Cooperative Multicriteria Games with Public and Private Criteria

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Cooperative Multicriteria Games with Public and Private Criteria; An Investigation of Core Concepts

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Abstract: A new class of cooperative multicriteria games is introduced which takes into account two different types of criteria: private criteria, which correspond to divisible and excludable goods, and public criteria, which in an allocation take the same value for each coalition member. The different criteria are not condensed by means of a utility function, but left in their own right. Moreover, the games considered are not single-valued, but each coalition can realize a set of vectors — representing the outcomes of each of the criteria — depending on several alternatives. Two core concepts are defined: the core and the dominance outcome core. The relation between the two concepts is studied and the core is axiomatized by means of consistency properties.

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1 Introduction

In matters of conflict, players frequently evaluate situations on the basis of several criteria. Still, games with multiple criteria and in particular *cooperative* games with multiple criteria have received relatively little attention in game theoretic literature. Some exceptions are Bergstresser and Yu (1977) and Lind (1996).

In the current paper, we introduce a new class of cooperative multicriteria games. Two fundamentally different types of criteria are considered: *private criteria* and *public criteria*. *Private* or divisible criteria share the characteristics of the criterion one usually works with when studying games with transferable utility, the characteristics of money: the amount obtained can be divided over coalition members so that one member consumes a different quantity than another member, and that which is consumed by one member cannot be consumed by another. In economic terms, these criteria are rival and excludable. *Public* or indivisible criteria have the same value for all members of a coalition; they are non-rival and non-excludable. Examples of such criteria are global warming, investment in medical research, or, on a different scale, the national rate of unemployment and its effect on the economy, political stability, and the safety in your country.

The introduction of public criteria is new to cooperative game theory, presumably because it is assumed that some central authority takes a (socially optimal) decision on such criteria. However, the value of a public criterion is often influenced by decisions made on private criteria by individual agents (think of pollution levels, for example). Hence, it seems that decisions on private and public criteria should not be treated separately. An integrated view on private and public criteria might expose the trade-offs faced by individuals not only between criteria in the same category, but also between criteria in different categories.

The 'value' of a coalition is usually interpreted as that which its members can guarantee themselves by joining forces. If multiple criteria are involved, then improvement in one criterion (number of fish caught) may well have detrimental effects on other criteria (environmental issues like biodiversity). So, the relative importance of different criteria plays a significant role. But the relative importance of two criteria may differ with their values. For example, rich countries attach relatively more importance to controlling pollution levels than to increasing production since production levels and pollution levels are already high. For developing countries with low production levels, however, increasing production is more important than controlling pollution levels. We believe that 'collapsing' the different criteria to one number by means of a utility function ignores some of the most interesting issues associated with multicriteria decision situations. By leaving the different criteria in their own right, one can investigate what kind of trade-offs players face between the criteria. Moreover, such an approach respects the incommensurability of some attributes: in many cases agents may be incapable of or morally opposed against aggregating the value of money and the value of — for instance — a human life to a common scale. In cooperative multicriteria games we therefore consider it natural to assign a set of vector values to each coalition, i.e., we consider characteristic correspondences instead of single valued characteristic functions and an obtainable 'value' is a vector that specifies the value of all the criteria for a particular alternative that is feasible to a coalition.

Cooperative multicriteria games with private and public criteria as defined and studied in the current paper generalize the games used in Bergstresser and Yu (1977) and Lind (1996). These authors do not discriminate between several types of criteria; they only use what we call private criteria. Moreover, the characteristic functions in their games are single-valued instead of set-valued.

After defining multicriteria cooperative games with private and public criteria, the obvious next step is the search for reasonable solutions to such games. This paper concentrates on core concepts, which rule out those outcomes which are in a sense unstable because subcoalitions of agents are able to reach agreements that are better for all their members. Taking into account the features of the model, the distinction between private and public criteria and the introduction of set-valued characteristic functions, we define two concepts: the dominance outcome core and the core.

Well-known axiomatizations of core concepts for single-criterion cooperative games (see Peleg (1985, 1986, 1987)) use a consistency or reduced game property. The consistency principle essentially means that if the grand coalition of players reaches an agreement, then no subcoalition of players has an incentive to renegotiate within the coalition after giving the players outside the coalition their part of the solution, because the proposed agreement is also a part of the solution of the reduced game played within the subcoalition.

The current paper investigates consistency properties of the proposed core for cooperative multicriteria games. We provide three axiomatic characterizations of the core that are based on the notion of consistency. One of these characterizations uses converse consistency, a property that postulates that a proposed agreement must be in the solution of a game if for every subcoalition it holds that the restriction of this agreement to the subcoalition is in the solution of the reduced game. A second axiomatization of the core uses a converse consistency requirement that restricts attention to subcoalitions of two players. The two axiomatizations of the core of cooperative multicriteria games that use converse consistency properties are similar to the axiomatizations of core concepts for cooperative games with or without transferable utility by Peleg (1985, 1986, 1987).

The third axiomatization of the core provided in this paper differs significantly from the previous two. It uses a new definition of reduced games, one that stresses the fact that there are players outside each subcoalition that cannot be ignored altogether by requiring players in a subcoalition to cooperate with at least one outside player. Consistency with respect to this new definition of reduced games is used to give an axiomatic characterization of the core for multicriteria games with an enlightenment property (see section 5) instead of converse consistency. It is shown by means of a counterexample that this characterization does not hold if the old definition of reduced games is used.

The set-up of the paper is as follows. Cooperative multicriteria games with private and public criteria are defined in section 2, along with the core and the dominance outcome core. In section 3 we prove that the dominance outcome core always contains the core and that both concepts coincide for games satisfying some additional assumptions. In the remainder of the paper, sections 4 and 5, we provide several axiomatizations of the core based on the notion of consistency. In section 4 converse consistency is used to characterize the core and in section 5 we give the new definition of reduced games that was mentioned before and use this to characterize the core without requiring converse consistency.

2 Definitions

For vectors $x, y \in \mathbb{R}^m$, we write $x \geqq y$ if $x_i \geq y_i$ for all $i = 1, \ldots, m$, $x \geq y$ if $x \geqq y$ and $x \neq y$, and $x > y$ if $x_i > y_i$ for all $i = 1, \ldots, m$. For a set $A \subseteq \mathbb{R}^m$, we define its Pareto edge by $\mathrm{Par}(A) := \{x \in A \mid \text{there is no } y \in A \text{ with } y > x\}$. The number of elements of a finite set A is denoted by $|A|$, the collection of all of its subsets by 2^A. For two subsets A and B of a vector space V we define $A + B = \{a + b \mid a \in A, b \in B\}$. For an arbitrary set A we denote by $\mathbb{R}^A$ the vector space of all real-valued functions on A.

Let U be an infinite set of players. A *cooperative multicriteria game with public and private* *criteria*, or a *game* for ease of notation, is described by

- A finite set D of divisible or private criteria;
- A finite set P of indivisible or public criteria;
- A finite, nonempty set $N \subset U$ of players;
- A correspondence $v : 2^N \setminus \{\emptyset\} \twoheadrightarrow \mathbb{R}^{D \cup P}$;

such that $D \cap P = \emptyset$, $D \cup P \neq \emptyset$ and $v(S) \neq \emptyset$ for each coalition $S \in 2^N \setminus \{\emptyset\}$. The sets D and P that define a certain game will not be mentioned explicitly and a game is simply denoted (N, v). For one-person coalitions we write $v(i)$ instead of $v(\{i\})$. Let Γ denote the set of games as defined above.

