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Potential maximizers and network formation^a

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Abstract

In this paper we study the formation of cooperation structures in superadditive cooperative TU-games. Cooperation structures are represented by hypergraphs. The formation process is modelled as a game in strategic form, where the payoffs are determined according to a weighted (extended) Myerson value. This class of solution concepts turns out to be the unique class resulting in weighted potential games. The argmax set of the weighted potential predicts the formation of the complete structure and structures payoff-equivalent to the complete structure. As by-products we obtain a representation theorem of weighted potential games in terms of weighted Shapley values and a characterization of the weighted (extended) Myerson values.

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1 Introduction

Several recent papers have modelled the distribution of payoffs in a cooperative game as a two-stage procedure. In the first stage, players decide on the extent and nature of cooperation with other players. During this period, players cannot enter into binding agreements of any kind, either on the nature of cooperation or on the subsequent division of payoffs. In the second period, the payoffs are given by an exogenously given allocation rule.

Perhaps, the first paper in this area was *Hart and Kurz* (1983). They analyze a situation where the first stage consists of a game where strategies of the players are announcements of players with whom a particular player wants to form a coalition. *Dutta, Nouweland and Tijs* (1996) and *Qin* (1996) focus attention on *Myerson's* (1977) *cooperation structures* rather than coalition structures. A cooperation structure is a graph whose vertices are identified with the players. A link between two players means that the players can carry on meaningful and direct negotiations with each other. These authors use a strategic game suggested by *Myerson* (1991), where each player announces the set of players with whom he or she wants to form a link. Then a link is formed between players i and j if and only if both i and j want to form a link with each other. *Dutta et al.* (1996) show that for a large class of allocation rules, the complete graph is the unique (up to payoff equivalence) undominated Nash equilibrium or coalition-proof Nash equilibrium.¹

In this paper, we consider the related problem of formation of *hypergraphs*. In a hypergraph the vertices are identified with the players and each edge represents a *conference*, that is a subset of players. Direct negotiations between players can only take place within conferences.² Note that since a graph can be viewed as a special hypergraph in which *each* conference is a pair of players, the model in the present paper is a generalization of the model of *Myerson* (1991). Our primary focus of interest is to see which hypergraphs

¹*Aumann and Myerson* (1988) consider an alternative formulation where in the first stage, cooperation structures form through a sequential process.

²See *Myerson* (1980) who proposed the use of hypergraphs to model cooperation possibilities between players.

turn out to result from *potential maximizing strategies*.³ We show that the complete hypergraph results from a potential maximizing strategy. Conversely, every potential maximizing strategy results in a cooperation structure which is payoff-equivalent to the complete hypergraph.

In the process of proving the main result, we also show that under an efficiency requirement, the only allocation rule which results in a conference formation game being a weighted potential game is the weighted (extended) Myerson value. We also find a representation theorem of weighted potential games in terms of weighted Shapley values and a characterization of the class of weighted (extended) Myerson values.

The plan of this paper is as follows. In section 2 we will show that a non-cooperative game is a weighted potential game if and only if its payoff function coincides with weighted Shapley values of particular cooperative games indexed by the set of strategy profiles. Section 3 deals with hypergraph communication situations and provides an axiomatic characterization of the class of weighted (extended) Myerson values using w -fairness and component efficiency. Conference formation games are discussed in section 4 and it is shown that the only solution concepts resulting in a weighted potential game are the weighted (extended) Myerson values. In section 5 we show that the argmax set of the weighted potential corresponds to the full cooperation structure and payoff-equivalent structures. We conclude in section 6.

2 Potential games

In *Ui* (1996) a representation theorem for potential games is given in terms of the Shapley value. In this section we will extend the result of *Ui* (1996) and provide a representation theorem for weighted potential games in terms of weighted Shapley values. We will first give some definitions.

A game in strategic form will be denoted by $\Gamma = (N; (S_i)_{i \in N}; (\pi_i)_{i \in N})$, where $N = \{1, \dots, n\}$ denotes the player set, S_i the strategy space of player $i \in N$, and $\pi = (\pi_i)_{i \in N}$ the payoff function which assigns to every strategy-tuple $s = (s_i)_{i \in N} \in \prod_{i \in N} S_i = S$ a

³See *Monderer and Shapley* (1996) who prove various properties of the class of weighted potential games. See also *Rosenthal* (1982).

vector in \mathbb{R}^N . For notational convenience we write $s_{-i} = (s_j)_{j \in N \setminus \{i\}}$ and $s_R = (s_i)_{i \in R}$.

Monderer and *Shapley* (1996) formally defined the class of weighted potential games. Let $w = (w_i)_{i \in N} \in \mathbb{R}_{++}^N$ be a vector of positive weights. A function $Q^w : \prod_{i \in N} S_i \rightarrow \mathbb{R}$ is called a w -potential for Γ if for every $i \in N$, every $s \in S$, and every $t_i \in S_i$ it holds that

$$\pi_i(s_i, s_{-i}) - \pi_i(t_i, s_{-i}) = w_i(Q^w(s_i, s_{-i}) - Q^w(t_i, s_{-i})). \quad (1)$$

The game Γ is called a w -potential game if it admits a w -potential. Γ is called a weighted potential game if Γ is a w -potential game for some weights $w \in \mathbb{R}_{++}^N$.

Monderer and *Shapley* (1996) point out that the argmax set of a weighted potential game does not depend on a particular choice of a weighted potential, and hence can be used as an equilibrium refinement. They also remark that this refinement is supported by some experimental results.⁴

The representation theorem in this section is in terms of cooperative games and weighted Shapley values. A cooperative game is an ordered pair (N, v) , where $N = \{1, \dots, n\}$ is the set of players, and v is a real-valued function on the family 2^N of all subsets of N with $v(\emptyset) = 0$. Denote the set of all cooperative games with player set N by TU^N .

