1}{n} \sum_{a\in A} \frac{\partial v_a}{\partial q_p}(\widehat{q}) &\geqq \frac{1}{n} \sum_{a\in A} \frac{\partial v_a}{\partial q_p}(q^*) = \frac{1}{n} c_p'(q_p^*) \\ &\geqq \frac{1}{n} c_p'(\widehat{q}_p) = \sum_{a\in E_p} \lambda_a^p \frac{\partial v_a}{\partial q_p}(\widehat{q}). \end{aligned}$$

This contradicts (22) for $p \in \mathcal{P}$.

From the above it follows that (21) has to hold if $\widehat{q} \leqq q^*$. I complete the proof of assertion (a) by showing that (21) cannot hold either.

I show that under the assumptions made it holds that for some $p \in \mathcal{P}$:

$$\sum_{a\in E_p} \lambda_a^p \frac{\partial v_a}{\partial q_p}(q^*) \leqq \sum_{a\in E_p} \lambda_a^p \frac{\partial v_a}{\partial q_p}(\widehat{q})$$

Indeed, if $\widehat{q} = q^*$ it is trivially concluded that $\frac{\partial v_a}{\partial q_p}(q^*) = \frac{\partial v_a}{\partial q_p}(\widehat{q})$ for every $a \in A$ and every $p \in \mathcal{P}$. This implies the desired conclusion that $\sum_{a\in A} \frac{\partial v_a}{\partial q_p}(q^*) \leqq \sum_{a\in A} \frac{\partial v_a}{\partial q_p}(\widehat{q})$.

Hence, suppose that $\widehat{q} \neq q^*$, implying $\widehat{q} < q^*$. Now since the function ∇v_a is monotone it follows immediately that for every $p \in \mathcal{P}$ the function $\sum_{a\in E_p} \lambda_a^p \nabla v_a$ is monotone as well. Hence, for every $p \in \mathcal{P}$

$$(q^* - \widehat{q}) \cdot \left(\sum_{a\in E_p} \lambda_a^p \nabla v_a(q^*) - \sum_{a\in E_p} \lambda_a^p \nabla v_a(\widehat{q})\right) < 0.$$

Since $q^* - \widehat{q} > 0$ it has to hold that $\sum_{a \in E_p} \lambda_a^p \frac{\partial v_a}{\partial q_p}(q^*) \leqq \sum_{a \in E_p} \lambda_a^p \frac{\partial v_a}{\partial q_p}(\widehat{q})$ for some $p \in \mathcal{P}$.

The derived property for $p \in \mathcal{P}$ now implies that

$$\begin{aligned} \frac{1}{n} \sum_{a \in A} \frac{\partial v_a}{\partial q_p}(q^*) &= \frac{1}{n} c_p' \left(q_p^*\right) \geqq \frac{1}{n} c_p' \left(\widehat{q}_p\right) \\ &= \sum_{a \in E_p} \lambda_a^p \frac{\partial v_a}{\partial q_p}(\widehat{q}) \geqq \sum_{a \in E_p} \lambda_a^p \frac{\partial v_a}{\partial q_p}(q^*) . \end{aligned}$$

This indeed contradicts (21). This completes the construction of the contradiction, implying that the initial hypothesis that $\widehat{q} \leqq q^*$ cannot be true. Thus, $\widehat{q}_{p'} > q_{p'}^*$ for some $p' \in \mathcal{P}$, hence, proving assertion (a).

Proof of (b)

To show the second assertion suppose that property (21) or property (22) holds for $p \in \mathcal{P}$ and suppose furthermore that for every $a \in A$ the utility function v_a is separable with respect to collective good $p \in \mathcal{P}$. Assume now to the contrary that $\widehat{q}_p \leqq q_p^*$.

Then it immediately follows that $c_p'(\widehat{q}_p) \leqq c_p'\left(q_p^*\right)$.

By the assumption that for every agent the Hessian is negative definite it immediately follows that $\frac{\partial^2 v_a}{\partial q_p^2}(q) < 0$ for every $a \in A$. Hence,

$$\sum_{a \in A} \frac{\partial v_a}{\partial q_p}(q^*) \leqq \sum_{a \in A} \frac{\partial v_a}{\partial q_p}(\widehat{q}) \text{ as well as } \sum_{a \in E_p} \lambda_a^p \frac{\partial v_a}{\partial q_p}(q^*) \leqq \sum_{a \in E_p} \lambda_a^p \frac{\partial v_a}{\partial q_p}(\widehat{q}) .$$

This follows from arguments equivalent to the ones used above to prove assertion (a) of the theorem. Now the rest of the proof of assertion (b) proceeds as above to a contradiction with the asserted conditions (21) and (22).

