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Herings, P.J.J.; Dreze, J.

Publication date:
1998

Citation for published version (APA):
Herings, P. J. J., & Dreze, J. (1998). *Continua of Underemployment Equilibria*. (CentER Discussion Paper; Vol. 1998-05). Microeconomics.

Continua of Underemployment Equilibria*

P. Jean-Jacques Herings† and Jacques H. Drèze‡

*We would like to thank Kenneth Arrow, Mordecai Kurz, Enrico Minelli, Heracles Polemarchakis, Ross Starr, and seminar participants in Venice, Stanford, and Bielefeld for helpful comments and suggestions. The usual disclaimer applies. The research of Jean-Jacques Herings has been made possible by a fellowship of the Royal Netherlands Academy of Arts and Sciences.

†P.J.J. Herings, Department of Econometrics and CentER, P.O. Box 90153, 5000 LE Tilburg, The Netherlands.

‡J.H. Drèze, CORE, 34 Voie du Roman Pays, B-1348, Louvain-la-Neuve, Belgium.

Abstract

In this paper the existence of unemployment is partly explained as being the result of coordination failures. This is achieved by considering a standard general equilibrium model and splitting the set of commodities in two groups. The first group contains commodities like gold. The prices of these commodities are fully flexible, even in the short run, and their markets always clear. The prices of the commodities in the second group are rigid in the short run (for instance labour services or some consumer goods) and households and firms may expect restricted supply possibilities. We show that such expectations are self-enforcing, even if all prices of commodities in the second group are competitive. In that case it is shown that as a result of coordination failures a continuum of equilibria results, among which an equilibrium with approximately no trade in the commodities of the second group, and a Walrasian equilibrium. In fact, these coordination failures also arise at other price systems, but then unemployment is the result of both a wrong price system and coordination failures. Moreover, some properties of the set of equilibria are analysed. Generically, there exists a continuum of non-indifferent equilibrium allocations. Under a condition implied by gross substitutability, there exists a continuum of equilibrium allocations in the neighbourhood of a competitive allocation. Examples show that the latter property may not hold in general.

JEL classification: C62, D51

Keywords: General Equilibrium; Underemployment; Coordination Failures; Indeterminacy

Contents

1 Introduction

The standard explanation for underutilisation of resources given by general equilibrium theory is that relative prices are wrong. For instance, if wages are too high, this may lead to an excess supply of labour, and consequently to unemployment. This, in turn, may lead to a lower total income of workers and a lower total demand for commodities. Consequently, also firms may face restricted possibilities for sales and underutilisation of resources. Seminal work that generalizes these lines of thought has been done in Bénassy (1975), Drèze (1975), and Younès (1975). For a recent survey of this work, see Bénassy (1993).

More recently, another explanation for the underutilisation of resources has been given in a game-theoretic framework by Roberts (1987a, 1987b, 1989a, 1989b). He considers a game that has several stages corresponding to the choice of prices and wages by firms and workers, the supply of labour and the demand for commodities by workers, and the actual hiring and production by firms. If firms expect that the total demand for their output is low, then they will hire only a limited amount of labour. This will have a negative impact on income of workers and thereby indeed lead to a low demand for outputs. Workers, expecting to be (partially) unemployed, supply limited amounts of labour and express low demands for commodities. In the work of Roberts it is shown that such expectations can be rational, even at Walrasian prices, and equilibria range from zero employment and zero output to the Walrasian equilibrium[1].

Results similar to those of Roberts date back even earlier. In the framework of a generalized game Heller and Starr (1979) obtain a continuum of myopic complete information equilibria ranging from an equilibrium with zero employment and zero output to the Walrasian equilibrium. In their generalized game prices are a priori given, and should be competitive for the result mentioned. There is only one stage where both firms and households make offers to buy and to sell simultaneously in all markets. The intuitions and even the conditions required, homothetic preferences and constant returns to scale production, are closely related to those in the models of Roberts.

In this paper it is shown that these results hold quite generally. Commodities are separated into two groups. The first group contains commodities like gold (or in a financial setting bonds or stocks) for which the price is fully flexible and therefore rationing cannot occur, even in the short run. The second group contains commodities like labour services that have rigid prices in the short run. Therefore, households and firms may expect restricted supply possibilities of these commodities due to coordination failures.

We will show that even if one takes the prices of the commodities in the second group e-

[1] On the other hand, Jones and Manuelli (1992) find that the results of Roberts are not robust against small changes in the formulation of the game.

qual to competitive values, there exists a continuum of underemployment equilibria, among which an equilibrium with approximately no trade in the markets for the group II commodities and a full employment equilibrium. The equilibrium with approximately no trade in the markets for the group II commodities is called an approximate no-trade equilibrium. This is somewhat misleading, since in general there is trade in the group I commodities. All unemployment resulting in the underemployment equilibria may be viewed as a result of coordination failures since the relative prices of the group II commodities are right, and the prices of the group I commodities are completely flexible. This makes the case where the fixed prices are competitive the most pure and illustrative case. Therefore, this case will be analysed in more detail.

A robust example with an empty set of group I commodities is constructed where the Walrasian equilibrium price system is unique, while at Walrasian prices there are only two different underemployment equilibrium allocations, the no-trade equilibrium allocation and the Walrasian equilibrium allocation. Although, there is still a continuum of underemployment equilibria in that example, i.e. a continuum of expectations, almost all of these equilibria lead to the same equilibrium allocation. Therefore, the question is addressed, given competitive prices for the group II commodities, whether in general one may expect a continuum of underemployment equilibrium allocations and, furthermore, whether there is a connected subset of the set of underemployment equilibria containing both an approximate no-trade equilibrium and a Walrasian equilibrium. Such a result would imply the existence of a continuum of allocations in a neighbourhood of the competitive allocation. The example makes clear that the latter property cannot be true in general. However, in the most interesting case where the set of group I commodities is non-empty it can be shown that generically in the initial endowments there is a continuum of underemployment equilibrium allocations, while for the case with an empty set of group I commodities a very weak condition guarantees this. Under somewhat stronger conditions it can also be guaranteed that a Walrasian equilibrium is connected to an approximate no-trade equilibrium.

2 The Model

For $m \in \mathbb{N}$, $\mathbb{R}^m_+$ is the non-negative orthant of $\mathbb{R}^m$, and $\mathbb{R}^m_{++}$ is the strictly positive orthant of $\mathbb{R}^m$. Vector inequalities will be denoted by $\leq, <, \ll, \geq, >$, and $\gg$.

An economy is denoted by $\mathcal{E} = ((X^h, \preceq^h, e^h)_{h \in H},\ (Y^f, (\theta^{fh})_{h \in H})_{f \in F},\ \tilde{p}^{\mathrm{II}}, \alpha, \beta)$. There are H households, indexed by $h \in H$, F firms, indexed by $f \in F$, and L commodities, indexed by $l \in L$.[2] Every household h has a consumption set X^h, a preference relation

[2] The use of H, F, and L for the number and the set of households, firms and commodities, respectively,

$\preceq^h$ on X^h, and an initial endowment $e^h \in \mathbb{R}^L$. The Cartesian product of the sets X^h is denoted by $\widetilde{X}$, so $\widetilde{X} = \prod_{h \in H} X^h$. Every firm f has a production possibility set Y^f. The set of total production possibilities, $\sum_{f \in F} Y^f$, is denoted by Y. The Cartesian product of the production possibility sets is denoted by $\widetilde{Y}$, so $\widetilde{Y} = \prod_{f \in F} Y^f$. Household h receives a share θ^{fh} of the profits of firm f.

The commodities are split into two groups, labeled I and II. Whenever such a label is attached to a symbol, it is meant to refer to the group of commodities indicated by the label. For instance, L^{I} will denote the number and the set of group I commodities. Without loss of generality, group I consists of the first L^{I} commodities. The prices of commodities in group I are assumed to be completely flexible, even in the short run. The markets for these commodities are organized in such a way that prices will immediately react to small changes in supply or demand. Examples are auctions (as for fish) or organized (commodity or stock) exchanges. The markets for these commodities are therefore never cleared by rationing in an equilibrium. The prices of commodities in group II on the contrary are fixed in the short run. Like many markets in the real world, small changes in supply or demand are not immediately reflected by a change in the price. Hence there is scope for rationing in the markets for these commodities, and agents in the economy may indeed expect rationing to occur in these markets. The prices of the commodities in group II are given by $\widetilde{p}^{\mathrm{II}} \in \mathbb{R}^{L^{\mathrm{II}}}_{++}$. We will normalize the prices such that $\sum_{l \in L^{\mathrm{II}}} \widetilde{p}^{\mathrm{II}}_l = 1$. Nothing precludes to take for $\widetilde{p}^{\mathrm{II}}$ the values corresponding to a Walrasian equilibrium price system, if such a price system exists. If group I is empty, then all prices are fixed in the short run. This is the case often studied in the fixed-price literature. Still, for many commodities, such an assumption seems too strong even in the short run. Therefore, we are mainly interested in the case where group I is non-empty. We will assume that group II is non-empty, since otherwise we are back in the standard competitive framework.

In general the total demand might not be equal to the total supply of commodities in group II at price system $\widetilde{p}^{\mathrm{II}}$, so households and firms may expect restrictions concerning their net demand or their net supply, following the lines of thought of the seminal contributions of Bénassy (1975), Drèze (1975), and Younès (1975). Both for households and for firms, restrictions on supply seem to occur much more frequently in western economies as has also been remarked by van der Laan (1980) and Kurz (1982). Many households are restricted in their supply of labour and many firms in their supply of outputs. Therefore, in this paper attention will be restricted to cases with rationing on the supply side of households and firms, while the demand side will never be rationed.

In the case of excess supplies, one needs a distributional rule to determine the final allocation that will result. Such a distributional rule is called a rationing system. In this

will not create ambiguities.

paper we will consider the case where each household and each firm has a fixed predetermined market share, which allows for uniform rationing as a special case. Our existence results hold a fortiori for more general rationing schemes admitting fixed predetermined market shares as a special case.

The vector $\alpha \in \mathbb{R}^{HL^{\mathrm{II}}}_{++}$ determines the market shares of the households (its components are denoted by α^h_l) and the vector $\beta \in \mathbb{R}^{FL^{\mathrm{II}}}_{++}$ (with components denoted by β^f_l) those of the firms. This rationing system implies that for every commodity $l \in L^{\mathrm{II}}$ there exists $r_l \in \mathbb{R}_+$ such that the supply possibilities for every household h of commodity l are given by $\alpha^h_l r_l$ and the supply possibilities for every firm f of commodity l are equal to $\beta^f_l r_l$. We could normalize α and β such that, for every $l \in L^{\mathrm{II}}$, $\sum_{h\in H} \alpha^h_l + \sum_{f\in F} \beta^f_l = 1$. Then α^h_l is the share of household h and β^f_l is the share of firm f in the total possible supply of commodity l. These vectors α and β only determine the supply possibilities of households and firms. Clearly, a household and a firm are completely free to demand a commodity and not to make use at all of the supply possibilities. The rationing system is treated like a black box. In reality these market shares are determined by all kind of factors that we will ignore in our model like the ability of suppliers to sell their products, the location of households and firms, or the existing relationships between them.

The expectations of available opportunities for a household h (firm f) on the various markets are described by a vector $\underline{z}^h \in -\mathbb{R}^{L^{\mathrm{II}}}_+$ ($\underline{y}^f \in \mathbb{R}^{L^{\mathrm{II}}}_+$), called the expected opportunities for household h (firm f). The vector of expected opportunities $(\underline{z}, \underline{y}) = (\underline{z}^1, \ldots, \underline{z}^H, \underline{y}^1, \ldots \underline{y}^F)$ describes the constraints expected in the economy. In equilibrium the expected opportunities are required to be rational. These expectations should therefore match the amounts allocated by the rationing system. So, for the case of the rationing system with market shares, the set of all expected opportunities that are relevant is given by the L^{II}-dimensional set $\underline{ZY}$, where

$$\begin{aligned}\underline{ZY} = \Big\{ &(\underline{z}, \underline{y}) \in -\mathbb{R}^{HL^{\mathrm{II}}}_+ \times \mathbb{R}^{FL^{\mathrm{II}}}_+ \;\Big|\\ &\exists r \in \mathbb{R}^{L^{\mathrm{II}}}_+,\ \forall h \in H,\ \forall f \in F,\ \underline{z}^h_l = -\alpha^h_l r_l,\ \underline{y}^f_l = \beta^f_l r_l, \quad l \in L^{\mathrm{II}} \Big\}.\end{aligned}$$

Firms are assumed to be profit maximizers. For every firm f, given expected opportunities $\underline{y}^f \in \mathbb{R}^{L^{\mathrm{II}}}_+$, the set of feasible production plans, $s^f(\underline{y}^f)$, is defined by

$$s^f(\underline{y}^f) = \left\{ y^f \in Y^f \,\middle|\, y^{f,\mathrm{II}} \leq \underline{y}^f \right\}.$$

Similarly, for every firm f, given a price system $p \in \mathbb{R}^L$ and expected opportunities $\underline{y}^f \in \mathbb{R}^{L^{\mathrm{II}}}_+$, the set of production plans maximizing profit, $\eta^f(p, \underline{y}^f)$, is defined by

$$\eta^f(p, \underline{y}^f) = \left\{ \hat{y}^f \in s^f(\underline{y}^f) \,\middle|\, p \cdot \hat{y}^f \geq p \cdot y^f,\ \forall y^f \in s^f(\underline{y}^f) \right\}.$$

If the set $\eta^f(p, \underline{y}^f)$ is non-empty, then the profit of firm f is defined by $\pi^f(p, \underline{y}^f) = p \cdot y^f$, for $y^f \in \eta^f(p, \underline{y}^f)$. If the set $\eta^f(p, \underline{y}^f)$ is non-empty for every firm f, then the wealth of a household h, w^h, is determined by the value of its initial endowments and the shares in the profits of the firms, $w^h = p \cdot e^h + \sum_{f \in F} \theta^{fh} \pi^f(p, \underline{y}^f)$. The budget set of a household h facing a price system $p \in \mathbb{R}^L$, having expected opportunities $\underline{z}^h \in -\mathbb{R}_+^{L^{\mathrm{II}}}$, and having wealth $w^h \geq p \cdot e^h$ is denoted by $\gamma^h(p, \underline{z}^h, w^h)$, so

$$\gamma^h(p, \underline{z}^h, w^h) = \left\{ x^h \in X^h \,\middle|\, p \cdot x^h \leq w^h \text{ and } x^{h,\mathrm{II}} - e^{h,\mathrm{II}} \geq \underline{z}^h \right\},$$

and its demand set $\delta^h(p, \underline{z}^h, w^h)$ is defined by

$$\delta^h(p, \underline{z}^h, w^h) = \left\{ \overline{x}^h \in \gamma^h(p, \underline{z}^h, w^h) \,\middle|\, x^h \preceq^h \overline{x}^h, \ \forall x^h \in \gamma^h(p, \underline{z}^h, w^h) \right\}.$$

The total excess demand in the economy, given $p \in \mathbb{R}^L$ and expected opportunities $(\underline{z}, \underline{y}) \in \underline{ZY}$, is defined by

$$\zeta(p, \underline{z}, \underline{y}) = \sum_{h \in H} \delta^h(p, \underline{z}^h, p \cdot e^h + \sum_{f \in F} \theta^{fh} \pi^f(p, \underline{y}^f)) - \sum_{h \in H} e^h - \sum_{f \in F} \eta^f(p, \underline{y}^f).$$

We are now in a position to give a definition of an underemployment equilibrium. This definition is also used in Drèze (1997).

Definition 2.1 (Underemployment equilibrium)
An underemployment equilibrium of the economy $\mathcal{E} = ((X^h, \preceq^h, e^h)_{h \in H}, (Y^f, (\theta^{fh})_{h \in H})_{f \in F}, \tilde{p}^{\mathrm{II}}, \alpha, \beta)$ is an element $(p^*, x^*, y^*, \underline{z}^*, \underline{y}^*) \in \mathbb{R}^L \times \widetilde{X} \times \widetilde{Y} \times \underline{ZY}$ satisfying

1. for every household $h \in H$, $x^{*h} \in \delta^h(p^*, \underline{z}^{*h}, p^* \cdot e^h + \sum_{f \in F} \theta^{fh} p^* \cdot y^{*f})$,
2. for every firm $f \in F$, $y^{*f} \in \eta^f(p^*, \underline{y}^{*f})$,
3. $\sum_{h \in H} x^{*h} - \sum_{h \in H} e^h - \sum_{f \in F} y^{*f} = 0$,
4. $p^{*\mathrm{II}} = \tilde{p}^{\mathrm{II}}$.

The set of all underemployment equilibria of an economy $\mathcal{E}$ is denoted by E. Notice that the definition of an underemployment equilibrium implies that the expected opportunities $(\underline{z}^*, \underline{y}^*)$ belong to $\underline{ZY}$. The expectations match the amounts determined by the rationing system.

The notion of Walrasian equilibrium fits easily in our framework. This is important since in many of our results we will be focussing on the possibility of coordination failures, and therefore non-Walrasian equilibria, at Walrasian prices.