Weighted Shapley values can easily be defined using unanimity games. For every $R \subseteq N$ the unanimity game (N, u_R) is defined by⁵

$$u_R(T) = \begin{cases} 1 & , \text{ if } R \subseteq T \\ 0 & , \text{ otherwise} \end{cases}. \quad (2)$$

Unanimity games were introduced by *Shapley* (1953). He showed that every cooperative game can be written as a linear combination of unanimity games in a unique way, $v = \sum_{R \subseteq N} \alpha_R u_R$, where $(\alpha_R)_{R \subseteq N}$ are called the unanimity coordinates of (N, v) .

Let $w = (w_i)_{i \in N} \in \mathbb{R}_{++}^N$ be a vector of positive weights. For all $R \subseteq N$ define $w_R := \sum_{i \in R} w_i$. The weighted Shapley value Φ^w of a cooperative game $(N, v) \in TU^N$ with unanimity coordinates $(\alpha_R)_{R \subseteq N}$ is then defined by

$$\Phi_i^w(N, v) = \sum_{R \subseteq N, i \in R} \frac{w_i}{w_R} \alpha_R. \quad (3)$$

⁴*Monderer* and *Shapley* (1996) point out that this may be a mere coincidence. See also *Van Huyck et al.* (1990) and *Crawford* (1991).

⁵ $R \subseteq T$ denotes that R is a subset of T , $R \subset T$ denotes that R is a strict subset of T .

To represent weighted potential games in terms of weighted Shapley values we need the following interaction between cooperative and non-cooperative games.⁶

Consider a player set $N = \{1, \dots, n\}$ and strategy space $S = \prod_{i \in N} S_i$. Assume that once the players have chosen a strategy profile $s \in S$ they face the cooperative game (N, v_s) . Furthermore, assume that the players have made a pre-play agreement on the allocation rule that determines their payoffs for any chosen cooperative game. If the players have agreed on allocation rule γ this implies that player i obtains $\gamma_i(N, v_s)$ if strategy profile $s \in S$ is played.

We will restrict ourselves to collections of cooperative games where the value of a coalition does not depend on the strategies of the players outside this coalition, $v_s(R)$ only depends on s_R . This means that we restrict ourselves to the following set of collections of cooperative games:

$$\mathcal{G}_{N,S} := \left\{ \{(N, v_s)\}_{s \in S} \in (TU^N)^S \mid v_s(R) = v_t(R) \text{ if } s_R = t_R \text{ for all } s, t \in S, R \subseteq N \right\}. \quad (4)$$

Denote the unanimity coordinates of the game v_s by $(\alpha_R^s)_{R \subseteq N}$. It can be shown that the condition in definition (4) can be rewritten in terms of these unanimity coordinates,

$$\mathcal{G}_{N,S} = \left\{ \{(N, v_s)\}_{s \in S} \in (TU^N)^S \mid \alpha_R^s = \alpha_R^t \text{ if } s_R = t_R \text{ for all } s, t \in S, R \subseteq N \right\}. \quad (5)$$

We can now state the main result of this section, which states that the class of weighted potential games can be represented in terms of weighted Shapley values.⁷

Theorem 2.1 Let $\Gamma = (N; (S_i)_{i \in N}; (\pi_i)_{i \in N})$ be a game in strategic form and $w \in \mathbf{R}_{++}^N$. Γ is a w -potential game if and only if there exists $\{(N, v_s)\}_{s \in S} \in \mathcal{G}_{N,S}$ such that

$$\pi_i(s) = \Phi_i^w(v_s), \text{ for all } i \in N \text{ and all } s \in S. \quad (6)$$

Proof: First we will prove the if-part of the theorem. Assume there exists $\{(N, v_s)\}_{s \in S} \in \mathcal{G}_{N,S}$ with $\pi_i(s) = \Phi_i^w(v_s)$, for all $i \in N$ and $s \in S$. Define

$$Q^w(s) := \sum_{R \subseteq N, R \neq \emptyset} \frac{\alpha_R^s}{w_R}. \quad (7)$$

⁶Note that we consider weighted potentials for *non-cooperative* games, as opposed to *Hart and Mas-Colell* (1989) who characterize weighted Shapley values using weighted potentials for *cooperative* games.

⁷If there is no ambiguity about the underlying player set we will simply write $\Phi^w(v)$ instead of $\Phi^w(N, v)$.

We will show that Q^w is a w -potential of Γ . Let $i \in N$, $s \in S$, and $t_i \in S_i$, then

$$\begin{aligned}
\pi_i(s) - \pi_i(t_i, s_{-i}) &= \Phi_i^w(v_s) - \Phi_i^w(v_{(t_i, s_{-i})}) \\
&= w_i \sum_{R \subseteq N, i \in R} \frac{\alpha_R^s}{w_R} - w_i \sum_{R \subseteq N, i \in R} \frac{\alpha_R^{(t_i, s_{-i})}}{w_R} \\
&= w_i \sum_{R \subseteq N, R \neq \emptyset} \frac{\alpha_R^s}{w_R} - w_i \sum_{R \subseteq N, R \neq \emptyset} \frac{\alpha_R^{(t_i, s_{-i})}}{w_R} \\
&= w_i (Q^w(s) - Q^w(t_i, s_{-i})),
\end{aligned}$$

where the third equality follows from (5).