This completes the proof of the theorem. ∎

The next proposition follows immediately from the proof of the proposition on the overprovision of the collective goods by reversing the inequalities. A proof is therefore omitted.

Proposition 4.10 (Underprovision) *Let Axiom 4.8 be satisfied. Suppose there exists an internal solution to the Pareto optimality problem (18), indicated by $q^* \in I^{\mathcal{P}}$. Furthermore, suppose that there exists a Nash equilibrium in the provision game Γ_Λ, indicated by $\widehat{q} \in I^{\mathcal{P}}$, which is an internal solution to the system of equations (15).*

(a) *If for every $p \in \mathcal{P}$*

$$\sum_{a \in E_p} \lambda_a^p \frac{\partial v_a}{\partial q_p}(q^*) < \frac{1}{n} \sum_{a \in A} \frac{\partial v_a}{\partial q_p}(q^*) , \tag{23}$$

or for every $p \in \mathcal{P}$

$$\sum_{a \in E_p} \lambda_a^p \frac{\partial v_a}{\partial q_p}(\widehat{q}) < \frac{1}{n} \sum_{a \in A} \frac{\partial v_a}{\partial q_p}(\widehat{q}), \tag{24}$$

then $\widehat{q}_{p'} < q^*_{p'}$ for some $p' \in \mathcal{P}$.

(b) *Let $p \in \mathcal{P}$. Suppose that property (23) or property (24) holds for $p \in \mathcal{P}$. If, additionally, for every $a \in A$ the utility function v_a is separable with respect to the collective good $p \in \mathcal{P}$ in the sense that for all $p' \in \mathcal{P}_{-p}$ $\frac{\partial^2 v_a}{\partial q_p \partial q_{p'}}(q) = 0$, $q \in I^{\mathcal{P}}$, then $\widehat{q}_p < q^*_p$.*

5 Endogenous coalition formation

In this section I consider a two-stage provision game with endogenous coalition formation. In the first stage provision coalitions are formed, while in the second stage these provision coalitions play the autonomous provision game analyzed in the previous section. One would expect that since coalitions are formed endogenously, pooling occurs and overprovision would be the natural outcome. With an example, however, I show that the resulting subgame perfect Nash equilibria are far from determined and that over- as well as underprovision can occur.

Formally, I introduce the two-stage provision game as follows:

First stage: coalition formation. All agents $a \in A$ select one and only one collective good $p_a \in \mathcal{P}$ for which that agent wants to participate in determining its provision level.[8]

The first stage thus results in a collection of provision coalitions denoted by $\widehat{\mathcal{L}} := \left\{ \widehat{E}_p \mid p \in \mathcal{P} \right\}$, where $\widehat{E}_p := \{a \in A \mid p_a = p\}$ for every $p \in \mathcal{P}$. Now $\widehat{\mathcal{L}}$ is a partition of A. (Hence, the collection $\widehat{\mathcal{L}}$ is a *coalition structure* in the sense of Aumann and Drèze [2].)

We assume that all constituents of the formed provision coalitions have equal weight. Using this we can also describe the resulting coalition structure by the weight system $\widehat{\Lambda}$ where for every $a \in A$:

$$\widehat{\lambda}_a^p = \begin{cases} \frac{1}{|\widehat{E}_p|} & \text{if } p = p_a \\ 0 & \text{otherwise.} \end{cases}$$

[8] Implicitly the assumption that each agent is member of exactly one provision coalition presupposes that obtaining such a membership is costly. It is clear that other models of coalition formation can be constructed, but it is unlikely that the conclusions will be significantly different from the ones derived here.

Second stage: autonomous provision. Given the weight system $\widehat{\Lambda}$ resulting from the first stage of the game, the formed provision coalitions play the autonomous provision game $\Gamma_{\widehat{\Lambda}}$. This results in some equilibrium provision schedule $\widehat{q}_{\widehat{\Lambda}} \in I^{\mathcal{P}}$, where $\widehat{q}_{\widehat{\Lambda},p} = 0$ if $\widehat{E}_p = \emptyset$.

I only consider subgame perfect Nash equilibria of the two-stage game described above. Using the previous insights we are able to determine the conditions under which such equilibria exist:

Proposition 5.1 *Let all conditions of Theorem 4.5 be satisfied. Then for the two-stage provision game described above there exists at least one subgame perfect Nash equilibrium.*

The proof of the proposition is trivial given Theorem 4.5 and is therefore omitted.