Example 2.1 Two neighbouring countries, A and B, negotiate to reduce CO_2 levels in the air. The marginal costs of reducing CO_2 levels increase as abatements increase: there are relatively cheap methods that can be used to reduce CO_2 levels at first, but to effect higher reductions, more expensive methods have to be employed as well. Suppose country A on its own can abate in a low-cost way by spending 100 to reduce the level of CO_2 in the air by 1, and it can abate more, a reduction of 3, at a cost of 600. Country B on its own can reduce the CO_2 level by 2 at a cost of 150 and by 7 at a cost of 900. If the countries cooperate, they can realize all the above mentioned possibilities but also profit from each other's expertise and abate relatively cheaper. They can reduce CO_2 levels by 3 at a cost of 200 and by 10 at a cost of 1200.

The cooperative multicriteria game describing this situation has one private criterion, minus the cost of the abatements, and one public criterion, the decrease in the CO_2 level in the air. The player set is $N = \{A, B\}$, the characteristic function is given by

$$\begin{aligned} v(A) &= \{(0,0), (-100,1), (-600,3)\}, \\ v(B) &= \{(0,0), (-150,2), (-900,7)\}, \\ v(\{A,B\}) &= v(A) \cup v(B) \cup \{(-200,3), (-1200,10)\}. \end{aligned}$$

Several subsets of Γ correspond to well-known classes of games.

Example 2.2 A game (N, v) with $P = \emptyset$, $|D| = 1$, and $|v(S)| = 1$ for each coalition $S \in 2^N \setminus \{\emptyset\}$ is essentially a TU-game.

Example 2.3 A game (N, v) with $P = \emptyset$ and $v(S)$ a compact and comprehensive (in the sense that $b \in v(S)$ and $0 \leqq a \leqq b$ implies $a \in v(S)$) subset of $\mathbb{R}^D_+$ is a multi-commodity game as studied by van den Nouweland *et al.* (1989).

The characteristic function of NTU-games is also set-valued and vector-valued, but describes for a coalition the payoff for each separate member, so that the value of a coalition S is a subset of $\mathbb{R}^S$. This differs from our cooperative multicriteria games, where the correspondence v maps the coalitions to a fixed vector space $\mathbb{R}^{D \cup P}$.

Cooperative multicriteria games with private and public criteria generalize the cooperative multicriteria games used by Bergstresser and Yu (1977) and Lind (1996) in the sense that these authors do not use set-valued characteristic functions and do not discriminate between different types of criteria.

In what follows, we need a definition of an allocation. In a game $(N, v) \in \Gamma$, an allocation takes an element of the set of values attainable by the grand coalition N and divides it among the players in accordance with the characteristics of the criteria: when restricted to divisible criteria everything is divided, whereas for indivisible criteria every player gets the same fixed amount. Before formally defining allocations, some more notation is needed.

Consider a game $(N, v) \in \Gamma$ and a vector $x = (x^i)_{i \in N}$ with $x^i \in \mathbb{R}^{D \cup P}$ for each $i \in N$. Let $S \in 2^N \setminus \{\emptyset\}$. Then x_S denotes the vector $(x^i)_{i \in S}$, i.e., x restricted to the components of the members of coalition S and $x(S)$ denotes the sum of the elements $(x^i)_{i \in S}$, $x(S) := \sum_{i \in S} x^i$. For a vector (or function) $y \in \mathbb{R}^{D \cup P}$ the restriction of y to P is denoted $y_{|P}$ and the restriction of y to D is denoted $y_{|D}$.

Definition 2.4 Given a game (N, v) an *allocation* is a vector $x = (x^i)_{i \in N}$ with $x^i \in \mathbb{R}^{D \cup P}$ for each player $i \in N$ that satisfies the requirement that there exists a $y \in v(N)$ for which

$$\begin{array}{rcll} \sum_{i \in N} x^i_{|D} & = & y_{|D} & \text{and} \\ x^i_{|P} & = & y_{|P} & \text{for each } i \in N. \end{array}$$

The set of allocations of (N, v) is denoted $A(N, v)$.

A coalition can *improve upon* an allocation if there is an outcome it can guarantee itself which — when distributed over its members in a feasible way — is at least as good for each member and better in some criterion for at least one coalition member. Formally, a coalition

$S \subseteq N$ can *improve upon* an allocation x if there exists a vector $y \in v(S)$ such that

$$\begin{array}{ccl} \sum_{i\in S} x^i_{|D} & \leqq & y_{|D} \quad \text{and} \\ x^i_{|P} & \leqq & y_{|P} \quad \text{for each } i \in S, \end{array}$$

where at least one of the inequalities is strict ($\leq$). Such a vector y is said to *dominate x via S*. An allocation in a game (N, v) is *individually rational* if one-player coalitions, i.e. individual players, cannot improve upon it and it is an *imputation* if neither N nor individual players can improve upon it. The set of individually rational allocations and the set of imputations of a game (N, v) are denoted by $IR(N, v)$ and $I(N, v)$, respectively.

A *solution concept σ* on the class Γ is a map that assigns to each game $(N, v) \in \Gamma$ a set of allocations $\sigma(N, v)$. Hence, $\sigma(N, v) \subseteq A(N, v)$ for all $(N, v) \in \Gamma$.

This paper concentrates on core concepts, i.e. concepts that rule out allocations that are in some sense unstable. We define two different core concepts.

Definition 2.5 The *core* $C(N, v)$ of a game (N, v) is the set of allocations upon which no coalition can improve:

$$\begin{array}{rcll} C(N,v) & = & \{x \in A(N,v) \mid & \text{there exist no } S \in 2^N \setminus \{\emptyset\} \text{ and } y \in v(S) \text{ s.t.} \\ & & & \sum_{i\in S} x^i_{|D} \ \leqq \ y_{|D} \quad \text{and} \\ & & & x^i_{|P} \ \leqq \ y_{|P} \quad \text{for each } i \in S \\ & & & \text{with at least one of the inequalities being strict } (\leq)\} \end{array}$$

Definition 2.6 The *dominance outcome core* $DOC(N, v)$ of a game (N, v) is the set of imputations for which there is no coalition S and another imputation y such that y^i is better than x^i for each player $i \in S$ and such that the players in S can jointly guarantee themselves at least what they get according to the allocation y:

$$\begin{array}{rcll} DOC(N,v) & = & \{x \in I(N,v) \mid & \text{there exist no } S \in 2^N \setminus \{\emptyset\}, y \in I(N,v), \text{ and } z \in v(S) \text{ s.t.} \\ & & & y^i \ \geq \ x^i \quad \text{for each } i \in S, \\ & & & \sum_{i\in S} y^i_{|D} \ \leqq \ z_{|D} \quad \text{and} \\ & & & y^i_{|P} \ \leqq \ z_{|P} \quad \text{for each } i \in S \qquad \} \end{array}$$

3 Relations between the Core and the Dominance Outcome Core

In this section we prove that the core of a game is always included in the dominance outcome core. Moreover, we prove that the core equals the dominance outcome core under some mild conditions.

