Evolutionary Selection of Behavioral Rules in a Cournot Model

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Evolutionary Selection of Behavioral Rules in a Cournot Model: A Local Bifurcation Analysis*

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Abstract

An evolutionary game theoretic model of Cournot competition with heterogeneous behavioral rules is analyzed. Individuals choose from a finite set of different behavioral rules. In general, a rule specifies the quantity to be produced in the current period as a function of past observations. More sophisticated rules can only be obtained by making extra information costs. The fraction of the population choosing a certain behavioral rule is updated every period on the basis of past performance of that rule. Two updating schedules analyzed in depth are the discrete choice model and the replicator dynamics.

The dynamical system consisting of the population dynamics coupled with the quantity dynamics of the Cournot game may feature local instability and exhibit the associated complicated dynamical phenomena. In particular, high evolutionary pressure or a small noise rate with respect to the choice of behavioral rules leads to highly irregular quantity dynamics converging to a strange attractor. *Journal of Economic Literature* Classification Numbers: C72, C73, D43, L13.

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1 Introduction

The significant influence of agents' motives and decision-making processes on the population's behavior was first recognized and described extensively by Schelling [22]. Schelling indicates that to cope with a population of heterogeneous agents it is necessary to consider the individual motives and decision-making processes in a highly simplified form. In particular, each agent may be considered as a routine with a well-defined way of functioning in relation to its environment. This paper introduces an evolutionary game theoretic model in which the environment is given by the agents who interact among themselves. In fact, each time period the agents are matched in pairs to play a Cournot game. To allow for heterogeneous agents we follow Brock and Hommes [7, 8] and let individuals choose their behavioral rule, i.e. routine, from a finite set of different rules. These rules are functions of past information and have a publicly available performance measure. Furthermore, the costs associated with a certain rule depend on the information and computational requirements of the rule. Individuals make a choice concerning the behavioral rules based upon their past performance and costs. This results in a nonlinear dynamical system with respect to the choice of behavioral rules that is coupled with the dynamics of the endogenous quantities supplied in the Cournot game.

Our concern is whether or not an individual may want to switch between behavioral rules over time. Consider the following scenario. All individuals in a population can either behave according to a sophisticated behavioral rule H_1 at small but positive costs or obtain a simple behavioral rule H_2 for free. Each time period all individuals are randomly matched in pairs to play a Cournot duopoly game. They use the behavioral rules to determine the quantity they supply in this stage game and they tend to choose the rule that yields the highest expected net profit. Suppose that if all players use the sophisticated rule H_1, all time paths of the endogenous variables, i.e. the quantities supplied in the Cournot game, would converge to a unique stable steady state, which is the well known Cournot-Nash equilibrium. Furthermore, assume that when all agents use the simple rule H_2, the Cournot-Nash equilibrium is unstable. If quantities start out close to the Cournot-Nash equilibrium and almost all players use the simple rule H_2, quantities will diverge from the equilibrium and average profits from using rule H_2 will decrease. As a result, the number of players willing to pay some information costs to use the sophisticated behavioral rule H_1 increases. As soon as enough agents have switched to H_1, quantities will be pushed back towards their steady state values. With quantities close to their steady state value, the average net profit of the simple predictor H_2 becomes higher, while the players using the sophisticated rule H_1 still have to pay the information costs. Consequently, most agents will switch to using the simple rule H_2 again, and the story repeats. In this paper we make the above argument rigorous by performing a local bifurcation analysis.

Another model in which agents are identified with a behavioral rule is discussed in Axelrod [2]. Axelrod describes two computer tournaments for which he invited various people to submit automata that would then compete against each other in the repeated Prisoner's Dilemma. The tit-for-tat automaton won both times. His next step was to simulate the effect of further tournaments. He began with an initial population in which each of the automata used in his second tournament was equally numerous. He then used a version of the replicator dynamics to make a guess at the profile of automata that would have been submitted in later tournaments. In the long run, the process settled on a polymorphic profile in which tit-for-tat had the largest share. The survival of the tit-for-tat automaton shows that cooperation in the Prisoner's Dilemma is sustainable.

Related issues concerning the selection and evolution of behavioral or learning rules are discussed in Björnerstedt and Schlag [6] and Kirchkamp and Schlag [14], respectively. Banerjee and Weibull [3] analyze the survival of nonrational agents in a strategic environment represented as a symmetric two-player game. Furthermore, see Vega-Redondo [27] for an evolutionary model of Cournotian oligopoly and Rhode and Stegeman [19] for a duopoly scenario. Extensive discussions on several evolutionary game theoretic models can be found in Samuelson [21], Vega-Redondo [26], and Weibull [28]. For a survey on dynamical systems and chaos theory we refer to Hofbauer and Sigmund [13].

The paper is organized as follows. In section 2 we introduce the general evolutionary model and the corresponding dynamical system. In addition, we state some local instability results. In section 3 and section 4 we analyze two typical examples where agents can choose between two different behavioral rules. For both examples we study two population dynamics: the discrete choice model and the replicator dynamics. Finally, section 5 concludes.

2 The Model

Consider an infinite population of players who are matched in pairs randomly each discrete-time period to play a symmetric Cournot duopoly game. However, instead of simultaneously choosing the supplied quantities directly, the players act according to behavioral rules that exactly prescribe the quantity to be supplied. Note that even though the model is formulated for the duopoly case, it can easily be generalized to allow for any finite number of players being randomly matched to play a symmetric

Cournot oligopoly game. For later convenience we start by briefly considering the traditional Cournot duopoly analysis.

2.1 Traditional Cournot Duopoly Analysis

Let x_i denote the quantity supplied by player $i = 1, 2$. In addition, let $P(x_i + x_j)$, $i \neq j$, denote the inverse demand function and let $C(x_i)$ denote the cost function, which is identical for both players. We assume that $P : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, there exists $\bar{x} < \infty$ such that $P(\bar{x}) = 0$, and P is both C^2, i.e. twice continuously differentiable, and strictly decreasing on $[0, \bar{x})$. Furthermore, we assume that for all $x \in [0, \bar{x})$ it holds that $P'(x) + x_i P''(x) \leq 0$. This means that each firm's marginal revenue is decreasing in the output of the other firm. Equivalently, each firm's profit function has nonpositive cross partials with respect to its own and the other firm's output. Finally, we impose that $P(0) > C'(0)$. This condition states that each firm would like to produce at least a small quantity if it were a monopoly. With respect to the cost function we assume that $C : \mathbb{R}_+ \to \mathbb{R}_+$ is C^2 and nondecreasing.

For player i the profit resulting from the above Cournot game is given by

$$\pi_i(x_i, x_j) = P(x_i + x_j)\, x_i - C(x_i).$$

We assume that the profit function is strictly concave in the firm's own output x_i. Profit maximizing behavior, taking the quantity supplied by the opponent as given, results in the following so-called reaction function for player i

$$x_i = R_i(x_j) = \arg\max_{x_i} \left[P(x_i + x_j)\, x_i - C(x_i) \right].$$

Due to symmetry, players i and j, $i \neq j$, have the same reaction function $R(\cdot)$. Furthermore, for both players the Cournot-Nash equilibrium quantity x^* corresponds to the solution of

$$x^* = R(x^*).$$

As shown by Novshek [17] and Rosen [20] such a solution x^* exists. The first-order condition for a maximum is

$$P(x) + x_i P'(x) - C'(x_i) = 0.$$

Taking the total differential of the first-order condition gives

$$\left(2P'(x) + x_i P''(x) - C''(x_i)\right) \partial x_i + \left(P'(x) + x_i P''(x)\right) \partial x_j = 0.$$

Rearranging terms we find that

$$R_i'(x_j) = \frac{\partial x_i}{\partial x_j} = -\frac{P'(x) + x_i P''(x)}{2P'(x) + x_i P''(x) - C''(x_i)}.$$

Again, symmetry implies that $R_i'(\cdot) = R_j'(\cdot)$. First, notice that the derivative of the reaction function is nonincreasing. The numerator is nonpositive because marginal revenue is decreasing in the other firm's output and the denominator is strictly negative because the profit function is assumed to be strictly concave. Second, the derivative of the reaction function is continuous because both the inverse demand function and the cost function are twice continuously differentiable. Tirole [25] states that a sufficient condition for the equilibrium x^* to be unique is that the derivative of the reaction function is unequal to 1 in absolute value over the relevant range of outputs, i.e. $|R'(x)| \neq 1$ for all $x \in [0, \bar{x})$. Since the derivative of the reaction function is continuous this implies that $|R'(x)| < 1$ for all $x \in [0, \bar{x})$ or $|R'(x)| > 1$ for all $x \in [0, \bar{x})$.

The traditional Cournot analysis refers to a static environment. However, in a dynamic setting the reaction function introduced above can be used to establish the so-called best-reply dynamics

$$x_{i,t} = R(x_{j,t-1}), i \neq j,$$

where $x_{i,t}$, $i = 1, 2$, denotes the quantity supplied by player i in period t. As shown by Rand [18] these best-reply dynamics may converge to the Cournot-Nash equilibrium, result in an exploding path of quantities, or even result in cyclic or chaotic behavior. In fact, the Cournot-Nash equilibrium is stable (unstable) under the best-reply dynamics if $|R'(x^*)| < 1$ ($|R'(x^*)| > 1$). The stability properties of the Cournot model under standard adjustment systems are analyzed more precisely in Hahn [12] and Seade [24].

In part of the paper we focus on the following linear-quadratic specification of the Cournot duopoly game. The inverse demand and cost function are given by

$$P(x_1 + x_2) = a - b(x_1 + x_2)$$

and

$$C(x_i) = cx_i - \frac{d}{2}x_i^2, i = 1, 2,$$

respectively. First, in order to have a strictly concave profit function we assume that $d < 2b$. Second, we require that at all times $x_i \leq \frac{c}{d}$ since the cost function is only upward sloping if this condition holds. The fraction $\frac{c}{d}$ can be interpreted as a capacity constraint. For the above specification of the inverse demand function and cost function the reaction function is given by

$$x_i = R(x_j) = \frac{a - c - bx_j}{2b - d}.$$

It can easily be calculated that in this case the Cournot-Nash equilibrium quantity, price, and profit are equal to

$$x^* = \frac{a-c}{3b-d}, \ \ P(2x^*) = \frac{a(b-d)+2bc}{3b-d}, \text{ and } \pi(x^*, x^*) = \left(b - \frac{d}{2}\right)\left(\frac{a-c}{3b-d}\right)^2,$$

respectively. Furthermore, we find that

$$\frac{\partial R(q)}{\partial q} = -\frac{b}{2b-d}.$$

This implies that the Cournot-Nash equilibrium in the linear-quadratic specification of the Cournot duopoly game is stable (unstable) under the best-reply dynamics if $d < b$ $(d > b)$. Note that the results stated in this paper do not arise from this particular nonlinear specification of the Cournot game.

2.2 Quantity Dynamics

As mentioned before the players act according to behavioral rules that exactly prescribe the quantity to be supplied in the Cournot game. In fact, the players can choose between K different behavioral rules $H_1, H_2, ..., H_K$. With each behavioral rule H_i, $i = 1, ..., K$, we associate an information set I_i and costs $T(I_i)$. An example of such an information set is the case where agents only need to know average industry output in the previous period. Another, more complete, information set may contain additional information about the structure of the underlying model or information about the best-reply mapping. It seems obvious that obtaining the latter information set requires more resources than the former. Therefore the costs $T(I_i)$ associated with obtaining this information set are higher.

Let $n_{i,t}$, $i = 1, , ..., K$, denote the fraction of players using rule H_i in period t and let $x_{i,t}$, $i = 1, ..., K$, denote the quantity supplied by a players using rule H_i in period t. Given a certain information set I_i the corresponding behavioral rule is given by

$$x_{i,t+1} = H_i(\mathbf{X}_t, \mathbf{N}_t),$$

where

$$\mathbf{X}_t = \begin{bmatrix} x_{1,t} & x_{1,t-1} & \cdots & x_{1,t-L} \\ x_{2,t} & x_{2,t-1} & & \cdots \\ \cdots & & \cdots & \cdots \\ x_{K,t} & \cdots & \cdots & x_{K,t-L} \end{bmatrix}$$

and

$$\mathbf{N}_t = \begin{bmatrix} n_{1,t} & n_{1,t-1} & \cdots & n_{1,t-L} \\ n_{2,t} & n_{2,t-1} & & \cdots \\ \cdots & & \cdots & \cdots \\ n_{K,t} & \cdots & \cdots & n_{K,t-L} \end{bmatrix}.$$

The above specification of $\mathbf{X}_t$ and $\mathbf{N}_t$ indicates that behavioral rules have a limited memory of $L+1$ periods. Note that, as the columns in $\mathbf{N}_t$ sum up to one, the last row of this matrix could be ignored. We include it, however, only for notational convenience. Together, the K heterogeneous behavioral rules constitute the quantity dynamics of the model. To make sure that the quantity dynamics are well-behaved in terms of dynamic implications we impose the following regularity conditions. Let $M_{m\times n}(S)$ denote the set of all $m\times n$ matrices with entries from the set S.