The next question I would like to address is whether there is always a subgame perfect Nash equilibrium of the two-stage provision game in which there is uniform overprovision of the collective goods. The public choice literature on related issues emphasizes that such overprovision should indeed emerge as the natural state. In my model one would expect that this effect would even be strengthened, since I assume the complete absence of a political decision structure and complete autonomy of the provision coalitions in the second stage of the game considered. However, I construct an example of a simple two-agent economy that shows that one should not expect overprovision to be the only case. In fact, I show that for different sets of parameter values any provision situation can be supported in this simple example, thus showing that a clear answer with regard to the question about overprovision cannot be given.

Example 5.2 Consider a two agent economy with two collective goods given by $A = \{a, b\}$ and $\mathcal{P} = \{1, 2\}$. For computational ease I assume that the collective commodity spaces are given by the unbounded nonnegative real space $\mathbb{R}_+$. (I emphasize that we can easily replace these commodity spaces by the unit interval as used in the previous analysis of the autonomous provision game.) For the two agents I assume the following linear utility structure:

$$\begin{aligned} v_a(q) &= \alpha\, q_1 + q_2, \text{where } \alpha > 1, \text{ and} \\ v_b(q) &= q_1 + \beta\, q_2, \text{where } \beta \geqq 0. \end{aligned}$$

I emphasize that this linear utility structure satisfies all regularity conditions that might be imposed on this economy, including the requirements on the preference structure introduced in Axioms 2.2 and 4.8.

Furthermore, for each collective good I assume that the provision costs are quadratic and given by $c_1(r) = \gamma_1 r^2$ and $c_2(r) = \gamma_2 r^2$.
I remark that for this economy the Pareto optimal provision levels are given by

$$q_1^* = \frac{1+\alpha}{2\,\gamma_1} \text{and } q_2^* = \frac{1+\beta}{2\,\gamma_2}. \tag{25}$$

I apply the backward induction technique to find the subgame perfect Nash equilibria of the two-stage provision game for this economy. As assumed in the second stage provision game Γ_Λ the two agents are always equally powerful within a formed provision coalition, i.e., we simply select $\lambda_a = \lambda_b = \frac{1}{2}$ in equation (15) if the coalition $\{1,2\}$ forms.
First, I remark that for the given weight system $\widehat{\Lambda}$ the Nash equilibrium in the second stage of the two-stage provision game is unique. I summarize the resulting Nash equilibria in the second stage of this two-stage game in the following table:

(p_a, p_b)	$\widehat{q}_1$	$\widehat{q}_2$
$(1,1)$	$\frac{1+\alpha}{2\,\gamma_1}$	0
$(1,2)$	$\frac{\alpha}{\gamma_1}$	$\frac{\beta}{\gamma_2}$
$(2,1)$	$\frac{1}{\gamma_1}$	$\frac{1}{\gamma_2}$
$(2,2)$	0	$\frac{1+\beta}{2\,\gamma_2}$

This implies that the first stage of the game can be represented in matrix form. In the following matrix, the entries are the utility values obtained by the two agents for the different equilibria resulting in the second stage of the game, given the strategy:

p_a \ p_b	**1**	**2**
1	$\frac{(3\alpha-1)(\alpha+1)}{8\gamma_1}, \frac{(3-\alpha)(\alpha+1)}{8\gamma_1}$	$\frac{\beta(2-\beta)}{2\gamma_2} + \frac{\alpha^2}{2\gamma_1}, \frac{\alpha(2-\alpha)}{2\gamma_1} + \frac{\beta^2}{2\gamma_2}$
2	$\frac{2\alpha-1}{2\gamma_1} + \frac{1}{2\gamma_2}, \frac{2\beta-1}{2\gamma_2} + \frac{1}{2\gamma_1}$	$\frac{(3-\beta)(\beta+1)}{8\gamma_2}, \frac{(3\beta-1)(\beta+1)}{8\gamma_2}$