To prove the only-if-part assume Γ is a w -potential game, with potential Q^w . Define for all $s \in S$ and all $R \subseteq N$

$$\alpha_R^s = \begin{cases} w_R \left\{ \sum_{i \in N} \left(\frac{\pi_i(s)}{w_i} \right) - (n-1)Q^w(s) \right\} & , \text{ if } R = N \\ w_R \left\{ -\frac{\pi_i(s)}{w_i} + Q^w(s) \right\} & , \text{ if } R = N \setminus \{i\}, i \in N \\ 0 & , \text{ otherwise} \end{cases} \quad (8)$$

which determine $v_s = \sum_{R \subseteq N} \alpha_R^s u_R$ for all $s \in S$.

We will show that $\{(N, v_s)\}_{s \in S} \in \mathcal{G}_{N,S}$. Let $R \subseteq N$, $s, t \in S$ with $s_R = t_R$. For $R = N$ or R with $|R| \leq n-2$ we immediately find that $\alpha_R^s = \alpha_R^t$. It remains to consider R with $|R| = n-1$. Let $i \in N$ and $R = N \setminus \{i\}$ then $\pi_i(s) - \pi_i(t) = w_i(Q^w(s) - Q^w(t))$ so

$$\alpha_R^s = w_R \left\{ -\frac{\pi_i(s)}{w_i} + Q^w(s) \right\} = w_R \left\{ -\frac{\pi_i(t)}{w_i} + Q^w(t) \right\} = \alpha_R^t.$$

So, $\{(N, v_s)\}_{s \in S} \in \mathcal{G}_{N,S}$.

Finally, we will show that for all $i \in N$ and $s \in S$ it holds that $\Phi_i^w(v_s) = \pi_i(s)$. Therefore, let $i \in N$ and $s \in S$. Then

$$\begin{aligned}
\Phi_i^w(v_s) &= w_i \sum_{R \subseteq N, i \in R} \frac{\alpha_R^s}{w_R} \\
&= w_i \left\{ \sum_{j \in N} \left(\frac{\pi_j(s)}{w_j} \right) - (n-1)Q^w(s) + \sum_{j \in N, j \neq i} \left(-\frac{\pi_j(s)}{w_j} + Q^w(s) \right) \right\} \\
&= w_i \left\{ \frac{\pi_i(s)}{w_i} \right\} = \pi_i(s).
\end{aligned}$$

This completes the proof. \square

Note that if Γ is a w -potential game then an associated potential is given by $Q^w(s) = \sum_{R \subseteq N, R \neq \emptyset} \frac{\alpha_R^s}{w_R}$ for all $s \in S$, where $(\alpha_R^s)_{R \subseteq N, s \in S}$ are the unanimity coordinates of $\{(N, v^s)\}_{s \in S} \in \mathcal{G}_{N,S}$ for which $\pi_i(s) = \Phi_i^w(v_s)$ for all $i \in N$ and all $s \in S$.⁸

3 Networks

In this section we will first introduce hypergraphs. After that we will discuss hypergraph communication situations and characterize a class of allocation rules for these situations.

A *hypergraph* is a pair (N, \mathcal{H}) with N the player set and \mathcal{H} a family of subsets of N . An element $H \in \mathcal{H}$ is called a *conference*. The interpretation of a hypergraph is as follows: communication between players in a hypergraph can only take place within a conference. Furthermore, communication via this conference cannot take place between a proper subset of this conference, i.e. all players of the conference have to participate in the communication. Note that a hypergraph is a generalization of a graph, which consists only of conferences with exactly two players.

Next we consider *hypergraph communication situations*, first introduced by *Myerson* (1980). Formally, a hypergraph communication situation is a triple (N, v, \mathcal{H}) , where (N, v) is a cooperative game and (N, \mathcal{H}) a hypergraph. By assuming that every player can communicate with himself we can restrict our attention to hypergraphs (N, \mathcal{H}) with $\mathcal{H} \subseteq \{H \in 2^N \mid |H| \geq 2\}$. We will denote the class of all these hypergraph communication situations with player set N by HCS^N .

In a hypergraph communication situation a coalition $S \subseteq N$ can effect communication in conferences in $\mathcal{H}(S) := \{H \in \mathcal{H} \mid H \subseteq S\}$. Further we define interaction sets of $(S, \mathcal{H}(S))$:

1. every $\{i\} \subseteq S$ is an interaction set.
2. every $H \in \mathcal{H}(S)$ is an interaction set.
3. if T_1 and T_2 are interaction sets with $T_1 \cap T_2 \neq \emptyset$, then $T_1 \cup T_2$ is an interaction set.

⁸It holds that $Q^w(s) = P^w(N, v_s)$, where P^w denotes the weighted potential for cooperative games as used in the characterization of the weighted Shapley values by *Hart* and *Mas-Colell* (1989).

We call a set $T \subseteq S$ an interaction component of S if T is an interaction set of $(S, \mathcal{H}(S))$ and there exists no interaction set T' of $(S, \mathcal{H}(S))$ with $T \subset T'$. We will denote the resulting partition of S in interaction components by S/\mathcal{H} .