Assumption Q1 The functions $\{H_i\}$ are all C^1, i.e. continuously differentiable. Furthermore, each behavioral rule $H_i : M_{K\times(L+1)}(\mathbb{R}_+) \times M_{K\times(L+1)}([0,1]) \to \mathbb{R}_+$, $i = 1,...,K$, has the Cournot-Nash quantity x^* as its unique steady state quantity, i.e. $x^* = H_i(\mathbf{X}^*, \mathbf{N}_t)$, for all $\mathbf{N}_t \in M_{K\times(L+1)}([0,1])$, where $\mathbf{X}^* \in M_{K\times(L+1)}(\{x^*\})$.

The latter assumption is a weak consistency assumption implying that the Cournot-Nash equilibrium x^* (or equivalently $\mathbf{X}^*$) is the unique steady state quantity of the complete dynamics of the Cournot duopoly model with heterogeneous behavioral rules: given that all players have been playing the Cournot-Nash equilibrium quantity for some time, each behavioral rule specifies to play the Cournot-Nash equilibrium in the current period. Particularly, it states that x^* is the unique steady state of each homogeneous behavioral rule. Assumption Q1 excludes the possibility that players "believe" in different steady state quantities. Allowing for this bias would introduce an additional source of instability in the model, not considered in this paper.

Remark 1 The Cournot duopoly game with heterogeneous behavioral rules described above can be interpreted as a symmetric $K \times K$ matrix game with a varying payoff matrix. This is caused by the fact that the quantity a player using a certain behavioral rule supplies, and consequently also the profit this player receives, typically varies over time.

Remark 2 The quantity dynamics of the model can alternatively be derived from the players having heterogeneous expectations with respect to the quantities supplied by their potential opponents in the next period. For economic models with heterogeneous beliefs we refer to Brock and Hommes [7, 8] and Cabrales and Hoshi [9].

2.3 Population Dynamics

We will now describe how the fractions $n_{i,t}$ evolve over time. The choice of a behavioral rule is based upon its past performance. Let $\pi_{i,j,t}(x_{i,t}, x_{j,t})$ denote the realized profit of a player supplying $x_{i,t}$ when he is matched with a player supplying $x_{j,t}$ in period t. More specifically,

$$\pi_{i,j,t}\left(x_{i,t},x_{j,t}\right)=P\left(x_{i,t}+x_{j,t}\right)x_{i,t}-C\left(x_{i,t}\right).$$

To keep the dynamics of the model analytically tractable, we only use the realized profit in the last period to determine the fitness measure of a behavioral rule. The probability of meeting a player supplying $x_{j,t}$ in period t is equal to $n_{j,t}$. Hence, the net expected profit of behavioral rule H_i in period t is

$$U_{i,t}=\sum_{j=1}^{K}n_{j,t}\pi_{i,j,t}\left(x_{i,t},x_{j,t}\right)-T\left(I_i\right)=\Pi^i\left(\mathbf{x}_t,\mathbf{n}_t\right)-T\left(I_i\right),$$

where $\Pi^i\left(\mathbf{x}_t,\mathbf{n}_t\right)$ is the expected profit of agents using behavioral rule H_i in period t and $T\left(I_i\right)$ are the associated information costs. In the above expression $\mathbf{x}_t=\left(x_{1,t},\ldots,x_{K,t}\right)'$ are the quantities corresponding to the different behavioral rules in period t and $\mathbf{n}_t=\left(n_{1,t},\ldots,n_{K,t}\right)'$ are the corresponding population fractions. Note that the expected profits, which determine the fitness measures $U_{i,t}$, are equal to the average profits of the behavioral rules. We assume that the fitness measures $U_{i,t}$ are publicly observable.

The fitness measures $U_{i,t}$ are used to determine the population fractions $n_{i,t+1}$. In general, such an adjustment process or dynamics in discrete time, describing how the population fractions evolve, is given by

$$n_{i,t+1}=h\left(U_{i,t},n_{i,t},\mathbf{U}_{-i,t},\mathbf{n}_{-i,t}\right),$$

with $\mathbf{U}_{-i,t}=\left(U_{1,t},...,U_{i-1,t},U_{i+1,t},...,U_{K,t}\right)'$ and $\mathbf{n}_{-i,t}=\left(n_{1,t},...,n_{i-1,t},n_{i+1,t},...,n_{K,t}\right)'$. To make sure that the population dynamics is well-behaved in terms of dynamic implications we assume that $h\left(\cdot,\cdot,\cdot,\cdot\right)$ is continuous, nondecreasing in the first argument, and such that the population state remains in the $(K-1)$-dimensional unit simplex Δ^K, which is a subset of the K-dimensional euclidean space $\mathbb{R}^K$. Given the fact that the dynamics remains in the unit simplex, the first regularity requirement is a sufficient condition for the existence of a steady state of the population dynamics. Note that dynamics in discrete time, contrary to dynamics in continuous time, automatically induce existence and uniqueness of a solution to the associated dynamical system through any initial population state in the simplex. The second regularity requirement reflects the assumption that successful behavior reproduces more rapidly. In section 2.3.1 and section 2.3.2 we introduce some commonly used specific population dynamics that satisfy these conditions and figure prominently in economic applications.

The preceding discussion shows that the evolutionary game theoretic model with K behavioral rules can be transformed into a $(2K-1)$-dimensional dynamical system. We need K equations to keep track of the quantity dynamics, i.e. the quantities supplied in the Cournot duopoly game by the players using the different behavioral rules. The other $K-1$ equations describe the population dynamics, i.e. the fractions of the population using the different behavioral rules. Consequently, the general dynamical system considered in this paper is the following

$$\begin{aligned} x_{i,t+1} &= H_i\left(\mathbf{X}_t, \mathbf{N}_t\right) \text{ for all } i = 1, ..., K \\ n_{i,t+1} &= h\left(U_{i,t}, n_{i,t}, \mathbf{U}_{-i,t}, \mathbf{n}_{-i,t}\right) \text{ for all } i = 1, ..., K-1. \end{aligned} \tag{1}$$

In the examples analyzed in section 3 and section 4 we focus on models with $K = 2$ behavioral rules. These evolutionary models give rise to (at most) 3-dimensional dynamical systems. This is no restriction since there are relatively few dynamical phenomena that are currently understood which only occur in dimension four or more.

Remark 3 The definition of the fitness measure can be generalized in a straightforward manner to include the performance of the behavioral rule over the last M periods. In case there is a delay in evaluating the behavioral rules, a fitness measure $U_{i,t}$ becomes

$$U_{i,t} = \sum_{m=0}^{M} w_{i,m} \sum_{j=1}^{K} n_{j,t-m}\pi_{i,j,t-m} - T\left(I_i\right) = \sum_{m=0}^{M} w_{i,m}\Pi^i\left(\mathbf{x}_{t-m}, \mathbf{n}_{t-m}\right) - T\left(I_i\right), \forall i,$$

where

$$\sum_{k=0}^{M} w_{i,k} = 1, \forall i, \text{ and } w_{i,k} \geq 0, \forall i, k.$$

Numerical simulations indicate that incorporating a delay yields similar results as working with the more basic definition of a fitness measure used in this paper.

2.3.1 Discrete Choice Model

This section contains a brief discussion of the discrete choice model. This model is treated extensively in Anderson, De Palma, and Thisse [1] and Manski and McFadden [15]. Brock and Hommes [7, 8] also focus on the discrete choice model. Other interesting applications of the discrete choice framework can be found in Goeree [11]. Furthermore, see Chen, Friedman, and Thisse [10] for an interesting discussion on the probabilistic choice approach.

Suppose that the utility associated with using behavioral rule H_i takes the form

$$\widetilde{U}_i = U_i + \frac{1}{\beta}\varepsilon_i,$$

where the ε_i's are IID. In fact, U_i is the deterministic part and $\frac{1}{\beta}\varepsilon_i$ captures the stochastic part of the utility of behavioral rule H_i. As explained by Anderson, De

Palma, and Thisse [1] the fraction of the population choosing behavioral rule H_i corresponds to the probability $\Pr\left(\widetilde{U}_i = \max_j \widetilde{U}_j\right)$. This captures the idea of bounded rationality since individuals do not necessarily select the rule that yields the highest utility. If the ε_i's are distributed according to the extreme value distribution this results in the so-called multinomial logit model, corresponding to the following updating of fractions

$$n_{i,t+1} = \frac{\exp\left[\beta U_{it}\right]}{\sum_{j=1}^{K} \exp\left[\beta U_{jt}\right]}, \forall i.$$

The parameter β is the intensity of choice measuring the inertia in switching behavioral rules. Notice that if $\beta = 0$ behavior is completely random: every behavioral rule is chosen with probability $\frac{1}{K}$. If $\beta \to \infty$ the most profitable strategy is played by the complete population. In fact, β can be interpreted as a measure of the natural selection with respect to the behavioral rules.

Note that in the case of the discrete choice model the population dynamics $h(\cdot,\cdot,\cdot,\cdot)$ even remains in the interior of the unit simplex. To be more precise, the discrete choice model does not satisfy the forward invariance property, see e.g. Nachbar [16]. Even though the discrete choice dynamics satisfies the no-extinction condition, this dynamics lacks the no-creation condition. However, as we generally view all strategies as being represented in the initial distribution the no-creation condition has little bite. The equilibrium fractions are easily seen to be

$$n_i^* = \frac{\exp\left[-\beta T\left(I_i\right)\right]}{\sum_{j=1}^{K} \exp\left[-\beta T\left(I_i\right)\right]}.$$

2.3.2 Replicator Dynamics

Another dynamics that figures prominently in economic applications is the discrete time replicator dynamics. In this case the fraction of agents playing behavioral rule H_i in period $t+1$ becomes

$$n_{i,t+1} = \frac{n_{i,t} F\left(U_{i,t}\right)}{\sum_{j=1}^{K} n_{j,t} F\left(U_{j,t}\right)},$$

where $F(\cdot) > 0$. In this specification $F\left(U_{i,t}\right)$ can be interpreted as the number of offspring of an individual using behavioral rule H_i in period t. Obviously, the replicator dynamics satisfies both the no-creation and the no-extinction condition. In addition, it is required that $F'(\cdot) > 0$. This implies that offspring increases with the payoff of a behavioral rule. As shown by Binmore, Gale, and Samuelson [4], Binmore and Samuelson [5], and Schlag [23], using the replicator dynamics to represent the population dynamics can be motivated in the context of a learning, aspiration, or imitation story. Furthermore, the simplicity of the replicator dynamics lends it considerable appeal. Note that at a steady state of these replicator dynamics we must have $n_i^* = 0$, $n_i^* = 1$, or

$$\sum_{j \in J} n_j^* F\left(U_j^*\right) = F\left(U_i^*\right), \forall i \in J,$$

where $J \subseteq \{1, \ldots, K\}$ is the set of behavioral rules that are being played with positive probability in the steady state. This implies that $F\left(U_i^*\right)$ has to be the same for all $i \in J$ and this means that information costs $T\left(I_i\right)$ have to be equal for all $i \in J$. If this is the case any set of fractions $\mathbf{n}^* = \left(n_1^*, \ldots, n_K^*\right)'$, with $\sum_{j \in J} n_j^* = 1$ and $n_i^* = 0$ for $i \notin J$ is a steady state. The case of equal information costs is considered in Proposition 1 in section 2.4.

In general, when information costs $T\left(I_i\right)$ are different for all behavioral rules, we have K distinct steady states. In each of these steady states only one behavioral rule is played with positive probability. Only the steady state in which the behavioral rule with the lowest information costs $T\left(I_i\right)$ is played with positive probability can be locally stable. In any other steady state it is profitable to choose a behavioral rule with lower information costs, since the system will remain at the Cournot-Nash equilibrium. When in a steady state only the behavior rule with the lowest information costs is played with positive probability, the (in)stability of a steady state of the complete system follows from the (in)stability of this behavioral rule.