From the matrix representation of the first stage of the two-stage provision game, I conclude that $\widehat{\mathcal{L}}_1 := \langle\{a\},\{b\}\rangle$ is a subgame perfect Nash equilibrium if and only if the following two inequalities are satisfied

$$\frac{\beta^2}{\gamma_2} \geqq \frac{3\,(1-\alpha)^2}{4\gamma_1} \tag{26}$$

$$\frac{\alpha^2}{\gamma_1} \geqq \frac{3\,(1-\beta)^2}{4\gamma_2} \tag{27}$$

Similarly, $\widehat{\mathcal{L}}_2 := \langle \{b\}, \{a\} \rangle$ is a subgame perfect Nash equilibrium if and only if the following two inequalities are satisfied

$$\frac{1}{\gamma_2} \geqq \frac{3(1-\alpha)^2}{4\gamma_1} \tag{28}$$

$$\frac{1}{\gamma_1} \geqq \frac{3(1-\beta)^2}{4\gamma_2} \tag{29}$$

With these inequalities I investigate equilibria for different sets of parameter values:

1. $\alpha = \beta = \frac{1}{2}$ and $\gamma_1 = \gamma_2 = \frac{1}{2}$.
For these parameter values there are two subgame perfect Nash equilibria. The first SPNE is given by $\widehat{\mathcal{L}}_1$ and the provision schedule $\widehat{q}_{\widehat{\mathcal{L}}_1} = (1, 1)$. The second SPNE is given by $\widehat{\mathcal{L}}_2$ and the provision schedule $\widehat{q}_{\widehat{\mathcal{L}}_2} = (2, 2)$. The Pareto optimal provision schedule for these parameter values is given by $\widetilde{q} = \left(\frac{3}{2}, \frac{3}{2}\right)$, implying that in the first SPNE there is uniform underprovision and in the second SPNE there is uniform overprovision.

2. $\alpha = \frac{1}{2}$, $\beta = 2$, $\gamma_1 = 1$ and $\gamma_2 = 4$.
For these parameter values there are two subgame perfect Nash equilibria. The first SPNE is given by $\widehat{\mathcal{L}}_1$ and the provision schedule $\widehat{q}_{\widehat{\mathcal{L}}_1} = \left(\frac{1}{2}, \frac{1}{2}\right)$. The second SPNE is given by $\widehat{\mathcal{L}}_2$ and the provision schedule $\widehat{q}_{\widehat{\mathcal{L}}_2} = \left(1, \frac{1}{4}\right)$. The Pareto optimal provision schedule for these parameter values is given by $\widetilde{q} = \left(\frac{3}{4}, \frac{3}{8}\right)$, implying that in the first SPNE the first collective good is underprovided, while the second collective good is overprovided. For the second SPNE this is exactly reversed.

Both parameter value sets analyzed above imply that a definitive answer with regard to over- and underprovision of collective goods cannot be given. Conclusions that there naturally is uniform overprovision should, therefore, be dismissed. ♦

6 Concluding remarks

In this paper I presented three game theoretic models of a coalitional provision mechanism for pure collective goods. From this analysis I have drawn two major conclusions.

First, coalitional provision of collective goods can in principle be guided to implement Pareto optimal levels of the collective goods by the choice of an appropriate tax-subsidy system. This tax-subsidy system has to be imposed by a central authority on the agents and the provision coalitions in the economy. There are two fundamental difficulties with such an implementation. The tax-subsidy system has a complex nature and is based on a Lindahl pricing scheme. The critique against Lindahl pricing therefore remains valid for this tax-subsidy system as well: It is simply too complex to have practical value. Furthermore, as shown by example, the implementation of such an optimal tax-subsidy system only has validity if the provision coalitions act perfectly competitive with regard to each other. If certain provision coalitions act differently, the provision of collective goods might be nonoptimal.

Second, if provision coalitions are autonomous and a central authority only imposes tax systems to achieve budget balance, there is no clear answer to the question whether there will be overprovision. It is intuitively expected that agents with the highest marginal utilities will pool into the appropriate provision coalition. In that fashion there should emerge overprovision in the resulting Nash equilibrium in the autonomous provision game. However, as shown in Section 5 there exist very simple examples of economies in which such a conclusion cannot be drawn. In fact spillover effects due to a different division of the agents over the provision coalitions make it impossible to draw *any* conclusion regarding over- or underprovision in equilibrium. In general there might result uniform overprovision, uniform underprovision, and mixed provision levels.

In this paper I have formulated a two stage extensive form game with endogenous coalition formation. For a local public goods economy Konishi and Weber [13] and Konishi, Le Breton and Weber [12] have considered a normal form provision game with endogenous coalition formation. However, these authors limit themselves to Tiebout type economies in which the collective goods provided by the coalitions have a strictly local nature. Extension of these models to address coalitional provision of global collective goods will be addressed in future research.

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