Definition 2.2 (Walrasian equilibrium)
An underemployment equilibrium $(p^*, x^*, y^*, \underline{z}^*, \underline{y}^*) \in \mathbb{R}^L \times \widetilde{X} \times \widetilde{Y} \times \underline{ZY}$ of the economy $\mathcal{E} = ((X^h, \preceq^h, e^h)_{h \in H}, (Y^f, (\theta^{fh})_{h \in H})_{f \in F}, \tilde{p}^{\mathrm{II}}, \alpha, \beta)$ is a Walrasian equilibrium if

1. for every household $h \in H$, $\underline{z}^{*h} < x^{*h} - e^h$,

2. for every firm $f \in F$, $y^{*f} < \underline{y}^{*f}$.

Since preference relations will be assumed convex, the definition of Walrasian equilibrium coincides with the usual one.

Consider two underemployment equilibria $(\overline{p}^*, \overline{x}^*, \overline{y}^*, \overline{\underline{z}}^*, \overline{\underline{y}}^*)$, $(\widehat{p}^*, \widehat{x}^*, \widehat{y}^*, \widehat{\underline{z}}^*, \widehat{\underline{y}}^*)$ of an economy $\mathcal{E}$. These two underemployment equilibria are said to be *potentially different* if $\overline{x}^* \neq \widehat{x}^*$ or if the different expectations of available opportunities lead to different sets of possible choices for at least one household, so if there exists a household h such that $\gamma^h(\overline{p}^*, \overline{\underline{z}}^{*h}, \overline{p}^* \cdot e^h + \sum_{f \in F} \theta^{fh} \overline{p}^* \cdot \overline{y}^{*f}) \neq \gamma^h(\widehat{p}^*, \widehat{\underline{z}}^{*h}, \widehat{p}^* \cdot e^h + \sum_{f \in F} \theta^{fh} \widehat{p}^* \cdot \widehat{y}^{*f})$. This seems to be the weakest reasonable definition of potentially different underemployment equilibria. Notice that it might be the case that all households get the same consumption bundle in two potentially different underemployment equilibria. Nevertheless, it is not unreasonable to make a distinction between two equilibria if the freedom of choice of some household is different. Notice that $\overline{\underline{z}}^* \neq \widehat{\underline{z}}^*$ is not a sufficient condition to get potentially different underemployment equilibria. If, for instance, $\overline{\underline{z}}_{l'}^{*h'} < \widehat{\underline{z}}_{l'}^{*h'} \leq -e_{l'}^{h'}$ and $\overline{\underline{z}}_l^{*h} = \widehat{\underline{z}}_l^{*h}$, $\forall h \in H \setminus \{h'\}$, $\forall l \in L \setminus \{l'\}$, whereas $\overline{p}^* = \widehat{p}^*$, $\overline{x}^* = \widehat{x}^*$, and $\overline{y}^* = \widehat{y}^*$, then the two underemployment equilibria $(\overline{p}^*, \overline{x}^*, \overline{y}^*, \overline{\underline{z}}^*, \overline{\underline{y}}^*)$ and $(\widehat{p}^*, \widehat{x}^*, \widehat{y}^*, \widehat{\underline{z}}^*, \widehat{\underline{y}}^*)$ are not potentially different. The fact that $X^{h'} \subset \mathbb{R}_+^L$ implies that the two sets of possible choices corresponding to $\overline{\underline{z}}^{h'}$ and $\widehat{\underline{z}}^{h'}$ are the same.

A stronger and more natural criterion for the distinction between two underemployment equilibria is given by the consideration of the consumption bundles of the households. The two underemployment equilibria $(\overline{p}^*, \overline{x}^*, \overline{y}^*, \overline{\underline{z}}^*, \overline{\underline{y}}^*)$ and $(\widehat{p}^*, \widehat{x}^*, \widehat{y}^*, \widehat{\underline{z}}^*, \widehat{\underline{y}}^*)$ are said to be *different* if there exists a household h such that $\overline{x}^{*h} \neq \widehat{x}^{*h}$. There is at least one household receiving a different consumption bundle. The way in which the production of the consumption bundles takes place or the prices against which trade takes place is of no concern for the notion of different underemployment equilibria. Clearly, two different underemployment equilibria are also potentially different.

A stronger criterion for the distinction between two underemployment equilibria is given by the consideration of the utility tuples of the households. Two underemployment equilibria $(\overline{p}^*, \overline{x}^*, \overline{y}^*, \overline{\underline{z}}^*, \overline{\underline{y}}^*)$ and $(\widehat{p}^*, \widehat{x}^*, \widehat{y}^*, \widehat{\underline{z}}^*, \widehat{\underline{y}}^*)$ are said to be *strongly different* if there exists a household h such that $\overline{x}^{*h} \succ^h \widehat{x}^{*h}$ or $\widehat{x}^{*h} \succ^h \overline{x}^{*h}$. Notice that two strongly different underemployment equilibria are also different.

3 Existence of a Continuum of Underemployment Equilibria

3.1 Assumptions

In this section we show the existence of a continuum of underemployment equilibria. We will make use of the following assumptions with respect to the economy $\mathcal{E}$.

A1. For every household $h \in H$, the consumption set X^h is non-empty, closed, convex, and $X^h \subseteq \mathbb{R}^L_+$.

A2. For every household $h \in H$, the preference relation $\preceq^h$ is complete, transitive, continuous, convex, and for every $\overline{x}^h \in X^h$ there exists $\widehat{x}^h \in X^h$ such that $\overline{x}^{h,\mathrm{II}} = \widehat{x}^{h,\mathrm{II}}$ and $\overline{x}^h \prec^h \widehat{x}^h$, and there exists $\widetilde{x}^h \in X^h$ such that $\overline{x}^{h,\mathrm{I}} = \widetilde{x}^{h,\mathrm{I}}$, $\overline{x}^{h,\mathrm{II}} < \widetilde{x}^{h,\mathrm{II}}$, and $\overline{x}^h \prec^h \widetilde{x}^h$.

A3. For every household $h \in H$, there is $x^h \in X^h$ such that $x^{h,\mathrm{I}} \ll e^{h,\mathrm{I}}$ and $x^{h,\mathrm{II}} = e^{h,\mathrm{II}}$, and for all $l' \in L^{\mathrm{II}}$ there is $x^h \in X^h$ such that $x^{h,\mathrm{I}} \leq e^{h,\mathrm{I}}$, $x^h_{l'} < e^h_{l'}$, and $x^h_l = e^h_l$, $\forall l \in L^{\mathrm{II}} \setminus \{l'\}$.

A4. For every firm $f \in F$, the production possibility set Y^f is closed, convex, $-\mathbb{R}^L_+ \subseteq Y^f$, $\theta^{fh} \geq 0$, $\forall h \in H$, and $\sum_{h \in H} \theta^{fh} = 1$. Moreover, $Y \cap -Y \subseteq \{0\}$.

A5. The price system and the rationing system satisfy $\widetilde{p}^{\mathrm{II}} \in \mathbb{R}^{L^{\mathrm{II}}}_{++}$ with $\sum_{l \in L^{\mathrm{II}}} \widetilde{p}^{\mathrm{II}}_l = 1$, $\alpha \in \mathbb{R}^{HL^{\mathrm{II}}}_{++}$, and $\beta \in \mathbb{R}^{FL^{\mathrm{II}}}_{++}$.

A6. For every household $h \in H$, the consumption set $X^h = \mathbb{R}^L_+$, the preference relation $\preceq^h$ can be represented by a utility function u^h, where u^h is twice differentiable on $\mathbb{R}^L_{++}$, $\partial u^h \gg 0$, $\partial^2 u^h$ is negative definite on $(\partial u^h)^\perp$,[3] and $u^h(e^h) \geq u^h(x^h)$, for every $x^h \in \mathbb{R}^L_+ \setminus \mathbb{R}^L_{++}$. For every firm $f \in F$, the production possibility set is described by a twice continuously differentiable function $g^f : \mathbb{R}^L \to \mathbb{R}$, so $Y^f = \{y^f \in Y^f \mid g^f(y^f) \leq 0\}$, and for any $\overline{y}^f$ on the production frontier $\{y^f \in Y^f \mid g^f(y^f) = 0\}$ it holds that $\partial^2 g^f$ is positive definite on $(\partial g^f)^\perp$.

A7. The economy $\mathcal{E}$ has a well-defined aggregate excess demand function $z : \mathbb{R}^L_{++} \times \underline{ZY} \to \mathbb{R}^L$. If $(p', -\underline{z}', \underline{y}') \leq (p, -\underline{z}, \underline{y})$ with $p'_{l'} = p_{l'}$, $\underline{z}'_{l'} = \underline{z}_{l'}$, and $\underline{y}'_{l'} = \underline{y}_{l'}$, then $z_{l'}(p', \underline{z}', \underline{y}') \leq z_{l'}(p, \underline{z}, \underline{y})$.[4]

The often made assumption in the fixed-price literature that $X^h = \mathbb{R}^L_+$ or that $X^h + \mathbb{R}^L_+ \subseteq X^h$ is replaced by the weaker assumption A1. Examples where the usual assumptions

[3] "$\perp$" denotes the orthogonal complement.

[4] For $l' \in L^{\mathrm{I}}$, $\underline{z}'_{l'} = \underline{z}_{l'}$ and $\underline{y}'_{l'} = \underline{y}_{l'}$ is trivially satisfied.

are not satisfied but ours are, concern group II commodities for which there is a clear physical upper bound on consumption in a given time interval, or commodities that can only be consumed together with a sufficient amount of another commodity. For instance, consumption at a remote place can only take place together with certain transportation services or some services cannot be supplied without sufficient education. Assumption A2 implies that there is non-satiation with respect to the group I commodities and with respect to the group II commodities, a much weaker requirement than monotonicity of preferences.

A preference relation $\preceq^h$ is said to be convex if $\overline{x}^h, \widehat{x}^h \in X^h$ and $\overline{x}^h \prec^h \widehat{x}^h$ implies $\overline{x}^h \prec^h \lambda\overline{x}^h + (1-\lambda)\widehat{x}^h$, $\forall \lambda \in [0,1)$.

The somewhat clumsy statement of Assumptions A2 and A3 guarantees that for the case $L^{\mathrm{II}} = 0$ we make the same assumptions as Debreu (1959). For the case $L^{\mathrm{II}} \geq 1$, our assumptions coincide with those of Debreu for an economy consisting of the first L^{I} commodities.

Assumption A6, which will be needed for part of the results, states the standard differentiability requirements on the primitive concepts, see for instance Mas-Colell (1985).

In addition to these primitive assumptions about individual agents, we shall need for our strongest result (Theorem 3.1.iii) an assumption akin to gross substitution. The assumption used in our proof of that result is a weaker form of the more intuitive Assumption A7. In the case of exchange economies, A7 could be stated for individual demands and would be preserved under aggregation. For this case Movshovich (1994) gives assumptions on primitive concepts implying a stronger form of A7. When individual incomes include profits, a lucid statement is only possible in terms of aggregate demand.

Assumption A7 states that the net demand for any one good does not increase when the prices and/or supply possibilities of other commodities are decreased. It is not required that the net demand for the other commodities increases. Actually, we only use that assumption starting from a competitive equilibrium, and still in weaker form. But we are unable to illustrate meaningfully what is gained by the weakening. For instance, the assumptions on individual primitives required to guarantee gross substitution at a competitive equilibrium imply gross substitution everywhere.

We could state A7 for correspondences, following Polterovich and Spivak (1983), but we use it in conjunction with A6, hence for functions, and therefore state it for functions.

3.2 The Existence Theorem

By Debreu (1959), (1) and (2) page 77, it follows that the set of attainable allocations of the economy $\mathcal{E}$, $A = \{(x,y) \in \widetilde{X} \times \widetilde{Y} \mid \sum_{h \in H} x^h - \sum_{h \in H} e^h - \sum_{f \in F} y^f = 0\}$, is compact. Let $b > 0$ be such that $\|(x,y)\|_\infty < b$, $\forall (x,y) \in A$. Since A is compact, such a b exists, and since $(e,0) \in A$ it follows that $b > \max_{h \in H, l \in L} e^h_l$. Observe that all potentially different underemployment equilibria are obtained when attention is restricted to expected opportunities $(\underline{z}, \underline{y}) \in \underline{ZY}$ satisfying, for every $l \in L^{\mathrm{II}}$, $\min\{-\underline{z}_l^h, \underline{y}_l^f \mid h \in H, f \in F\} \leq b$. The set of underemployment equilibria sustained by such expectations is denoted by $\widehat{E}$.

The extend to which the market for a commodity $l \in L^{\mathrm{II}}$ is employed in an underemployment equilibrium $(p^*, x^*, y^*, \underline{z}^*, \underline{y}^*)$ in $\widehat{E}$ will be measured by the number $\upsilon_l \in [0, 1]$, where

$$\upsilon_l = \frac{1}{b} \min\{-\underline{z}_l^{*h}, \underline{y}_l^{*f} \mid h \in H, f \in F\}.$$

If $\upsilon_l = 0$, then the market for commodity l has collapsed completely and no supply is expected to take place. If $\upsilon_l = 1$, then no binding constraints on supply are expected in the market for commodity l. We will need this measure of employment to distinguish between different underemployment equilibria.

Theorem 3.1

Let $\mathcal{E} = ((X^h, \preceq^h, e^h)_{h \in H}, (Y^f, (\theta^{fh})_{h \in H})_{f \in F}, \tilde{p}^{\mathrm{II}}, \alpha, \beta)$ *be an economy with* $H \geq 2$ *and* $L^{\mathrm{I}} \geq 1$.

(i) *Under A1-A5, the set of underemployment equilibria* $\widehat{E}$ *owns a component* $\widehat{E}^{\mathrm{c}}$ *which:*
contains a continuum of potentially different underemployment equilibria;
includes an underemployment equilibrium with $\max_{l \in L^{\mathrm{II}}} \upsilon_l = \upsilon$ *for all* $\upsilon \in (0, 1]$.

(ii) *Under A1-A6, generically in initial endowments,* $\widehat{E}$ *owns a component* $\widehat{E}^{\mathrm{c}}$ *which:*
contains a continuum of strongly different underemployment equilibria.

(iii) *Under A1-A7, if* $\tilde{p}^{\mathrm{II}} = p^{*\mathrm{II}}$ *with* $(p^*, x^*, y^*, \underline{z}^*, \underline{y}^*)$ *a Walrasian equilibrium,* $\widehat{E}$ *owns a component* $\widehat{E}^{\mathrm{c}}$ *which:*
ranges from an approximate no-trade equilibrium at prices $p \leq p^*$ *to the competitive equilibrium* $(p^*, x^*, y^*, \underline{z}^*, \underline{y}^*)$.

Proof

See Appendix.

3.3 Interpretation of the Theorem

Theorem 3.1.i states that there is a connected set of underemployment equilibria ranging from an underemployment equilibrium with arbitrarily low trade in the group II commodities to an equilibrium without rationing in the market for at least one group II commodity. The markets for the group I commodities are in equilibrium without rationing. This means that there are many different expectations leading to an underemployment equilibrium, ranging from the expectations that no household and no firm will supply a positive amount of any group II commodity, to the expectations that at least in one market for group II commodities free trade without rationing is possible. There exists an underemployment equilibrium $(p^*, x^*, y^*, \underline{z}^*, \underline{y}^*) \in \widehat{E}^{\mathrm{c}}$ with $x^{*,\mathrm{II}}$ arbitrarily close to e^{II}, and $y^{*,\mathrm{II}}$, $\underline{z}^*$, and $\underline{y}^*$ all arbitrarily close to zero, so with all υ_l arbitrarily close to zero. Furthermore, there exists an underemployment equilibrium $(p^*, x^*, y^*, \underline{z}^*, \underline{y}^*) \in \widehat{E}^{\mathrm{c}}$ where for some $l \in L^{\mathrm{II}}$ it holds that no household and no firm faces binding expected opportunities in the market for commodity l, so $x_l^{*h} - e_l^h > \underline{z}_l^{*h}$, $\forall h \in H$, and $y_l^{*f} < \underline{y}_l^{*f}$, $\forall f \in F$, and υ_l is equal to one. These two "extreme" equilibria are contained in a connected set of underemployment equilibria and this connected set contains a continuum of potentially different equilibria. Recall that two underemployment equilibria are potentially different if at least one household has a different set of possible choices. We will show by means of Example 4.1 in Section 4 that it is possible that there is no underemployment equilibrium in the set $\widehat{E}$ with υ_l exactly equal to 0 for all l.

The notion of potentially different is rather weak, since it is not claimed that potentially different underemployment equilibria are in fact different. Theorem 3.1.ii makes clear that generically this is the case. Keeping everything fixed, except initial endowments, there exists a subset $\overline{\Omega}$ of $\mathbb{R}^{HL}_{++}$ such that the closure of $\mathbb{R}^{HL}_{++} \setminus \overline{\Omega}$ in $\mathbb{R}^{HL}_{++}$ has Lebesgue measure zero, and for every specification of initial endowments $(e^1, \ldots, e^H) \in \overline{\Omega}$, there is a continuum of strongly different underemployment equilibria. Generically in initial endowments, there is a continuum of different utilities that households can have in an underemployment equilibrium, irrespective of the prices of group II commodities being compatible with competitive values or not. If those prices have competitive values, then the Walrasian equilibrium is one of the underemployment equilibria. It might however be that the continuum of allocations is close to the no-trade equilibrium and not to the competitive equilibrium. This case is dismal from an economic point of view, because arbitrarily small perturbations away from competitive expectations would then lead to a severe depression.