Conform this partition we define the value of coalition $S \subseteq N$ in (N, v, \mathcal{H}) by

$$v^{\mathcal{H}}(S) := \sum_{C \in S/\mathcal{H}} v(C).$$

We call $(N, v^{\mathcal{H}})$ the hypergraph-restricted game. An allocation rule γ is a function that assigns to every $(N, v, \mathcal{H}) \in HCS^N$ an element of \mathbb{R}^N . If there is no ambiguity about the game (N, v) we will write $\gamma(\mathcal{H})$ instead of $\gamma(N, v, \mathcal{H})$. For a positive weight-vector $w = (w_i)_{i \in N} \in \mathbb{R}_{++}^N$ the *weighted (extended) Myerson value*, μ^w , is the allocation rule which assigns to every (N, v, \mathcal{H}) the w -weighted Shapley value of the hypergraph-restricted game $(N, v^{\mathcal{H}})$,

$$\mu^w(N, v, \mathcal{H}) := \Phi^w(N, v^{\mathcal{H}}).$$

We will characterize the w -weighted (extended) Myerson value by two properties, *component efficiency* and *w-fairness*. Consider for an allocation rule γ these two properties:

Component efficiency: For all hypergraph communication situations $(N, v, \mathcal{H}) \in HCS^N$ it holds for all $C \in N/\mathcal{H}$:

$$\sum_{i \in C} \gamma_i(\mathcal{H}) = v(C).$$

w-Fairness: For all $(N, v, \mathcal{H}) \in HCS^N$, all $H \subseteq N$ and all $i, j \in H$

$$\frac{1}{w_i} (\gamma_i(\mathcal{H}) - \gamma_i(\mathcal{H} \setminus \{H\})) = \frac{1}{w_j} (\gamma_j(\mathcal{H}) - \gamma_j(\mathcal{H} \setminus \{H\})).$$

Component efficiency states that the players in an interaction component divide the value $v(C)$ amongst themselves. The property w -fairness is an extension of the fairness property of Myerson (1980), who characterized the (extended) Myerson value by the properties component efficiency and fairness.

The following lemma shows that the w -weighted (extended) Myerson value satisfies the two properties component efficiency and w -fairness. In the proof we use some results of Kalai and Samet (1988). They showed that the w -weighted Shapley value satisfies the

dummy property, *additivity*, and *partnership consistency*. The *dummy property* states that $\Phi_i^w(N, v) = v(\{i\})$ for all (N, v) with $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subseteq N \setminus \{i\}$. *Additivity* states that $\Phi^w(N, v + z) = \Phi^w(N, v) + \Phi^w(N, z)$ for all cooperative games (N, v) and (N, z) . To describe *partnership consistency* we need the notion of *partnership*. A coalition $S \subseteq N$ is a *partnership* in (N, v) if for all $T \subset S$ and all $R \subseteq N \setminus S$, $v(R \cup T) = v(R)$. Partnership consistency of Φ^w states that for every partnership S in (N, v) it holds that

$$\Phi_i^w(v) = \Phi_i^w(\Phi_S^w(v) u_S), \text{ for every } i \in S,$$

where $\Phi_S^w(v) = \sum_{j \in S} \Phi_j^w(v)$.

Lemma 3.1 The w -weighted (extended) Myerson value, μ^w , satisfies component efficiency and w -fairness.

Proof: First we will show that μ^w satisfies component efficiency. Let $(N, v, \mathcal{H}) \in HCS^N$ and C an interaction component of (N, \mathcal{H}) . We define two games (N, v^C) and $(N, v^{N \setminus C})$. For all $T \subseteq N$ let

$$\begin{aligned} v^C(T) &:= v^{\mathcal{H}}(T \cap C), \\ v^{N \setminus C}(T) &:= v^{\mathcal{H}}(T \setminus C). \end{aligned}$$

Since C is an interaction component of (N, \mathcal{H}) it holds that $v^{\mathcal{H}} = v^C + v^{N \setminus C}$. Since all $i \in C$ are dummy players in the game $(N, v^{N \setminus C})$, we conclude from the dummy player property of the w -weighted Shapley value, that $\Phi_i^w(v^{N \setminus C}) = 0$ for all $i \in C$. In the same way we find for all $i \in N \setminus C$ that $\Phi_i^w(v^C) = 0$. Using this and the additivity of the w -weighted Shapley values we find

$$\begin{aligned} \sum_{i \in C} \Phi_i^w(v^{\mathcal{H}}) &= \sum_{i \in C} \Phi_i^w(v^C) + \sum_{i \in C} \Phi_i^w(v^{N \setminus C}) \\ &= \sum_{i \in C} \Phi_i^w(v^C) = \sum_{i \in N} \Phi_i^w(v^C) = v^C(N) = v^{\mathcal{H}}(C) = v(C), \end{aligned}$$

where the fourth equality follows from the efficiency of the w -weighted Shapley value.

Secondly, we will show that the w -weighted (extended) Myerson value satisfies w -fairness. Let $(N, v, \mathcal{H}) \in HCS^N$ and $H \in \mathcal{H}$. Define $\mathcal{H}' := \mathcal{H} \setminus \{H\}$ and $v' := v^{\mathcal{H}} - v^{\mathcal{H}'}$.

For all $T \subseteq N$ with $H \not\subseteq T$ we then have

$$v'(T) = \sum_{R \in T/\mathcal{H}} v(R) - \sum_{R \in T/\mathcal{H}'} v(R) = 0$$

since $T/\mathcal{H} = T/\mathcal{H}'$. This means that H is a partnership in v' . From partnership consistency of Φ^w , it follows for all $i \in H$ that

$$\Phi_i^w(v') = \Phi_i^w \left(\left(\sum_{j \in H} \Phi_j^w(v') \right) u_H \right) = \frac{w_i}{\sum_{j \in H} w_j} \left(\sum_{j \in H} \Phi_j^w(v') \right)$$

So, for all $i, j \in H$

$$\frac{\Phi_i^w(v')}{w_i} = \frac{\Phi_j^w(v')}{w_j}.$$

From this we find

$$\frac{\mu_i^w(\mathcal{H}) - \mu_i^w(\mathcal{H}')}{w_i} = \frac{\Phi_i^w(v')}{w_i} = \frac{\Phi_j^w(v')}{w_j} = \frac{\mu_j^w(\mathcal{H}) - \mu_j^w(\mathcal{H}')}{w_j},$$

where the first and third equalities follow from the definition of the game (N, v') and the additivity of the w -weighted Shapley values. Hence, μ^w satisfies w -fairness. \square

The following theorem shows that the w -weighted (extended) Myerson value is the unique rule that is component efficient and w -fair.