In the remaining part of the paper we mainly focus on the replicator dynamics with deterministic noise, as discussed by Young and Foster [29]. The population dynamics then looks like

$$n_{i,t+1} = (1 - K\delta) \frac{n_{i,t} F\left(U_{i,t}\right)}{\sum_{j=1}^{K} n_{j,t} F\left(U_{j,t}\right)} + \delta,$$

where $\delta < \frac{1}{K}$ is assumed to be small. The parameter δ is called deterministic noise and its interpretation is as follows. Each period t a fraction of $K\delta$ firms leaves the market and is replaced by new firms. In their first period these new firms choose one of the existing behavioral rules randomly. A law of large numbers argument can be used to show that each behavioral rule is chosen by a fraction of $\frac{1}{K}$ of the new firms. The presence of the noise implies that all steady states have to be interior: the steady states in which only one behavioral rule is played are destroyed. The interior steady states have the property that $\delta < n_i^* < 1 - (K-1)\delta$ for all $i = 1, ..., K$. In this paper we will primarily be concerned with the case $K = 2$, where there is one behavioral rule with zero information costs and one behavioral rule with positive information costs T. Furthermore, we take $F\left(U_{i,t}\right) = \alpha + U_{i,t}$, where α is the so-called background (lifetime) birthrate. Some straightforward calculations show that the unique steady state fraction of people playing the behavioral rule with information costs T is equal to

$$n^* = \frac{1}{2T}\left(\delta\left(2\left(\alpha+\Pi^*\right)-T\right)+T-\sqrt{\delta^2\left(2\left(\alpha+\Pi^*\right)-T\right)^2+\left(1-2\delta\right)T^2}\right),$$

where Π^* is the profit, exclusive of information costs, made by all the players in the equilibrium of the dynamical system. In fact, it can be shown that $\delta < n^* < \frac{1}{2}$.

2.4 Local Instability Results

Assumption Q1 states that x^* is the unique steady state quantity of the complete $(2K-1)$-dimensional dynamical system that describes the Cournot duopoly model with heterogeneous behavioral rules. Consider such a model with players uniformly distributed over the space of behavioral rules and all fractions of players fixed at $\frac{1}{K}$, i.e. $x_{i,t+1} = H_i\left(\mathbf{X}_t, \bar{\mathbf{N}}_t\right)$ for all $i = 1, ..., K$, where $\bar{\mathbf{N}}_t \in M_{K\times(L+1)}\left(\left\{\frac{1}{K}\right\}\right)$. We assume the following.

Assumption Q2 The steady state quantity x^* (or equivalently $\mathbf{X}^*$) of the model with players uniformly distributed over the behavioral rules is hyperbolic (for all $i = 1, ..., K$ the linearization of $x_{i,t+1} = H_i\left(\mathbf{X}_t, \bar{\mathbf{N}}_t\right)$ has no eigenvalues on the unit circle) and there exists an i such that $x_{i,t+1} = H_i\left(\mathbf{X}_t, \bar{\mathbf{N}}_t\right)$ is locally unstable at x^*.

This assumption means that when all fractions of players using behavioral rule H_i are fixed at $\frac{1}{K}$, the corresponding quantity dynamics has an unstable steady state x^*. Let $(\mathbf{X}^*, \mathbf{N}^*)$ denote the equilibrium of the complete dynamical system.

Assumption Q3 With respect to the behavioral rules assume that for all H_i, $i = 1, ..., K$, it holds that

$$\left(\left.\frac{\partial H_i}{\partial \mathbf{N}_t}\right|_{(\mathbf{X}^*,\mathbf{N}^*)}\right)_{k,j} = \left(\left.\frac{\partial H_i}{\partial \mathbf{N}_t}\right|_{(\mathbf{X}^*,\mathbf{N}^*)}\right)_{l,j} \quad \text{for all } k,l = 1,...,K \text{ and } j = 1,...,L+1.$$

Assumption Q3 puts some restriction on the class of behavioral rules we allow for in Proposition 1. However, the above requirement does not exclude any common behavioral rules. In particular, it still allows for the best-reply and imitation behavioral rules stated explicitly in the remaining part of this paper. The first local instability result says that the unstable steady state of the dynamical system with all population fractions fixed at $\frac{1}{K}$ cannot be stabilized by switching of behavioral rules.

Proposition 1 *Assume Q1, Q2, and Q3. With respect to the behavioral rules assume that for all H_i information costs $T(I_i) = 0$. Furthermore, assume that the population* *dynamics is described by the discrete choice model, the replicator dynamics, or the replicator dynamics with deterministic noise. Then the complete dynamical system with behavioral rules given by $x_{i,t+1} = H_i(\mathbf{X}_t, \mathbf{N}_t)$ has a locally unstable steady state.*

Proof. We show that the linearization of $x_{i,t+1} = H_i(\mathbf{X}_t, \mathbf{N}_t)$ at x^* (or equivalently $\mathbf{X}^*$) is the same as the linearization of $x_{i,t+1} = H_i\left(\mathbf{X}_t, \bar{\mathbf{N}}_t\right)$ at x^*. Because $T(I_i) = 0$ for all $i = 1, ..., K$ it follows that for the population dynamics described by the discrete choice model it holds that $n_i^* = \frac{1}{K}$ for all $i = 1, ..., K$. Furthermore, this distribution of players over behavioral rules is also a steady state of the population dynamics in case it is described by the replicator dynamics or the replicator dynamics with deterministic noise. Consider the linearization of $x_{i,t+1} = H_i(\mathbf{X}_t, \mathbf{N}_t)$ around the steady state $(\mathbf{X}^*, \mathbf{N}^*)$ of the dynamical system

$$\begin{aligned}
\delta x_{i,t+1} &= \mathbf{e}^T \left[\left.\frac{\partial H_i}{\partial \mathbf{X}_t}\right|_{(\mathbf{X}^*,\mathbf{N}^*)} \otimes \delta \mathbf{X}_t \right] \mathbf{e} + \mathbf{e}^T \left[\left.\frac{\partial H_i}{\partial \mathbf{N}_t}\right|_{(\mathbf{X}^*,\mathbf{N}^*)} \otimes \delta \mathbf{N}_t \right] \mathbf{e} \\
&= \mathbf{e}^T \left[\left.\frac{\partial H_i}{\partial \mathbf{X}_t}\right|_{(\mathbf{X}^*,\mathbf{N}^*)} \otimes \delta \mathbf{X}_t \right] \mathbf{e} \\
&\quad + \mathbf{e}^T \left[\left.\frac{\partial H_i}{\partial \mathbf{N}_t}\right|_{(\mathbf{X}^*,\mathbf{N}^*)} \otimes \mathbf{N}_t \right] \mathbf{e} - \frac{1}{K} \mathbf{e}^T \left[\left.\frac{\partial H_i}{\partial \mathbf{N}_t}\right|_{(\mathbf{X}^*,\mathbf{N}^*)} \right] \mathbf{e},
\end{aligned}$$

where $\delta x_{i,t+1} = x_{i,t+1} - x^*$, $\mathbf{e}^T \in M_{1\times K}(\{1\})$, $\mathbf{e} \in M_{(L+1)\times 1}(\{1\})$ and $\otimes$ denotes the direct product. Because we assumed that

$$\left(\left.\frac{\partial H_i}{\partial \mathbf{N}_t}\right|_{(\mathbf{X}^*,\mathbf{N}^*)} \right)_{k,j} = \left(\left.\frac{\partial H_i}{\partial \mathbf{N}_t}\right|_{(\mathbf{X}^*,\mathbf{N}^*)} \right)_{l,j} \quad \text{for all } k, l = 1, ..., K \text{ and } j = 1, ..., L+1,$$

$\delta x_{i,t+1}$ can be rewritten as follows

$$\begin{aligned}
\delta x_{i,t+1} &= \mathbf{e}^T \left[\left.\frac{\partial H_i}{\partial \mathbf{X}_t}\right|_{(\mathbf{X}^*,\mathbf{N}^*)} \otimes \delta \mathbf{X}_t \right] \mathbf{e} \\
&\quad + \sum_{j=1}^{L+1} \left(\left.\frac{\partial H_i}{\partial \mathbf{N}_t}\right|_{(\mathbf{X}^*,\mathbf{N}^*)} \right)_{k,j} - \frac{1}{K} \sum_{k=1}^{K} \sum_{j=1}^{L+1} \left(\left.\frac{\partial H_i}{\partial \mathbf{N}_t}\right|_{(\mathbf{X}^*,\mathbf{N}^*)} \right)_{k,j}.
\end{aligned}$$

In the above expression the last two terms on the right-hand side of the equality sign cancel. Therefore, we are left with

$$\delta x_{i,t+1} = \mathbf{e}^T \left[\left.\frac{\partial H_i}{\partial \mathbf{X}_t}\right|_{(\mathbf{X}^*,\mathbf{N}^*)} \otimes \delta \mathbf{X}_t \right] \mathbf{e}.$$

This is exactly the linearization of $x_{i,t+1} = H_i\left(\mathbf{X}_t, \bar{\mathbf{N}}_t\right)$ at x^*. ■

A second local instability result arises when there are different information costs. Assume w.l.o.g. that $T(I_1) > ... > T(I_K) \geq 0$. We replace assumptions Q2 and Q3 by assumptions Q2' and Q3', respectively.

Assumption Q2' When all players use the cheapest behavioral rule H_K, the steady state x^* (or equivalently $\mathbf{X}^*$) is hyperbolic and locally unstable.

In an economic setting it is natural to take the cheapest behavioral rule to be some myopic or naive rule of thumb. Such a behavioral rule typically includes a time lag, does not incorporate any knowledge about the way the market works, and lacks information on the equilibrium quantities. Furthermore, a myopic rule is likely to include persistent forecast errors with respect to the behavior of other market participants, or not to take this behavior into account at all. It is not difficult to imagine that markets are unstable under such a behavioral rule.

Let R_t denote the set of all entries from the matrix $\mathbf{X}_t$ and let $\mathbf{r}$ denote a K-dimensional vector with all K components r_k, $k = 1, ..., K$, taken from the set R_t, i.e. $\mathbf{r} \in R_t^K$. Define for all behavioral rules H_i, $i = 1, ..., K$, and for all time lags $l = 0, ..., L$ a function $h_{i,t-l} : R_t \to \mathbb{R}_+$. Furthermore, for all $i = 1, ..., K$ and for all $l = 0, ..., L$ let $\mathbf{h}_{i,t-l}$ denote the K-dimensional vector with the kth component equal to $h_{i,t-l}(r_k)$.

Assumption Q3' With respect to each behavioral rule H_i, $i = 1, ..., K$, assume that it is independent of $\mathbf{N}_t$, i.e. $H_i(\mathbf{X}_t, \mathbf{N}_t) = \hat{H}_i(\mathbf{X}_t)$, or can be rewritten as follows

$$H_i(\mathbf{X}_t, \mathbf{N}_t) = \tilde{H}_i(\mathbf{n}_t \cdot \mathbf{h}_{i,t}, \mathbf{n}_{t-1} \cdot \mathbf{h}_{i,t-1}, ..., \mathbf{n}_{t-L} \cdot \mathbf{h}_{i,t-L}),$$

where $\mathbf{n}_{t-l} \cdot \mathbf{h}_{i,t-l}$ denotes the inproduct of $\mathbf{n}_{t-l} = (n_{1,t-l}, ..., n_{K,t-l})'$ and $\mathbf{h}_{i,t-l}$.

Again, this assumption does not exclude any common behavioral rules. The second local instability result is as follows.

Proposition 2 *Assume Q1, Q2', and Q3'. With respect to the behavioral rules assume that $T(I_1) > ... > T(I_K) \geq 0$. Furthermore, assume that the population dynamics is described by the discrete choice model. When the intensity of choice β is sufficiently large, then the steady state x^* (or equivalently $\mathbf{X}^*$) of the complete dynamical system is locally unstable.*

Proof. All behavioral rules $H_i, i = 1, ..., K$, are $L + 2$th order difference equations

$$x_{i,t+1} = H_i(\mathbf{X}_t, \mathbf{N}_t) = F_i\left(\mathbf{x}_t, \mathbf{x}_{t-1}, ..., \mathbf{x}_{t-(L+1)}\right),$$

where $L+1$ is the maximum number of lags of the behavioral rule H_i. Let x^* be the steady state quantity. First, consider the behavioral rules that do depend on $\mathbf{N}_t$. Because $\sum_{i=1}^{K} n_{i,t-l} = 1$, for all $l = 0, ..., L$, we know that in the steady state x^* (or equivalently $\mathbf{X}^*$) it holds that $\mathbf{n}_{t-l} \cdot \mathbf{h}_{i,t-l} = h_{i,t-L}(x^*)$, for all these behavioral rules and for all $l = 0, ..., L$. Consequently, $F_i(\mathbf{x}^*, ..., \mathbf{x}^*, \mathbf{x}) = \mathbf{x}^*$ for all these rules and for all $\mathbf{x}$. Hence, the partial derivative $F'_{i,L+2}(\mathbf{x}^*, ..., \mathbf{x}^*, \mathbf{x}) = 0$ for all these rules. Second, it is trivial that this conclusion also holds for the behavioral rules that do not depend on $\mathbf{N}_t$. As the population dynamics is described by the discrete choice model we know that the steady state fractions are given by

$$n_i^* = \frac{\exp\left[-\beta T(I_i)\right]}{\sum_{j=1}^{K} \exp\left[-\beta T(I_j)\right]}, i = 1, ..., K.$$

Consider a behavioral rule H_i

$$x_{i,t+1} = H_i(\mathbf{X}_t, \mathbf{N}^*) = G(\mathbf{x}_t, \mathbf{x}_{t-1}, ..., \mathbf{x}_{t-L}),$$

describing the quantity dynamics when the fractions of players using behavioral rules $H_1, ..., H_K$ are fixed at the steady state values of the complete dynamical system. The dynamical system with fixed population fractions is an $L+1$th order difference equation. Using $F'_{i,L+2}(\mathbf{x}^*, ..., \mathbf{x}^*, \mathbf{x}) = 0$ for all i, it can be shown that for each behavioral rule the eigenvalues of the Jacobian matrix at the steady state of the $(L+2)$-dimensional system are $\lambda = 0$ and the same $L+1$ eigenvalues corresponding to the Jacobian at the steady state of the dynamical system with fixed population fractions.