Proposition 3.1 *For each game $(N, v) \in \Gamma$ it holds that $C(N, v) \subseteq DOC(N, v)$.*

Proof. Let $(N, v) \in \Gamma$. If $C(N, v) = \emptyset$ we are done. So, assume $C(N, v) \neq \emptyset$ and let $x = (x^i)_{i \in N} \in C(N, v)$. Then

- $x^i \in \mathbb{R}^{D \cup P}$ for each player $i \in N$ and
- there exists a $y \in v(N)$ such that $\begin{array}{lcll} \sum_{i \in N} x^i_{|D} & = & y_{|D} & \\ x^i_{|P} & = & y_{|P} & \text{for each } i \in N. \end{array}$
- Since $x \in C(N, v)$, neither N nor individual players can improve upon it.

Consequently, $x \in I(N, v)$. Now suppose $x \notin DOC(N, v)$. Then let $S \in 2^N \setminus \{\emptyset\}, y \in I(N, v)$, and $z \in v(S)$ be such that

$$\begin{array}{lcll} y^i & \geq & x^i & \text{for each } i \in S; \\ \sum_{i \in S} y^i_{|D} & \leqq & z_{|D}; & \\ y^i_{|P} & \leqq & z_{|P} & \text{for each } i \in S. \end{array}$$

Hence, there exist an $S \in 2^N \setminus \{\emptyset\}$, $z \in v(S)$ such that

$$\begin{array}{lclcll} \sum_{i \in S} x^i_{|D} & \leqq & \sum_{i \in S} y^i_{|D} & \leqq & z_{|D} & \\ x^i_{|P} & \leqq & y^i_{|P} & \leqq & z_{|P} & \text{for each } i \in S \end{array}$$

with at least one strict inequality since $y^i \geq x^i$ for all $i \in S$. This contradicts $x \in C(N, v)$. □

In general, the core is not equal to the dominance outcome core. In the following example both cores do not coincide.

Example 3.2 Consider a three-player, bicriteria game (N, v) where the first criterion is divisible and the second public. Define $v(i) = \{(1, 10)\}$ for all $i \in N$ and $v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = v(\{1, 2, 3\}) = \{(3, 10)\}$. Then $I(N, v) = DOC(N, v) = \{(x^1, x^2, x^3) \mid x^1 = x^2 = x^3 = (1, 10)\}$. However, $C(N, v) = \emptyset$, since every two-player coalition can improve upon the unique imputation. For instance, for $S = \{1, 2\}$ and $y = (3, 10) \in v(S)$: $x^1_1 + x^2_1 < 3 = y_1$ and $x^1_2 = x^2_2 = 10 = y_2$.

Under some restrictions, however, the two cores coincide.

Proposition 3.3 *Let $(N, v) \in \Gamma$ be a game for which the following four properties hold:*

1. Comprehensiveness of $v(i)$ for each $i \in N$ and of $v(N)$:

$$\begin{cases} \text{for all } i \in N \quad \text{and all } a \in v(i) & : \ \{x \in \mathbb{R}^{D\cup P} \mid x \leqq a\} \ \subseteq \ v(i) \\ \qquad\qquad\qquad \text{for all } a \in v(N) & : \ \{x \in \mathbb{R}^{D\cup P} \mid x \leqq a\} \ \subseteq \ v(N) \end{cases}$$

2. Compactness conditions:

$$\begin{cases} \text{for all } i \in N \quad \text{and all } a \in v(i) & : \ (\{a\} + \mathbb{R}_+^{D\cup P}) \cap v(i) \quad \text{is compact} \\ \qquad\qquad\qquad \text{for all } a \in v(N) & : \ (\{a\} + \mathbb{R}_+^{D\cup P}) \cap v(N) \quad \text{is compact} \end{cases}$$

3. Nonlevelness of $v(i)$ for each $i \in N$ and of $v(N)$:

$$\begin{cases} \text{for all } i \in N \quad \text{and all } a, b \in \mathrm{Par}(v(i)) & : \ \text{if } a \geqq b, \quad \text{then } a = b \\ \qquad\qquad\qquad \text{for all } a, b \in \mathrm{Par}(v(N)) & : \ \text{if } a \geqq b, \quad \text{then } a = b \end{cases}$$

4. A superadditivity condition:

For each $S \in 2^N \setminus \{\emptyset\}, y \in v(S)$, and $z^i \in v(i)$ for each $i \in N \setminus S$ it holds that if $y_{|P} \geqq z^i_{|P}$ for all $i \in N \setminus S$, then $a \in v(N)$, where $a \in \mathbb{R}^{D\cup P}$ is defined as follows:

$$\begin{aligned} a_{|D} &= y_{|D} + \textstyle\sum_{i \in N\setminus S} z^i_{|D} \\ a_{|P} &= y_{|P} \end{aligned}$$

Then $C(N, v) = DOC(N, v)$.

Remark 3.4 The definition of nonlevel sets given above is a standard definition. In the proof of the proposition it is convenient to use the following equivalent formulation:

$$\begin{cases} \text{for all } i \in N, \ \text{all } b \in \mathrm{Par}(v(i)), & \text{and all } a \in \mathbb{R}^{D\cup P} \ : \ \text{if } a \geq b, \ \text{then } a \notin v(i) \\ \qquad\qquad\quad \text{for all } b \in \mathrm{Par}(v(N)) & \text{and all } a \in \mathbb{R}^{D\cup P} \ : \ \text{if } a \geq b, \ \text{then } a \notin v(N) \end{cases}$$

Proof (prop. 3.3). By proposition 3.1: $C(N, v) \subseteq DOC(N, v)$. To prove that $DOC(N, v) \subseteq C(N, v)$, let $x = (x^i)_{i\in N} \in I(N, v)$ and assume that $x \notin C(N, v)$. Then there exist $S \in 2^N \setminus \{\emptyset, N\}$ and $y \in v(S)$ such that

$$\begin{cases} \sum_{i\in S} x^i_{|D} & \leqq \ y_{|D} \\ \quad x^i_{|P} & \leqq \ y_{|P} \quad \text{for each } i \in S \end{cases} \tag{1}$$

where at least one inequality is strict ($\leq$). For each $i \in N \setminus S$, let $z^i \in v(i)$ be such that $z^i_{|P} \leqq y_{|P}$ and $z^i \in \mathrm{Par}(v(i))$, the Pareto edge of $v(i)$. Such z^i exist: let $i \in N \setminus S$ and $a \in v(i)$, which is possible by nonemptiness of $v(i)$. Either $a_{|P} \leqq y_{|P}$ or, using comprehensiveness of $v(i)$, one can lower the coordinates in $\{k \in P \mid a_k > y_k\}$ without leaving $v(i)$. So let $b \in v(i)$ be such that $b_{|P} \leqq y_{|P}$. By assumption the set $\{c \in v(i) \mid c \geqq b\}$ is compact and hence the set $\{c \in v(i) \mid c \geqq b, c_{|P} = b_{|P}\}$ is compact. Define $u \in \mathbb{R}^{D\cup P}$ such that $u_k = 1$ for $k \in D$ and $u_k = 0$ for $k \in P$. By nonemptiness and compactness of $\{c \in v(i) \mid c \geqq b, c_{|P} = b_{|P}\}$, we know that

$$\alpha^* := \max\{\alpha \in \mathbb{R}_+ \mid b + \alpha u \in \{c \in v(i) \mid c \geqq b, c_{|P} = b_{|P}\}\} \tag{2}$$

exists. We claim that $b + \alpha^* u \in \mathrm{Par}(v(i))$. Suppose to the contrary, that $b + \alpha^* u$ is not on the Pareto edge of $v(i)$. Then $d > b+\alpha^* u$ for some $d \in v(i)$. In particular, $d_k > b_k+\alpha^* u_k = b_k+\alpha^*$ for each $k \in D$. Take $\beta = \min\{d_k-b_k \mid k \in D\}$. Then $\beta > \alpha^*$ and $b+\beta u \leqq d$. By comprehensiveness of $v(i)$ it follows that $b+\beta u \in v(i)$. Also $b+\beta u \in \{c \in v(i) \mid c \geqq b, c_{|P} = b_{|P}\}$. Hence by (2), $\beta \leq \alpha^*$ must hold. This yields a contradiction. So $b + \alpha^* u \in \{c \in v(i) \mid c \geqq b, c_{|P} = b_{|P}\} \cap \mathrm{Par}(v(i))$. Since $b_{|P} \leqq y_{|P}$, we can now define the desired z^i by $z^i := b + \alpha^* u$.