Theorem 3.1 The w -weighted (extended) Myerson value μ^w is the unique rule that satisfies component efficiency and w -fairness.

Proof: From lemma 3.1 we know that μ^w satisfies component efficiency and w -fairness. We only need to show here that μ^w is the unique solution concept which satisfies these properties.

Suppose there are two rules γ^1 and γ^2 which satisfy component efficiency and w -fairness. Let (N, v, \mathcal{H}) be a communication situation with a minimum number of conferences such that $\gamma^1(\mathcal{H}) \neq \gamma^2(\mathcal{H})$. By component efficiency it follows that $\mathcal{H} \neq \emptyset$. Let $H \in \mathcal{H}$ and $\{i, j\} \subseteq H$. From w -fairness of γ^1 we then find

$$\frac{1}{w_i} \left(\gamma_i^1(\mathcal{H}) - \gamma_i^1(\mathcal{H} \setminus \{H\}) \right) = \frac{1}{w_j} \left(\gamma_j^1(\mathcal{H}) - \gamma_j^1(\mathcal{H} \setminus \{H\}) \right).$$

Using this, the minimality of \mathcal{H} , and the w -fairness of γ^2 respectively, we find

$$\begin{aligned} w_j \gamma_i^1(\mathcal{H}) - w_i \gamma_j^1(\mathcal{H}) &= w_j \gamma_i^1(\mathcal{H} \setminus \{H\}) - w_i \gamma_j^1(\mathcal{H} \setminus \{H\}) \\ &= w_j \gamma_i^2(\mathcal{H} \setminus \{H\}) - w_i \gamma_j^2(\mathcal{H} \setminus \{H\}) \\ &= w_j \gamma_i^2(\mathcal{H}) - w_i \gamma_j^2(\mathcal{H}). \end{aligned}$$

So

$$\frac{\gamma_i^1(\mathcal{H}) - \gamma_i^2(\mathcal{H})}{w_i} = \frac{\gamma_j^1(\mathcal{H}) - \gamma_j^2(\mathcal{H})}{w_j}.$$

This expression is valid for all pairs $\{i, j\}$ for which there exists an $H \in \mathcal{H}$ with $\{i, j\} \subseteq H$. Hence, it is also valid for all pairs $\{s, t\}$ that are in the same interaction component.

Let $C \in N/\mathcal{H}$ and $i \in C$. For all $j \in C$ we now have

$$\frac{1}{w_j} (\gamma_j^1(\mathcal{H}) - \gamma_j^2(\mathcal{H})) = \frac{1}{w_i} (\gamma_i^1(\mathcal{H}) - \gamma_i^2(\mathcal{H})).$$

Let $d := \frac{1}{w_i} (\gamma_i^1(\mathcal{H}) - \gamma_i^2(\mathcal{H}))$. Then for all $j \in C$: $\gamma_j^1(\mathcal{H}) - \gamma_j^2(\mathcal{H}) = w_j d$. Component efficiency of γ^1 and γ^2 gives us

$$\sum_{j \in C} \gamma_j^1(\mathcal{H}) = \sum_{j \in C} \gamma_j^2(\mathcal{H}) = v(C).$$

Thus,

$$0 = \sum_{j \in C} (\gamma_j^1(\mathcal{H}) - \gamma_j^2(\mathcal{H})) = \sum_{j \in C} w_j d.$$

Since $w \in \mathbb{R}_{++}^N$ it follows that $d = 0$. Since C was chosen arbitrarily, we conclude that $\gamma^1(\mathcal{H}) = \gamma^2(\mathcal{H})$. \square

4 Network formation

In this section we will model the process that leads to the formation of a conference structure as a game in strategic form. The game is a generalization of the linking game as formulated by *Myerson* (1991) and discussed by *Dutta, Noweland* and *Tijs* (1996). Furthermore, we will show that in this game, the only component efficient allocation rules that result in a weighted potential game are the weighted (extended) Myerson values.

Let γ be an allocation rule and (N, v) a cooperative game. Define the *conference formation game* $\Gamma(N, v, \gamma)$ determined by the tuple $(N; (S_i)_{i \in N}; (f_i^\gamma)_{i \in N})$ where for all $i \in N$

$$S_i := \{T \mid T \subseteq 2^{N \setminus \{i\}}\}$$

represents the strategy set of player i . A strategy of player i denotes the set of coalitions player i wants to join to form conferences. A strategy profile $s = (s_1, \dots, s_n) \in \prod_{i \in N} S_i$, induces a set of conferences $\mathcal{H}(s)$ given by

$$\mathcal{H}(s) := \{H \mid |H| \geq 2; H \setminus \{i\} \in s_i, i \in H\}.$$

The interpretation is that a conference is formed if and only if all players in this conference are willing to form it. The payoff function $f^\gamma = (f_i^\gamma)_{i \in N}$ is then defined as the allocation rule applied to the conference structure formed,

$$f^\gamma(s) = \gamma(\mathcal{H}(s)).$$

In case there is no ambiguity about the underlying cooperative game we will simply write $\Gamma(\gamma)$ instead of $\Gamma(N, v, \gamma)$. In the remainder we will consider an arbitrary game (N, v) .