For β large, the steady state fraction using the cheapest behavioral rule approaches 1, i.e. $n_K^*(\beta) \approx 1$, whereas $n_j^*(\beta) \approx 0$ for all $j \neq K$. Consider the dynamical system describing the quantity dynamics when all players use behavioral rule H_K. By assumption Q2', the steady state of this dynamical system is locally unstable. Since for β large, the system with fixed population fractions gets C^1-close to the system in which all players use behavioral rule H_K, we conclude that for β sufficiently large, at the steady state x^* the system with fixed population fractions is locally unstable. This implies that the complete dynamical system is also locally unstable at the steady state x^*. ■

3 Best-Reply versus Rational Players

In this section we consider a simple, but typical, example of the general model introduced in section 2. In fact, we analyze the case where there are $K = 2$ behavioral rules available in the population. Namely, a best-reply behavioral rule and a so-called rational behavioral rule. Let n_t be the fraction of rational players in period t and let x_t and y_t denote the quantities supplied by the best-reply players and the rational players in period t, respectively.

With respect to the best-reply players we assume that they know average industry output in the previous period. Since average industry output in period t is equal to $\overline{x}_t = n_t y_t + (1 - n_t) x_t$, the best-reply players supply the quantity

$$x_{t+1} = R(\bar{x}_t) = R(n_t y_t + (1 - n_t) x_t)$$

in period $t+1$. From the above expression it follows that the introduction of a certain amount of rational players may stabilize the best-reply dynamics. In fact, we have $\frac{dR(\bar{x}_t)}{dx_t} = (1 - n_t) R'(\overline{x}_t)$. Consequently, if the fraction n_t is large enough the resulting best-reply dynamics may become stable, even though the original best-reply dynamics is not ($|R'(x^*)| > 1$).

Rational players reason more subtle than best-reply players. We assume that at time $t+1$ they know the fraction of people playing according to the rational behavioral rule in that period. Rational players can therefore also be interpreted as perfect foresight players. In addition, they are able to calculate the quantity x_{t+1} supplied by the best-reply players in period $t+1$. Their optimal strategy is implicitly defined as follows

$$y_{t+1} = R(n_{t+1} y_{t+1} + (1 - n_{t+1}) x_{t+1}).$$

The strategy of the rational players is something like a Nash equilibrium in a game which is "contaminated" with a number of best-reply players. Because of the extensive computational and informational requirements of the rational behavioral rule it seems logical to associate costs $T > 0$ with this rule. Since we assumed $R(\cdot)$ to be downward sloping the rational behavioral rule can be determined explicitly and be written as

$$y_{t+1} = G(x_{t+1}, n_{t+1}).$$

Note that both rules discussed above are unbiased as they have the Cournot-Nash equilibrium quantity x^* as their unique steady state quantity. Furthermore, x^* is also the unique steady state quantity of the complete dynamics of the model with heterogeneous behavioral rules. In fact, this dynamical system is given by

$$\begin{aligned} x_{t+1} &= f(x_t, y_t, n_t) = R(n_t y_t + (1 - n_t) x_t) \\ y_{t+1} &= g(x_t, y_t, n_t) = G(f(x_t, y_t, n_t), h(x_t, y_t, n_t)) \\ n_{t+1} &= h(x_t, y_t, n_t), \end{aligned}$$

where the mapping $h(x_t, y_t, n_t)$ represents the population dynamics. With respect to the above dynamical system it should be noted that

$$x_{t+1} = R(n_t y_t + (1 - n_t) x_t) = y_t.$$

This means that the only difference between the rational strategy and the best-reply strategy is that the latter always lags one period behind. Therefore, we can reduce the dimension of the dynamical system by one and write everything in terms of x_t. The resulting 2-dimensional dynamical system is

$$\begin{aligned} x_{t+1} &= R\left(n_t G\left(x_t, n_t\right) + \left(1 - n_t\right) x_t\right) \\ n_{t+1} &= h\left(x_t, G\left(x_t, n_t\right), n_t\right). \end{aligned} \tag{2}$$

Before specifying the population dynamics $h\left(x_t, G\left(x_t, n_t\right), n_t\right)$ we make the following observation with respect to the dynamical system (2).

Lemma 3 *Consider the dynamical system (2). The eigenvalues of the Jacobian matrix of the linearized system, evaluated at the equilibrium* (x^*, n^*)*, are* $\lambda_1 = \frac{(1-n^*)R'(x^*)}{1-n^*R'(x^*)}$ *and* $\lambda_2 = \left.\frac{\partial h(x_t, G(x_t,n_t), n_t)}{\partial n_t}\right|_{(x^*,n^*)}$.

Proof. To determine the eigenvalues of the Jacobian matrix evaluated in the equilibrium (x^*, n^*), we consider the entries in the first row of this matrix. It is easily seen that these entries are given by

$$J_{11} = \left(n^* G_1\left(x^*, n^*\right) + \left(1 - n^*\right)\right) R'\left(x^*\right) \text{ and } J_{12} = n^* G_2\left(x^*, n^*\right) R'\left(x^*\right),$$

where G_i is the derivative of the function $G(\cdot, \cdot)$ with respect to its ith variable. To compute both $G_1(x^*, n^*)$ and $G_2(x^*, n^*)$ we need to take the total differential of

$$y_{t+1} = R\left(n_t y_{t+1} + \left(1 - n_{t+1}\right) x_{t+1}\right),$$

which results in

$$\begin{aligned} \partial y_{t+1} &= R'\left(n_{t+1} y_{t+1} + \left(1 - n_{t+1}\right) x_{t+1}\right) \\ &\quad \times \left(\left(1 - n_{t+1}\right) \partial x_{t+1} + n_{t+1} \partial y_{t+1} + \left(y_{t+1} - x_{t+1}\right) \partial n_{t+1}\right). \end{aligned}$$

Rearranging terms gives

$$\frac{\partial y_{t+1}}{\partial x_{t+1}} = G_1\left(x_{t+1}, n_{t+1}\right) = \frac{\left(1 - n_{t+1}\right) R'\left(n_{t+1} y_{t+1} + \left(1 - n_{t+1}\right) x_{t+1}\right)}{1 - n_{t+1} R'\left(n_{t+1} y_{t+1} + \left(1 - n_{t+1}\right) x_{t+1}\right)}$$

and

$$\frac{\partial y_{t+1}}{\partial n_{t+1}} = G_2\left(x_{t+1}, n_{t+1}\right) = \frac{\left(y_{t+1} - x_{t+1}\right) R'\left(n_{t+1} y_{t+1} + \left(1 - n_{t+1}\right) x_{t+1}\right)}{1 - n_{t+1} R'\left(n_{t+1} y_{t+1} + \left(1 - n_{t+1}\right) x_{t+1}\right)}.$$

Evaluating the above expressions in the equilibrium (x^*, n^*) results in

$$G_1\left(x^*, n^*\right) = \frac{\left(1 - n^*\right) R'\left(x^*\right)}{1 - n^* R'\left(x^*\right)} \text{ and } G_2\left(x^*, n^*\right) = 0.$$

With respect to the Jacobian matrix this implies

$$J_{11} = \left(n^* \frac{(1-n^*) R'(x^*)}{1 - n^* R'(x^*)} + (1-n^*)\right) R'(x^*) = \frac{(1-n^*) R'(x^*)}{1 - n^* R'(x^*)} \text{ and } J_{12} = 0.$$

Consequently, the eigenvalues of the Jacobian matrix of the linearized system, evaluated in the equilibrium (x^*, n^*), are $\lambda_1 = J_{11} = \frac{(1-n^*)R'(x^*)}{1-n^*R'(x^*)}$ and $\lambda_2 = J_{22} = \left.\frac{\partial h(x_t, G(x_t,n_t),n_t)}{\partial n_t}\right|_{(x^*,n^*)}$. ■

3.1 Discrete Choice Model

Suppose that the population fractions of players using a certain behavioral rule are updated according to the discrete choice model. In this case the dynamical system (2) becomes

$$\begin{aligned} x_{t+1} &= R(n_t G(x_t, n_t) + (1-n_t) x_t) \qquad (3)\\ n_{t+1} &= h(x_t, G(x_t, n_t), n_t) = \\ &= \frac{\exp[\beta(\Pi^r(x_t, G(x_t, n_t), n_t) - T)]}{\exp[\beta(\Pi^r(x_t, G(x_t, n_t), n_t) - T)] + \exp[\beta\Pi^{br}(x_t, G(x_t, n_t), n_t)]}, \end{aligned}$$

where

$$\Pi^r(x_t, G(x_t, n_t), n_t) = n_t \pi(G(x_t, n_t), G(x_t, n_t)) + (1-n_t)\pi(G(x_t, n_t), x_t)$$

and

$$\Pi^{br}(x_t, G(x_t, n_t), n_t) = n_t \pi(x_t, G(x_t, n_t)) + (1-n_t)\pi(x_t, x_t)$$

are the average profit of the rational players and the best-reply players in period t, respectively. In the above definitions subscripts have been omitted for notational convenience. Furthermore, the parameter β is the intensity of choice. We will study what happens to the dynamics of (3) as the intensity of choice β increases. Note that by definition we have that $\Pi^r(x_t, G(x_t, n_t), n_t) \geq \Pi^{br}(x_t, G(x_t, n_t), n_t)$.

3.1.1 Local Stability Analysis

The first step in analyzing the behavior of the dynamical system (3) is to investigate the local stability properties of the equilibrium. These properties are summarized in Proposition 4.

Proposition 4 *Consider the dynamical system (3). Let $R'(x^*) < -1$. The equilibrium of (3) is (x^*, n^*), where x^* is the Cournot-Nash equilibrium quantity and $n^* = \frac{1}{1+\exp[\beta T]}$. Then given T there exists a value β^* of β such that the equilibrium (x^*, n^*) is locally stable for all $\beta < \beta^*$ and unstable for all $\beta > \beta^*$. Moreover, $\beta^* = \frac{1}{T}\ln\left[\frac{R'(x^*)-1}{R'(x^*)+1}\right]$.*

Proof. Straightforward calculations show that (x^*, n^*) is the equilibrium of (3). According to Lemma 3 the eigenvalues of (3) are $\lambda_1 = \frac{(1-n^*)R'(x^*)}{1-n^*R'(x^*)}$ and $\lambda_2 = \frac{\partial h(x_t, G(x_t,n_t),n_t)}{\partial n_t}\Big|_{(x^*,n^*)}$. We start by considering the eigenvalue λ_2. Note that $h(x_t, G(x_t, n_t), n_t)$ only depends upon n_t through the profit functions. Furthermore, it holds that

$$\begin{aligned}\frac{\partial \Pi^r (x_t, G(x_t, n_t), n_t)}{\partial n_t} &= n_t \left(\frac{\partial \pi(y,y)}{\partial y} G_2 + \frac{\partial \pi(y,y)}{\partial x} G_2 \right) \\ &\quad + (1-n_t) \frac{\partial \pi(y,x)}{\partial y} G_2 \\ &\quad + \pi(G(x_t,n_t), G(x_t,n_t)) - \pi(G(x_t,n_t), x_t),\end{aligned}$$

which is equal to 0 in equilibrium because $G_2(x^*, n^*) = 0$. A similar argument can be used to show that

$$\frac{\partial \Pi^{br} (x_t, G(x_t, n_t), n_t)}{\partial n_t}\Big|_{(x^*,n^*)} = 0.$$

From the above reasoning it follows immediately that $\lambda_2 = 0$. As a result, we have that the eigenvalue λ_1 is the only nonzero eigenvalue. Hence, the equilibrium is locally stable (unstable) if

$$\lambda_1 = \frac{(1-n^*) R'(x^*)}{1-n^* R'(x^*)} > (<) - 1.$$

Since $n^* = \frac{1}{1+\exp[\beta T]}$ this condition is equivalent to $0 < \beta < \beta^* \, (\beta > \beta^*)$, where $\beta^* = \frac{1}{T} \ln \left[\frac{R'(x^*)-1}{R'(x^*)+1} \right]$. ■

Proposition 4 states that the equilibrium of (3) loses stability for high values of the intensity of choice β. A high intensity of choice means that there is little inertia in switching between behavioral rules. As a result, most players will switch to the myopic behavioral rule whenever the dynamical system is close to the equilibrium. Namely, close to the equilibrium the myopic rule results in a higher profit because there are no information costs associated with this rule. Since the equilibrium is unstable under this myopic behavioral rule we are left with the above result.