Under which circumstances is there a continuum of underemployment allocations near a competitive allocation? For such a result to be true, it is necessary that $\widetilde{p}^{\mathrm{II}}$ be compatible with a competitive equilibrium. Theorem 3.1.iii shows that the component of underemployment equilibria containing an approximate no-trade equilibrium also contains a Walrasian equilibrium if A7 is invoked. From this it follows by a simple argument that there is an underemployment equilibrium with $\min_{l \in L^{\mathrm{II}}} \upsilon_l$ equal to any $\upsilon \in (0, 1]$. Values of υ close to one correspond to approximately Walrasian equilibria.

Theorem 3.1 is striking since it even holds in the circumstances that are most favourable for competitive equilibrium: all prices of group II commodities equal to competitive values and, in a world with time and uncertainty, all future commodities belong to group I. The intuition behind Theorem 3.1 is the same as the explanation given in the introduction for the results of Roberts. If firms expect that the total demand for output is low, then they will hire only a limited amount of labour. This has a negative impact on the income of workers and thereby indeed leads to a low demand for outputs. As Drèze (1997) argues, this reasoning can be given empirical underpinning. Theorem 3.1 shows that this reasoning can also be verified formally. For the result to hold one also needs downwards rigidity of the prices of the group II commodities. Otherwise, excess supplies of group II commodities could lead to lower prices of these commodities. However, Theorem 3.1 makes clear that also at those lower prices, there is again scope for coordination failures. It may be difficult to get out of a situation with coordination failures. All the households and firms together would have to revise their expectations simultaneously.

Following the arguments of Drèze (1997), Theorem 3.1 has even more important economic consequences. For instance, it makes clear that the observation of excess supply is not sufficient to infer the existence of price and wage distortions. Indeed, Theorem 3.1.i and 3.1.ii hold for any price system for the group II commodities, whereas the prices of the group I commodities are completely flexible. When prices or wages are not at competitive values, their distorting effects can even be magnified by coordination failures as expressed in Theorem 3.1.i. Because of the multiplicity of underemployment equilibria, the modelling of dynamics becomes crucial, and history will play an important role. In the presence of coordination failures, price and wage dynamics might not be very helpful in getting back to a competitive allocation of resources and can even worsen the situation. For a further development of these arguments in two macroeconomic models, the Real Business Cycle model and the Barro-Grossman-Malinvaud model, see Drèze (1997).

3.4 The Case Without Group I Commodities

Since much of the fixed-price literature considers the case $L^{\mathrm{I}} = 0$ and this case is probably the clearest for illustrative purposes, we treat it in somewhat more detail here. The proof of Theorem 3.1.i will be given for $H \geq 1$ and $L^{\mathrm{I}} \geq 0$. Even if $L^{\mathrm{I}} = 0$ there will be a continuum of potentially different equilibria. In an economy with only one household and no production, there are no different underemployment equilibria.

Example 4.2 is a robust example of a Cobb-Douglas economy with $H = 2$ and $L^{\mathrm{I}} = 0$ such that in every underemployment equilibrium all households keep their initial endowments. By Theorem 3.1.ii such an example cannot be given if L^{I} is greater than or equal to one. But the example is not fully convincing since the price system considered is rather extreme. Suppose the prices of the group II commodities are compatible with a Walrasian equilibrium price system for the entire economy. Is it still possible that all underemployment equilibria, except the ones corresponding to the Walrasian equilibrium, are utility equivalent at Walrasian prices? The question will be answered affirmatively by means of the robust Example 4.3.

Assume that at every underemployment equilibrium with $\max_{l \in L} \upsilon_l = 1$ there is trade on some of the markets. Since there is a component of the set of underemployment equilibria connecting such an underemployment equilibrium to an approximate no-trade equilibrium, this implies the existence of a continuum of different underemployment equilibria. The weak conditions of Theorem 3.2 are such that trade in some of the markets occurs at every underemployment equilibrium with $\max_{l \in L} \upsilon_l = 1$. By $0_{-l'}$ for some $l' \in L$ we will denote expected opportunities of no supply possibilities in the market for every commodity in L being different from l', and no rationing in the market for commodity l', so $\underline{z}^h = 0_{-l'}$ implies that $\underline{z}^h_l = 0$, $\forall l \in L \setminus \{l'\}$, and $\underline{z}^h_{l'} =$ "$-\infty$", and $\underline{y}^f = 0_{-l'}$ implies that $\underline{y}^f_l = 0$, $\forall l \in L \setminus \{l'\}$, and $\underline{y}^f_{l'} =$ "$+\infty$".

Theorem 3.2

Let the economy $\mathcal{E} = ((X^h, \preceq^h, e^h)_{h \in H}, (Y^f, (\theta^{fh})_{h \in H})_{f \in F}, \widetilde{p}^{\mathrm{II}}, \alpha, \beta)$ with $L^{\mathrm{I}} = 0$ satisfy A1-A5. If, for every $l \in L$, there exists $h \in H$ such that $e^h \notin \delta^h(\widetilde{p}^{\mathrm{II}}, 0_{-l}, \widetilde{p}^{\mathrm{II}} \cdot e^h)$ or there exists $f \in F$ such that $0 \notin \eta^f(\widetilde{p}^{II}, 0_{-l})$, then the set $\widehat{E}$ of underemployment equilibria of $\mathcal{E}$ owns a component $\widehat{E}^{\mathrm{c}}$ which contains a continuum of strongly different underemployment equilibria. It includes an underemployment equilibrium with $\max_{l \in L} \upsilon_l = \upsilon$ for all $\upsilon \in (0, 1]$.

Proof

See Appendix.

The additional condition we need to have a continuum of strongly different underemployment equilibria is very weak. Indeed, we require that for every commodity there exists a household or a firm that is willing to supply it if it expects no restrictions on the supply of this commodity, but it expects not to have supply possibilities of any other commodity, whereas the household does not receive any profit income. The assumption means that for every labour service there is a household willing to supply it, while for every intermediate product or consumer good there is a firm willing to supply it, under the conditions that the household expects to be fully restricted in the supply of all other labour services (and land, etc.) and receives no profit income, while the firm expects to be fully restricted in the supply of any other output. Requiring this at Walrasian prices would considerably weaken the assumption, since Walrasian prices are already balanced in some sense. Moreover, that we only need the assumption in the case households or firms expect to be fully restricted in the supply of all other commodities is also pleasant, since it means that supplying the commodity under consideration is the only way to achieve a positive income.

The proof of Theorem 3.1.iii will be given for $H \geq 1$ and $L^{\mathrm{I}} \geq 0$. Even if $L^{\mathrm{I}} = 0$ the underemployment equilibria range from an approximate no-trade equilibrium to a competitive equilibrium under A1-A7.

3.5 Relation to the Literature

In the early fixed-price literature, where $L^{\mathrm{I}} = 0$ and there is no rationing in the market for an a priori numeraire commodity, Silvestre (1980) gives a robust example with a continuum of equilibria when the fixed price is taken equal to a competitive price. Although the example is robust, it is also possible to give robust examples with a unique equilibrium. The continuum of Silvestre is obtained by varying the rationing system. For instance, in case there is an excess supply in a market and there are two suppliers of the corresponding commodity, both facing binding rationing, then any distribution of the demand among the suppliers may yield an equilibrium. Indeed, if in the set-up of the fixed-price literature a rationing system is specified, then the results of Laroque and Polemarchakis (1978) imply that there is generically a finite number of equilibria and the results of Herings (1996b), Chapter 11, that there is generically an odd number of equilibria.

For the case $L^{\mathrm{I}} = 0$ and no production, van der Laan (1982) states a result similar to Theorem 3.1.i, using arguments from combinatorial optimization. Using that approach such a result was indeed shown to be correct in Herings (1993). For that much simpler case the result is even true for $v = 0$. In Herings (1993) it is shown as well that the connectedness property a component possesses, is very strong and can be used to generate all kinds of interesting corollaries.

In Citanna, Crès, and Villanacci (1995) the case $L^{\mathrm{I}} = 0$ without production is considered. Their main result claims that given a commodity $l \in L$ and an amount of employment $v \in [0, 1]$ there exists a competitive price system p^* inducing an underemployment equilibrium for which $v_l = v$ in an economy with $\tilde{p}^{\mathrm{II}} = p^*$. Unfortunately, this result is not correct as is demonstrated by Example 4.3. Such a result would be true, even with the competitive price system p^* fixed in advance, if expectations of restricted demand opportunities are taken in consideration, see Herings (1996a).

4 Three Examples

In Section 3 it has been promised that we would give an example of an economy without an underemployment equilibrium at which $v_l = 0$ for all $l \in L^{\mathrm{II}}$.

Example 4.1
Consider the economy $\mathcal{E} = ((\mathbb{R}^2_+), \preceq^1, (1, 1), (Y^f, 1), 1, \alpha, \beta)$, where $\preceq^1$ is represented by the utility function $u^1(x^1_1, x^1_2) = x^1_1 x^1_2$, $Y^1 = \{y^1 \in \mathbb{R}^2_+ \mid y^1_2 \leq 0,\ y^1_1 \leq \sqrt{-y^1_2}\}$, $L^{\mathrm{I}} = 1$, and $L^{\mathrm{II}} = 1$. The rationing system (α, β) can be chosen arbitrarily (satisfying A5). This

example satisfies A1-A5. Therefore we know by Theorem 3.1.i that there exists a connected set of underemployment equilibria that contains an underemployment equilibrium with $\max_{l \in L^{\mathrm{II}}} \upsilon_l = \upsilon_2 = \upsilon$, for all $\upsilon \in (0,1]$. Solving the firm's profit maximization problem yields that for every $p_1 \in \mathbb{R}_+$, for every $\underline{y}^1_2 \in \mathbb{R}_+$, $\eta^1((p_1,1),\underline{y}^1_2) = \{p_1/2, -(p_1)^2/4\}$ and $\pi^1((p_1,1),\underline{y}^1_2) = (p_1)^2/4$. Since the firm never wants to supply commodity 2, it is never affected by the opportunities expected in this market.

Let the household be constrained by $x^1_2 - 1 \geq -\upsilon$. If it supplies υ to the firm, then $p_1 = 2\sqrt{\upsilon}$ is required for profit maximization. At that price, the unconstrained demand of the household is $x_1 = (1+\frac{p_1}{2})^2/(2p_1)$, $x_2 = (1+\frac{p_1}{2})^2/2$. Hence, $x_2 - 1 < -\upsilon = 1-(p_1)^2/4$ iff $p_1 \leq \frac{2}{3}$, or equivalently $\upsilon \leq \frac{1}{9}$, in which case the constraint is binding. There is a continuum of strongly different equilibria for $\upsilon \in (0, \frac{1}{9}]$ with $p_1 = 2\sqrt{\upsilon}$; but there is no equilibrium at $\upsilon = 0$, since this would imply $p_1 = 0$ and excess demand of good 1.

If an input vector subject to supply rationing is used to produce an output not subject to supply rationing and desired by consumers, then technology and tastes should be such that there exists a relative price for the output at which it is neither supplied nor demanded, given the prices and expected opportunities for the other goods. It is difficult to formulate assumptions on primitives that imply such a property, which should be related to the existence of a finite rate of transformation of inputs into outputs.

The following, robust, example of a Cobb-Douglas economy shows that it is possible that in every underemployment equilibrium all households keep their initial endowments.

Example 4.2

Consider the economy $\mathcal{E} = ((\mathbb{R}^2_+, \preceq^h, (1,1))_{h\in H}, (Y^f, (\theta^{fh})_{h\in H})_{f\in F}, (1/5, 4/5), \alpha, \beta)$, where $L^{\mathrm{I}} = 0$, $\preceq^1$ is represented by the utility function $u^1(x^1_1, x^1_2) = (x^1_1)^2 x^1_2$, and $\preceq^2$ by the utility function $u^2(x^2_1, x^2_2) = x^2_1 (x^2_2)^2$, $Y^f = -\mathbb{R}^L_+$, $f \in F$, so we consider an economy without production possibilities, or, alternatively, an economy where all production has already taken place. Furthermore, $((\theta^{fh})_{h\in H})_{f\in F}$, α, and β can be chosen arbitrarily (but should of course satisfy A4 and A5). Notice that A1-A5 are satisfied by this economy, so we know by Theorem 3.1.i that there exists a connected set of underemployment equilibria ranging from an approximate no-trade equilibrium to an equilibrium without rationing in at least one market and containing a continuum of potentially different underemployment equilibria. Let us compute the set of underemployment equilibria directly. If both households expect no restrictions with respect to the supply of commodity 2, then the optimal consumption bundle of household 1 is $(10/3, 5/12)$ and the optimal consumption bundle of household 2 is $(5/3, 10/12)$. This means that both households supply commodity 2 in exchange for commodity 1. Notice that irrespective of the supply possibilities consumers expect for both commodities, at the price system $(1/5, 4/5)$ they will supply commodity 2 in exchange for commodity 1, unless they expect both to be fully rationed on the supply of commodity 2. It follows that there is a continuum of underemployment equilibria, given by $(\tilde{p}^{\mathrm{II}}, e, 0, \underline{z}^*, \underline{y}^*)$ with $(\underline{z}^*, \underline{y}^*) \in \underline{ZY}$, where both households keep the initial endowments, $\underline{z}_2^{*1} = \underline{z}_2^{*2} = 0$, and $\underline{z}_1^*$ is arbitary. All equilibria are characterized by full rationing of the supply of commodity 2, $\upsilon_2 = 0$, whereas the expected opportunities for supply of commodity 1 do not really matter. The impossibility of supply of commodity 2 leads to zero net demand of commodity 1 for both households. Although the expectations with respect to the market for commodity 1 do not really matter for the equilibrium allocation, there is a continuum of potentially different underemployment equilibria ranging from the expectations that no supply is possible in the market for commodity 1, $\upsilon_1 = 0$, to the expectations that there are no supply restrictions in the market for commodity 1, $\upsilon_1 = 1$.

Although the underemployment equilibria in Example 4.2 are potentially different, they are different in a weak sense. They are not different or strongly different as defined in Section 2. But the example is not fully convincing since the price system considered is rather extreme. Example 4.3 shows that it is possible that all underemployment equilibria, except the ones corresponding to the unique Walrasian equilibrium, are utility equivalent at Walrasian prices.

Example 4.3

Consider the economy $\mathcal{E} = ((\mathbb{R}^3_+, \preceq^h, e^h)_{h \in H}, (Y^f, (\theta^{fh})_{h \in H})_{f \in F}, (1/3, 1/3, 1/3), \alpha, \beta)$, where $L^{\mathrm{I}} = 0$, the prefence relation of household 3 is represented by the utility function $u^3 : \mathbb{R}^3_+ \to \mathbb{R}$ defined by $u^3(x^3) = x_1^3 x_2^3 x_3^3$ and its initial endowments are given by $e^3 = (c, c, c)$. For any price system such that $\sum_{l \in L} p_l = 1$ it holds that $d^3(p) = (c/(3p_1), c/(3p_2), c/(3p_3))$. The preferences of the other households are such that their (unrationed) demand as a function of the price system is differentiable, and their demands at price system $(1/3, 1/3, 1/3)$ sum up to their total initial endowments. Then it follows easily that taking c large enough guarantees that the Walrasian equilibrium price system $(1/3, 1/3, 1/3)$ is unique. Notice that if $p = (1/3, 1/3, 1/3)$, household 3 wants to keep its initial endowments and is therefore never affected by expectations concerning the supply possibilities in any of the markets. The rather artificial household 3 is used only to show that the rather strange phenomena of the example are not related to the issue of uniqueness of the Walrasian equilibrium. Household 3 can be dispensed with entirely if so desired.

Households 1 and 2 have preferences and initial endowments as indicated in Figures 2a and 2b. For those figures it is assumed that $p = (1/3, 1/3, 1/3)$. Using Walras' law, the amount of the third commodity can be determined by measuring the distance to the fourty-five degree line. There is no production, $Y^f = -\mathbb{R}^3_+$, $f \in F$. The rationing system is determined by arbitrarily chosen $\alpha \in \mathbb{R}^{HL}_{++}$ and $\beta \in \mathbb{R}^{FL}_{++}$. The example is constructed in such a way that A1-A5 are satisfied by this economy. So we know by Theorem 3.1.i that there exists a connected set of underemployment equilibria ranging from an approximate no-trade equilibrium to an equilibrium without rationing in at least one market and containing a continuum of potentially different underemployment equilibria. Moreover, one of the underemployment equilibria is given by the Walrasian equilibrium since we consider Walrasian prices.

