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De Waegenaere, A.M.B.; Wakker, P.P.

Publication date:
1997

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Anja De Waegenaere & Peter Wakker

CentER for Economic Research, Tilburg University,

P.O. Box 90153, Tilburg 5000 LE, The Netherlands

May 9, 1997

Abstract

This paper introduces the signed Choquet integral, i.e., a nonmonotonic generalization of the Choquet integral. Applications to welfare theory, multi-period optimization, and asset pricing are described.

Keywords: Choquet integral, comonotonicity, arbitrage, time preference.

1 Introduction

This paper presents a non-monotonic generalization of the Choquet integral. The Choquet integral provides a method of integration for nonadditive measures, and originates from Choquet (1954). It was introduced in decision theory by Schmeidler (1982, 1989), and it has since proved to be extremely useful. In decision under uncertainty, the context of Schmeidler's paper, it leads to a generalization of expected utility that allows for nonadditivity of probability (Gilboa, 1987; Wakker, 1989a,b; Nakamura, 1990). Theories based on the Choquet integral are called "rank-dependent," and can explain Ellsberg and Allais paradox behavior. They have been used by Tversky & Kahneman (1992) to develop a theoretical model for their prospect theory (Kahneman & Tversky, 1979); a similar model was developed independently by Luce (1991) and Luce & Fishburn (1991).

For decision under risk, Choquet integration was used by Quiggin (1981), Yaari (1987), Chateauneuf (1996), and others. Chateauneuf et al. (1996) uses Choquet integration to price financial assets on markets with frictions. The Choquet integral also proved useful in other areas of decision theory. In welfare theory, it generalizes the Gini index and can incorporate equity considerations (Weymark, 1981; Lopes, 1984; Ebert, 1987; Yaari, 1988; Porath & Gilboa, 1994). In fuzzy set theory, decision-theoretic foundations have been obtained (Wakker, 1990b; Murofushi & Sugeno, 1989). The Choquet integral has also been used in game theory (Gilboa & Schmeidler, 1995; Haller, 1995; Lo, 1995; Mukerji & Shin, 1996) and multi-period optimization (Gilboa, 1989; Shalev, 1994). Its mathematics is explained by Fishburn (1988) and Denneberg (1994).

One characteristic property of Choquet integrals is monotonicity, i.e., increases of the input lead to higher integral values. While monotonicity is imperative in many contexts, it is naturally violated in other contexts. For instance, a welfare allocation $(10, 10, 10, 10)$ ($10 for person 1, ..., $10 for person 4) is sometimes preferred to an allocation $(11, 12, 13, 15)$ if envy and conflict can arise in the latter (Crosby, 1976; Tversky & Griffin, 1991). In multi-period optimization, Hsee & Abelson (1991) found that the majority of subjects prefer a constant low income to a decreasing income ending with the same low income, even though the latter dominates the former in each time point. Kahneman et al. (1993) found that subjects violate temporal monotonicity when choosing between aversive episodes such as immersing hands in cold water. This paper presents a theory that accommodates the described violations of monotonicity.

Section 2 introduces signed Choquet integrals, that generalize Choquet integrals by permitting violations of monotonicity. The section presents necessary and sufficient conditions for a functional to be a signed Choquet integral, and also provides a characterization in terms of preference conditions. Section 3 extends classical concavity and convexity results to signed Choquet integrals. Sections 4 and 5 describe applications to asset pricing in a context with uncertainty, and multi-period optimization, respectively. Proofs are presented in the Appendix.

2 A characterization of signed Choquet integrals

We consider a finite set $\{1,\ldots,n\}$. Elements of $\mathbb{R}^n$ are denoted as x or as $(x_1,\ldots,x_n)$; they represent functions from $\{1,\ldots,n\}$ to $\mathbb{R}$. In the context of decision under uncertainty, elements of $\{1,\ldots,n\}$ are *states (of nature)*. One state is true, the others are not true, and it is not known which state is true. Functions x are *acts*, an act x yielding x_j if j is the true state of nature. Because it is unknown which state is true, it is unknown which outcome an act will yield. In the context of welfare theory, elements of $\{1,\ldots,n\}$ designate persons, and x designates an allocation that yields x_j for person j. In the context of multi-period optimization, elements of $\{1,\ldots,n\}$ designate periods, and x_j is the consumption/income in period j. Other interpretations can be developed. This section, more or less arbitrarily, adopts the terminology of decision under uncertainty; of course, its results are equally relevant for the other contexts.