In order to prove that weighted (extended) Myerson values are the only allocation rules that are component efficient and that lead to conference formation games which are weighted potential games, we need two lemmas.

Lemma 4.1 Let γ be an allocation rule and $w \in \mathbb{R}_{++}^N$. If the conference formation game $\Gamma(\gamma)$ is a w -potential game, then for all hypergraphs (N, \mathcal{H}) , all $H \subseteq N$ and all $i, j \in H$

$$\frac{1}{w_i} (\gamma_i(\mathcal{H}) - \gamma_i(\mathcal{H} \setminus \{H\})) = \frac{1}{w_j} (\gamma_j(\mathcal{H}) - \gamma_j(\mathcal{H} \setminus \{H\})). \quad (9)$$

Proof: Since $\Gamma(\gamma)$ is a w -potential game, $\Gamma(\gamma)$ has a w -potential P^w . We will show that γ satisfies equation (9).

Let (N, \mathcal{H}) be a hypergraph, so $(N, v, \mathcal{H}) \in HCS^N$. Define for all $k \in N$,

$$s_k := \{H \setminus \{k\} \mid H \in \mathcal{H}, k \in H\}.$$

Then it holds that $\mathcal{H}(s) = \mathcal{H}$. Let $H \in \mathcal{H}$ and $i \in H$, then for all $j \in H \setminus \{i\}$ we get

$$\begin{aligned} P^w(s_i \setminus \{H \setminus \{i\}\}, s_j, s_{-ij}) &= P^w(s_i \setminus \{H \setminus \{i\}\}, s_j \setminus \{H \setminus \{j\}\}, s_{-ij}) \\ &= P^w(s_i, s_j \setminus \{H \setminus \{j\}\}, s_{-ij}), \end{aligned}$$

since the three strategy tuples all result in the formation of the same conferences, the conferences in $\mathcal{H} \setminus \{H\}$, and hence, they all result in the same payoffs.

Using this we find for all $i, j \in H$

$$\begin{aligned} \frac{1}{w_i}(\gamma_i(\mathcal{H}) - \gamma_i(\mathcal{H} \setminus \{H\})) &= \frac{1}{w_i}(f_i^\gamma(s) - f_i^\gamma(s_i \setminus \{H \setminus \{i\}\}, s_{-i})) \\ &= P^w(s) - P^w(s_i \setminus \{H \setminus \{i\}\}, s_{-i}) \\ &= P^w(s) - P^w(s_j \setminus \{H \setminus \{j\}\}, s_{-j}) \\ &= \frac{1}{w_j}(f_j^\gamma(s) - f_j^\gamma(s_j \setminus \{H \setminus \{j\}\}, s_{-j})) \\ &= \frac{1}{w_j}(\gamma_j(\mathcal{H}) - \gamma_j(\mathcal{H} \setminus \{H\})). \end{aligned}$$

This completes the proof. \square

The following lemma shows that the conference formation game corresponding to an arbitrary cooperative game with a weighted (extended) Myerson value used as an allocation rule is a weighted potential game.

Lemma 4.2 The conference formation game $\Gamma(\mu^w)$ is a w -potential game.

Proof: Consider the following set of cooperative games, indexed by the set of strategy profiles of $\Gamma(\mu^w)$, $\{(N, v^{\mathcal{H}(s)})\}_{s \in S}$. Let $R \subseteq N$ and $s = (s_R, s_{N \setminus R}) \in S$. Since $v^{\mathcal{H}(s)}(R) = \sum_{C \in R/\mathcal{H}(s)} v(C)$ and $R/\mathcal{H}(s)$ does not depend on $s_{N \setminus R}$ it follows that $v^{\mathcal{H}(s)}(R)$ does not depend on $s_{N \setminus R}$. This implies that $\{(N, v^{\mathcal{H}(s)})\}_{s \in S} \in \mathcal{G}_{N,S}$. Since $f_i^{\mu^w}(s) = \mu_i^w(\mathcal{H}(s)) = \Phi_i^w(v^{\mathcal{H}(s)})$ by definition, it follows by theorem 2.1 that $\Gamma(\mu^w)$ is a w -potential game. \square

If we combine the results of the lemmas above we can prove the next theorem.

Theorem 4.1 Let γ be a solution concept that is component efficient. The conference formation game $\Gamma(\gamma)$ is a weighted potential game if and only if γ coincides with a weighted (extended) Myerson value for all hypergraphs (N, \mathcal{H}) .

Proof: Suppose that the conference formation game $\Gamma(\gamma)$ is a weighted potential game. From lemma 4.1 it follows that there exist weights w for which γ satisfies equation (9). Since γ is component efficient, it then follows, by the proof of theorem 3.1, that γ coincides with μ^w for all hypergraphs (N, \mathcal{H}) .