Note that the equilibrium of (3) is locally stable for all values of the intensity of choice $\beta > 0$ in case $-1 \leq R'(x^*) \leq 0$. Furthermore, in case information costs $T = 0$ we have $\lambda_1 = \frac{\frac{1}{2}R'(x^*)}{1-\frac{1}{2}R'(x^*)}$. This means that in the model without information costs the equilibrium is stable for all values $\beta > 0$.

Finally, Proposition 4 could be reformulated with the roles of the intensity of choice β and the information costs T interchanged. In fact, given β there exists a value $T^* = \frac{1}{\beta} \ln \left[\frac{R'(x^*)-1}{R'(x^*)+1} \right]$ of T such that the equilibrium is locally stable for all $T < T^*$ and unstable for all $T > T^*$.

3.1.2 A Bifurcation Scenario

This section deals with the linear-quadratic specification of the Cournot duopoly model. For later convenience we rewrite the dynamical system in terms of deviations $X_t = x_t - x^*$ and $Y_t = y_t - x^*$ from the Cournot-Nash equilibrium quantity x^*. It can easily be shown that

$$Y_t = G(X_t, n_t) = -\frac{b(1-n_t)}{(2+n_t)b - d} X_t.$$

Consequently, the linear-quadratic specification of the dynamical system (3) written in deviations becomes

$$\begin{aligned} X_{t+1} &= -\frac{b(1-n_t)}{(2+n_t)b-d} X_t \\ n_{t+1} &= \frac{1}{1+\exp\left[\beta\left(T - \left(b-\frac{d}{2}\right)\left(\frac{3b-d}{(2+n_t)b-d}\right)^2 X_t^2\right)\right]}. \end{aligned} \tag{4}$$

Write F for the 2-dimensional map (4). The derivation of the profit functions for the linear-quadratic specification of the model, which are used to determine the population dynamics of (4), can be found in Appendix A. We have to keep in mind that production levels are always nonnegative and smaller than the capacity constraint, that is $-x^* \leq X_t \leq \frac{c}{d} - x^*$ or, equivalently, $-\frac{a-c}{3b-d} \leq X_t \leq \frac{c}{d} - \frac{a-c}{3b-d}$. An important feature of the dynamical system (4) is that it is symmetric with respect to the line $X_t = 0$. This property will prove to be convenient later on. In fact, it enables us to determine the secondary bifurcation analytically.

The model with best-reply versus rational players, discrete choice dynamics, and the linear-quadratic specification of the Cournot duopoly game is, upon a transformation of variables, mathematically equivalent with a case studied in Brock and Hommes [7, 8]. They study the evolution of equilibrium prices in a cobweb model with rational versus naive expectations. The state variable in their model is the price p_t and the parameters (next to the intensity of choice β and information costs T) are marginal supply $\widehat{b}$ and marginal demand B. In particular, if we transform the state variable p_t according to $X_t = \sqrt{\frac{2}{2b-d}} p_t$ and additionally assume that $\widehat{b} = b$ and $B = 2b - d$ we end up exactly with the model presented here. The equivalence is shown in more detail in Appendix B.

We will now study how the dynamical behavior of (4) changes as β increases. Proposition 5 states what happens when the equilibrium (X^*, n^*), with $X^* = 0$ and $n^* = \frac{1}{1+\exp[\beta T]}$, loses stability.

Proposition 5 *Consider the dynamical system (4). Let $b < d < 2b$. Then the primary bifurcation, i.e. a period doubling bifurcation, occurs at $\beta^* = \frac{1}{T}\ln\left[\frac{3b-d}{d-b}\right]$. In* *fact, for values* $\beta < \beta^*$ *the equilibrium* $\left(0, \frac{1}{1+\exp[\beta T]}\right)$ *is locally stable and for values* $\beta > \beta^*$ *the equilibrium is unstable and a locally stable period 2 cycle exists. This period 2 cycle is given by*

$$\left\{\left(\sqrt{\frac{(\beta-\beta^*)\,T}{2\,(2b-d)\,\beta}}, \frac{d-b}{2b}\right)', \left(-\sqrt{\frac{(\beta-\beta^*)\,T}{2\,(2b-d)\,\beta}}, \frac{d-b}{2b}\right)'\right\}.$$

The secondary bifurcation, where the period 2 cycle loses stability, occurs at $\beta^{**} = \beta^* + \frac{b}{(d-b)T}$.

Proof. First, the local stability result and the primary bifurcation value follow directly from Proposition 4. At $\beta = \beta^*$ it holds that the eigenvalues of the Jacobian matrix, evaluated at the equilibrium, are equal to $\lambda_1 = -1$ and $\lambda_2 = 0$. Consequently, a period doubling bifurcation occurs. Second, due to the aforementioned symmetry we are looking for a period 2 cycle of the form

$$\left\{(X_1, n)', (X_2, n)'\right\},$$

where

$$X_2 = \frac{-b\,(1-n)}{(2+n)\,b-d} X_1.$$

Since the same equality must hold with X_1 and X_2 interchanged, we get

$$\frac{-b\,(1-n)}{(2+n)\,b-d} = -1,$$

implying $n = \frac{d-b}{2b}$. Writing $X = X_1 = -X_2 > 0$ and using the second equation of the dynamical system (4) we find that X_1 is the positive solution of

$$n = \frac{1}{1+\exp\left[\beta\left(T + 2b\,(2n-1)\,X^2\right)\right]}.$$

Substituting $n = \frac{d-b}{2b}$ and solving this equation gives the solution provided in Proposition 5. Let F^2 denote the second iterate of the map F, i.e. the map F composed with itself two times. Third, to study the stability of this period 2 cycle we consider the Jacobian matrix of F^2, evaluated in the period 2 cycle $\left\{(X_1, n)', (X_2, n)'\right\}$. This Jacobian matrix is given by

$$\mathbf{J}_{F^2} = \begin{pmatrix} 1 - \frac{2(d-b)T(\beta-\beta^*)}{b} & \sqrt{\frac{8(\beta-\beta^*)T}{(2b-d)\beta}}\frac{b-(d-b)(\beta-\beta^*)T}{3b-d} \\ -\sqrt{\frac{\beta(\beta-\beta^*)T}{2(2b-d)}}\frac{(d-b)(2b-d)(3b-d)(b-(d-b)(\beta-\beta^*)T)}{b^3} & -\frac{T(\beta-\beta^*)(d-b)(2b-(d-b)(\beta-\beta^*)T)}{b^2} \end{pmatrix}.$$

It can be checked that the eigenvalues of this matrix are complex conjugates and that the determinant of $\mathbf{J}_{F^2}$ is given by

$$\det \mathbf{J}_{F^2} = \frac{(d-b)^2\,(\beta-\beta^*)^2\,T^2}{b^2}.$$

At β^{**} the above determinant is equal to 1 and both eigenvalues lie on the unit circle. In fact, both eigenvalues are equal to -1. Finally, because $\det \mathbf{J}_{F^2}$ is increasing in β the period 2 cycle is locally stable if $\beta < \beta^{**}$ and unstable if $\beta > \beta^{**}$. ■

Proposition 5 indicates a source of complicated dynamics for intermediate values of the intensity of choice β. Namely, at β^{**} a so-called 1 : 2 strong resonance Hopf bifurcation occurs. An important feature of this Hopf bifurcation is the emergence of four period 4 cycles, two locally stable and two unstable. Therefore, we have coexistence of low periodic attractors. It depends upon the initial state (X_0, n_0) to which of the two stable period 4 cycles, which are symmetric with respect to the line $X_t = 0$, the equilibrium path settles down. The basin of attraction of an attractor is the set of initial states (X_0, n_0) converging to the attractor. In case the dynamical system exhibits small stochastic noise, coexisting attractors can be a source of complicated dynamical behavior since orbits may jump irregularly from one basin of attraction to the other. For a more detailed discussion of this special Hopf bifurcation we refer to Brock and Hommes [8].

For large values of the intensity of choice β, the dynamical behavior can be complicated due to the occurrence of strange attractors. Figure 1 shows the attractors of the model for the parameter values $a = 17$, $b = 1$, $c = 10$, $d = \frac{5}{4}$, $T = 1$, and for different values of β. For this numerical example the primary period doubling bifurcation occurs at $\beta^* = \ln 7$ and the 1 : 2 strong resonance Hopf bifurcation of the period 2 cycle occurs at $\beta^{**} = 4 + \ln 7$. As β increases further, apparently both stable period 4 cycles turn into 4-piece chaotic attractors after a cascade of infinitely many period doubling bifurcations. One of these 4-piece chaotic attractors is depicted in Figure 1. Like the two stable period 4 cycles, the two coexisting 4-piece chaotic attractors are also symmetric with respect to the line $X_t = 0$. As β increases even further, our numerical simulations show the existence of a strange attractor.

Figure 2 shows the time series of X_t and n_t for the dynamical system associated with the numerical example when β equals 25. These time series clearly reveal what is going on. For some time almost all agents use the best-reply strategy, i.e. n_t is close to 0, and the production levels are close to the Cournot-Nash equilibrium, i.e. X_t is close to 0. However, since the best-reply dynamics is unstable, the dynamics move away from the steady state and output begins to fluctuate erratically. At a certain time they move so wildly that it becomes worthwhile to spend money on the rational behavioral rule. Then almost all agents suddenly switch behavioral rules and n_t becomes close to 1. This stabilizes the system and output returns to its equilibrium level, which in turn makes the best-reply strategy more profitable. As a result, n_t converges to 0 very quickly again.

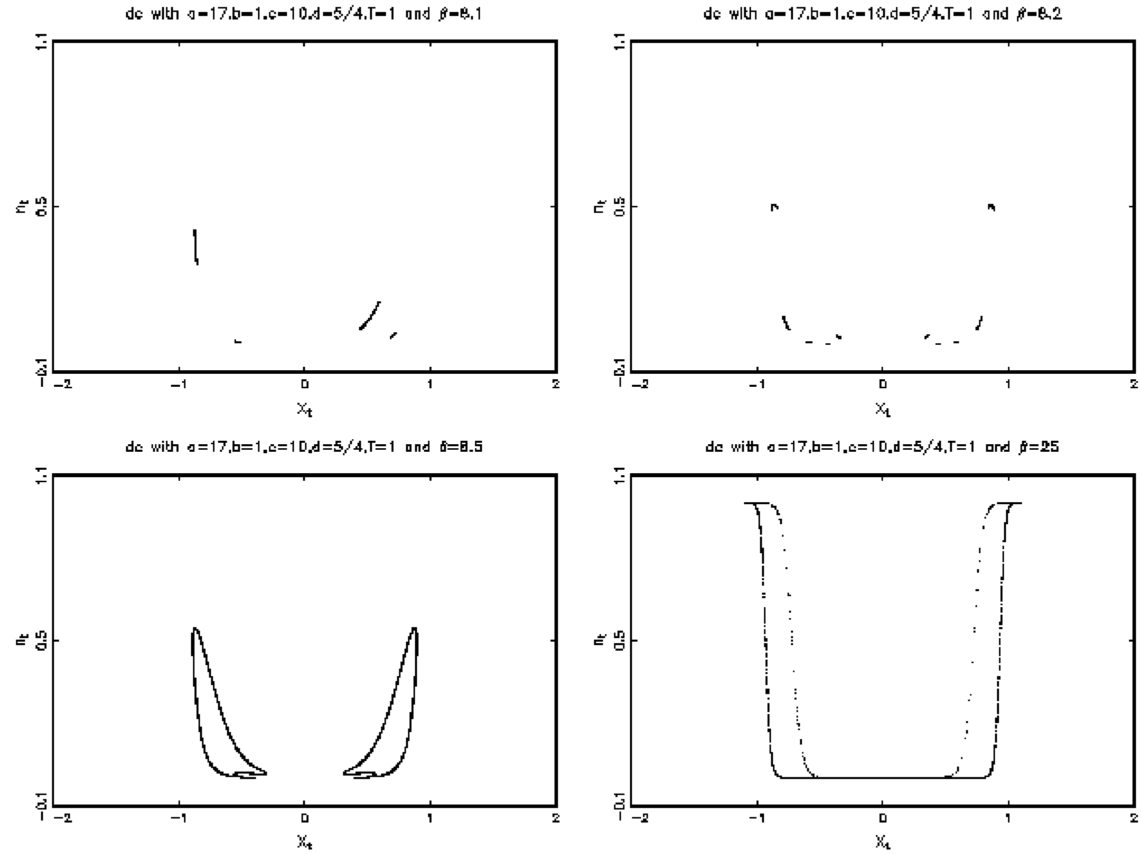

Figure 1: Attractors for the model with best-reply versus rational players and discrete choice dynamics.