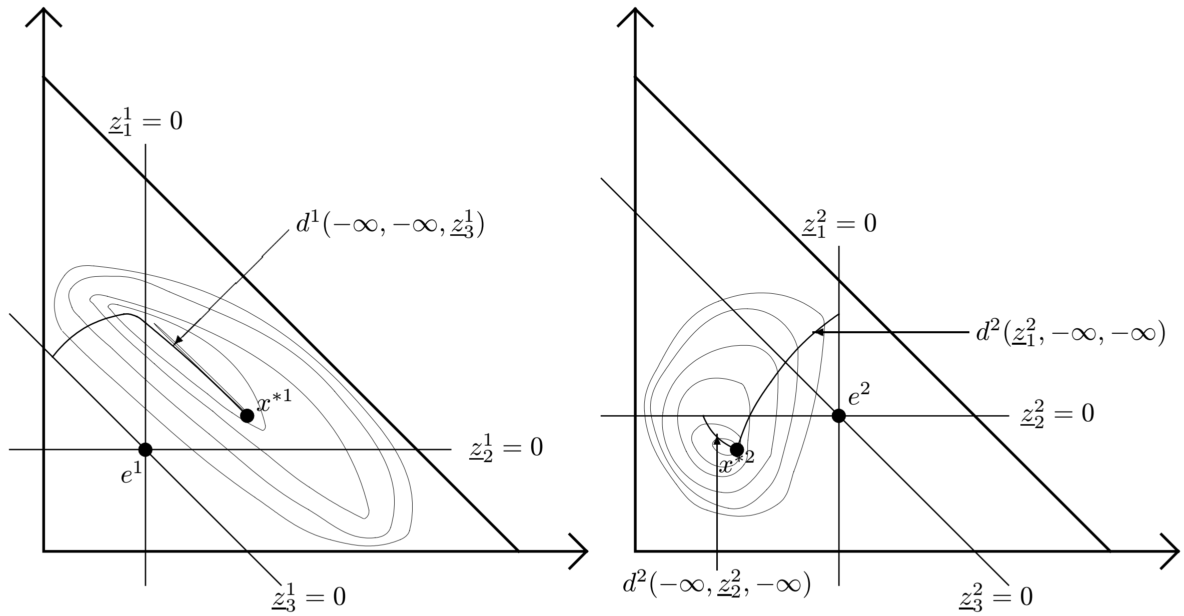

Figure 2a. Preferences of household 1.

Figure 2b. Preferences of household 2.

We will again compute the set of underemployment equilibria. For $h \in H$, let $d^h(\underline{z}^h)$ denote the solution to $\max u^h(x^h)$ subject to $\sum_{l \in L} x_l^h \leq \sum_{l \in L} e_l^h$, and $\underline{z}_l^h \leq x_l^h - e_l^h, \forall l \in L$. For this price system there exists a unique Walrasian equilibrium, where the consumption bundles of households 1 and 2 are indicated by x^{*1} and x^{*2} in Figures 2a and 2b. In Figure 2a the demand of household 1 at expected opportunities that are such that the household is not rationed in the markets for commodities 1 and 2, and expects supply possibilities $\underline{z}_3^1$ in the market for commodity 3, is depicted $(d^1(-\infty, -\infty, \underline{z}_3^1))$. Similarly, in Figure 2b the demand of household 2 at expected opportunities that are such that the household is not rationed in the markets for commodities 1 and 3, and supply possibilities $\underline{z}_2^2$ are expected in the market for commodity 2, is depicted $(d^2(-\infty, \underline{z}_2^2, -\infty))$, as well as the demand of household 2 at expected opportunities that are such that the household is not rationed in the markets for commodities 2 and 3, and expects supply possibilities $\underline{z}_1^2$ in the market for commodity 1 $(d^2(\underline{z}_1^2, -\infty, -\infty))$. For this example we claim that there are only two different underemployment equilibria. The first one is given by the no-trade equilibrium where everyone keeps his initial endowment, and the second one by the Walrasian equilibrium. The claim will be proved in three steps.

Step 1. It is shown that there is no underemployment equilibrium close to (but not equal to) the Walrasian equilibrium. If $\underline{z}_3^1$ is in absolute value smaller than but close to the Walrasian equilibrium value, $x_3^{*1} - e_3^1$, then $d_1^1(\underline{z}_1^1, \underline{z}_2^1, \underline{z}_3^1) + d_2^1(\underline{z}_1^1, \underline{z}_2^1, \underline{z}_3^1) \approx x_1^{*1} + x_2^{*1}$ and $d_1^1(\underline{z}_1^1, \underline{z}_2^1, \underline{z}_3^1) < x_1^{*1}$. (If $\underline{z}_3^1 \leq x_3^{*1} - e_3^1$, then $d^1(\underline{z}_1^1, \underline{z}_2^1, \underline{z}_3^1) = x^{*1}$, so the only way an underemployment equilibrium may result is that household 2 expects no restrictions with respect to supply in any market, which yields the Walrasian equilibrium). However, by no choice of $\underline{z}_1^2$ and $\underline{z}_2^2$ it holds that

$$d_1^2(\underline{z}_1^2, \underline{z}_2^2, \underline{z}_3^2) + d_2^2(\underline{z}_1^2, \underline{z}_2^2, \underline{z}_3^2) \approx e_1^1 + e_2^1 + e_1^2 + e_2^2 - x_1^{*1} - x_2^{*1} \text{ and } d_1^2(\underline{z}_1^2, \underline{z}_2^2, \underline{z}_3^2) > x_1^{*2}.$$

Therefore, there is no underemployment equilibrium being close to the Walrasian equilibrium.

Step 2. It is shown that $\underline{z}_1^2 = \underline{z}_3^1 = 0$ yields the no-trade equilibrium, irrespective of the value of $\underline{z}_2^2$. This step is obvious and yields the connected set of underemployment equilibria ranging from the no-trade equilibrium to an equilibrium without rationing in the market for commodity 2. However, also in the latter equilibrium there is no trade, although it is potentially different from the former no-trade equilibrium.

Step 3. It is shown that there is no underemployment equilibrium $(\tilde{p}^{\text{II}}, x^*, y^*, \underline{z}^*, \underline{y}^*)$, where $\underline{z}_1^{*2} \neq 0$ or $\underline{z}_3^{*1} \neq 0$, not being equal to the Walrasian equilibrium. Clearly, either $\underline{z}_3^{*1}$ is binding for household 1, or $\underline{z}_3^{*1}$ is not binding for household 1. In the latter case, household 1 demands x^{*1}, which is only compatible with a Walrasian equilibrium. Consequently, the only way to obtain an underemployment equilibrium not being equal to the Walrasian equilibrium is by having a binding $\underline{z}_3^{*1}$ for household 1. Let $a \in -\mathbb{R}_+$ be such that $d_1^1(-\infty, -\infty, a) = e_1^1$, i.e. $a \approx x_3^{*1} - e_3^1$. Suppose $\underline{z}_3^{*1} \leq a$, then $d_1^1(\underline{z}_1^{*1}, \underline{z}_2^{*1}, \underline{z}_3^{*1}) + d_2^1(\underline{z}_1^{*1}, \underline{z}_2^{*1}, \underline{z}_3^{*1}) \approx x_1^{*1} + x_2^{*1}$ and $d_1^1(\underline{z}_1^{*1}, \underline{z}_2^{*1}, \underline{z}_3^{*1}) < x_1^{*1}$, which leads to a contradiction as in Step 1. Consequently, $\underline{z}_3^{*1} > a$ and is binding for household 1. Now, either $\underline{z}_1^{*1}$ is binding for household 1, or $\underline{z}_1^{*1}$ is non-binding for household 1. In the latter case, using $\underline{z}_3^{*1} > a$, it follows that $d_1^1(\underline{z}^{*1}) < e_1^1$. However, $d_1^2(\underline{z}^{*2}) \leq e_1^2$, a contradiction. Consequently, $\underline{z}_1^{*1}$ is binding for household 1. If $\underline{z}_1^{*1} < 0$, then $d^1(\underline{z}^{*1}) < e_1^1$, leading to a contradiction as before. So, $\underline{z}_1^{*1} = 0$ and $d_2^1(\underline{z}^{*2}) > e_2^1$ unless $\underline{z}_3^{*1} = 0$. If $\underline{z}_1^{*1} = 0$, then $\underline{z}_1^{*2} = 0$ and $d_2^2(\underline{z}^{*2}) > e_2^1$ unless $\underline{z}_3^{*2} = 0$. So, we obtain a contradiction with $d_2^1(\underline{z}^{*1}) + d_2^2(\underline{z}^{*2}) = e_2^1 + e_2^2$, unless $\underline{z}_1^{*1} = \underline{z}_1^{*2} = \underline{z}_3^{*1} = \underline{z}_3^{*2} = 0$, which leads to a no-trade equilibrium.

In Example 4.3 the conditions of Theorem 3.2 cannot be satisfied, since there are only two different underemployment equilibria. In this example it holds that $\delta^1(\tilde{p}^{\text{II}}, 0_{-2}, \tilde{p}^{\text{II}} \cdot e^1) = \{e^1\}$, whereas $\delta^2(\tilde{p}^{\text{II}}, 0_{-2}, \tilde{p}^{\text{II}} \cdot e^2) = \{e^2\}$. For this situation to occur it is necessary that household 2 that is a supplier of commodity 2 at the Walrasian equilibrium, is no longer willing to supply this commodity if it expects no supply possibilities of commodities 1 and 3.

Also the extension of Theorem 3.1.iii to $L^{\text{I}} = 0$ does not apply. Let us denote the excess demand correspondence, which is a function now, by z, and let us surpress the dependence on p, which is constant, and $\underline{y}$, which is redundant. Let $(p^*, x^*, \underline{z}^*)$ denote the competitive equilibrium for the economy of Example 4.3. Now consider $-\underline{z} \leq -\underline{z}^*$ that arises from keeping $\underline{z}_2$ at competitive values, and decreasing $\underline{z}_1$ and $\underline{z}_3$ away from competitive values. Since $\underline{z}^*$ is non-binding, initially these decrements have no effect, $z(\underline{z}) = 0$, and A7 is satisfied. Without loss of generality, either $\underline{z}_1$ or $\underline{z}_3$ becomes binding after a sufficiently large decrement. In both cases it is easily verified that $z_2(\underline{z}) > \max_{l \in \{1,3\}} z_l(\underline{z})$, contradicting A7 and contradicting also the weaker version of A7 used in the proof. Indeed, limited expected supply possibilities in the market for commodity 3 lead to a strongly positive demand for commodity 2 by household 1. The demand for commodity 2 increases even more than the demand for commodity 3, a strong complementarity. Analogously, limited expected supply possibilities in the market for commodity 1 lead to a strongly positive demand for commodity 2 by household 2. Again, the demand for commodity 2 increases even more than the demand for commodity 1. Theorem 3.1.iii shows that if one rules out such strong complementarities, then the situation of Example 4.3 does not occur and there is a continuum of strongly different equilibrium allocations ranging from an approximate no-trade equilibrium to a competitive equilibrium.

Appendix: Proofs

A first step in the proof is to show that the production possibility correspondences and budget correspondences are continuous.

We compactify the consumption sets and the production possibility sets using the number b as defined in Subsection 3.2, so $\widehat{X}^h = \{x^h \in X^h \mid \|x^h\|_\infty \leq b\}$ and $\widehat{Y}^f = \{y^f \in Y^f \mid \|y^f\|_\infty \leq b\}$. It follows from a standard argument that there is no loss of generality in using the compactified consumption and production sets when studying the existence of underemployment equilibria. The feasible production plans, supply, budget, and demand correspondences derived from $\widehat{X}^h$ and $\widehat{Y}^h$ are denoted by $\widehat{s}^f$, $\widehat{\eta}^f$, $\widehat{\gamma}^h$, and $\widehat{\delta}^h$, respectively. Let us define the set P of prices, expected opportunities, and wealths by

$$P = \{(p, \underline{z}^h, w^h) \in \mathbb{R}^L_+ \times -\mathbb{R}^{L^{\mathrm{II}}}_+ \times \mathbb{R} \mid p \cdot e^h \leq w^h, \text{ and } p^{\mathrm{I}} > 0 \text{ or } p^{\mathrm{II}} \cdot \underline{z}^h < 0\}.$$

Lemma A.1

Let the economy $\mathcal{E}$ satisfy A1-A5. Then the production possibility correspondence $\widehat{s}^f : \mathbb{R}^{L^{\mathrm{II}}}_+ \to \mathbb{R}^L$ of firm f is compact-valued, convex-valued and continuous, and the budget correspondence $\widehat{\gamma}^h : P \to \mathbb{R}^L$ of household h is compact-valued, convex-valued, and continuous.

Proof

Compact-valuedness and convex-valuedness of $\hat{s}^f$ are trivial. First we show the upper hemi-continuity of the production possibility correspondence. Let some $\underline{y}^f \in \mathbb{R}_+^{L^{\mathrm{II}}}$ be given, let $(\underline{y}^{fn})_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}_+^{L^{\mathrm{II}}}$ converging to $\underline{y}^f$, and let the sequence $(y^{fn})_{n\in\mathbb{N}}$ be such that $y^{fn} \in \hat{s}^f(\underline{y}^{fn})$. Clearly, $(y^{fn})_{n\in\mathbb{N}}$ remains in a compact set. Therefore, it has a converging subsequence, also denoted by $(y^{fn})_{n\in\mathbb{N}}$, converging to, say, $\overline{y}^f \in \hat{Y}^f$. It has to be shown that $\overline{y}^f \in \hat{s}^f(\underline{y}^f)$. Since $y^{fn} \leq \underline{y}^{fn}$, it follows that $\overline{y}^f \leq \underline{y}^f$. Consequently, $\overline{y}^f \in \hat{s}^f(\underline{y}^f)$ and $\hat{s}^f$ is upper hemi-continuous.

Next lower hemi-continuity of the production possibility correspondence is shown. Let some $\underline{y}^f \in \mathbb{R}_+^{L^{\mathrm{II}}}$ be given, let $(\underline{y}^{fn})_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}_+^{L^{\mathrm{II}}}$ converging to $\underline{y}^f$, and let $\overline{y}^f$ be an element of $\hat{s}^f(\underline{y}^f)$. The correspondence $\hat{s}^f$ is lower hemi-continuous at $\underline{y}^f$ if there is a sequence $(y^{fn})_{n\in\mathbb{N}}$ in $\mathbb{R}^L$ such that $y^{fn} \in \hat{s}^f(\underline{y}^{fn})$ and $y^{fn} \to \overline{y}^f$. Let the sets $\overline{L}$ and $\underline{L}$ be defined by

$$\overline{L} = \{l \in L^{\mathrm{II}} \mid \overline{y}_l^f > 0\},$$
$$\underline{L} = \{l \in L^{\mathrm{II}} \mid \overline{y}_l^f \leq 0\}.$$

For $n \in \mathbb{N}$, let $\lambda^{fn} \in [0,1]$ be defined by

$$\lambda^{fn} = \min\left\{\min_{l\in\overline{L}} \frac{\underline{y}_l^{fn}}{\overline{y}_l^f}, 1\right\}.$$

For $n \in \mathbb{N}$, let y^{fn} be defined by

$$y^{fn} = \lambda^{fn}\overline{y}^f.$$

It holds that $y^{fn} \in \hat{Y}^f$ since $0 \in \hat{Y}^f$ and $\hat{Y}^f$ is convex. Moreover, for $l \in \overline{L}$ it holds that $y_l^{fn} = \lambda^{fn}\overline{y}_l^f \leq \frac{\underline{y}_l^{fn}}{\overline{y}_l^f}\overline{y}_l^f = \underline{y}_l^{fn}$, and for $l \in \underline{L}$ it holds that $y_l^{fn} \leq 0 \leq \overline{y}_l^{fn}$. So, $y^{fn} \in \hat{s}^f(\underline{y}^{fn})$. Notice that $\lambda^{fn} \to \min\left\{\min_{l\in\overline{L}} \frac{\underline{y}_l^f}{\overline{y}_l^f}, 1\right\} = 1$. Therefore, it follows that $y^{fn} \to \overline{y}^f$.

Compact-valuedness and convex-valuedness of $\hat{\gamma}^h$ are trivial. Let us show upper hemi-continuity of the budget correspondence. Let some $(\overline{p}, \underline{z}^h, \overline{w}^h) \in P$ be given, let $(p^n, \underline{z}^{hn}, w^{hn})_{n\in\mathbb{N}}$ be a sequence in P converging to $(\overline{p}, \underline{z}^h, \overline{w}^h)$, and let the sequence $(x^{hn})_{n\in\mathbb{N}}$ be such that $x^{hn} \in \hat{\gamma}^h(p^n, \underline{z}^{hn}, w^{hn})$. Clearly, $(x^{hn})_{n\in\mathbb{N}}$ remains in a compact set. Therefore, it has a converging subsequence, also denoted by $(x^{hn})_{n\in\mathbb{N}}$, converging to, say, $\overline{x}^h \in \hat{X}^h$. It has to be shown that $\overline{x}^h \in \hat{\gamma}^h(\overline{p}, \underline{z}^h, \overline{w}^h)$. Since $p^n \cdot x^{hn} \leq w^{hn}$ it follows that $\overline{p} \cdot \overline{x}^h \leq \overline{w}^h$. Since $x^{hn,\mathrm{II}} - e^{h,\mathrm{II}} \geq \underline{z}^{hn}$ it follows that $\overline{x}^{h,\mathrm{II}} - e^{h,\mathrm{II}} \geq \underline{z}^h$. Consequently, $\overline{x}^h \in \hat{\gamma}^h(\overline{p}, \underline{z}^h, \overline{w}^h)$ and $\hat{\gamma}^h$ is upper hemi-continuous.