The reverse statement follows by lemma 4.2. \square

5 Potential maximizing strategies

In this section we will consider potential maximizing strategies in the conference formation game $\Gamma(\mu^w)$. Throughout this section we will assume that the underlying cooperative game (N, v) is *superadditive*, i.e. $v(R \cup T) \geq v(R) + v(T)$, for all disjoint $R, T \subseteq N$. We will show that the strategy resulting in the complete conference structure, the structure with all subsets of players in the set of conferences, is a potential maximizing strategy. Furthermore, we will show that all potential maximizing strategies result in the same payoff as the strategy corresponding to the full cooperation structure.⁹

First we need some notation to denote the structures that will result according to the conference formation game with a weighted (extended) Myerson value used as allocation rule. Let $\bar{s} = (\bar{s}_1, \dots, \bar{s}_n)$ be the strategy tuple with $\bar{s}_i = 2^{N \setminus \{i\}}$ for all $i \in N$. This strategy implies that player i is willing to cooperate with all subsets of the other players. The corresponding set of conferences will be denoted by $\bar{\mathcal{H}} := \mathcal{H}(\bar{s}) = \{T \in 2^N \mid |T| \geq 2\}$. A set of conferences \mathcal{H} is called *essentially complete* with regard to solution concept γ iff \mathcal{H} and $\bar{\mathcal{H}}$ are payoff-equivalent, i.e. $\gamma(\mathcal{H}) = \gamma(\bar{\mathcal{H}})$. To facilitate the proof of the main theorem in this section we will first prove two lemmas. The first lemma states that a player is never worse off forming an additional conference, extending a result of *Myerson (1977)*.

Lemma 5.1 Let $(N, v, \mathcal{H}) \in HCS^N$, $H \subseteq N$ and $w \in \mathbb{R}_{++}^N$. For all $i \in H$ it holds that

$$\mu_i^w(\mathcal{H} \cup \{H\}) \geq \mu_i^w(\mathcal{H}).$$

⁹These results generalize analogous results of *Qin (1996)* who was concerned with *Myerson values* and potential games.

Proof: Let $v' := v^{\mathcal{H} \cup \{H\}} - v^{\mathcal{H}}$. The superadditivity of v implies that $v'(R) \geq 0$ for all $R \subseteq N$ since every component in $(N, (\mathcal{H} \cup \{H\})(R))$ is the union of one or more components in $(N, \mathcal{H}(R))$. For all R with $H \not\subseteq R$ it follows that $v^{\mathcal{H} \cup \{H\}}(R) = v^{\mathcal{H}}(R)$ and hence $v'(R) = 0$.

Let $i \in H$. Since $v'(R) = 0$ for all $R \subseteq N \setminus \{i\}$ we have

$$v'(R \cup \{i\}) \geq v'(R), \text{ for all } R \subseteq N \setminus \{i\}. \quad (10)$$

From *Weber* (1988) it follows that there exists a probability distribution p_i^w on $2^{N \setminus \{i\}}$ such that

$$\Phi_i^w(v') = \sum_{R \subseteq N \setminus \{i\}} p_i^w(R) (v'(R \cup \{i\}) - v'(R)). \quad (11)$$

Combining equations (10) and (11) completes the proof since $v' = v^{\mathcal{H} \cup H} - v^{\mathcal{H}}$ and Φ^w satisfies additivity. \square

The following lemma considers a specific deviation of a player, say i . If player i deviates to a strategy which is a superset of his original strategy and this deviation does not influence his payoff, then the payoffs of all the other players remain unchanged as well.

Lemma 5.2 Let $w \in \mathbb{R}_{++}^N$. Then for all $i \in N$, all $s_{-i} \in S_{-i}$, and all $s_i, s'_i \in S_i$ with $s'_i \subseteq s_i$, it holds that if $f_i^{\mu^w}(s_i, s_{-i}) = f_i^{\mu^w}(s'_i, s_{-i})$ then $f^{\mu^w}(s_i, s_{-i}) = f^{\mu^w}(s'_i, s_{-i})$.

Proof: If $\mathcal{H}(s_i, s_{-i}) = \mathcal{H}(s'_i, s_{-i})$, then the statement in the theorem is obviously true. Otherwise, since $s'_i \subseteq s_i$ there exist $k \in \mathbb{N}$ and $H_1, \dots, H_k \in 2^N$ with $i \in H_j$ for all $j \in \{1, \dots, k\}$ such that $\mathcal{H}(s_i, s_{-i}) = \mathcal{H}(s'_i, s_{-i}) \cup \{H_1, \dots, H_k\}$. Define $\mathcal{H}_0 := \mathcal{H}(s'_i, s_{-i})$ and for all $j \in \{1, \dots, k\}$

$$\mathcal{H}_j := \mathcal{H}(s'_i, s_{-i}) \cup \{H_1, \dots, H_j\}.$$

Since $\mu_i^w(\mathcal{H}_0) = \mu_i^w(\mathcal{H}_k)$, it follows from lemma 5.1 that $\mu_i^w(\mathcal{H}_{j-1}) = \mu_i^w(\mathcal{H}_j)$ for all $j \in \{1, \dots, k\}$.

For $j \in \{1, \dots, k\}$, define $v' := v^{\mathcal{H}_j} - v^{\mathcal{H}_{j-1}}$. Since $\mu_i^w(\mathcal{H}_j) = \mu_i^w(\mathcal{H}_{j-1})$ it follows from (10) and (11) that

$$v'(R \cup \{i\}) = v'(R), \text{ for all } R \subseteq N \setminus \{i\}, \quad (12)$$

since $p_i^w(R) > 0$ for all $R \subseteq N \setminus \{i\}$.¹⁰ Consider an arbitrary $l \in N \setminus \{i\}$ and $S \subseteq N \setminus \{l\}$. Using equation (12) and the fact that $v'(T) = 0$ for every T with $H_j \not\subseteq T$ we have

$$v'(R \cup \{l\}) = v'((R \cup \{l\}) \setminus \{i\}) = 0 \quad (13)$$

and

$$v'(R) = v'(R \setminus \{i\}) = 0. \quad (14)$$

It follows that $\Phi_l^w(v') = 0$ and hence, by the additivity of the weighted Shapley values that

$$\mu_l^w(\mathcal{H}_j) = \Phi_l^w(v^{\mathcal{H}_j}) = \Phi_l^w(v^{\mathcal{H}_{j-1}}) = \mu_l^w(\mathcal{H}_{j-1}). \quad (15)$$

We conclude that

$$f^{\mu^w}(s'_i, s_{-i}) = \mu^w(\mathcal{H}_0) = \mu^w(\mathcal{H}_1) = \dots = \mu^w(\mathcal{H}_k) = f^{\mu^w}(s_i, s_{-i}). \quad (16)$$

This completes the proof. \square

We can now state our main theorem, dealing with the potential maximizers in conference formation games that are weighted potential games.