By the superadditivity condition the vector $a \in \mathbb{R}^{D\cup P}$ with $a_{|D} = y_{|D} + \sum_{i\in N\setminus S} z^i_{|D}$ and $a_{|P} = y_{|P}$ is an element of $v(N)$. Using the comprehensiveness of $v(N)$ and the compactness assumption on $v(N)$, it follows in a similar manner as demonstrated above, that the set $\{c \in v(N) \mid c \geqq a, c_{|P} = a_{|P}\}$ contains an element b on the Pareto edge of $v(N)$. Take such a $b \in v(N)$. This b is used to construct an imputation $\hat{x}$ that dominates imputation x via coalition S. Define $\hat{x} = (\hat{x}^i)_{i\in N} \in (\mathbb{R}^{D\cup P})^N$ as follows:

$$\begin{array}{rcll} \hat{x}^i_{|P} & = & b_{|P} & \text{for each } i \in N \\ \hat{x}^i_{|D} & = & z^i_{|D} + \frac{1}{|N\setminus S|}(b - y - \sum_{i\in N\setminus S} z^i)_{|D} & \text{for each } i \in N \setminus S \\ \hat{x}^i_{|D} & = & x^i_{|D} + \frac{1}{|S|}(y - \sum_{i\in S} x^i)_{|D} & \text{for each } i \in S \end{array}$$

Notice that

- $\sum_{i\in N} \hat{x}^i_{|D} = b_{|D}$ and $\hat{x}^i_P = b_{|P}$ for all $i \in N$. Since $b \in v(N)$, it follows that $\hat{x}$ is an allocation;

- Since b is on the Pareto edge of $v(N)$, using the nonlevelness of $v(N)$ yields that the allocation $\hat{x}$ cannot be improved upon by the grand coalition N;

- Since $y_{|D} \geqq \sum_{i\in S} x^i_{|D}$ and $b_{|P} = a_{|P} = y_{|P} \geqq x^i_{|P}$ for all $i \in S$, we have that $\hat{x}^i \geqq x^i$ for each player $i \in S$. Also, $x \in I(N, v)$ by assumption. Hence, singleton coalitions $\{i\}$ with $i \in S$ cannot improve upon $\hat{x}$;

- Since $b \geqq a$, $a_{|D} = y_{|D}+\sum_{i\in N\setminus S} z^i_{|D}$, and $b_{|P} = a_{|P} = y_{|P} \geqq z^i_{|P}$ for each $i \in N \setminus S$, we have that $\hat{x}^i \geqq z^i$ for each $i \in N \setminus S$. Using the nonlevelness of $v(i)$ and the fact that z^i lies on the Pareto edge of $v(i)$, we derive that singleton coalitions $\{i\}$ with $i \in N \setminus S$ cannot improve upon $\hat{x}$.

From the four points above we deduce that $\hat{x} \in I(N, v)$. Moreover, $y \in v(S), \sum_{i \in S} \hat{x}^i_{|D} = y_{|D}$, and $\hat{x}^i_{|P} = b_{|P} = y_{|P}$ for each $i \in S$. Thus, by (1) and the construction of $\hat{x}$: $\hat{x}^i \geq x^i$ for each $i \in S$ (recall that $\hat{x}^i_{|P} = \hat{x}^j_{|P}$ for all players i, j). Conclude that $x \notin DOC(N, v)$. Hence, $DOC(N, v) \subseteq C(N, v)$, which completes the proof. $\square$

4 Axiomatizations of the Core with Converse Consistency

In this section we study some properties of the core and provide several axiomatizations, all based on the notions of consistency and converse consistency. The consistency principle essentially means that if the grand coalition of players reaches an agreement, then no subcoalition of players has an incentive to renegotiate within the subcoalition after giving the players outside it their part of the solution, because the proposed agreement is also in the solution of the reduced game played within the subcoalition. The converse consistency axiom requires that a proposed agreement must be in the solution of a game if for every subcoalition it holds that the restriction of this agreement to that subcoalition is in the solution of the reduced game. Hence, it provides information about the solution of a game, given information about the solution of its reduced games, justifying the name 'converse' consistency. The axiomatizations are similar to those of Peleg (1985, 1986, 1987).

Definition 4.1 Let $(N, v) \in \Gamma, x \in A(N, v)$, and $S \in 2^N \setminus \{\emptyset, N\}$. The *reduced game* (S, v^x_S) of (N, v) with respect to allocation x and coalition S is the game defined by

$$\begin{array}{rcl} v^x_S(S) & = & v(N) - \tilde{x}(N \setminus S) \\ v^x_S(T) & = & \cup_{Q \subseteq N \setminus S} \; (v(T \cup Q) - \tilde{x}(Q)) \quad \text{for all } T \in 2^S \setminus \{\emptyset, S\}, \end{array}$$

where $\tilde{x} = (\tilde{x}^i)_{i \in N} \in (\mathbb{R}^{D \cup P})^N$ is defined for all $i \in N$ by

$$\tilde{x}^i_k = \begin{cases} x^i_k & \text{if} \quad k \in D \\ 0 & \text{if} \quad k \in P. \end{cases}$$

The interpretation of the reduced game is as follows. Suppose the group of all players initially agrees on an allocation x, and the players in $N \setminus S$ withdraw from the decision-making process taking their agreed-upon share of the private goods with them. Then, if the agents in S reconsider, they are facing the game v^x_S, because in their negotiations they take into account that they can cooperate with some of the players in $N \setminus S$ as long as those are given their shares of the private goods. Note that the players who leave the decision-making process are not guaranteed anything about the public criteria. Since these criteria are public, their level will ultimately be determined by the players who still take part in the decision-making process. Hence, players who leave this process take a risk, but if the solution concept is consistent, then the remaining players will not change their minds about the initially agreed-upon levels of the public criteria. This is similar to the treatment of public goods in van den Nouweland *et al.* (1998).