Acts x and y are *comonotonic* if there are no i, j such that $x_i > x_j$ and $y_i < y_j$. For any permutation ρ on $\{1,\ldots,n\}$, the *comonotonic cone* C_ρ associated with ρ is defined as $\{x \in \mathbb{R}^n : x_{\rho(1)} \geq \cdots \geq x_{\rho(n)}\}$. Hence, ρ assigns to each rank-number j the state of nature that has the j-th place in the rank-ordering with respect to outcomes. A subset of

$I\!R^n$ is *comonotonic* if every pair of acts is comonotonic, which holds if and only if there is a permutation ρ such that the subset is contained in the comonotonic cone C_ρ (Wakker, 1989b, Lemma VI.3.3). We consider general set functions $v : 2^{\{1,\ldots,n\}} \to I\!R$; it is permitted that v takes negative values. v is a *capacity* if $v(\emptyset) = 0$, $v(\{1,\ldots,n\}) = 1$, and v satisfies monotonicity with respect to set inclusion, i.e., $A \supset B \Rightarrow v(A) \geq v(B)$; then $v(A) \geq 0$ for all A. Capacities generalize probability measures because additivity with respect to disjoint union is not imposed. A *signed capacity* v drops the monotonicity requirement, i.e., $v(\emptyset) = 0$ and $v(\{1,\ldots,n\}) = 1$ are the only requirements. Thus, a signed capacity can take negative values.

Let v be an arbitrary set function. For any $x \in I\!R^n$, we define $\int x\,dv$, analogously to the Choquet integral, as follows:

(i) Take a permutation ρ that is *compatible* with x, i.e., $x_{\rho(1)} \geq \cdots \geq x_{\rho(n)}$.

(ii) Define

$$\pi_{\rho(j)} := v(\{\rho(1),\ldots,\rho(j)\}) - v(\{\rho(1),\ldots,\rho(j-1)\}) \tag{1}$$

for all j (thus $\pi_{\rho(1)} = v(\{\rho(1)\}) - v(\emptyset)$).

(iii) $\int x\,dv = \sum_{i=1}^n \pi_j x_j$ is the *signed Choquet integral* of x with respect to v.

If $x_i = x_j$ for some $i \neq j$, then the rank-ordering ρ is not uniquely defined. It is elementarily verified that then any rank-ordering ρ that is compatible with x gives the same result, so that the signed Choquet integral is well-defined after all. Note also that the integral remains the same if we replace v by $v' = v - c$ for any constant c. In particular, one can take $v' = v - v(\emptyset)$, i.e., one can restrict attention to set functions assigning 0 to the empty set. The numbers π_j are called *decision weights*. In general, they can well be negative; they are all nonnegative if and only if v is monotonic with respect to set inclusion. Because $v(\{1,\ldots,n\}) = 1$ for signed capacities, the signed Choquet integral of $(\alpha,\ldots,\alpha)$ with respect to a signed capacity is α for all $\alpha \in I\!R$.

In the following theorem we characterize signed Choquet integrals, generalizing Schmeidler's (1986) representation (see also Anger, 1977, Theorem 3) to arbitrary set functions.

The continuity condition imposed on the functional V in the theorem can be considerably weakened. For instance, it suffices to impose continuity at one point $(\alpha, \ldots, \alpha)$, or some boundedness or measurability restriction. The characteristic property of the signed Choquet integral is *comonotonic additivity*, which for a general functional V means that $V(x+y) = V(x) + V(y)$ whenever x and y are comonotonic. (If $V(x+y) = V(x) + V(y)$ for all x, y, then we call V *additive*.)

THEOREM 1 *$V : I\!R^n \to I\!R$ is a signed Choquet integral if and only if:*

(1) V is continuous.

(2) V satisfies comonotonic additivity. □

Next we characterize preference relations that can be represented by signed Choquet integrals when the set function is a signed capacity. A binary relation $\succeq$ on $I\!R^n$ is a *weak order* if it is complete ($x \succeq y$ or $y \succeq x$ for all x, y) and transitive. It is *continuous* if $\{x \in I\!R^n : x \succeq y\}$ and $\{x \in I\!R^n : x \preceq y\}$ are closed for all $y \in I\!R^n$. It satisfies *comonotonic additivity* if $x \succeq y$ implies $x + z \succeq y + z$ for all comonotonic x, y, z. $(\alpha, \ldots, \alpha)$ is a *certainty equivalent* of x if it is equal in preference to x. $\succeq$ is *constant-monotonic* if $\alpha > \beta \Rightarrow (\alpha, \cdots, \alpha) \succ (\beta, \cdots, \beta)$ for all real α, β. The assumption will imply that the set function assigns value 1 to the state space. A function V *represents* $\succeq$ if $x \succeq y \Leftrightarrow V(x) \geq V(y)$.

COROLLARY 2 *Let $\succeq$ be a binary relation on $I\!R^n$. Then $\succeq$ can be represented by a signed Choquet integral with respect to a signed capacity, if and only if:*

(1) $\succeq$ is a weak order;

(2) $\succeq$ is continuous;

(3) $\succeq$ is constant-monotonic;

(4) $\succeq$ satisfies comonotonic additivity.

Further, the representing signed Choquet integral is uniquely determined. □

3 Concavity and convexity

Throughout this section, we assume that V is a signed Choquet integral with respect to a set function v. We consider V as a function on $\mathbb{R}^n$, and convexity and concavity properties of V refer to mixtures of elements from $\mathbb{R}^n$ (and not to mixtures of set functions). We call a set function v *concave* if

$$v(A \cup B) + v(A \cap B) \leq v(A) + v(B) \tag{2}$$

for all A, B. v is *convex* if the reversed inequality holds.