Finally, lower hemi-continuity of the budget correspondence is shown. Let some $(\overline{p}, \underline{z}^h, \overline{w}^h) \in P$ be given, let $(p^n, \underline{z}^{hn}, w^{hn})_{n\in\mathbb{N}}$ be a sequence in P converging to $(\overline{p}, \underline{z}^h, \overline{w}^h)$, and let $\overline{x}^h$ be an element of $\hat{\gamma}^h(\overline{p}, \underline{z}^h, \overline{w}^h)$. The correspondence $\hat{\gamma}^h$ is lower hemi-continuous at $(\overline{p}, \underline{z}^h, \overline{w}^h)$

if there is a sequence $(x^{h^n})_{n\in\mathbb{N}}$ in $\mathbb{R}^L$ such that $x^{h^n} \in \widehat{\gamma}^h(p^n, \underline{z}^{h^n}, w^{h^n})$ and $x^{h^n} \to \overline{x}^h$. Let the sets $\overline{L}$ and $\underline{L}$ be defined by

$$\overline{L} = \{l \in L^{\mathrm{II}} \,|\, \overline{x}^h_l - e^h_l < 0\},$$
$$\underline{L} = \{l \in L^{\mathrm{II}} \,|\, \overline{x}^h_l - e^h_l \geq 0\}.$$

Now two cases have to be considered, $\overline{p} \cdot \overline{x}^h < \overline{w}^h$ and $\overline{p} \cdot \overline{x}^h = \overline{w}^h$.

Case 1. $\overline{p} \cdot \overline{x}^h < \overline{w}^h$. Let $\widehat{x}^h \in \widehat{X}^h$ be chosen such that $\widehat{x}^{h,\mathrm{I}} \leq e^{h,\mathrm{I}}$ and $\widehat{x}^{h,\mathrm{II}} = e^{h,\mathrm{II}}$. For $n \in \mathbb{N}$, let $\lambda^{h^n} \in [0,1]$ be defined by

$$\lambda^{h^n} = \min\left\{\min_{l\in\overline{L}} \frac{\underline{z}^{h^n}_l}{\overline{x}^h_l - e^h_l}, 1\right\}. \tag{1}$$

For $n \in \mathbb{N}$, let x^{h^n} be defined by

$$x^{h^n} = \lambda^{h^n}\overline{x}^h + (1 - \lambda^{h^n})\widehat{x}^h.$$

It holds that $x^{h^n} \in \widehat{X}^h$ by convexity of $\widehat{X}^h$. Moreover, using that $\overline{p} \cdot \overline{x}^h < \overline{w}^h$ and $p^n \cdot \widehat{x}^h \leq p^n \cdot e^h \leq w^{h^n}$, it holds for n sufficiently large that

$$p^n \cdot x^{h^n} = \lambda^{h^n} p^n \cdot \overline{x}^h + (1 - \lambda^{h^n}) p^n \cdot \widehat{x}^h \leq \lambda^{h^n} w^{h^n} + (1 - \lambda^{h^n}) w^{h^n} = w^{h^n}.$$

Furthermore, for $l \in \overline{L}$,

$$x^{h^n}_l - e^h_l = \lambda^{h^n}(\overline{x}^h_l - e^h_l) \geq \frac{\underline{z}^{h^n}_l}{\overline{x}^h_l - e^h_l}(\overline{x}^h_l - e^h_l) = \underline{z}^{h^n}_l,$$

and for $l \in \underline{L}$,

$$x^{h^n}_l - e^h_l \geq 0 \geq \underline{z}^{h^n}_l.$$

So, for n sufficiently large, $x^{h^n} \in \widehat{\gamma}^h(p^n, \underline{z}^{h^n}, w^{h^n})$. Notice that

$$\lambda^{h^n} \to \min\left\{\min_{l\in\overline{L}} \frac{\overline{\underline{z}}^h_l}{\overline{x}^h_l - e^h_l}, 1\right\} = 1,$$

so it follows that $x^{h^n} \to \overline{x}^h$.

Case 2. $\overline{p} \cdot \overline{x}^h = \overline{w}^h$. Let $\widehat{x}^h \in \widehat{X}^h$ be such that $\widehat{x}^{h,\mathrm{I}} \ll e^{h,\mathrm{I}}$, and $\widehat{x}^{h,\mathrm{II}} = e^{h,\mathrm{II}}$. Choose $\widetilde{x}^h \in \widehat{X}^h$ as follows. If $\overline{p}_{l'} > 0$ for some $l' \in L^{\mathrm{I}}$, then let $\widetilde{x}^h$ be equal to $\widehat{x}^h$. Otherwise, there is $l'' \in L^{\mathrm{II}}$ such that $\overline{p}^{\mathrm{II}}_{l''} \cdot \overline{\underline{z}}^{h,\mathrm{II}}_{l''} < 0$. Then let $\widetilde{x}^h$ be such that $\widetilde{x}^{h,\mathrm{I}} \leq e^{h,\mathrm{I}}$, $\widetilde{x}^h_{l''} = e^h_{l''} - \varepsilon$ with $\varepsilon < -\overline{\underline{z}}^h_{l''}$, and $\widetilde{x}^h_l = e^h_l$, $\forall l \in L^{\mathrm{II}} \setminus \{l''\}$. It follows from A3 as well as the convexity of $\widehat{X}^h$, that indeed $\widetilde{x}^h$ can be chosen in the way described above. Notice that $\overline{p} \cdot \widetilde{x}^h < \overline{w}^h$ and $\overline{\underline{z}}^h_{l''} < \widetilde{x}^h_{l''} - e^h_{l''}$. Clearly, there exists $\overline{n} \in \mathbb{N}$ such that for all $n \geq \overline{n}$, $p^n \cdot \widetilde{x}^h < w^{h^n}$ and $\underline{z}^{h^n}_{l''} < \widetilde{x}^h_{l''} - e^h_{l''}$. For $n \geq \overline{n}$, let $\lambda^{h^n} \in [0,1]$ be defined as in (1). For $n \geq \overline{n}$, let x^{h^n} be defined by

$$x^{h^n} = \mu^{h^n}(\lambda^{h^n}\overline{x}^h + (1 - \lambda^{h^n})\widehat{x}^h) + (1 - \mu^{h^n})\widetilde{x}^h,$$

where μ^{h^n} is given by $\mu^{h^n} = 1$ if $p^n \cdot (\lambda^{h^n}\overline{x}^h + (1-\lambda^{h^n})\widehat{x}^h) \leq w^{h^n}$, and $\mu^{h^n} = (w^{h^n} - p^n \cdot \widetilde{x}^h)/(p^n \cdot (\lambda^{h^n}\overline{x}^h + (1-\lambda^{h^n})\widehat{x}^h - \widetilde{x}^h))$, otherwise. Notice that $\mu^{h^n} \in (0,1)$ in the latter case. As before, it is easy to verify that $x^{h^n} \in \widehat{\gamma}^h(p^n, \underline{z}^{h^n}, w^{h^n})$, and that $\lambda^{h^n} \to 1$ and $\mu^{h^n} \to 1$. So it follows that $x^{h^n} \to \overline{x}^h$ and that $\widehat{\gamma}^h$ is lower hemi-continuous. Q.E.D.

Lemma A.1 extends the lemma in Drèze (1975), page 304, in several respects. First, we have included production. Secondly, our assumptions with respect to the consumption sets, preference relations, and initial endowments are weaker. Thirdly, we cannot limit ourselves to the case where $w^h = p \cdot e^h$, since the possibility of production may lead to profit income for households. For the same reasons, Theorem 2.2 of Herings (1996a), page 67, is generalized. Lemma A.1 leads to upper hemi-continuity of demand and supply correspondences and continuity of profit functions.

Lemma A.2

Let the economy $\mathcal{E}$ satisfy A1-A5. Then the supply correspondence $\widehat{\eta}^f : \mathbb{R}^L \times \mathbb{R}_+^{L^{\text{II}}} \to \mathbb{R}^L$ of firm f and the demand correspondence $\widehat{\delta}^h : P \to \mathbb{R}^L$ of household h are compact-valued, convex-valued, and upper hemi-continuous. The profit function $\widehat{\pi}^f : \mathbb{R}^L \times \mathbb{R}_+^{L^{\text{II}}} \to \mathbb{R}$ of firm f is continuous.

Proof

This follows from Lemma A.1 and an application of the maximum theorem. Q.E.D.

Some other properties of $\widehat{\eta}^f$ and $\widehat{\delta}^h$ are readily seen. For instance the boundary behaviour that $\underline{z}_l^h = 0$ implies $x_l^h \geq e_l^h$ for every $x^h \in \widehat{\delta}^h(p, \underline{z}^h, w^h)$, and $\underline{y}_l^f = 0$ implies $y_l^f \leq 0$ for every $y^f \in \widehat{\eta}^f(p, \underline{y}^f)$. Using the definition of $\widehat{\gamma}^h(p, \underline{z}^h, w^h)$, $p \cdot x^h \leq w^h$ for every $x^h \in \widehat{\delta}^h(p, \underline{z}^h, w^h)$.

Now we construct a correspondence $\widehat{\zeta}$ such that its zero points correspond to all potentially different underemployment equilibria. Denote the minimal market share in the market for a commodity $l \in L^{\text{II}}$ by $\underline{\alpha}_l$, so $\underline{\alpha}_l = \min\{\alpha_l^h, \beta_l^f \mid h \in H, f \in F\}$.

The m-dimensional unit cube is given by $Q^m = \{q \in \mathbb{R}^m \mid 0 \leq q_i \leq 1, i = 1, \ldots, m\}$. Let $(\phi_1, \phi_2) : Q^{L^{\text{II}}} \to \underline{ZY}$ be the function that associates to $q \in Q^{L^{\text{II}}}$ the expected opportunities

$$\left(\frac{-\alpha_l^h b}{\underline{\alpha}_l} q_l\right)_{h \in H, l \in L^{\text{II}}}, \left(\frac{\beta_l^f b}{\underline{\alpha}_l} q_l\right)_{f \in F, l \in L^{\text{II}}},$$

where $\phi_1(q)$ determines the expected opportunities of the households and $\phi_2(q)$ the expected opportunities of the firms. So, for $l \in L^{\text{II}}$, $q_l \in [0,1]$ parametrizes the expected opportunities in the market for commodity l, $(\frac{-\alpha_l^1 b}{\underline{\alpha}_l} q_l, \ldots, \frac{\beta_l^F b}{\underline{\alpha}_l} q_l)$. The expected opportunities range from $(0,0)$ if $q_l = 0$, to a vector $(\underline{z}, \underline{y})$ satisfying $\min\{-\underline{z}_l^h, \underline{y}_l^f \mid h \in H, f \in F\} = b$. The parameter q_l coincides with υ_l as defined in Subsection 3.2.

The correspondence $\widehat{\zeta} : \mathbb{R}^L_+ \times Q^{L^{\mathrm{II}}} \to \mathbb{R}^L$ is defined by

$$\widehat{\zeta}(p,q) = \sum_{h \in H} \widehat{\delta}^h(p, \phi_1^h(q), p \cdot e^h + \sum_{f \in F} \theta^{fh} \widehat{\pi}^f(p, \phi_2^f(q))) - \sum_{h \in H} e^h - \sum_{f \in F} \widehat{\eta}^f(p, \phi_2^f(q)).$$

The restriction of $\widehat{\zeta}$ to the set $(\mathbb{R}^{L^{\mathrm{I}}}_+ \times \{\widetilde{p}^{\mathrm{II}}\}) \times Q^{L^{\mathrm{II}}}$ is denoted by $\widehat{\zeta}_{|\widetilde{p}^{\mathrm{II}}}$. It holds that $\widehat{\zeta}_{|\widetilde{p}^{\mathrm{II}}}$ is a compact-valued and convex-valued correspondence that is upper hemi-continuous everywhere, except at the point $((0, \widetilde{p}^{\mathrm{II}}), 0)$.

The set of zero points of $\widehat{\zeta}_{|\widetilde{p}^{\mathrm{II}}}$ is denoted by $\widehat{Z}_0 = \{(p,q) \in \mathbb{R}^L_+ \times Q^{L^{\mathrm{II}}} \mid p^{\mathrm{II}} = \widetilde{p}^{\mathrm{II}}$ and $0 \in \widehat{\zeta}(p,q)\}$. The correspondence $\widehat{\psi} : \widehat{Z}_0 \to \mathbb{R}^L \times \widetilde{X} \times \widetilde{Y} \times \underline{ZY}$ is defined by relating the set

$$\{p\} \times \left(\left(\prod_{h \in H} \widehat{\delta}^h\Big(p, \phi_1^h(q), p \cdot e^h + \sum_{f \in F} \theta^{fh} \widehat{\pi}^f(p, \phi_2^f(q))\Big) \times \prod_{f \in F} \widehat{\eta}^f\Big(p, \phi_2^f(q)\Big)\right) \cap A\right) \times \{(\phi_1(q), \phi_2(q))\}$$

to $(p,q) \in \widehat{Z}_0$. Then $\widehat{\psi}(\widehat{Z}_0)$ is the set of all potentially different underemployment equilibria of $\mathcal{E}$, $\widehat{\psi}(\widehat{Z}_0) = \widehat{E}$.

To prove Theorem 3.1.i we will use a fixed point theorem. In fact, Browder's fixed point theorem (see Browder (1960)), and the extension of it to correspondences as stated in Theorem A.3 (see Mas-Colell (1974), Theorem 3, page 230) will be needed in the proof.

Theorem A.3 (Browder's fixed point theorem)
Let S be a non-empty, compact, convex subset of $\mathbb{R}^m$ and let $\varphi : S \times [0,1] \to S$ be a compact-valued, convex-valued, upper hemi-continuous correspondence. Then the set $F_\varphi = \{(s, \lambda) \in S \times [0,1] \mid s \in \varphi(s, \lambda)\}$ contains a component F^{c}_φ such that $(S \times \{0\}) \cap F^{\mathrm{c}}_\varphi \neq \emptyset$ and $(S \times \{1\}) \cap F^{\mathrm{c}}_\varphi \neq \emptyset$.

The m-dimensional unit simplex is denoted by $S^m = \{s \in \mathbb{R}^{m+1}_+ \mid \sum_{i=1}^{m+1} s_i = 1\}$ and, for $\varepsilon \geq 0$, the subset of the cube satisfying that each of its elements has at least one component greater than or equal to ε by $Q^m(\varepsilon) = \{q \in Q^m \mid \|q\|_\infty \geq \varepsilon\}$. Obviously, $Q^m(0) = Q^m$. Now, for $\varepsilon \geq 0$, an artificial correspondence $\widetilde{\zeta} : S^{L^{\mathrm{I}}} \times Q^{L^{\mathrm{II}}}(\varepsilon) \to \mathbb{R}^L$ is considered. To prove Theorem 3.1.i we take $\widetilde{\zeta}(s,q)$ equal to $\widehat{\zeta}(s_1, \ldots, s_{L^{\mathrm{I}}}, s_{L^{\mathrm{I}}+1}\widetilde{p}^{\mathrm{II}}, q)$. The set $\widetilde{Z}_- = \widetilde{\zeta}^{-1}(-\mathbb{R}^L_+) = \{(s,q) \in S^{L^{\mathrm{I}}} \times Q^{L^{\mathrm{II}}}(\varepsilon) \mid \widetilde{\zeta}(s,q) \cap -\mathbb{R}^L_+ \neq \emptyset\}$ has a very special structure as the following result shows.

Lemma A.4
Let $\varepsilon \geq 0$ and $p^{\mathrm{II}} \in \mathbb{R}^{L^{\mathrm{II}}}_{++}$ be given. Let $\widetilde{\zeta} : S^{L^{\mathrm{I}}} \times Q^{L^{\mathrm{II}}}(\varepsilon) \to \mathbb{R}^L$ be a compact-valued, convex-valued, upper hemi-continuous correspondence satisfying that for every $(s,q) \in S^{L^{\mathrm{I}}} \times Q^{L^{\mathrm{II}}}(\varepsilon)$, for every $z \in \widetilde{\zeta}(s,q)$, $(s_1, \ldots, s_{L^{\mathrm{I}}}, s_{L^{\mathrm{I}}+1}p^{\mathrm{II}}) \cdot z \leq 0$, and, for $l \in L^{\mathrm{II}}$, $q_l = 0$ implies $z_l \geq 0$. Then $\widetilde{Z}_-$ has a component $\widetilde{Z}^{\mathrm{c}}_-$ such that for every $\upsilon \in [\varepsilon, 1]$ there is $(s^\upsilon, q^\upsilon) \in \widetilde{Z}^{\mathrm{c}}_-$ with $\|q^\upsilon\|_\infty = \upsilon$.