Theorem 5.1 Let w be a vector of positive weights and let P^w be a weighted potential for the conference formation game $\mathcal{H}(\mu^w)$. Then $\bar{s} \in \operatorname{argmax} P^w$. Further, if $t \in \operatorname{argmax} P^w$ then $\mathcal{H}(t)$ is essentially complete for μ^w .

Proof: Let $i \in N$, $s_i \in S_i$ and $s_{-i} \in S_{-i}$. Define the following conference sets: $\mathcal{H}^1 := \mathcal{H}(\bar{s}_i, s_{-i})$ and $\mathcal{H}^2 := \mathcal{H}(s_i, s_{-i})$. From $s_i \subseteq \bar{s}_i$ we conclude that $\mathcal{H}^2 \subseteq \mathcal{H}^1$. Furthermore, note that if $H \in \mathcal{H}^1 \setminus \mathcal{H}^2$, then $i \in H$. If we apply lemma 5.1 repeatedly for all $H \in \mathcal{H}^1 \setminus \mathcal{H}^2$ then

$$f_i^{\mu^w}(\bar{s}_i, s_{-i}) = \mu_i^w(\mathcal{H}^1) \geq \mu_i^w(\mathcal{H}^2) = f_i^{\mu^w}(s_i, s_{-i}). \quad (17)$$

We conclude that \bar{s} is a weakly dominant strategy.

Consider the n-tuple of weakly dominant strategies \bar{s} and an arbitrary n-tuple of strategies t . Construct a sequence $(s^i)_{i=0}^n$ with $s^i = (\bar{s}_1, \dots, \bar{s}_i, t_{i+1}, \dots, t_n)$. This construction implies that $s^0 = t$ and $s^n = \bar{s}$. Since \bar{s} is a weakly dominant strategy it holds

¹⁰This follows for example from the expression for weighted Shapley values of *Kalai and Samet* (1988).

for all $i \in \{0, \dots, n-1\}$ that $\mu_{i+1}^w(\mathcal{H}(s^{i+1})) \geq \mu_{i+1}^w(\mathcal{H}(s^i))$, so $P^w(s^{i+1}) \geq P^w(s^i)$. Thus

$$P^w(\bar{s}) = P^w(s^n) \geq P^w(s^{n-1}) \geq \dots \geq P^w(s^1) \geq P^w(s^0) = P^w(t). \quad (18)$$

This completes the proof of the first part of the theorem.

Furthermore, since $P^w(\bar{s}) \geq P^w(t)$ for all strategy-tuples $t \in S$ it follows that if t is a potential maximizing strategy then $P^w(\bar{s}) = P^w(t)$. But then every inequality in (18) has to hold with equality for this strategy-tuple t . Since

$$P^w(s^k) - P^w(s^{k-1}) = \frac{1}{w_k} \left(\mu_k^w(\mathcal{H}(s^k)) - \mu_k^w(\mathcal{H}(s^{k-1})) \right) \geq 0$$

for all $k \in \{1, \dots, n\}$ it follows that $\mu_k^w(\mathcal{H}(s^k)) = \mu_k^w(\mathcal{H}(s^{k-1}))$ for all $k \in \{1, \dots, n\}$. From lemma 5.2 we then conclude that $\mu^w(\mathcal{H}(s^k)) = \mu^w(\mathcal{H}(s^{k-1}))$ for all $k \in \{1, \dots, n\}$ and hence,

$$\mu^w(\mathcal{H}(t)) = \mu^w(\mathcal{H}(s^0)) = \dots = \mu^w(\mathcal{H}(s^n)) = \mu^w(\mathcal{H}(\bar{s})).$$

We conclude that if $t \in \operatorname{argmax} P^w$ then it holds that $\mathcal{H}(t)$ is essentially complete for μ^w . \square

6 Concluding remarks

In this paper we have extended the model of *Myerson* (1991), who introduced a strategic form game that describes a link formation process resulting in a graph. Here we describe a strategic form game, called the conference formation game, resulting in a hypergraph.

In this paper we restrict ourselves to superadditive games and to conference formation games that are weighted potential games. It turns out that, under an efficiency requirement, the class of allocation rules generating conference formation games that are weighted potential games is the class of weighted (extended) Myerson values. Furthermore, the argmax set of the conference games that are weighted potential games predicts the formation of the full cooperation structure or a structure that is payoff equivalent to the full cooperation structure.

Although we concentrated on conference formation games, most of the results in this paper also hold for the original game of *Myerson* (1991) dealing with links rather than

conferences. This is shown in a much more technical way, not using the representation theorem of section 2, in *Dutta, Nouweland and Tijs (1995)*, a preliminary version of *Dutta et al. (1996)*.

Our results point in the direction of the formation of a full cooperation structure. However, if we drop the superadditivity assumption or consider another solution concept, this need not be the case. *Aumann and Myerson (1988)* derive different results for a game where the cooperation structure is formed sequentially rather than simultaneously.

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