Let us consider some axioms that are used in the remainder of this section. A solution concept σ on Γ satisfies:

- **One Person Efficiency (OPE)** if for each game $(N, v) \in \Gamma$ with $|N| = 1$ it holds that $\sigma(N, v) = IR(N, v)$;
- **Individual Rationality (IR)** if for each game $(N, v) \in \Gamma$ it holds that $\sigma(N, v) \subseteq IR(N, v)$;
- **Inclusion of Imputation Set for two-player Games ($\mathbf{II_2}$)** if for every two-player game $(N, v) \in \Gamma$ it holds that $\sigma(N, v) \supseteq I(N, v)$;
- **Restricted Nonemptiness (r-NEM)** if for each game $(N, v) \in \Gamma$ it holds that if $C(N, v) \neq \emptyset$, then $\sigma(N, v) \neq \emptyset$;
- **Consistency (CONS)** if for each game $(N, v) \in \Gamma$ it holds that $x \in \sigma(N, v)$ implies $x_S \in \sigma(S, v_S^x)$ for each coalition $S \in 2^N \setminus \{\emptyset, N\}$;
- **Converse Consistency (COCONS)** if for each game $(N, v) \in \Gamma$ with $|N| \geq 2$ and each allocation $x \in A(N, v)$ it holds that if $x_S \in \sigma(S, v_S^x)$ for each $S \in 2^N \setminus \{\emptyset, N\}$, then $x \in \sigma(N, v)$;
- **Converse Consistency for Two-Player Reductions ($\mathbf{COCONS_2}$)** if for each game $(N, v) \in \Gamma$ with $|N| \geq 3$ and each allocation $x \in A(N, v)$ it holds that if $x_S \in \sigma(S, v_S^x)$ for each $S \in 2^N \setminus \{\emptyset, N\}$ with $|S| = 2$, then $x \in \sigma(N, v)$.

The next proposition states that the core satisfies all these axioms.

Proposition 4.2 *The core satisfies OPE, IR, II_2, r-NEM, CONS, COCONS, and $COCONS_2$.*

Proof. It is obvious that the core satisfies OPE, IR, II_2 and r-NEM.

To prove that the core satisfies CONS, let $(N,v) \in \Gamma, x \in C(N,v)$, and $S \in 2^N \setminus \{\emptyset, N\}$. Suppose that $x_S \notin C(S, v_S^x)$. Then there exist a coalition $T \in 2^S \setminus \{\emptyset\}$ and a vector $z \in v_S^x(T)$ such that

$$\begin{array}{lcl} \sum_{i \in T} x^i_{|D} & \leqq & z_{|D} \\ x^i_{|P} & \leqq & z_{|P} \quad \text{for all } i \in T \end{array}$$

with at least one strict inequality ($\leq$). Since $z \in v_S^x(T)$, there exist a $Q \subseteq N \setminus S$ and $y \in v(T \cup Q)$ such that $z = y - \tilde{x}(Q)$. Observe that by definition of the reduced game, $Q = N \setminus S$ if $T = S$. Now we have

$$\begin{array}{ccccccc} \sum_{i \in T \cup Q} x^i_{|D} & = & \sum_{i \in T} x^i_{|D} + \sum_{i \in Q} \tilde{x}^i_{|D} & \leqq & (z + \tilde{x}(Q))_{|D} & = & y_{|D} \\ x^i_{|P} & \leqq & z_{|P} & = & (y - \tilde{x}(Q))_{|P} & = & y_{|P} \quad \text{for all } i \in T \cup Q \end{array}$$

where at least one of the inequalities is strict ($\leq$). But then x cannot be in the core of (N,v), since $T \cup Q$ can improve upon it. Hence $x_S \in C(S, v_S^x)$ and the core satisfies CONS.

To prove that the core satisfies COCONS$_2$, Let $(N,v) \in \Gamma$ with $|N| \geq 3$ and $x \in A(N,v)$ such that $x_S \in C(S, v_S^x)$ for every two-player coalition $S \in 2^N \setminus \{\emptyset, N\}$. We will prove that no coalition of players can improve upon x, and hence $x \in C(N,v)$.

Suppose that N can improve upon x. Then, for some $y \in v(N)$:

$$\begin{array}{lcl} \sum_{i \in N} x^i_{|D} & \leqq & y_{|D} \\ x^i_{|P} & \leqq & y_{|P} \quad \text{for all } i \in N \end{array}$$

where at least one of the inequalities is strict ($\leq$). Let $S \in 2^N \setminus \{\emptyset, N\}$ have two players. Then, for a y as mentioned above it holds that

$$\begin{array}{ccccc} \sum_{i \in S} x^i_{|D} & \leqq & y_{|D} - \sum_{i \in N \setminus S} x^i_{|D} & = & (y - \tilde{x}(N \setminus S))_{|D} \\ x^i_{|P} & \leqq & y_{|P} & = & (y - \tilde{x}(N \setminus S))_{|P} \quad \text{for all } i \in S \end{array}$$

where at least one of the inequalities is strict ($\leq$). Since $y - \tilde{x}(N \setminus S) \in v_S^x(S)$, we find that S can improve upon x_S in (S, v_S^x). This contradicts $x_S \in C(S, v_S^x)$. We conclude that N cannot improve upon x in (N,v).

Now, let $T \in 2^N \setminus \{\emptyset, N\}$. To prove that T cannot improve upon x in (N,v), let $i \in T, j \in N \setminus T$, and $S := \{i,j\}$. Then $x_S \in C(S, v_S^x)$, so in particular $\{i\}$ cannot improve upon x_S in (S, v_S^x). Consequently, for $T \setminus \{i\} \subseteq N \setminus S$, there is no $z \in v((T \setminus \{i\}) \cup \{i\}) - \tilde{x}(T \setminus \{i\}) \subseteq v_S^x(i)$ such that $x^i_{|D} \leqq z_{|D}$ and $x^i_{|P} \leqq z_{|P}$, where at least one of the inequalities is strict ($\leq$). So there is no $y \in v(T)$ such that

$$\begin{array}{lcl} \sum_{k \in T} x^k_{|D} = x^i_{|D} + \sum_{k \in T \setminus \{i\}} x^k_{|D} & \leqq & y_{|D} \\ x^k_{|P} & \leqq & y_{|P} \quad \text{for all } k \in T \end{array}$$

where at least one of the inequalities is strict ($\leq$). Consequently, T cannot improve upon x on (N, v).

We conclude that $x \in C(N, v)$ and that the core satisfies COCONS$_2$.

Notice that COCONS is not implied by COCONS$_2$, since COCONS$_2$ is not applicable to games $(N, v) \in \Gamma$ with $|N| = 2$. The proof that the core satisfies COCONS, however, is similar to the proof that it satisfies COCONS$_2$ and is therefore omitted. $\square$

Our next proposition lays the basis for the first axiomatization of the core.

Proposition 4.3 *Let ϕ and ψ be two solution concepts on Γ. If ϕ satisfies OPE and CONS and ψ satisfies OPE and COCONS, then $\phi(N, v) \subseteq \psi(N, v)$ for each game $(N, v) \in \Gamma$.*

Proof. The proof is by induction on the number of players. First, let $(N, v) \in \Gamma$ have only one player. Then $\phi(N, v) = \psi(N, v)$ by OPE. Next, assume that the claim holds for each game with at most $n \in \mathbb{N}$ players and let $(N, v) \in \Gamma$ have $n + 1$ players. Let $x \in \phi(N, v)$. By CONS of ϕ: $x_S \in \phi(S, v_S^x)$ for every $S \in 2^N \setminus \{\emptyset, N\}$. By induction $\phi(S, v_S^x) \subseteq \psi(S, v_S^x)$ for every $S \in 2^N \setminus \{\emptyset, N\}$. Using COCONS of ψ we obtain $x \in \psi(N, v)$. $\square$

Applying this proposition twice gives us the following axiomatization of the core.