Proof

Since $\widetilde{\zeta}$ is a compact-valued, upper hemi-continuous correspondence, $\widetilde{\zeta}(S^{L^{\mathrm{I}}} \times Q^{L^{\mathrm{II}}}(\varepsilon))$ is compact, and therefore there exists a compact, convex set Z satisfying $\widetilde{\zeta}(S^{L^{\mathrm{I}}} \times Q^{L^{\mathrm{II}}}(\varepsilon)) \subseteq Z$. The compact-valued, convex-valued, upper hemi-continuous correspondences $\varphi^1 : Z \to S^{L^{\mathrm{I}}}$, $\varphi^2 : Z \to S^{L^{\mathrm{II}}-1}$, and $\varphi^3 : S^{L^{\mathrm{I}}} \times S^{L^{\mathrm{II}}-1} \times [\varepsilon, 1] \to Z$ are defined by

$$\begin{aligned} \varphi^1(z) &= \{\overline{s} \in S^{L^{\mathrm{I}}} \mid \textstyle\sum_{l \in L^{\mathrm{I}}} \overline{s}_l z_l + \overline{s}_{L^{\mathrm{I}}+1} p^{\mathrm{II}} \cdot z^{\mathrm{II}} \geq \sum_{l \in L^{\mathrm{I}}} s_l z_l + s_{L^{\mathrm{I}}+1} p^{\mathrm{II}} \cdot z^{\mathrm{II}},\ \forall s \in S^{L^{\mathrm{I}}}\}, \quad z \in Z, \\ \varphi^2(z) &= \{\overline{t} \in S^{L^{\mathrm{II}}-1} \mid \overline{t} \cdot z^{\mathrm{II}} \geq t \cdot z^{\mathrm{II}},\ \forall t \in S^{L^{\mathrm{II}}-1}\}, \quad z \in Z, \\ \varphi^3(s,t,\lambda) &= \widetilde{\zeta}(s, \lambda \tfrac{t}{\|t\|_\infty}), \quad (s,t,\lambda) \in S^{L^{\mathrm{I}}} \times S^{L^{\mathrm{II}}-1} \times [\varepsilon, 1]. \end{aligned}$$

It follows that the correspondence $\varphi : Z \times S^{L^{\mathrm{I}}} \times S^{L^{\mathrm{II}}-1} \times [\varepsilon, 1] \to Z \times S^{L^{\mathrm{I}}} \times S^{L^{\mathrm{II}}-1}$ defined by

$$\varphi(z,s,t,\lambda) = \varphi^3(s,t,\lambda) \times \varphi^1(z) \times \varphi^2(z), \quad (z,s,t,\lambda) \in Z \times S^{L^{\mathrm{I}}} \times S^{L^{\mathrm{II}}-1} \times [\varepsilon, 1],$$

is a compact-valued, convex-valued, and upper hemi-continuous correspondence, and the set $Z \times S^{L^{\mathrm{I}}} \times S^{L^{\mathrm{II}}-1}$ is non-empty, compact, and convex. By Theorem A.3 it follows that the set $F_\varphi = \{(z,s,t,\lambda) \in Z \times S^{L^{\mathrm{I}}} \times S^{L^{\mathrm{II}}-1} \times [\varepsilon, 1] \mid (z,s,t) \in \varphi(z,s,t,\lambda)\}$ contains a component F^{c}_φ such that $(Z \times S^{L^{\mathrm{I}}} \times S^{L^{\mathrm{II}}-1} \times \{\varepsilon\}) \cap F^{\mathrm{c}}_\varphi \neq \emptyset$ and $(Z \times S^{L^{\mathrm{I}}} \times S^{L^{\mathrm{II}}-1} \times \{1\}) \cap F^{\mathrm{c}}_\varphi \neq \emptyset$. The connectedness of F^{c}_φ therefore yields that, for every $\upsilon \in [\varepsilon, 1]$, $(Z \times S^{L^{\mathrm{I}}} \times S^{L^{\mathrm{II}}-1} \times \{\upsilon\}) \cap F^{\mathrm{c}}_\varphi \neq \emptyset$. Let some $(z^*, s^*, t^*, \lambda^*) \in F^{\mathrm{c}}_\varphi$ be given. So,

$$(z^*, s^*, t^*, \lambda^*) \in \varphi^3(s^*, t^*, \lambda^*) \times \varphi^1(z^*) \times \varphi^2(z^*) = \widetilde{\zeta}(s^*, \lambda^* \frac{t^*}{\|t^*\|_\infty}) \times \varphi^1(z^*) \times \varphi^2(z^*).$$

Therefore, $(s^*_1, \ldots, s^*_{L^{\mathrm{I}}}) \cdot z^{*\mathrm{I}} + s^*_{L^{\mathrm{I}}+1} p^{\mathrm{II}} \cdot z^{*\mathrm{II}} \leq 0$. Using that $s^* \in \varphi^1(z^*)$ it follows by taking s equal to the l-th, respectively $(l+1)$-th, unit vector that $z^{*\mathrm{I}}_l \leq 0$, $\forall l \in L^{\mathrm{I}}$, and $p^{\mathrm{II}} \cdot z^{*\mathrm{II}} \leq 0$.

Suppose $\max_{l \in L^{\mathrm{II}}} z^*_l > 0$. Since $p^{\mathrm{II}} \in \mathbb{R}^{L^{\mathrm{II}}}_{++}$ and $p^{\mathrm{II}} \cdot z^{*\mathrm{II}} \leq 0$, there exists $l' \in L^{\mathrm{II}}$ with $z^*_{l'} < 0$. From $\max_{l \in L^{\mathrm{II}}} z^*_l > 0$, $z^*_{l'} < 0$, and $t^* \in \varphi^2(z^*)$ it follows that $t^*_{l'} = 0$, implying that $z^*_{l'} \geq 0$, a contradiction. Consequently, $\max_{l \in L^{\mathrm{II}}} z^*_l \leq 0$. We have shown that $z^* \in -\mathbb{R}^L_+$. The function $g : Z \times S^{L^{\mathrm{I}}} \times S^{L^{\mathrm{II}}-1} \times [\varepsilon, 1] \to S^{L^{\mathrm{I}}} \times Q^{L^{\mathrm{II}}}(\varepsilon)$ is defined by

$$g(z,s,t,\lambda) = (s, \lambda \frac{t}{\|t\|_\infty}), \quad (z,s,t,\lambda) \in Z \times S^{L^{\mathrm{I}}} \times S^{L^{\mathrm{II}}-1} \times [\varepsilon, 1],$$

and the set $\widetilde{Z}^{\mathrm{c}}_-$ is defined by $\widetilde{Z}^{\mathrm{c}}_- = g(F^{\mathrm{c}}_\varphi)$. Clearly, for every $(s,q) \in \widetilde{Z}^{\mathrm{c}}_-$, $\widetilde{\zeta}(s,q) \cap -\mathbb{R}^L_+ \neq \emptyset$. The set $\widetilde{Z}^{\mathrm{c}}_-$ is connected by the connectedness of F^{c}_φ and the continuity of g. For every $\upsilon \in [\varepsilon, 1]$, there exists $(z^\upsilon, s^\upsilon, t^\upsilon) \in Z \times S^{L^{\mathrm{I}}} \times S^{L^{\mathrm{II}}-1}$ such that $(z^\upsilon, s^\upsilon, t^\upsilon, \upsilon) \in F^{\mathrm{c}}_\varphi$, so $g(z^\upsilon, s^\upsilon, t^\upsilon, \upsilon) = (s^\upsilon, \upsilon \frac{t}{\|t\|_\infty}) = (s^\upsilon, q^\upsilon) \in \widetilde{Z}^{\mathrm{c}}_-$. Obviously, $\|q^\upsilon\|_\infty = \upsilon$. Q.E.D.

The correspondence $\widetilde{\zeta}$ has a continuum of points with a non-positive vector in its image set. These points range from a point on the boundary of $Q^{L^{\mathrm{II}}}(\varepsilon)$ with every component less than or equal to ε to a point on the boundary of $Q^{L^{\text{II}}}(\varepsilon)$ where at least one component equals one.

We are now in a position to give a proof of Theorem 3.1.i. One of the problems we have to deal with is the possible lack of upper hemi-continuity of $\widehat{\zeta}$ at a point $((0, \widetilde{p}^{\text{II}}), 0)$.

Proof of Theorem 3.1.i

For $\varepsilon \geq 0$, the correspondence $\widetilde{\zeta}^{\varepsilon} : S^{L^{\text{I}}} \times Q^{L^{\text{II}}}(\varepsilon) \to \mathbb{R}^L$ is defined by $\widetilde{\zeta}^{\varepsilon}(s, q) = \widehat{\zeta}(s_1, \ldots, s_{L^{\text{I}}}, s_{L^{\text{I}}+1}\widetilde{p}^{\text{II}}, q)$.

Let some $\varepsilon > 0$ be given. Notice that $(s_1, \ldots, s_{L^{\text{I}}}) > 0$ or $s_{L^{\text{I}}+1}\widetilde{p}^{\text{II}} \gg 0$. In the latter case, $q \in Q^{L^{\text{II}}}(\varepsilon)$ implies $s_{L^{\text{I}}+1}\widetilde{p}^{\text{II}} \cdot \phi_2(q) < 0$. So, by Lemma A.2 it follows that $\widetilde{\zeta}^{\varepsilon}$ is compact-valued, convex-valued, and upper hemi-continuous. Since $\widetilde{\zeta}^{\varepsilon}$ satisfies all conditions of Lemma A.4, the set $(\widetilde{\zeta}^{\varepsilon})^{-1}(-\mathbb{R}^L_+)$ has a component $\widetilde{Z}^{\text{c}}_{-}$ such that for every $\upsilon \in [\varepsilon, 1]$ there is $(s^{\upsilon}, q^{\upsilon}) \in \widetilde{Z}^{\text{c}}_{-}$ with $\|q^{\upsilon}\|_{\infty} = \upsilon$.

We show that $\widetilde{Z}^{\text{c}}_{-} = (\widetilde{\zeta}^{\varepsilon})^{-1}(\{0\})$. Let $(s^*, q^*) \in \widetilde{Z}^{\text{c}}_{-}$ be given. Then there is $z \in \widetilde{\zeta}^{\varepsilon}(s^*, q^*) \cap -\mathbb{R}^L_+ = \widehat{\zeta}(s^*_1, \ldots, s^*_{L^{\text{I}}}, s^*_{L^{\text{I}}+1}\widetilde{p}^{\text{II}}, q^*) \cap -\mathbb{R}^L_+$. Let $p^* \in \mathbb{R}^L_+$, $\underline{y}^{*f} \in \mathbb{R}^{L^{\text{II}}}_+$, $f \in F$, $\underline{z}^{*h} \in -\mathbb{R}^{L^{\text{II}}}_+$, $h \in H$, and $w^{*h} \in [p^* \cdot e^h, \infty)$, $h \in H$, be defined by $p^* = (s^*_1, \ldots, s^*_{L^{\text{I}}}, s^*_{L^{\text{I}}+1}\widetilde{p}^{\text{II}})$, $\underline{y}^{*f} = \phi^f_1(q^*)$, $\underline{z}^{*h} = \phi^h_2(q^*)$, and $w^{*h} = p^* \cdot e^h + \sum_{f \in F} \theta^{fh} \widehat{\pi}^f(p^*, \underline{y}^{*f})$. Then there is $x^{*h} \in \widehat{\delta}^h(p^*, \underline{z}^{*h}, w^{*h})$, $h \in H$, $y^f \in \widehat{\eta}^f(p^*, \underline{y}^{*f})$, $f \in F$, such that $\sum_{h \in H} x^{*h} - \sum_{h \in H} e^h - \sum_{f \in F} y^f = z$. Let y^{*1} be defined by $y^{*1} = y^1 + z$, and y^{*f}, $f \in F \setminus \{1\}$, by $y^{*f} = y^f$. It remains to be shown that $y^{*1} \in \widehat{\eta}^f(p^*, \underline{y}^{*f})$. Since $(x^*, y^*) \in A$, it follows by the convexity of $\preceq^h$ that $x^{*h} \in \delta^h(p^*, \underline{z}^{*h}, w^{*h})$, $h \in H$. Then non-satiation with respect to group II commodities and convexity of $\preceq^h$ implies $p^* \cdot x^{*h} = w^{*h}$, $h \in H$, and therefore $p^* \cdot z = 0$. So $p^* \cdot y^{*1} = p^* \cdot y^1$. Since there is no rationing on the demand side, it is obvious that $y^{*1} \in \widehat{s}^1(\underline{y}^{*f})$, so it holds that $y^{*1} \in \widehat{\eta}^f(p^*, \underline{y}^{*f})$.

For $n \in \mathbb{N}$, take $\varepsilon = \frac{1}{n}$ and denote the resulting component of $(\widetilde{\zeta}^{\varepsilon})^{-1}(\{0\})$ by $\widetilde{Z}^{\text{c}}_{-}(n)$. By Hildenbrand (1974), Proposition 1, page 16, the sequence $\{\widetilde{Z}^{\text{c}}_{-}(n)\}_{n \in \mathbb{N}}$ has a convergent subsequence which we also denote by $\{\widetilde{Z}^{\text{c}}_{-}(n)\}_{n \in \mathbb{N}}$. By Mas-Colell (1985), Theorem A.5.1.(ii), page 10, the closed limit of the sequence $\{\widetilde{Z}^{\text{c}}_{-}(n)\}_{n \in \mathbb{N}}$, denoted by $\overline{\widetilde{Z}}^{c}_{-}$, is connected since every $\widetilde{Z}^{\text{c}}_{-}(n)$ is connected. Since $\|q\|_{\infty} \geq \frac{1}{n}$ for every $(s, q) \in \widetilde{Z}^{\text{c}}_{-}(n)$, it holds that the set $\overline{\overline{Z}}^{\text{c}}_{-} = \overline{\widetilde{Z}}^{\text{c}}_{-} \setminus (S^{L^{\text{I}}} \times \{0\})$ is connected. For every $\upsilon \in (0, 1]$ there is $(s^{\upsilon}, q^{\upsilon}) \in \overline{\overline{Z}}^{\text{c}}_{-}$ with $\|q^{\upsilon}\|_{\infty} = \upsilon$.

Let $(\overline{s}, \overline{q})$ be an element of $\overline{\overline{Z}}^{c}_{-}$. Then there exists a sequence of points $\{(s^n, q^n)\}_{n \in \mathbb{N}}$ such that $\|q^n\|_{\infty} > 0$, $\widetilde{\zeta}^0(s^n, q^n) = 0$, and $(s^n, q^n) \to (\overline{s}, \overline{q})$. We show that $\overline{s}_{L^{\text{I}}+1} > 0$. Suppose $\overline{s}_{L^{\text{I}}+1} = 0$. Then $\widetilde{\zeta}^0(\overline{s}, \overline{q}) = \widehat{\zeta}((\overline{s}_1, \ldots, \overline{s}_{L^{\text{I}}}, 0), \overline{q})$, and since $\overline{s}_l > 0$ for some $l \in L^{\text{I}}$, it follows by upper hemi-continuity of $\widehat{\zeta}$ that $0 \in \widehat{\zeta}((\overline{s}_1, \ldots, \overline{s}_{L^{\text{I}}}, 0), \overline{q}) \subseteq \zeta((\overline{s}_1, \ldots, \overline{s}_{L^{\text{I}}}, 0), \phi_1(\overline{q}), \phi_2(\overline{q}))$. This leads to a contradiction, because the non-satiation with respect to group II commodities implies $\zeta((\overline{s}_1, \ldots, \overline{s}_{L^{\text{I}}}, 0), \phi_1(\overline{q}), \phi_2(\overline{q})) = \emptyset$. Consequently, $\overline{s}_{L^{\text{I}}+1} > 0$, for every

$(\overline{s},\overline{q}) \in \overline{\overline{Z}}^{\mathrm{c}}_{-}$.

The function $g : \overline{\overline{Z}}^{\mathrm{c}}_{-} \to \mathbb{R}^L \times Q^{L^{\mathrm{II}}}$ is defined by

$$g(s,q) = ((\frac{s_1}{s_{L^{\mathrm{I}}+1}}, \ldots, \frac{s_{L^{\mathrm{I}}}}{s_{L^{\mathrm{I}}+1}}, \widetilde{p}^{\mathrm{II}}), q), \quad (s,q) \in \overline{\overline{Z}}^{\mathrm{c}}_{-}.$$

If $(\overline{p},\overline{q}) \in g(\overline{\overline{Z}}^{\mathrm{c}}_{-})$, then there exists a sequence of points $\{(p^n,q^n)\}_{n\in\mathbb{N}}$ such that $\|q^n\|_\infty > 0$, $0 \in \widehat{\zeta}(p^n,q^n)$, and $(p^n,q^n) \to (\overline{p},\overline{q})$, and the upper hemi-continuity of $\widehat{\zeta}$ at such a point $(\overline{p},\overline{q})$ implies $0 \in \widehat{\zeta}(\overline{p},\overline{q})$. The set $\widehat{Z}^{\mathrm{c}}_0$ is defined by $\widehat{Z}^{\mathrm{c}}_0 = g(\overline{\overline{Z}}^{\mathrm{c}}_{-})$. It is immediate that $\widehat{Z}^{\mathrm{c}}_0$ is connected. The set $\widehat{E}^{\mathrm{c}}$ is defined by $\widehat{E}^{\mathrm{c}} = \widehat{\psi}(\widehat{Z}^{\mathrm{c}}_0)$. We finish the proof by showing that $\widehat{E}^{\mathrm{c}}$ is connected and that it contains a continuum of potentially different equilibria.