3.2 Replicator Dynamics

Now we consider the case where the population dynamics is modelled according to the replicator dynamics with deterministic noise instead of the discrete choice model. This transforms the dynamical system (2) into

$$
\begin{aligned}
x_{t+1} &= R\left(n_t G\left(x_t, n_t\right) + \left(1 - n_t\right) x_t\right) && (5)\\
n_{t+1} &= \frac{n_t\left(1 - 2\delta\right)\left(\alpha + \Pi^r\left(x_t, G\left(x_t, n_t\right), n_t\right) - T\right)}{\alpha + n_t\left(\Pi^r\left(x_t, G\left(x_t, n_t\right), n_t\right) - T\right) + \left(1 - n_t\right)\Pi^{br}\left(x_t, G\left(x_t, n_t\right), n_t\right)} + \delta.
\end{aligned}
$$

In this section we study what happens to the dynamical behavior of (5) as the deterministic noise δ goes to zero.

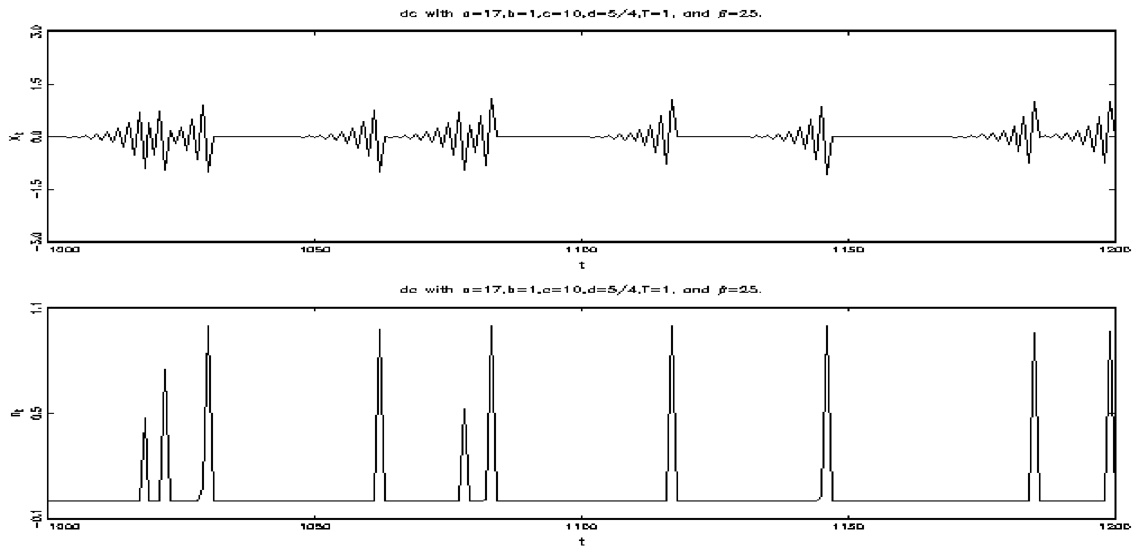

Figure 2: Time series for the model with best-reply versus rational players and discrete choice dynamics for $\beta = 25$, corresponding to the strange attractor.

3.2.1 Local Stability Analysis

The local stability analysis of the equilibrium of the dynamical system (5) is summarized in Proposition 6.

Proposition 6 *Consider the dynamical system (5). Let* $0 < \delta < \frac{1}{2}$, $T > 0$, *and* $R'(x^*) < -1$. *The equilibrium of (5) is* (x^*, n^*), *where* x^* *is the Cournot-Nash equilibrium quantity and*

$$n^* = \frac{1}{2T}\left(\delta\left(2\left(\alpha+\Pi^*\right)-T\right)+T-\sqrt{\delta^2\left(2\left(\alpha+\Pi^*\right)-T\right)^2+\left(1-2\delta\right)T^2}\right).$$

Furthermore, the equilibrium (x^*, n^*) *is locally stable (unstable) for* $\delta > \delta^*$ $(\delta < \delta^*)$, *where*

$$\delta^* = \frac{\left(R'\left(x^*\right)^2-1\right)T}{2R'\left(x^*\right)\left(T\left(R'\left(x^*\right)+1\right)-2\left(\alpha+\Pi^*\right)\right)}.$$

Proof. Straightforward calculations show that (x^*, n^*) is the equilibrium of (5). According to Lemma 3 the eigenvalues of the Jacobian matrix of the dynamical system (2), evaluated at the equilibrium (x^*, n^*), are $\lambda_1 = \frac{(1-n^*)R'(x^*)}{1-n^*R'(x^*)}$ and $\lambda_2 = \left.\frac{\partial h(x_t, G(x_t,n_t), n_t)}{\partial n_t}\right|_{(x^*,n^*)}$. First, consider the eigenvalue λ_2. In case the population dynamics are represented by the replicator dynamics with deterministic noise it holds that

$$0 \;<\; \lambda_2 = (1-2\delta)\,\frac{\left(\alpha+\Pi^*\right)\left(\alpha+\Pi^*-T\right)}{\left(\alpha+\Pi^*-n^*T\right)^2} < \frac{\left(\alpha+\Pi^*\right)\left(\alpha+\Pi^*-T\right)}{\left(\alpha+\Pi^*-n^*T\right)^2}$$

$$< \frac{(\alpha + \Pi^*)(\alpha + \Pi^* - T)}{\left(\alpha + \Pi^* - \frac{1}{2}T\right)^2} < 1.$$

Note that in the above derivation we use $\delta < n^* < \frac{1}{2}$. Second, because the eigenvalue λ_2 is always strictly between 0 and 1, the equilibrium (x^*, n^*) is locally stable (unstable) if

$$\lambda_1 = \frac{(1 - n^*) R'(x^*)}{1 - n^* R'(x^*)} > (<) - 1.$$

Substituting the expression for n^* and solving for the deterministic noise δ gives the condition as stated in Proposition 6. ■

Proposition 6 says that the equilibrium of (5) loses stability for low values of the deterministic noise δ. A low deterministic noise means that there are hardly any players who choose a behavioral rule at random. As a result, almost all players will use the myopic behavioral rule whenever the dynamical system is close to the equilibrium. Namely, close to the equilibrium the myopic rule results in a higher profit because there are no information costs associated with this rule. Since the equilibrium is unstable under this myopic behavioral rule we are left with the result mentioned in Proposition 6.

Note that the equilibrium (x^*, n^*) of (5) is locally stable for all values of the deterministic noise δ in case $-1 \leq R'(x^*) \leq 0$. Furthermore, in case there are no information costs the equilibrium is stable for all values of the deterministic noise $\delta > 0$.

Again, Proposition 6 can be reformulated with the roles of the deterministic noise δ and the information costs T interchanged. In fact, given δ there exists a value

$$T^* = -\frac{4\delta R'(x^*)(\alpha + \Pi^*)}{R'(x^*)(R'(x^*) - 2\delta(R'(x^*) + 1)) - 1}$$

of T such that the equilibrium is locally stable for all $T < T^*$ and unstable for all $T > T^*$.

3.2.2 A Bifurcation Scenario

In this section we consider the linear-quadratic specification of the Cournot duopoly model where the replicator dynamics with deterministic noise describes the population dynamics. Contrary to the dynamical system (4), where the discrete choice model describes the population dynamics, the current linear-quadratic specification does not exhibit any symmetry. This is due to the more complicated structure of the replicator dynamics compared to the discrete choice model. As a result, we have to turn to numerical simulations immediately in order to analyze a bifurcation scenario. In these simulations we let $a = 17$, $b = 1$, $c = 10$, $d = 1.1$, $T = 1$, and $\alpha = 0$. To be consistent we have performed the simulations with the model rewritten in deviations from the Cournot-Nash quantity. However, we have not included the resulting dynamical system because of the complicated and nonintuitive expression for the population dynamics. Note that the quantity dynamics of the current model are represented by the same equation as the quantity dynamics in (4). The equilibrium fraction of the population dynamics is given by

$$n^* = \frac{1}{2}\left(11\delta + 1 - \sqrt{(11\delta)^2 + (1 - 2\delta)}\right).$$

From Proposition 6 it can be concluded that the dynamical system undergoes the primary bifurcation, i.e. a period doubling bifurcation, at $\delta^* = \frac{19}{2180}$. As a result, a locally stable period 2 cycle emerges. Generically, the secondary bifurcation is a Hopf bifurcation, resulting in an invariant set consisting of two closed curves. In this numerical example the period 2 cycle undergoes a Hopf bifurcation at $\delta^{Hopf} \approx 0.00107$. Figure 3 gives four attractors of this dynamical system for the numerical example with the parameters $a = 17$, $b = 1$, $c = 10$, $d = 1.1$, $T = 1$, $\alpha = 0$, and different values of δ.

Figure 4 shows the corresponding time series for $\delta = 0.00060$. The time series can be interpreted similarly as in the case of the discrete choice dynamics. If the dynamical system is near the Cournot-Nash equilibrium quantity, the fraction of players using the best-reply behavioral rule will increase. This causes the dynamical system to become unstable and drives the quantities away from the Cournot-Nash equilibrium quantity, making it profitable for the players to use the rational behavioral rule. Consequently, the fraction of rational players will increase and the dynamical system becomes stable again, i.e. quantities return to the Cournot-Nash equilibrium. The main difference between the discrete choice dynamics and the replicator dynamics is that the latter responds less fast to differences in the fitness of the behavioral rules. This can be concluded from comparing Figure 2 and Figure 4.

4 Imitators versus Best-Reply Players

As a second example we confront best-reply players with imitators. Just like the model in section 3, best-reply players respond optimally to the average industry output in the previous period. Imitators copy the quantity produced by the best-reply players in the previous period. As a consequence, the imitation rule is relatively cheap compared to the best-reply rule since it does not require any knowledge of the

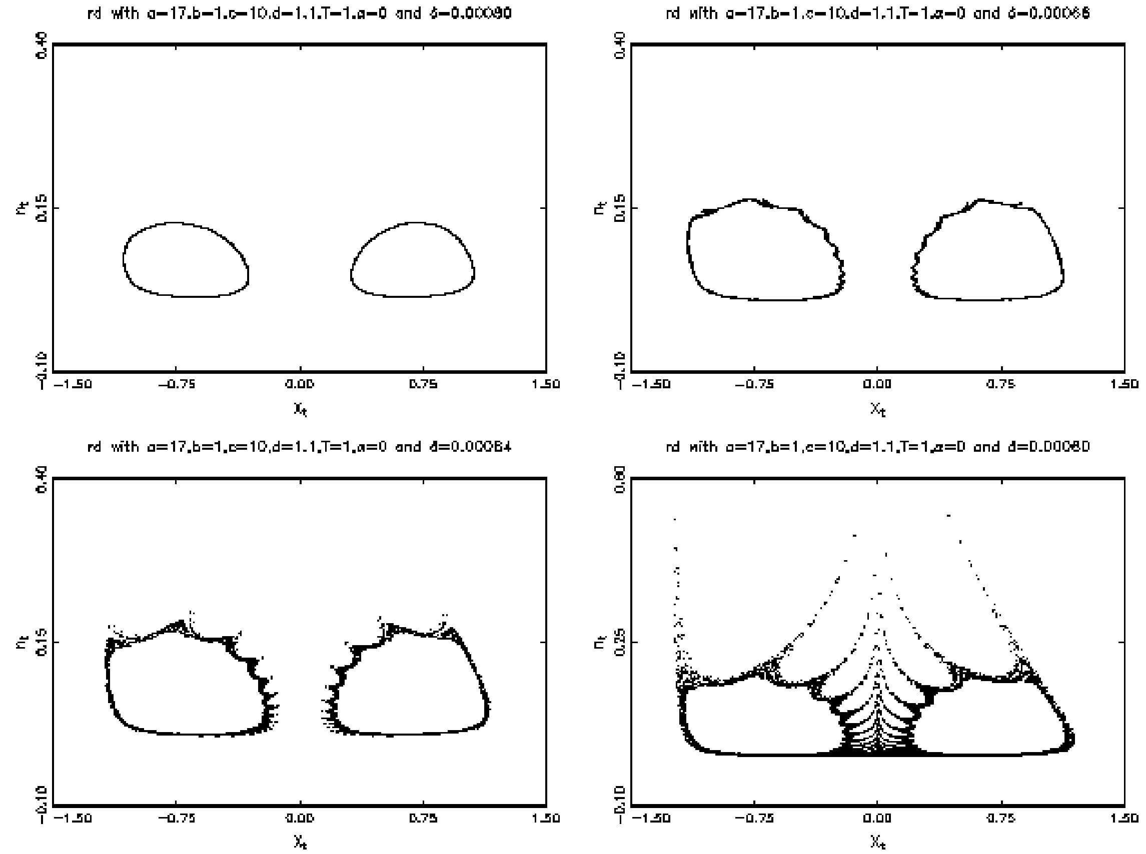

Figure 3: Attractors for the model with best-reply versus rational players and replicator dynamics.