Theorem 4.4 *A solution concept σ on Γ satisfies OPE, CONS, and COCONS, if and only if σ is the core.*

Proof. The core satisfies the three axioms according to proposition 4.2. Let σ be a solution concept on Γ that also satisfies the axioms. Now apply proposition 4.3. Since σ satisfies OPE and CONS and the core satisfies OPE and COCONS, we find that $\sigma(N, v) \subseteq C(N, v)$ for each $(N, v) \in \Gamma$. Since the core satisfies OPE and CONS and σ satisfies OPE and COCONS, we find that $C(N, v) \subseteq \sigma(N, v)$ for each $(N, v) \in \Gamma$. Hence, $\sigma(N, v) = C(N, v)$ for all $(N, v) \in \Gamma$. $\square$

According to our next result, if a solution concept σ on Γ satisfies individual rationality and consistency, then it is included in the core.

Proposition 4.5 *Let σ be a solution concept on Γ that satisfies IR and CONS. Then $\sigma(N, v) \subseteq C(N, v)$ for each game $(N, v) \in \Gamma$.*

Proof. Let $(N, v) \in \Gamma$. We discern three cases.

- If $|N| = 1$, then $\sigma(N, v) \subseteq IR(N, v) = C(N, v)$ by IR of σ;
- If $|N| = 2$, let $x \in \sigma(N, v)$. Individual players cannot improve upon x by IR of σ. It remains to show that N cannot improve upon x. Suppose to the contrary that N *can* improve upon x. Then there exists a vector $y \in v(N)$ such that

  $$\begin{array}{lcl} \sum_{i\in N} x^i_{|D} & \leqq & y_{|D} \\ x^i_{|P} & \leqq & y_{|P} \quad \text{for all } i \in N \end{array}$$

  where at least one of the inequalities is strict ($\leq$). Let $i \in N$. Then

  $$\begin{array}{lcccl} x^i_{|D} & \leqq & y_{|D} - \sum_{j\in N\setminus\{i\}} x^j_{|D} & = & (y - \tilde{x}(N \setminus \{i\}))_{|D} \\ x^i_{|P} & \leqq & y_{|P} & = & (y - \tilde{x}(N \setminus \{i\}))_{|P} \end{array}$$

  where at least one of the inequalities is strict ($\leq$). Since $y - \tilde{x}(N\setminus\{i\}) \in v(N) - \tilde{x}(N\setminus\{i\}) = v^x_{\{i\}}(i)$, it follows that $x^i \notin IR(\{i\}, v^x_{\{i\}})$. By IR of σ, $x^i \notin \sigma(\{i\}, v^x_{\{i\}})$. But $x \in \sigma(N, v)$ and CONS of σ imply that $x^i \in \sigma(\{i\}, v^x_{\{i\}})$, a contradiction. Hence, one has to conclude that N cannot improve upon x in (N, v).

  This leads to the conclusion that $\sigma(N, v) \subseteq C(N, v)$ for two-player games $(N, v) \in \Gamma$;
- If $|N| \geq 3$, let $x \in \sigma(N, v)$. By CONS of σ, $x_S \in \sigma(S, v^x_S)$ for each $S \in 2^N \setminus \{\emptyset\}$ with $|S| = 2$. By the previous step, $\sigma(S, v^x_S) \subseteq C(S, v^x_S)$ for such two-player coalitions S. Using COCONS_2 of the core, it follows that $x \in C(N, v)$. □

In the part of the proof of proposition 4.5 where we indicate that the grand coalition N in a two-player game (N, v) cannot improve upon an allocation $x \in \sigma(N, v)$ the use of summation signs and notations like $N \setminus \{i\}$ seems unnecessarily complicated, since $N \setminus \{i\}$ consists of only one player. We adopt the more general notation, however, because with this notation it is easily seen that it also proves that the grand coalition cannot improve upon an allocation $x \in \sigma(N, v)$ in games with an arbitrary number of players.

Our next axiomatization applies the converse consistency axiom for two-player reductions.

Theorem 4.6 *A solution concept σ on Γ satisfies IR, II_2, CONS, and $COCONS_2$ if and only if σ is the core.*

Proof. The core satisfies the four axioms by proposition 4.2. Let σ be a solution concept on Γ that also satisfies the axioms. Proposition 4.5 shows that $\sigma(N, v) \subseteq C(N, v)$ for every $(N, v) \in \Gamma$.

It remains to show that $C(N,v) \subseteq \sigma(N,v)$ for each $(N,v) \in \Gamma$. We consider three separate cases in which the game has one, two, or more than two players. First we consider two-player games, since this result is required for the argumentation in one-player games.

- If $|N| = 2$, we know that $\sigma(N,v) \subseteq C(N,v)$ from proposition 4.5 and $C(N,v) = I(N,v) \subseteq \sigma(N,v)$ by II_2 of σ. So $C(N,v) = \sigma(N,v)$;

- Consider a one player game $(\{i\}, v)$ and let $x^i \in C(\{i\}, v)$. Consider $j \in U \setminus \{i\}$ and the game $(\{i,j\}, w) \in \Gamma$ defined by $w(i) = w(\{i,j\}) = v(i)$ and $w(j) = \{a\}$ with $a_{|D} = 0$ and $a_{|P} = x^i_{|P}$. Denote the allocation in $(\{i,j\}, w) \in \Gamma$ which gives x^i to player i and a to player j by (x^i, a). Then $(x^i, a) \in C(\{i,j\}, w) = \sigma(\{i,j\}, w)$. Also, $(\{i\}, w_{\{i\}}^{(x^i,a)}) = (\{i\}, v)$, since $w_{\{i\}}^{(x^i,a)}(i) = w(\{i,j\}) - \tilde{a} = v(i)$. By CONS of σ, $x^i \in \sigma(\{i\}, w_{\{i\}}^{(x^i,a)}) = \sigma(\{i\}, v)$. Hence, $C(N,v) \subseteq \sigma(N,v)$ if $|N| = 1$;

- If $|N| \geq 3$, let $x \in C(N,v)$. By CONS of the core: $x_S \in C(S, v_S^x) = \sigma(S, v_S^x)$ whenever $|S| = 2$, hence $x \in \sigma(N,v)$ by COCONS_2 of σ.

We conclude that $\sigma(N,v) = C(N,v)$ for all games $(N,v) \in \Gamma$. □

5 An Axiomatization of the Core with Enlightening

In the proofs of theorems 4.4 and proposition 4.5, we showed that a solution concept σ on Γ satisfies $\sigma(N,v) \subseteq C(N,v)$ for each game $(N,v) \in \Gamma$ by assuming that σ satisfies consistency and some form of individual rationality or one person efficiency, i.e, an assumption that focusses on individual players. The other inclusion, $C(N,v) \subseteq \sigma(N,v)$ was harder to prove. In the previous section two notions of converse consistency were used to establish this part. In the article of Peleg (1985) on an axiomatization of the core of NTU games, it was shown that — given an infinite set of potential agents from which the finite player sets are drawn — the converse consistency axiom can be replaced by a (restricted) nonemptiness axiom to establish inclusion of the core in σ. The same is observed in axiomatizations of equilibria in noncooperative games (cf. Peleg and Tijs (1996) and Norde *et al.* (1996)), where properties like restricted nonemptiness, individual rationality, consistency and converse consistency are studied in a different set-up. Peleg and Tijs (1996) prove that if a solution concept on a set of noncooperative games satisfies consistency and a requirement on single player games, it is a subset of the Nash equilibrium set. If, in addition, a converse consistency property is imposed, the solution concept coincides with the set of Nash equilibria. Norde *et al.* (1996) show that in mixed extensions of finite noncooperative games converse consistency can be replaced by nonemptiness.