By Lemma A.2 and the continuity of the functions ϕ_1 and ϕ_2 it follows that $\widehat{\psi}$ is a compact-valued, convex-valued, and upper hemi-continuous correspondence. Suppose $\widehat{E}^{\mathrm{c}}$ is not connected, then there exist two disjoint, non-empty sets E^1 and E^2 such that E^1 and E^2 are both closed in $\widehat{E}^{\mathrm{c}}$ and $E^1 \cup E^2 = \widehat{E}^{\mathrm{c}}$. Therefore, by the upper hemi-continuity of $\widehat{\psi}$, it holds that $\widehat{\psi}^{-1}(E^1)$ and $\widehat{\psi}^{-1}(E^2)$ are closed in $\widehat{Z}^{\mathrm{c}}_0$. Suppose $\overline{q} \in \widehat{\psi}^{-1}(E^1) \cap \widehat{\psi}^{-1}(E^2)$. Let $\xi^1, \xi^2 \in \widehat{\psi}(\overline{q})$ be such that $\xi^1 \in E^1$ and $\xi^2 \in E^2$. Then $\lambda\xi^1 + (1-\lambda)\xi^2 \in \widehat{\psi}(\overline{q})$, $\forall \lambda \in [0,1]$, since $\widehat{\psi}(\overline{q})$ is convex, so ξ^2 is an element of the component in $\widehat{E}^{\mathrm{c}}$ containing ξ^1, a contradiction to the construction of the sets E^1 and E^2. Consequently, $\widehat{\psi}^{-1}(E^1) \cap \widehat{\psi}^{-1}(E^2) = \emptyset$. Moreover, $\widehat{\psi}^{-1}(E^1) \cup \widehat{\psi}^{-1}(E^2) = \widehat{Z}^{\mathrm{c}}_0$, while both $\widehat{\psi}^{-1}(E^1)$ and $\widehat{\psi}^{-1}(E^2)$ are closed in $\widehat{Z}^{\mathrm{c}}_0$. So $\widehat{Z}^{\mathrm{c}}_0$ is not connected, a contradiction. This concludes the proof that $\widehat{E}^{\mathrm{c}}$ is connected.

There is an underemployment equilibrium $(p^*, x^*, y^*, \underline{z}^*, \underline{y}^*) \in \widehat{E}^{\mathrm{c}}$ such that for some $l' \in L^{\mathrm{II}}$ it holds that $\min\{-\underline{z}^{*h}_{l'} \mid h \in H\} = b$. Let some $h \in H$ be given. By A3 there is $\overline{x}^h \in X^h$ such that $\overline{x}^{h,\mathrm{I}} \leq e^{h,\mathrm{I}}$, $\overline{x}^h_{l'} < e^h_{l'}$, and $\overline{x}^h_l = e^h_l$, $\forall l \in L^{\mathrm{II}} \setminus \{l'\}$. The connectedness of $\widehat{E}^{\mathrm{c}}$ implies that for every $\overline{b} \in (0, e^h_{l'} - \overline{x}^h_{l'}]$ there exists an underemployment equilibrium $(p^*, x^*, y^*, \underline{z}^*, \underline{y}^*) \in \widehat{E}^{\mathrm{c}}$ such that $\min\{-\underline{z}^{*h}_{l'} \mid h \in H\} = \overline{b}$. Now, let $\widehat{x}^h$ be the convex combination of $\overline{x}^h$ and e^h satisfying $e^h_{l'} - \widehat{x}^h_{l'} = \overline{b}$. Then $\widehat{x}^h$ belongs to the budget set of household h in the underemployment equilibrium related to $\overline{b}$, whereas $\widehat{x}^h + \varepsilon(\overline{x}^h - e^h)$ is not an element of this set for every $\varepsilon > 0$. So, underemployment equilibria associated with different values of $\overline{b}$ are potentially different. Q.E.D.

Proof of Theorem 3.1.ii

Let some $l' \in L^{\mathrm{II}}$ be given. For $e \in \mathbb{R}^{HL}_{++}$, $\mathcal{E}_{l'}(e) = ((X^h_{l'}, \preceq^h_{l'}, (e^h_l)_{l \in L^{\mathrm{I}} \cup \{l'\}})_{h\in H}, (Y^f_{l'}, (\theta^{fh})_{h\in H})_{f\in F})$ is the projection of $\mathcal{E}$ on the coordinates corresponding to the commodities in $L^{\mathrm{I}} \cup \{l'\}$, fixing the other coordinates at the values of the initial endowments or at zero. So $X^h_{l'} = \mathbb{R}^{L^{\mathrm{I}}+1}_{+}$, $\preceq^h_{l'}$ is defined by $\overline{x}^h \preceq^h_{l'} \widehat{x}^h$ for $\overline{x}^h, \widehat{x}^h \in X^h_{l'}$ if $\overline{\overline{x}}^h \preceq^h_{l'} \widehat{\widehat{x}}^h$ with $\overline{\overline{x}}^h = \overline{x}^h_l$, $l \in L^{\mathrm{I}} \cup \{l'\}$, $\widehat{\widehat{x}}^h_l = \widehat{x}^h_l$, $l \in L^{\mathrm{I}} \cup \{l'\}$, and $\overline{\overline{x}}^h_l = e^h_l$ and $\widehat{\widehat{x}}^h_l = e^h_l$ otherwise, and $Y^f_{l'} = \{(y^f_1, \ldots, y^f_{L^{\mathrm{I}}}, y^f_{l'}) \in \mathbb{R}^{L^{\mathrm{I}}+1} \mid (y^f_1, \ldots, y^f_{L^{\mathrm{I}}}, 0, y^f_{l'}, 0) \in Y^f\}$. For all $h \in H$, fix the initial endowments of commodities $l \in L^{\mathrm{II}} \setminus \{l'\}$ and denote this $H(L^{\mathrm{II}}-1)$-dimensional vector by $\overline{e}(-l')$. Similarly, the initial endowments corresponding to the commodities in $L^{\mathrm{I}} \cup \{l'\}$ are denoted by the $H(L^{\mathrm{I}}+1)$-dimensional vector $e(l')$. It can be shown as in Laroque (1978), Proposition 3.1, page 1131, and Appendix, Proposition A4, page 1152, that there is a full measure subset $\overline{\Omega}(\overline{e}(-l'))$ of $\mathbb{R}^{H(L^{\mathrm{I}}+1)}_{++}$ such that, for every $e(l') \in \overline{\Omega}(\overline{e}(-l'))$, for every competitive equilibrium of the economy $\mathcal{E}_{l'}(e(l'), \overline{e}(-l'))$, there is trade in the market for every commodity $l \in L^{\mathrm{I}} \cup \{l'\}$. It follows by a standard argument that $\overline{\Omega}_{l'}$, the set of initial endowments $e \in \mathbb{R}^{HL}_{++}$ for which in every competitive equilibrium of the resulting economy $\mathcal{E}_{l'}(e)$ there is non-zero trade in the market for every commodity in $L^{\mathrm{I}} \cup \{l'\}$, is open. Moreover, $\cup_{\overline{e}(-l') \in \mathbb{R}^{H(L^{\mathrm{II}}-1)}} \{(e(l'), \overline{e}(-l')) \mid e(l') \in \overline{\Omega}(\overline{e}(-l'))\} \subseteq \overline{\Omega}_{l'}$. Therefore, $\overline{\Omega}_{l'}$ is an open set of full measure, and $\overline{\Omega} = \cap_{l' \in L^{\mathrm{II}}} \overline{\Omega}_{l'}$ is an open set of full measure.

Let $e \in \overline{\Omega}$ be given and let $\widehat{E}^{\mathrm{c}}$ be a component of the set of underemployment equilibria of $\mathcal{E} = ((X^h, \preceq^h, e^h)_{h \in H}, (Y^f, (\theta^{fh})_{h \in H})_{f \in F}, \tilde{p}^{\mathrm{II}}, \alpha, \beta)$ which includes an underemployment equilibrium with $\max_{l \in L^{\mathrm{II}}} \upsilon_l = \upsilon$ for all $\upsilon \in (0, 1]$. By Theorem 3.1.i such a component exists.

Suppose there are not two strongly different underemployment equilibria. For every $\upsilon \in (0, 1]$ there is an underemployment equilibrium in $\widehat{E}^{\mathrm{c}}$ with $\max_{l \in L^{\mathrm{II}}} \upsilon_l = \upsilon$, allocation $(x(\upsilon), y(\upsilon))$, where $x^h(\upsilon) \sim^h x^h(1)$, $x^{h,\mathrm{II}}(\upsilon) - e^{h,\mathrm{II}} \geq \phi^h_1(q(\upsilon))$, and $y^{f,\mathrm{II}}(\upsilon) \leq \phi^f_2(q(\upsilon))$, with $\|q(\upsilon)\|_\infty = \upsilon$. The allocation $(x(0), y(0))$ is defined as a limit point of the sequence $(x(1/n), y(1/n))_{n \in \mathbb{N}}$. It follows by market equilibrium, $x^{h,\mathrm{II}}(\upsilon) - e^{h,\mathrm{II}} \geq \phi^h_1(q(\upsilon))$, $\upsilon \in (0, 1]$, and $y^{f,\mathrm{II}}(\upsilon) \leq \phi^f_2(q(\upsilon))$, $\upsilon \in (0, 1]$, that $x^{h,\mathrm{II}}(0) = e^{h,\mathrm{II}}$ and $y^{f,\mathrm{II}}(0) = 0$. By the closedness of A it follows that $(x(0), y(0)) \in A$.

We show that there is $h' \in H$ such that $x^{h'}(1) \neq x^{h'}(0)$ or there is $f' \in F$ such that $y^{f'}(1) \neq y^{f'}(0)$. Suppose, on the contrary, that $x^h(1) = x^h(0)$ for all $h \in H$ and $y^f(1) = y^f(0)$ for all $f \in F$. Let $l' \in L^{\mathrm{II}}$ be such that there is no rationing in the market for commodity l' in some underemployment equilibrium in $\widehat{E}^{\mathrm{c}}$. Then it follows that $((p_l(1))_{l \in L^{\mathrm{I}} \cup \{l'\}}, (x_l(1))_{l \in L^{\mathrm{I}} \cup \{l'\}}, (y_l(1))_{l \in L^{\mathrm{I}} \cup \{l'\}})$ is a competitive equilibrium for the economy $\mathcal{E}_{l'}(e)$. Since $e \in \overline{\Omega}$, there is non-zero trade in the market for commodity l', a contradiction. Consequently, there is $h' \in H$ such that $x^{h'}(1) \neq x^{h'}(0)$ or there is $f' \in F$ such that $y^{f'}(1) \neq y^{f'}(0)$.

Now consider the truncated economy $\overline{\mathcal{E}} = ((\overline{X}^h, \preceq^h, e^h)_{h \in H}, (\overline{Y}^f, (\theta^{fh})_{h \in H})_{f \in F})$, where $\overline{X}^h = \{x^h \in X^h \mid x^{h,\mathrm{II}} - e^{h,\mathrm{II}} \geq \phi^h_1(q(1))\}$ and $\overline{Y}^f = \{y^f \in Y^f \mid y^{f,\mathrm{II}} \leq \phi^f_2(q(1))\}$. Clearly, $(p(1), x(1), y(1))$ is a competitive equilibrium for $\overline{\mathcal{E}}$ and therefore $(x(1), y(1))$ is a Pareto optimal allocation in $\overline{\mathcal{E}}$. However, for every $\lambda \in (0, 1)$, $(\lambda x(0) + (1-\lambda)x(1), \lambda y(0) + (1-\lambda)y(1))$ is a feasible allocation (using that trivially $x^{h,\mathrm{II}}(0) - e^{h,\mathrm{II}} \geq \phi^h_1(q(1))$ and $y^{f,\mathrm{II}}(0) \leq \phi^f_2(q(1))$) for $\overline{\mathcal{E}}$ that satisfies $\lambda x^h(0) + (1-\lambda)x^h(1) \succeq^h x^h(1)$ for all $h \in H$. Moreover, $\lambda x^{h'}(0) + (1-\lambda)x^{h'}(1) \succ^{h'} x^{h'}(1)$ or $\lambda y^{f'}(0) + (1-\lambda)y^{f'}(1)$ in the interior of $Y^{f'}$, contradicting the Pareto optimality of the allocation $(x(1), y(1))$ in $\overline{\mathcal{E}}$. Consequently, there are two strongly different underemployment equilibria in $\widehat{E}^{\mathrm{c}}$, and, by the connectedness of $\widehat{E}^{\mathrm{c}}$, there is a continuum of strongly different underemployment equilibria in $\widehat{E}^{\mathrm{c}}$. Q.E.D.

We generalize the assumptions of Theorem 3.1.iii. To avoid unnecessary technicalities, we consider the case where $\widehat{\zeta}$ is a function, denoted by $\widehat{z}$. We parametrize relevant price systems and expectations of available opportunities by means of a vector $q \in Q^L$. The first L^{I} components of q are used to parametrize the prices of the first L^{I} commodities, and the last L^{II} components to parametrize the expected opportunities for the group II commodities. Let $p^* \gg 0$ be a competitive price system for the economy $\mathcal{E}$. The function $\overline{p} : Q^L \to \mathbb{R}^L$ is defined by $\overline{p}_l(q) = p^*_l q_l$ if $l \in L^{\mathrm{I}}$, and $\overline{p}_l(q) = p^*_l$ if $l \in L^{\mathrm{II}}$.

The function $\overline{z} : Q^L \to \mathbb{R}^L$ is defined by

$$\overline{z}(q) = \widehat{z}(\overline{p}(q), q^{\mathrm{II}}), \quad q \in Q^L.$$

Notice that $\overline{p}(q)$ depends on q^{I} only. Let B^L denote the boundary of Q^L where all components are positive and at least one is equal to 1, so $B^L = \{q \in Q^L \mid \exists l \in L,\ q_l = 1$ and $q \gg 0\}$. We say that $\overline{z}$ satisfies the boundary condition if

$$\forall q \in B^L,\ \overline{z}(q) = 0 \text{ or } \exists l' \in L \text{ such that } q_{l'} > \min_{l \in L} q_l \text{ and } \overline{z}_{l'}(q) < \max_{l \in L} \overline{z}_l(q). \tag{2}$$

We prove Theorem 3.1.iii with A7 replaced by the weaker A7' below[5].

A7'. For at least one Walrasian equilibrium $(p^*, x^*, y^*, \underline{z}^*, \underline{y}^*)$ of $\mathcal{E}$ the function $\overline{z}$ satisfies Condition (2).

Proof of Theorem 3.1.iii

Let some $\varepsilon > 0$ be given. First we show the existence of a connected set $\overline{Z}^{\mathrm{c}}_{-}$ such that for every $\lambda \in [\varepsilon, 1]$ there is $q \in \overline{Z}^{\mathrm{c}}_{-}$ inducing an underemployment equilibrium with $\sum_{l \in L} q_l = \lambda L$.

We extend $\overline{z}$ to a subset of the set $R = \{r \in \mathbb{R}^L \mid \varepsilon \le \sum_{l \in L} r_l \le L\}$. Let $\rho : R \to Q^L$ be the projection function that projects r on the set $\{q \in Q^L \mid \sum_{l \in L} q_l = \sum_{l \in L} r_l\}$ by minimizing the Euclidean distance to this set. Let the continuous, compact-valued correspondence $\varphi : R \to Q^L$ be defined by $\varphi(r) = \{q \in Q^L \mid \sum_{l \in L} q_l = \sum_{l \in L} r_l\}$ and the continuous function $g : R \times Q^L \to \mathbb{R}$ by $g(r, q) = -\sum_{l \in L}(r_l - q_l)^2$. Then the correspondence that assigns to $r \in R$ the set of points $\overline{q} \in \varphi(r)$ maximizing $g(r, q)$ on $\varphi(r)$ is an upper hemi-continuous, compact-valued correspondence by the maximum theorem. Since $\varphi(r)$ is convex for every $r \in R$ it follows that there is a unique maximizer. It is clear that

[5] A7 leads to the following property: $\forall q \in B^L$, if $q_{l'} = 1$, then $\overline{z}_{l'}(q) \le 0$. Let some $q \in B^L$ be given. If $\overline{z}(q) = 0$, then Condition (2) is satisfied. If $\overline{z}(q) \neq 0$, then q is not the vector of all ones. Let $l' \in L$ be such that $q_{l'} = 1$. Then $q_{l'} > \min_{l \in L} q_l$, and $\overline{z}_{l'}(q) \le 0 < \max_{l \in L} \overline{z}_l(q)$.

the correspondence coincides with ρ, so ρ is a continuous function. Using the first-order conditions it follows that if $\rho(r) = q$, then either $\sum_{l \in L} r_l = L$ and $\rho(r) = 1$ or $\sum_{l \in L} r_l < L$ and there is $\lambda \in \mathbb{R}$, $\mu_l \geq 0$, $l \in L$, $\nu_l \geq 0$, $l \in L$, such that, for every $l \in L$, $q_l = r_l - \lambda + \mu_l - \nu_l$, $\mu_l q_l = 0$ and $\nu_l(q_l - 1) = 0$.