underlying Cournot model. Now let x_t denote the quantity supplied by the best-reply players, z_t the quantity supplied by the imitators, and n_t the fraction of the population using the best-reply behavioral rule in period t. Consequently, the complete dynamics of the model are given by

$$\begin{aligned} x_{t+1} &= R\left(nx_t + (1-n)\, z_t\right) \\ z_{t+1} &= x_t \\ n_{t+1} &= h\left(x_t, z_t, n_t\right). \end{aligned} \tag{6}$$

Note that this is a 3-dimensional dynamical system which dimension cannot be reduced. Furthermore, the Cournot-Nash equilibrium quantity x^* is not the unique steady state quantity of the imitation rule. In fact, any quantity is a steady state of the imitation rule. The Cournot-Nash equilibrium quantity is, however, still the unique steady state quantity of the complete dynamical system. For this reason we do not have to worry about Assumption Q1 not being satisfied. Before specifying

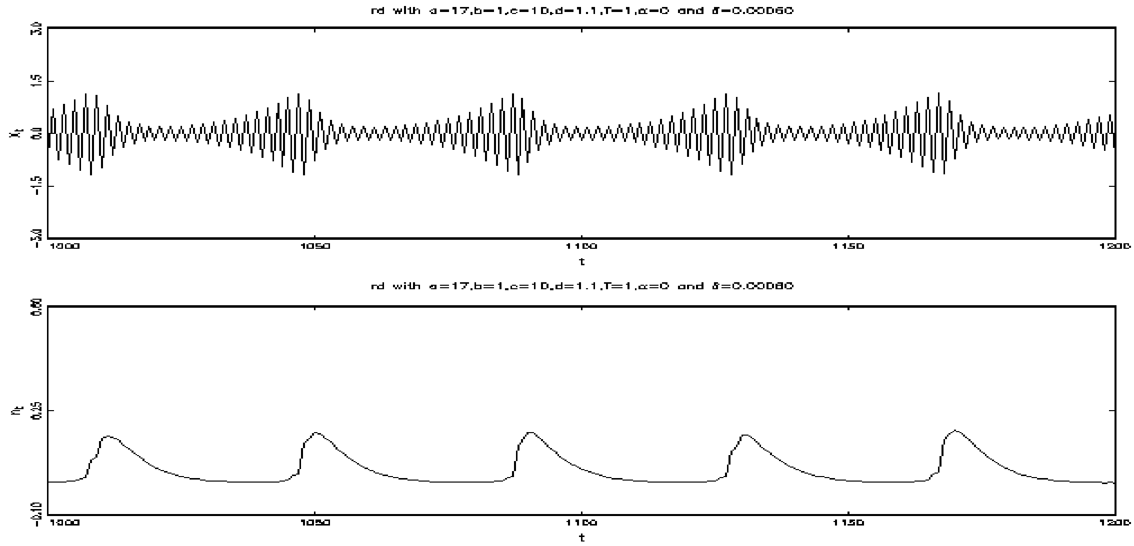

Figure 4: Time series for the model with best-reply versus rational players and replicator dynamics for $\delta = 0.00060$, corresponding to the strange attractor.

the population dynamics $h(x_t, z_t, n_t)$ we have the following result with respect to the dynamical system (6).

Lemma 7 *Consider the dynamical system (6). The eigenvalues of the Jacobian matrix of the linearized system, evaluated at the equilibrium (x^*, x^*, n^*), are*

$$\lambda_{1,2} = \frac{1}{2} n^* R'(x^*) \pm \frac{1}{2} \sqrt{(n^* R'(x^*))^2 + 4(1 - n^*) R'(x^*)}$$

and $\lambda_3 = \left. \frac{\partial h(x_t, z_t, n_t)}{\partial n_t} \right|_{(x^*, x^*, n^*)}$.

Proof. It can easily be shown that the Jacobian matrix, evaluated at the equilibrium (x^*, x^*, n^*), is given by

$$\mathbf{J} = \begin{pmatrix} n^* R'(x^*) & (1 - n^*) R'(x^*) & 0 \\ 1 & 0 & 0 \\ J_{31} & J_{32} & \left. \frac{\partial h(x_t, z_t, n_t)}{\partial n_t} \right|_{(x^*, x^*, n^*)} \end{pmatrix}.$$

The corresponding eigenvalues are exactly as stated in Lemma 7. ■

4.1 Discrete Choice Model

Let the population fractions be updated according to the discrete choice model again. In this case the model with imitators and best-reply players, as represented by the dynamical system (6), becomes

$$
\begin{aligned}
x_{t+1} &= R\left(nx_t + (1-n)\,z_t\right) \\
z_{t+1} &= x_t \\
n_{t+1} &= h\left(x_t, z_t, n_t\right) = \frac{\exp\left[\beta\left(\Pi^{br}\left(x_t, z_t, n_t\right) - T\right)\right]}{\exp\left[\beta\left(\Pi^{br}\left(x_t, z_t, n_t\right) - T\right)\right] + \exp\left[\beta \Pi^{im}\left(x_t, z_t, n_t\right)\right]},
\end{aligned}
\tag{7}
$$

where $\Pi^{br}(x_t, z_t, n_t)$ and $\Pi^{im}(x_t, z_t, n_t)$ are the average profit of the best-reply players and the imitators in period t, respectively. We will study what happens to the dynamics of (7) as the intensity of choice β increases.

4.1.1 Local Stability Analysis

We have the following result with respect to the local stability of the equilibrium of the dynamical system (7).

Proposition 8 *Consider the dynamical system (7). Let $-8 < R'(x^*) < -1$. The equilibrium of (7) is (x^*, x^*, n^*), where x^* is the Cournot-Nash equilibrium quantity and $n^* = \frac{1}{1+\exp[\beta T]}$. Then given T there exists a value β^* of β such that the equilibrium (x^*, x^*, n^*) is locally stable for all $\beta < \beta^*$ and unstable for all $\beta > \beta^*$. Moreover,*

$$
\beta^* = \frac{1}{T}\ln\left[\frac{1}{-R'(x^*) - 1}\right].
$$

Proof. Straightforward calculations show that (x^*, x^*, n^*) is the equilibrium of (7). First, it can easily be checked that the eigenvalue $\lambda_3 = 0$. Second, the other two eigenvalues λ_1 and λ_2 are complex conjugates if and only if

$$
\left(n^* R'(x^*)\right)^2 + 4\left(1 - n^*\right) R'(x^*) < 0.
$$

Rearranging terms shows that this condition is equivalent to

$$
-R'(x^*) < 4\frac{1-n^*}{(n^*)^2}.
$$

Because of the information costs $T > 0$ we know that $n^* \in \left(0, \frac{1}{2}\right]$. Consequently, the above condition is always satisfied if $R'(x^*) > -8$. The equilibrium (x^*, x^*, n^*) loses its stability when the eigenvalues $\lambda_{1,2}$ are on the unit circle, i.e.

$$
\lambda_1 \times \lambda_2 = -\left(1 - n^*\right) R'(x^*) = 1 \Leftrightarrow n^* = \frac{1 + R'(x^*)}{R'(x^*)}.
$$

Substituting $n^* = \frac{1}{1+\exp[\beta T]}$ and solving for β gives the condition as stated in Proposition 8. ■

Note that the equilibrium of (7) is locally stable for all values of the intensity of choice $\beta > 0$ in case $-1 \le R'(x^*) \le 0$.

4.1.2 A Bifurcation Scenario

In case we consider a linear inverse demand function and quadratic cost functions, the dynamical system (7) rewritten in deviations becomes

$$\begin{aligned} X_{t+1} &= -\frac{b}{2b-d}\left(n_t X_t + \left(1-n_t\right) Z_t\right) \\ Z_{t+1} &= X_t \\ n_{t+1} &= \frac{1}{1+\exp\left[\beta\left(T - b\left(Z_t - X_t\right)\left(n_t X_t + \left(1-n_t\right) Z_t\right) + \left(\frac{1}{2}d - b\right)\left(Z_t^2 - X_t^2\right)\right)\right]}, \end{aligned} \tag{8}$$

where $X_t = x_t - x^*$ and $Z_t = z_t - x^*$ are deviations from the Cournot-Nash equilibrium quantity x^*. As indicated by Proposition 8 a Hopf bifurcation occurs at $\beta^* = \frac{1}{T}\ln\left[\frac{2b-d}{d-b}\right]$. In fact, the equilibrium of the dynamical system becomes unstable and an invariant closed curve emerges. Because the primary bifurcation is a Hopf bifurcation we cannot use the symmetry of the dynamical system (8) to determine the secondary bifurcation analytically. The invariant set resulting after the Hopf bifurcation consists of infinitely many points which cannot be written down explicitly. Consequently, it is not possible to determine the Jacobian matrix. As β increases further the closed curve breaks up and a strange attractor is created. Figure 5 gives pictures of the attractors for parameter values $a = 17$, $b = 1$, $c = 10$, $d = 1.1$, $T = 1$, and for different values of β. For this numerical example the primary bifurcation occurs at $\beta^* = 2\ln 3$.

4.2 Replicator Dynamics

Consider the model with imitators versus best-reply players and the population dynamics described by the replicator dynamics with deterministic noise. This results in a specification of the dynamical system (6) given by

$$\begin{aligned} x_{t+1} &= R\left(nx_t + \left(1-n\right) z_t\right) \\ z_{t+1} &= x_t \\ n_{t+1} &= \frac{n_t\left(1-2\delta\right)\left(\alpha + \Pi^{br}\left(X_t, Z_t, n_t\right) - T\right)}{\alpha + n_t\left(\Pi^{br}\left(X_t, Z_t, n_t\right) - T\right) + \left(1-n_t\right)\Pi^{im}\left(X_t, Z_t, n_t\right)} + \delta \end{aligned} \tag{9}$$

In this section we study what happens to the dynamical behavior of (9) as the deterministic noise δ approaches zero.

4.2.1 Local Stability Analysis

The local stability analysis of the equilibrium of the dynamical system (9) is summarized in Proposition 9.

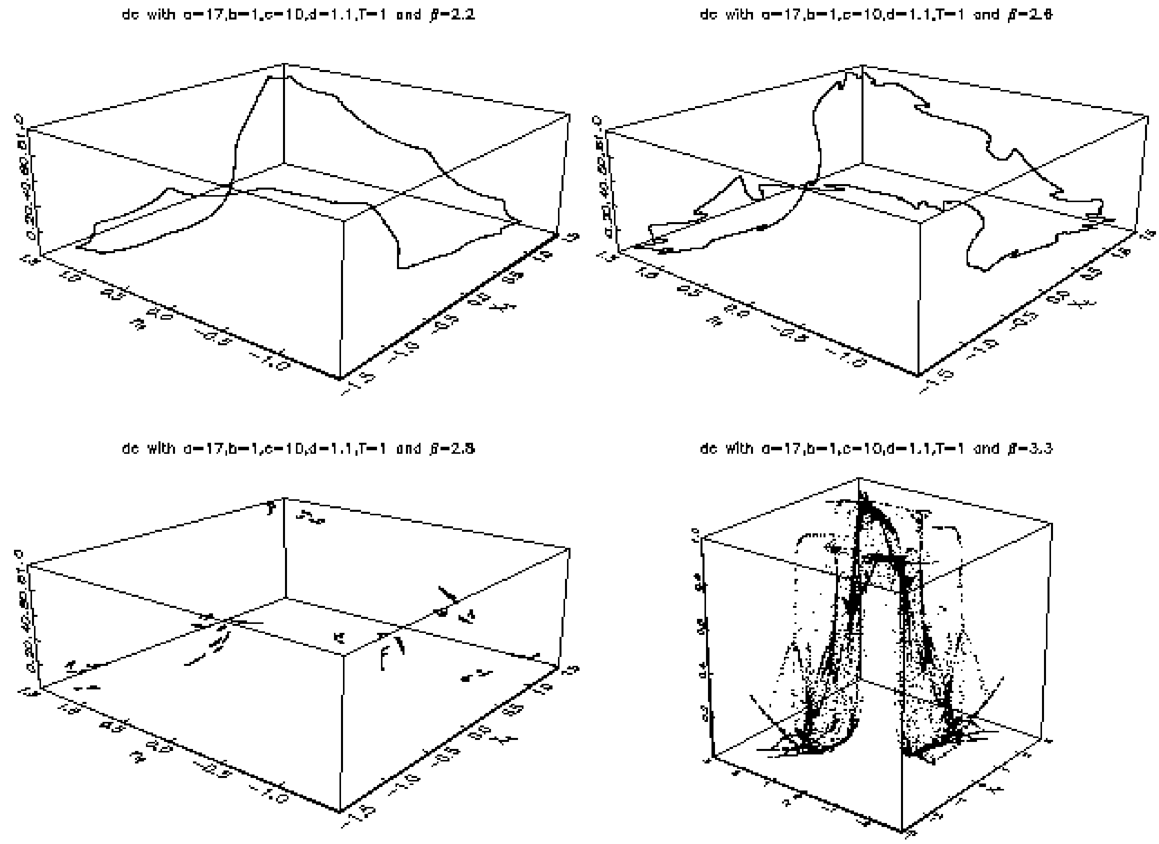

Figure 5: Attractors for the model with imitators versus best-reply players and discrete choice dynamics.