In the current section we slightly modify the definition of reduced games of the previous section and show that the core can be axiomatized by means of restricted nonemptiness, consistency with respect to the new type of reduced games, and individual rationality. A similar definition of reduced games is used in Voorneveld and van den Nouweland (1997) to provide a new axiomatization of the core for games with transferable utility.

The section concludes with an example showing that converse consistency cannot be replaced with restricted nonemptiness if the definition of reduced games from section 4 is used.

Definition 5.1 Let $(N, v) \in \Gamma, x \in A(N, v)$, and $S \in 2^N \setminus \{\emptyset, N\}$. The *reduced game* $(S, \overline{v}_S^x)$ of (N, v) with respect to allocation x and coalition S is the game defined by:

$$\begin{array}{rcl} \overline{v}_S^x(S) & = & v(N) - \tilde{x}(N \setminus S) \\ \overline{v}_S^x(T) & = & \cup_{Q \subseteq N \setminus S, Q \neq \emptyset} \left(v(T \cup Q) - \tilde{x}(Q)\right) \quad \text{for all } T \in 2^S \setminus \{\emptyset, S\}. \end{array}$$

The difference between this definition of a reduced game and the one in definition 4.1 is that we require the set Q in the specification of $\overline{v}_S^x(T)$ to be nonempty. This reflects the intuition that, although attention is restricted to the players in S, the players in $N \setminus S$ do not *leave* the game, but strongly influence the game from behind the scenes. The remaining players don't ignore those in $N \setminus S$, but always cooperate with at least some of them.

With the reduction as given in definition 5.1, we obtain a new consistency axiom $\overline{\text{CONS}}$. A solution concept σ on Γ satisfies:

- $\overline{\textbf{CONS}}$ if for each game $(N, v) \in \Gamma$ it holds that $x \in \sigma(N, v)$ implies $x_S \in \sigma(S, \overline{v}_S^x)$ for each coalition $S \in 2^N \setminus \{\emptyset, N\}$.

The core satisfies $\overline{\text{CONS}}$. This is shown in the following proposition, along with other statements concerning the core and $\overline{\text{CONS}}$.

Proposition 5.2 *The following claims are true:*

1. *The core satisfies* $\overline{\mathit{CONS}}$*;*

2. *Consider a game* $(N, v) \in \Gamma$ *with* $|N| \geq 3$*. If* $x \in IR(N, v)$ *and* $x_S \in C(S, \overline{v}_S^x)$ *for each* $S \in 2^N \setminus \{\emptyset, N\}$ *with* $|S| = 2$*, then* $x \in C(N, v)$*;*

3. *Let σ be a solution concept on Γ that satisfies IR and $\overline{\text{CONS}}$. Then $\sigma(N,v) \subseteq C(N,v)$ for each $(N,v) \in \Gamma$.*

Proof.

1. The proof that the core satisfies $\overline{\text{CONS}}$ is similar to the proof that the core satisfies consistency in proposition 4.2;

2. Suppose $x \in IR(N,v)$ and $x_S \in C(S, \overline{v}_S^x)$ for each $S \in 2^N \setminus \{\emptyset, N\}$ with $|S| = 2$. Then individual players cannot improve upon x because $x \in IR(N,v)$. To show that N and other coalitions $T \in 2^N$ with $|T| \geq 2$ cannot improve upon x, apply the arguments used in the proof that the core satisfies COCONS_2 in proposition 4.2;

3. Let $(N,v) \in \Gamma$. The proof that $\sigma(N,v) \subseteq C(N,v)$ if $|N| \in \{1,2\}$ is completely analogous to the corresponding part of the proof of proposition 4.5. If $|N| \geq 3$, let $x \in \sigma(N,v)$. By $\overline{\text{CONS}}$ of σ, $x_S \in \sigma(S, \overline{v}_S^x)$ for each $S \in 2^N \setminus \{\emptyset, N\}$ with $|S| = 2$. Hence, using the previous step of this proof, we find that $x_S \in C(S, \overline{v}_S^x)$ for each $S \in 2^N \setminus \{\emptyset, N\}$ with $|N| = 2$. By IR of σ, $x \in \sigma(N,v) \subseteq IR(N,v)$. Then, by part 2 of the current proposition, it follows that $x \in C(N,v)$. □

The main result of this section is the following axiomatization of the core.

Theorem 5.3 *A solution concept σ on Γ satisfies IR, $\overline{\text{CONS}}$, and r-NEM if and only if σ is the core.*

Proof. We have already seen that the core satisfies the three axioms. Let σ be a solution concept on Γ that also satisfies the three axioms. From Proposition 5.2, part 3, we know that $\sigma(N,v) \subseteq C(N,v)$ for each $(N,v) \in \Gamma$. It remains to show that $C(N,v) \subseteq \sigma(N,v)$ for each $(N,v) \in \Gamma$.

Let $(N,v) \in \Gamma$. If $C(N,v) = \emptyset$ we are done, so assume $C(N,v) \neq \emptyset$, and let $x = (x^i)_{i \in N} \in C(N,v)$. Also, let $n \in U \setminus N$ and define a game $(N \cup \{n\}, w) \in \Gamma$ as follows:

$$
\begin{array}{llll}
w(n) & = & \{y \in \mathbb{R}^{D \cup P} \mid \text{there exists a } k \in D \text{ s.t. } y_k < 0\} & \\
 & \cup & \{y \in \mathbb{R}^{D \cup P} \mid \text{there exists a } k \in P \text{ s.t. } y_k < x_k^i\} & \\
w(i) & = & \{y \in \mathbb{R}^{D \cup P} \mid \text{there exists a } k \in D \cup P \text{ s.t. } y_k < x_k^i\} & \text{for } i \in N \\
w(S \cup \{n\}) & = & v(S) & \text{for } S \subseteq N, S \neq \emptyset \\
w(S) & = & v(S) & \text{for } S \subseteq N, |S| \geq 2.
\end{array}
$$

(Recall that for public criteria $k \in P$ one has that $x^i_k = x^j_k$ for all players $i, j \in N$. Consequently, it does not matter which player $i \in N$ is chosen in the definition of $w(n)$ above.)

We show that $C(N \cup \{n\}, w) = \{(x, d)\}$, where (x, d) is the allocation that gives $x^i \in \mathbb{R}^{D \cup P}$ to each player $i \in N$ and $d \in \mathbb{R}^{D \cup P}$ to player n, with $d_{|D} = 0$ and $d_{|P} = x^i_{|P}$ (for arbitrary $i \in N$, as above). Obviously, $(x, d) \in C(N \cup \{n\}, w)$. Now, let $(b^i)_{i \in N} \times \{b^n\} \in C(N \cup \{n\}, w)$. Using the definitions of $(w(j))_{j \in N \cup \{n\}}$, we see that it must hold that $b^i \geqq x^i$ for each player $i \in N$ and $b^n \geqq d$, to make sure that individual players in $N \cup \{n\}$ cannot improve upon $(b^i)_{i \in N} \times \{b^n\}$. If one or more of these inequalities are strict, then

$$\begin{array}{ccccc} \sum_{i \in N \cup \{n\}} b^i_{|D} & \geqq & \sum_{i \in N} x^i_{|D} + d_{|D} & = & \sum_{i \in N} x^i_{|D} \\ b^i_{|P} & \geqq & x^i_{|P} & & \text{for each player } i \in N \cup \{n\}, \end{array}$$

with at least one strict inequality. This would contradict $(x, d) \in C(N \cup \{n\}, w)$. Hence, $(b^i)_{i \in N} = (x^i)_{i \in N}$ and $b^n = d$ and this proves that (x, d) is the unique core element of $(N \cup \{n\}, w)$.