The set Δ is defined by $\Delta = \{\delta \in \mathbb{R}^L \mid \sum_{l \in L} \delta_l = 0 \text{ and } \delta_l \geq -1, \forall l \in L\}$. Then $\delta + \lambda \mathbf{1} \in R$ for every $\delta \in \Delta$ and $\lambda \in [\varepsilon, 1]$, with $\mathbf{1}$ the vector of all ones. The continuous function $\varphi^1 : \Delta \times [\varepsilon, 1] \to \mathbb{R}^L$ is defined by $\varphi^1(\delta, \lambda) = \overline{z}(\rho(\delta + \lambda 1))$. Since φ^1 is a continuous function, the set $\varphi^1(\Delta \times [\varepsilon, 1])$ is compact, and therefore there exists a compact, convex set Z satisfying $\varphi^1(\Delta \times [\varepsilon, 1]) \subseteq Z$. The compact-valued, convex-valued, upper hemi-continuous correspondence $\varphi^2 : Z \to \Delta$ is defined by

$$\varphi^2(z) = \left\{ \overline{\delta} \in \Delta \mid \sum_{l \in L} \overline{\delta}_l z_l \geq \sum_{l \in L} \delta_l z_l, \ \forall \delta \in \Delta \right\}, \quad z \in Z.$$

It follows that the correspondence $\varphi : Z \times \Delta \times [\varepsilon, 1] \to Z \times \Delta$ defined by

$$\varphi(z, \delta, \lambda) = \varphi^1(\delta, \lambda) \times \varphi^2(z), \quad (z, \delta, \lambda) \in Z \times \Delta \times [\varepsilon, 1],$$

is a compact-valued, convex-valued, and upper hemi-continuous correspondence, and the set $Z \times \Delta$ is non-empty, compact, and convex. By Theorem A.3 it follows that the set $F_\varphi = \{(z, \delta, \lambda) \in Z \times \Delta \times [\varepsilon, 1] \mid (z, \delta) \in \varphi(z, \delta, \lambda)\}$ contains a component F_φ^{c} such that $(Z \times \Delta \times \{\varepsilon\}) \cap F_\varphi^{\mathrm{c}} \neq \emptyset$ and $(Z \times \Delta \times \{1\}) \cap F_\varphi^{\mathrm{c}} \neq \emptyset$. The connectedness of F_φ^{c} therefore yields that, for every $v \in [\varepsilon, 1]$, $(Z \times \Delta \times \{v\}) \cap F_\varphi^{\mathrm{c}} \neq \emptyset$. Let some $(z^*, \delta^*, \lambda^*) \in F_\varphi^{\mathrm{c}}$ be given. So, $(z^*, \delta^*, \lambda^*) \in \varphi^1(\delta^*, \lambda^*) \times \varphi^2(z^*)$. Let us define $q^* = \rho(\delta^* + \lambda^* \mathbf{1})$ and $p^* = \overline{p}(q^*)$.

Suppose $\max_{l \in L} z_l^* > 0$. There is $l^1 \in L$ such that $z_{l^1}^* = \max_{l \in L} z_l^*$ and $p_{l^1}^* > 0$. Otherwise, $\delta^* \in \varphi^2(z^*)$ implies $\delta_l^* = -1$ for all $l \in L$ with $p_l^* > 0$, and hence $q_l^* \leq q_{l^1}^* = 0$ where $p_{l^1}^* = 0$, so $\sum_{l \in L} q_l^* = 0$, a contradiction. Then, since $p^* \cdot z^* \leq 0$, there is $l^2 \in L$ such that $z_{l^2}^* < 0$ and $p_{l^2}^* > 0$. This implies $\delta_{l^2}^* = -1$. It follows that $q^* \gg 0$, since $q_l^* = 0$ for some $l \in L$ implies that $q_{l^2}^* = 0$, so $l^2 \in L^{\mathrm{II}}$, and $z_{l^2}^* \geq 0$, which gives a contradiction. Without loss of generality we can assume that $\delta_{l^1}^* > 0$. Using that $\delta_{l^1}^* > 0$, $\delta_{l^2}^* = -1$ and $q^* \gg 0$, it follows from the first-order conditions for the projection that $q_{l^1}^* = 1$. Moreover, for every $l' \in L$, if $z_{l'}^* < \max_{l \in L} z_l^*$, then $\delta_{l'}^* = -1$, so $q_{l'}^* = \min_{l \in L} q_l^*$. This contradicts A7', unless $q^* = \mathbf{1}$. Consequently, $q^* = \mathbf{1}$ or $\max_{l \in L} z_l^* \leq 0$.

Since p^* is Walrasian it holds that $\overline{z}(\mathbf{1}) = 0$. The function $g : Z \times \Delta \times [\varepsilon, 1] \to Q^L$ is defined by

$$g(z, \delta, \lambda) = \delta + \lambda \mathbf{1}, \quad (z, \delta, \lambda) \in Z \times \Delta \times [\varepsilon, 1],$$

and the set $\overline{Z}_-^{\mathrm{c}}$ is defined by $\overline{Z}_-^{\mathrm{c}} = g(F_\varphi^{\mathrm{c}})$. We have shown that for every $q \in \overline{Z}_-^{\mathrm{c}}$, $\overline{z}(q) \in -\mathbb{R}_+^L$. As in the proof of Theorem 3.1.i it follows that $\overline{z}(q) = 0$. The set $\overline{Z}_-^{\mathrm{c}}$ is connected by the connectedness of F_φ^{c} and the continuity of g. For every $\lambda \in [\varepsilon, 1]$, there exists $(z^\lambda, \delta^\lambda, \lambda) \in F_\varphi^{\mathrm{c}}$, so $g(z^\lambda, \delta^\lambda, \lambda) = (\delta^\lambda + \lambda \mathbf{1}) = q^\lambda \in \overline{Z}_-^{\mathrm{c}}$. Obviously, $\sum_{l \in L} q_l^\lambda = \lambda L$.

For $n \in \mathbb{N}$, take $\varepsilon = \frac{1}{n}$ and denote the resulting component of $\{q \in Q^L \mid \sum_{l \in L} q_l \geq \varepsilon$ and $\overline{z}(q) = 0\}$ that contains $\mathbf{1}$ by $\overline{Z}_0^{\mathrm{c}}(n)$. Obviously, $\overline{Z}_0^{\mathrm{c}}(n^1) \subset \overline{Z}_0^{\mathrm{c}}(n^2)$ if $n^1 < n^2$. By Mas-Colell (1985), Theorem A.5.1.(ii), page 10, the closed limit of the sequence $\{\overline{Z}_0^{\mathrm{c}}(n)\}_{n \in \mathbb{N}}$, denoted by $\overline{Z}_0^{\mathrm{c}}$, is connected. For every $\lambda \in (0, 1]$ it holds that there is $q^\lambda \in \overline{Z}_0^{\mathrm{c}}$ with $\sum_{l \in L} q_l^\lambda = \lambda L$, and by continuity of $\overline{z}$ at any such point, it follows that $\overline{z}(q^\lambda) = 0$. Moreover, since for every $\lambda \in (0, 1]$ there is $q^\lambda \in \overline{Z}_0^{\mathrm{c}}$ with $\sum_{l \in L} q_l^\lambda = \lambda L$ it holds that for every $\upsilon \in (0, 1]$ there is $\overline{q}^\upsilon \in \overline{Z}_0^{\mathrm{c}}$ with $\max_{l \in L^{\mathrm{II}}} \overline{q}_l^\upsilon = \upsilon$, and for every $\upsilon \in (0, 1]$ there is $\widehat{q}^\upsilon \in \overline{Z}_0^{\mathrm{c}}$ with $\min_{l \in L^{\mathrm{II}}} \widehat{q}_l^\upsilon = \upsilon$. Let the set of underemployment equilibria $\widehat{E}^{\mathrm{c}}$ be defined by

$$\widehat{E}^{\mathrm{c}} = \widehat{\psi}(\{(\overline{p}(q), q^{\mathrm{II}}) \in \mathbb{R}_+^L \times Q^{L^{\mathrm{II}}} \mid q \in \overline{Z}_0^{\mathrm{c}} \setminus \{0\}\})$$

As in the proof of Theorem 3.1.i it follows that $\widehat{E}^{\mathrm{c}}$ is connected, whereas the properties given above imply that for every $\upsilon \in (0, 1]$ there is an underemployment equilibrium in $\widehat{E}^{\mathrm{c}}$ with $\max_{l \in L^{\mathrm{II}}} \upsilon_l = \upsilon$ and for every $\upsilon \in (0, 1]$ there is an underemployment equilibrium in $\widehat{E}^{\mathrm{c}}$ with $\min_{l \in L^{\mathrm{II}}} \upsilon_l = \upsilon$. The set $\widehat{E}^{\mathrm{c}}$ ranges from an approximate no-trade equilibrium at prices $p \leq p^*$ to the competitive equilibrium $(p^*, x^*, y^*, \underline{z}^*, \underline{y}^*)$. Q.E.D.

Proof of Theorem 3.2

By Theorem 3.1.i, $\widehat{E}$ has a component $\widehat{E}^{\mathrm{c}}$ which includes an underemployment equilibrium with $\max_{l \in L^{\mathrm{II}}} \upsilon_l = \upsilon$ for all $\upsilon \in (0, 1]$. If there are two different underemployment equilibria in $\widehat{E}^{\mathrm{c}}$, then there is a continuum of different underemployment equilibria in $\widehat{E}^{\mathrm{c}}$ by the connectedness of $\widehat{E}^{\mathrm{c}}$.

Suppose there are not two different underemployment equilibria in $\widehat{E}^{\mathrm{c}}$. Then, for every $\upsilon \in (0, 1]$ there is an underemployment equilibrium in $\widehat{E}^{\mathrm{c}}$ with $\max_{l \in L^{\mathrm{II}}} \upsilon_l = \upsilon$ and allocation $(x(\upsilon), y(\upsilon))$, where $x(\upsilon) = x(1)$, $x^h(\upsilon) - e^h \geq \phi_1^h(q(\upsilon))$, $h \in H$, and $y^f(\upsilon) \leq \phi_2^f(q(\upsilon))$, $f \in F$, with $\|q(\upsilon)\|_\infty = \upsilon$. Now, for every $\upsilon \in (0, 1]$, $x^h(1) - e^h \geq \phi_1^h(q(\upsilon))$, implying that $x^h(1) \geq e^h$, $h \in H$. Moreover, for every $\upsilon \in (0, 1]$, $\sum_{h \in H} x^h(1) = \sum_{h \in H} e^h + \sum_{f \in F} y^f(\upsilon) \leq \sum_{h \in H} e^h + \sum_{f \in F} \phi_2^f(q(\upsilon))$, implying that $x^h(1) = e^h$, $h \in H$, and $\sum_{f \in F} y^f(\upsilon) = 0$, $\forall \upsilon \in (0, 1]$.

Suppose there is $f' \in F$ such that $y^{f'}(1) \neq 0$. By choosing $y^f = 0$, $f \in F \setminus \{f'\}$, it follows that $y^{f'}(1) + \sum_{f \in F \setminus \{f'\}} y^f = y^{f'}(1) \in Y$, and by choosing $y^{f'} = 0$ it follows that $\sum_{f \in F \setminus \{f'\}} y^f(1) + y^{f'} = -y^{f'}(1) \in Y$. So, $0 \neq y^{f'}(1) \in Y \cap -Y \subseteq \{0\}$, a contradiction. Consequently, $y^f(1) = 0$, $f \in F$.

Let $l' \in L$ be such that there is no rationing in the market for commodity l' at the underemployment equilibrium $(\widetilde{p}^{\mathrm{II}}, x(1), y(1), \underline{z}(1), \underline{y}(1))$. There is $h \in H$ such that $e^h \notin \delta^h(\widetilde{p}^{\mathrm{II}}, 0_{-l'}, \widetilde{p}^{\mathrm{II}} \cdot e^h)$ or there is $f \in F$ such that $0 \notin \eta^f(\widetilde{p}^{\mathrm{II}}, 0_{-l'})$. In the latter case there is $\overline{y}^f \in s^f(0_{-l'})$ such that $p \cdot \overline{y}^f > 0$. The convex combination $\lambda \overline{y}^f + (1 - \lambda) y^f(1) = \lambda \overline{y}^f$ belongs to $s^f(\underline{y}^f(1))$ for λ sufficiently small since $\underline{y}_{l'}^f(1) \geq b$, while $\widetilde{p}^{\mathrm{II}} \cdot \lambda \overline{y}^f > 0$, a contradiction to $\widetilde{p}^{\mathrm{II}} \cdot y^f(1) = \widetilde{p}^{\mathrm{II}} \cdot 0 = 0$. In the former case there is $\overline{x}^h \in \gamma^h(\widetilde{p}^{\mathrm{II}}, 0_{-l'}, \widetilde{p}^{\mathrm{II}} \cdot e^h)$ such that $\overline{x}^h \succ^h e^h$.

Since $\underline{z}^h_{l'}(1) \leq -b \leq -e^h_{l'}$ it follows that $\gamma^h(\widetilde{p}^{\mathrm{II}}, 0_{-l'}, \widetilde{p}^{\mathrm{II}} \cdot e^h) \subseteq \gamma^h(\widetilde{p}^{\mathrm{II}}, \underline{z}^h(1), \widetilde{p}^{\mathrm{II}} \cdot e^h) = \gamma^h(\widetilde{p}^{\mathrm{II}}, \underline{z}^h(1), \widetilde{p}^{\mathrm{II}} \cdot e^h + \sum_{f \in F} \theta^{fh} \widetilde{p}^{\mathrm{II}} \cdot y^f(1))$. This leads to a contradiction with $x^h(1) = e^h$. Consequently, the hypothesis that there are not two different underemployment equilibria in $\widehat{E}^{\mathrm{c}}$ is false, and there is a continuum of different underemployment equilibria in $\widehat{E}^{\mathrm{c}}$.

The existence of a continuum of strongly different underemployment equilibria in $\widehat{E}^{\mathrm{c}}$ follows immediately if there is $h \in H$ such that $e^h \notin \delta^h(\widetilde{p}^{\mathrm{II}}, 0_{-l'}, \widetilde{p}^{\mathrm{II}} \cdot e^h)$ since $\gamma^h(\widetilde{p}^{\mathrm{II}}, 0_{-l'}, \widetilde{p}^{\mathrm{II}} \cdot e^h) \subseteq \gamma^h(\widetilde{p}^{\mathrm{II}}, \underline{z}^h(1), \widetilde{p}^{\mathrm{II}} \cdot e^h + \sum_{f \in F} \theta^{fh} \widetilde{p}^{\mathrm{II}} \cdot y^f(1))$, so $e^h \notin \delta^h(\widetilde{p}^{\mathrm{II}}, \underline{z}^h(1), \widetilde{p}^{\mathrm{II}} \cdot e^h + \sum_{f \in F} \theta^{fh} \widetilde{p}^{\mathrm{II}} \cdot y^f(1))$ and $e^h \prec^h x^h(1)$. If such a household h does not exist, then by assumption there is $f \in F$ such that $0 \notin \eta^f(\widetilde{p}^{\mathrm{II}}, 0_{-l'})$. It follows that $0 \notin \eta^f(\widetilde{p}^{\mathrm{II}}, \underline{y}^f(1))$, and $\pi^f(\widetilde{p}^{\mathrm{II}}, \underline{y}^f(1)) > 0$. Let $h \in H$ be such that $\theta^{fh} > 0$. Then there is an open neighbourhood O of e^h such that $\widetilde{p}^{\mathrm{II}} \cdot x^h < \widetilde{p}^{\mathrm{II}} \cdot e^h + \sum_{f \in F} \theta^{fh} \pi^f(\widetilde{p}^{\mathrm{II}}, \underline{y}^f(1))$, $\forall x^h \in O$, and by non-satiation with respect to group II commodities at the initial endowment there is $\overline{x}^h \in O \cap \widehat{X}^h$ such that $e^h < x^h$ and $e^h \prec^h x^h$. Clearly, $\overline{x}^h \in \gamma^h(\widetilde{p}^{\mathrm{II}}, \underline{z}^h(1), \widetilde{p}^{\mathrm{II}} \cdot e^h + \sum_{f \in F} \theta^{fh} \pi^f(\widetilde{p}^{\mathrm{II}}, \underline{y}^f(1)))$, so $x^h(1) \succ e^h$, and it follows that there is a continuum of strongly different underemployment equilibria.

Q.E.D.

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