Proposition 9 *Consider the dynamical system (9). Let* $0 < \delta < \frac{1}{2}$*,* $T > 0$*, and* $-8 < R'(x^*) < -1$*. The equilibrium of (9) is* (x^*, x^*, n^*)*, where* x^* *is the Cournot-Nash equilibrium quantity and*

$$n^* = \frac{1}{2T}\left(\delta\left(2\left(\alpha + \Pi^*\right) - T\right) + T - \sqrt{\delta^2\left(2\left(\alpha + \Pi^*\right) - T\right)^2 + \left(1 - 2\delta\right)T^2}\right).$$

Furthermore, the equilibrium (x^*, x^*, n^*) *is locally stable (unstable) for* $\delta > \delta^*$ $(\delta < \delta^*)$*, where*

$$\delta^* = \frac{\left(R'(x^*) + 1\right)T}{R'(x^*)\left(\left(R'(x^*) + 2\right)\left(\alpha + \Pi^*\right) - \left(R'(x^*) + 1\right)T\right)}.$$

Proof. Straightforward calculations show that (x^*, x^*, n^*) is the equilibrium of (9). It can easily be checked that the equilibrium (x^*, x^*, n^*) loses stability when the eigenvalues λ_1 and λ_2, which are complex conjugates, cross the unit circle. This happens when

$$n^* = \frac{1 + R'(x^*)}{R'(x^*)}.$$

Substituting the expression for n^* and solving for δ gives the condition as stated in Proposition 9. ■

Note that the equilibrium (x^*, x^*, n^*) of (9) is locally stable for all values of the deterministic noise δ in case $-1 \leq R'(x^*) \leq 0$.

4.2.2 A Bifurcation Scenario

Due to the fact that the replicator dynamics represents the population dynamics, the dynamical system (9) does not exhibit any symmetry. This implies that we have to turn to numerical simulations immediately to analyze a bifurcation scenario. For the same reason as in section 3 we have not included the dynamical system representing the linear-quadratic specification of the model with the replicator dynamics describing the population dynamics. Let the parameters $a = 17$, $b = 1$, $c = 10$, $d = 1.1$, $T = 1$, and $\alpha = 0$. As indicated by Proposition 9 a Hopf bifurcation occurs at $\delta^* = \frac{9}{490}$. Figure 6 gives attractors for the numerical example with the above parameter values and different values of the deterministic noise δ.

5 Concluding Remarks

In this paper we have considered an evolutionary game theoretic model dealing with the selection of behavioral rules in a Cournot duopoly game. Contrary to most evolutionary game theory, which concentrates on the selection of equilibrium actions in matrix games, we have focussed on a setting in which different types of behavioral rules are selected. The population dynamics representing the evolutionary selection of behavioral rules is coupled with the quantity dynamics arising from the interplay between these different behavioral rules. Once again, note that the results in this paper can easily be generalized to Cournot oligopoly games.

The behavior of the resulting dynamical system depends critically on the stability of the quantity dynamics. First, in case the cheapest behavioral rule is stable, the evolutionary process converges to a situation where most agents use this behavioral rule and produce quantities equal to the Cournot-Nash equilibrium quantity. Second, if the cheapest behavioral rule is unstable, complicated dynamical behavior may occur. In particular, high evolutionary pressure or a small noise rate with respect to the choice of behavioral rules leads to highly irregular quantity dynamics converging to a strange attractor.

Two typical examples have been analyzed to illustrate the complicated dynamical behavior in detail. The fractions of the population using each of the two behavioral

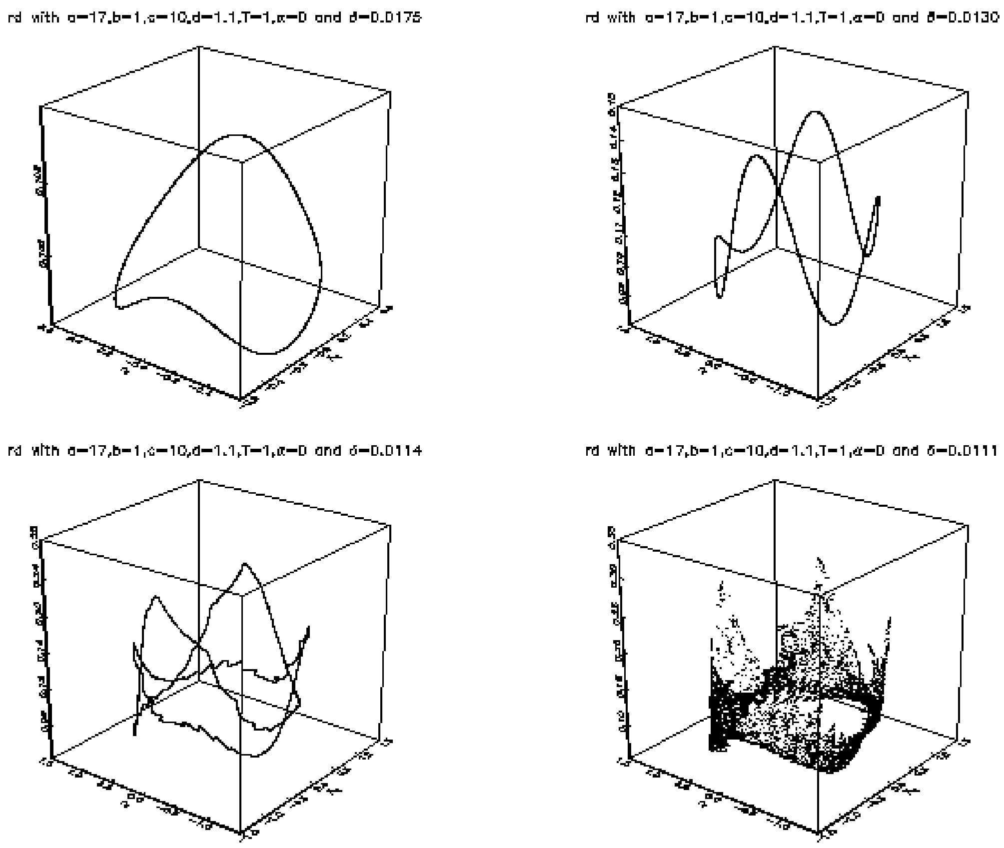

Figure 6: Attractors for the model with imitators versus best-reply players and replicator dynamics.

rules were determined by the discrete choice dynamics or the replicator dynamics with deterministic noise. In case the population dynamics is described by the discrete choice model, we found that if the simple behavioral rule is unstable and natural selection with respect to the behavioral rules is strong enough, strange dynamical behavior may occur. In case we use the replicator dynamics to describe the updating of the population fractions, we observe complicated dynamical behavior if the simple rule is unstable and deterministic noise is small enough. Note that the nonlinearity causing this erratic behavior arises from the interplay between population dynamics and quantity dynamics and not from a particular nonlinear specification of the underlying Cournot duopoly model.

The period doubling bifurcation and the different kind of Hopf bifurcations discussed in this paper are so-called local bifurcations. These local bifurcations lead to the creation of new periodic orbits near a steady state or near a periodic orbit. A topic of further research is the analysis of global bifurcations, i.e. homoclinic bifurcations, in this kind of models. Global bifurcations have important consequences for the global dynamical behavior of the model.

Appendices

A Derivation of Profit Functions

Consider the symmetric Cournot duopoly game with a linear demand function and quadratic cost functions, as specified in section 2.1. In that case, the profit of an individual supplying quantity x_1, when he is matched with an individual supplying x_2, is given by

$$\begin{aligned}\pi(x_1, x_2) &= P(x_1, x_2)\, x_1 - c(x_1) = (a - b(x_1 + x_2))\, x_1 - cx_1 + \frac{d}{2} x_1^2 \\ &= (a - c - bx_2)\, x_1 + \left(\frac{d}{2} - b\right) x_1^2.\end{aligned}$$

Now suppose the individuals in the population can choose between the behavioral rules H_1 and H_2, which prescribe quantities x_1 and x_2, respectively. Let n denote the fraction of the population supplying x_1. Then, in the case of random matching between individuals, the average profit of someone supplying x_1 can be written as

$$\begin{aligned}\Pi^1(x_1, x_2, n) &= n\pi(x_1, x_1) + (1-n)\,\pi(x_1, x_2) \\ &= (a - c)\, x_1 + \left(\frac{d}{2} - b\right) x_1^2 - b\,(nx_1 + (1-n)\, x_2)\, x_1.\end{aligned}$$

Rewritten in terms of deviations $X_1 = x_1 - x^*$ and $X_2 = x_2 - x^*$ from the Cournot-Nash equilibrium quantity x^*, the above expression becomes

$$\begin{aligned}\Pi^1(X_1, X_2, n) &= \left(b - \frac{d}{2}\right)\left(\frac{a-c}{3b-d}\right)^2 + \left(\frac{d}{2} - b\right) X_1^2 \\ &\quad - b\,(nX_1 + (1-n)\, X_2)\left(X_1 + \frac{a-c}{3b-d}\right).\end{aligned}$$

Using a similar argument we can find the average profit of someone supplying x_2. In fact, this profit is given by

$$\begin{aligned}\Pi^2(X_1, X_2, n) &= \left(b - \frac{d}{2}\right)\left(\frac{a-c}{3b-d}\right)^2 + \left(\frac{d}{2} - b\right) X_2^2 \\ &\quad - b\,(nX_1 + (1-n)\, X_2)\left(X_2 + \frac{a-c}{3b-d}\right).\end{aligned}$$

Note that only the difference between the average profits $\Pi^1(X_1, X_2, n)$ and $\Pi^2(X_1, X_2, n)$ matters for the discrete choice dynamics. Therefore, an important quantity is

$$\begin{aligned}\Pi^1(X_1, X_2, n) - \Pi^2(X_1, X_2, n) &= \left(\frac{d}{2} - b\right)\left(X_1^2 - X_2^2\right) \\ &\quad + b\left(nX_1 + (1-n)X_2\right)(X_1 - X_2).\end{aligned}$$

B Equivalence of the Cournot and Cobweb Model

In this appendix we show that the Cournot duopoly model discussed in section 3.1 is mathematically equivalent to the cobweb model with naive versus rational expectations introduced by Brock and Hommes [7, 8].

Consider the Cournot model with rational players and best-reply players. In case the discrete choice model describes the population dynamics, this Cournot model is represented by the dynamical system (4), that is

$$\begin{aligned}X_{t+1} &= -\frac{b(1-n_t)}{(2+n_t)b-d}X_t \\ n_{t+1} &= \frac{1}{1+\exp\left[\beta\left(T - \left(b-\frac{d}{2}\right)\left(\frac{3b-d}{(2+n_t)b-d}\right)^2 X_t^2\right)\right]}.\end{aligned}$$

The corresponding cobweb model of Brock and Hommes [7, 8], with a linear demand and supply function, can be represented by

$$\begin{aligned}p_{t+1} &= -\frac{\widehat{b}(1-m_t)}{2B+\widehat{b}(1+m_t)}p_t \\ m_{t+1} &= \tanh\left(\frac{\beta}{2}\left[\frac{\widehat{b}}{2}\left(\frac{\widehat{b}(1-m_t)}{2B+\widehat{b}(1+m_t)}+1\right)^2 p_t^2\right] - T\right),\end{aligned}$$

where $\widehat{b}$ and B are the marginal demand and supply parameters, p_t is the market clearing price in period t, and $m_t = n_{1,t} - n_{2,t} = 2n_t - 1$ is the difference in the population fractions. This means that if $m_t = 1$ all agents are rational and if $m_t = -1$ all agents use naive expectations. Rewriting their dynamical system in terms of n_t gives

$$\begin{aligned}p_{t+1} &= -\frac{\widehat{b}(1-n_t)}{B+\widehat{b}n_t}p_t \\ n_{t+1} &= \frac{1}{1+\exp\left[\beta\left(T - \frac{\widehat{b}}{2}\left(\frac{\widehat{b}+B}{B+\widehat{b}n_t}\right)^2 p_t^2\right)\right]}.\end{aligned}$$

Clearly, if we take the parameters $\widehat{b}=b$ and $B=2b-d$ their dynamical system becomes

$$\begin{aligned} p_{t+1} &= -\frac{b\left(1-n_t\right)}{\left(2+n_t\right)b-d}p_t \\ n_{t+1} &= \frac{1}{1+\exp\left[\beta\left(T-\frac{b}{2}\left(\frac{3b-d}{(2+n_t)b-d}\right)^2 p_t^2\right)\right]}, \end{aligned}$$

which is equivalent to (4) up to the following transformation of variables

$$X_t=\sqrt{\frac{b}{2b-d}}p_t.$$

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