Also, we claim that $(N, \overline{w}^{(x,d)}_N) = (N, v)$. Namely,

$$\begin{array}{lllllll} \overline{w}^{(x,d)}_N(N) & = & w(N \cup \{n\}) - \tilde{d} & = & w(N \cup \{n\}) - 0 & = & v(N) \\ \overline{w}^{(x,d)}_N(S) & = & w(S \cup \{n\}) - \tilde{d} & = & w(S \cup \{n\}) - 0 & = & v(S) \quad \text{for } S \notin \{\emptyset, N\}. \end{array}$$

By r-NEM of σ we know that $\sigma(N \cup \{n\}, w) \neq \emptyset$ and we already saw that $\sigma(N \cup \{n\}) \subseteq C(N \cup \{n\}, w) = \{(x, d)\}$. So, $\sigma(N \cup \{n\}, w) = \{(x, d)\}$. Hence, by $\overline{\text{CONS}}$ of σ: $x = (x, d)_N \in \sigma(N, \overline{w}^{(x,d)}_N) = \sigma(N, v)$. This proves that $C(N, v) \subseteq \sigma(N, v)$, finishing our proof. □

The main step in the proof, showing that $C(N, v) \subseteq \sigma(N, v)$ for each game $(N, v) \in \Gamma$, proceeds by 'enlightening' core elements. In this procedure, one considers a game with a nonempty core and an arbitrary allocation in this core. Then, a game is constructed with a player set that strictly includes the players of the original game in such a way that this larger game has a unique core element and such that this new, enlarged, game and its unique core element reduced to the original player set are the original game and core element. Restricted nonemptiness is then used to derive the desired inclusion.

We conclude by showing that the analogon of theorem 5.3 does not hold if we replace $\overline{\text{CONS}}$ by consistency with respect to the old definition of reduced games. In particular, we construct a solution concept σ on Γ that satisfies IR, CONS, and r-NEM, which is not equal to the core.

Let $\mathcal{T} \subset \Gamma$ be the class of games with a nonempty core, one divisible criterion, and zero public criteria:

$$\mathcal{T} := \{(N, v) \in \Gamma \mid C(N, v) \neq \emptyset, |D| = 1, P = \emptyset\}.$$

Since for each game $(N, v) \in \mathcal{T}$ the core is nonempty, there is only one criterion, and v takes nonempty values (see section 2), we conclude that the function v is bounded from above. Hence, the function $\sup v$, where $\sup v(S)$ is the supremum of $v(S)$ for each $S \in 2^N \setminus \{\emptyset\}$, is well-defined. Define a solution concept σ on Γ as follows:

$$\sigma(N, v) = \begin{cases} C(N, v) & \text{if} \quad (N, v) \notin \mathcal{T} \\ \{Nu(N, \sup v)\}, \text{ the nucleolus of } (N, \sup v) & \text{if} \quad (N, v) \in \mathcal{T} \end{cases}$$

If $(N, v) \in \mathcal{T}$, then $C(N, v) = C(N, \sup v)$. The game $(N, \sup v)$ is a TU-game. Recall (cf. Schmeidler (1969)) that the nucleolus of a TU-game with a nonempty core is always included in the core. The solution concept σ satisfies r-NEM because the nucleolus exists for TU-games.

To prove IR of σ, we distinguish between $(N, v) \in \mathcal{T}$ and $(N, v) \notin \mathcal{T}$. If $(N, v) \notin \mathcal{T}$, it is clear that $\sigma(N, v) = C(N, v) \subseteq IR(N, v)$ by IR of the core. If $(N, v) \in \mathcal{T}$, then $C(N, v) = C(N, \sup v)$. Consequently,

$$\sigma(N, v) = \{Nu(N, \sup v)\} \subseteq C(N, \sup v) = C(N, v) \subseteq IR(N, v)$$

by IR of the core and inclusion of the nucleolus in the core if the core of a TU-game is nonempty (Schmeidler (1969)).

The solution concept σ also satisfies CONS. If $(N, v) \notin \mathcal{T}$, then $(S, v_S^x) \notin \mathcal{T}$ for each $x \in A(N, v)$ and $S \in 2^N \setminus \{\emptyset, N\}$ and hence it follows from consistency of the core that $x_S \in \sigma(S, v_S^x)$ for each $x \in \sigma(N, v)$ and $S \in 2^N \setminus \{\emptyset, N\}$. So, suppose $(N, v) \in \mathcal{T}$, so that $\sigma(N, v) = \{Nu(N, \sup v)\}$. Let $S \in 2^N \setminus \{\emptyset, N\}$ and $x \in \sigma(N, v)$, i.e., $x = Nu(N, \sup v)$.

Notice, first of all, that the reduced game (S, v_S^x) is again an element of $\mathcal{T}$. It is clear that the reduced game has no public and exactly one private criterion. Also, $x \in \sigma(N, v) = \{Nu(N, \sup v)\} \subseteq C(N, \sup v) = C(N, v)$ and the core satisfies CONS. This shows that $x_S \in C(S, v_S^x)$ and, hence, $C(S, v_S^x) \neq \emptyset$.

We know by consistency of the nucleolus for TU-games (cf. Peleg (1986)) that x_S is the nucleolus of (S, w), where the reduced game w is defined by

$$\begin{aligned} w(S) &= (\sup v)(N) - x(N \setminus S) \\ w(T) &= \max_{Q \subseteq N \setminus S} \{(\sup v)(T \cup Q) - x(Q)\} \quad \text{for } T \in 2^S \setminus \{\emptyset, S\}. \end{aligned}$$

Notice that

$$\begin{aligned} w(S) &= (\sup v)(N) - x(N \setminus S) \\ &= \sup(v(N) - x(N \setminus S)) \\ &= \sup v_S^x(S), \end{aligned}$$

and for $T \in 2^S \setminus \{\emptyset, S\}$:

$$\begin{aligned} w(T) &= \max_{Q \subseteq N \setminus S} \{(\sup v)(T \cup Q) - x(Q)\} \\ &= \max_{Q \subseteq N \setminus S} \{\sup(v(T \cup Q) - x(Q))\} \\ &= \sup \cup_{Q \subseteq N \setminus S} \{v(T \cup Q) - x(Q)\} \\ &= \sup v_S^x(T). \end{aligned}$$

So $x_S = Nu(S, w) = Nu(S, \sup v_S^x) \in \sigma(S, v_S^x)$, completing our proof that σ satisfies CONS.

To show that $\sigma \neq C$, consider the two-player game $(\{1,2\}, v) \in \mathcal{T}$ with $v(1) = v(2) = \{0\}$ and $v(\{1,2\}) = \{1\}$. Then $\sigma(\{1,2\}, v) = \{Nu(\{1,2\}, v)\} = \{(\frac{1}{2}, \frac{1}{2})\} \neq C(\{1,2\}, v) = \{(x^1, x^2) \in \mathbb{R}^2 \mid x^1 \geq 0, x^2 \geq 0, x^1 + x^2 = 1\}$.

As an aside, notice that the solution concept σ also satisfies OPE. This follows from OPE of the core and $\sigma(N, v) = C(N, v)$ if $|N| = 1$. This implies that in theorem 4.4 the converse consistency axiom cannot be replaced by restricted nonemptiness and individual rationality.

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