It is elementarily verified that V satisfies *positive homogeneity*, i.e., $V(\lambda x) = \lambda V(x)$ for all $\lambda > 0$. V is *subadditive* if $V(x+y) \leq V(x) + V(y)$ for all x, y.

THEOREM 3 *The following four statements are equivalent.*

(1) V is convex.

(2) V is subadditive.

(3) V is the maximum of dominated linear functionals.

(4) v is concave. □

The equivalence of (1) and (4) adapts Proposition 3 of Schmeidler (1986) to signed Choquet integrals. The following lemma prepares for Theorem 5 and is given in the main text because it may have interest of its own.

LEMMA 4 *Consider two signed Choquet integrals V and V', with respect to set functions v and v', respectively, satisfying $v(\emptyset) = v'(\emptyset) = 0$ and $v(\{1, \ldots, n\}) = v'(\{1, \ldots, n\})$. Then $V \geq V' \Leftrightarrow v \geq v'$.* □

Lemma 4 will mostly be used in the special case where v' is an additive set function.

THEOREM 5 *Assume that $v(\emptyset) = 0$. Then the set function v is concave if and only if $V(x) = \max\{\int x \, d\mu | \mu \in \mathcal{P}(v)\}$, where $\mathcal{P}(v)$ is the set of additive set functions μ that lie below v everywhere and satisfy $\mu(\{1, \ldots, n\}) = v(\{1, \ldots, n\})$.* □

Theorem 5 has been used by De Waegenaere et al. (1996). They describe a general equilibrium model for asset trading on markets with dealers in which the equilibrium price of an asset equals the signed Choquet integral of its payoff with respect to an equilibrium set function. Theorem 5 then allows to rewrite the equilibrium pricing rule in such a way that the total cost of a portfolio can be written as a linear part (the price of the portfolio) plus a subadditive part (the spread charged by the dealer).

OBSERVATION 6 *Theorems 3 and 5 can be applied to* $-V$ *and* $-v$*, resulting in substitution of the terms concave for convex, superadditive* ($V(x+y) \geq V(x) + V(y)$) *for subadditive, minimum for maximum, dominating for dominated, above for below, and vice versa.* □

4 An application to asset pricing

In the context of asset pricing, the elements of $\{1, 2, \ldots, n\}$ are states of nature. At date zero, assets can be traded. At a later date, date one, exactly one of the states of nature will turn out to be true, and the assets yield a payoff depending on the true state. Therefore, the payoff of an asset at date one can be represented by a vector $A \in I\!R^n$.

A fundamental assumption made in asset pricing is that asset prices are arbitrage free, i.e., they do not allow for *arbitrage possibilities*. Roughly speaking, there is an arbitrage possibility if there exists a portfolio that has a non-positive price and yields non-negative payoffs with certainty (i.e., whatever state will be true) and a positive payoff in at least one state of nature. More precisely, suppose that there are J assets with payoff vectors $A^j \in I\!R^n$ and prices $q_j \in I\!R$. Let $A \in I\!R^{n \times J}$ and $q \in I\!R^J$ denote the matrix of date one asset payoffs and the row vector of date zero asset prices, respectively. Then the asset prices are arbitrage free if there does not exist a portfolio $z = (z_1, z_2, \ldots, z_J) \in I\!R^J$ such that $qz \leq 0$ and $(Az)_s \geq 0$ for all $s \in \{1, 2, \ldots, n\}$ with at least one strict inequality. The assumption that asset prices are arbitrage free turns out to be extremely useful for asset pricing models. It implies that there exists a probability measure (not necessarily unique) such that the price of an asset equals the expected value of its payoff with respect to this measure discounted by the price of a riskless bond with payoff one in each state. This probability measure is called a risk neutral probability measure. A risk neutral probability measure can easily be determined from observed asset prices and, consequently, any portfolio can be priced by simply calculating the expected value of its discounted payoff.

For frictionless markets, the assumption that asset prices are arbitrage free is completely justified. Indeed, when an arbitrage possibility were to exist on a frictionless market then nothing would prevent the agents from exploiting this arbitrage possibility and hence the market will not be in equilibrium. In realistic settings, however, asset trading is often subject to constraints such as leverage constraints or no-overinsurance constraints, and mostly occurs through the intermediation of brokers or dealers charging a price for their intermediation. The existence of such frictions can disturb the no-arbitrage relation of asset prices because agents may simply not be able to exploit a potential arbitrage possibility. This implies that many of the models that are used in practice should in fact not be used under the presence of frictions, since they rely heavily on the no-arbitrage relation of asset prices. Hence, there is a need for an alternative to the traditional no-arbitrage pricing rule for the case of markets with frictions. De Waegenaere et al. (1996) develops a general equilibrium model for asset trading on markets with dealers charging bid-ask spreads. Theorem 7 is used there to prove that an equilibrium set function v exists on the state space such that the equilibrium asset prices equal the signed Choquet integral of their payoff with respect to this set function.

The following example illustrates how one obtains a pricing rule that is applicable to markets with frictions by replacing the Lebesque integral appearing in no-arbitrage pricing by the more general signed Choquet integral. The standard general equilibrium model on incomplete asset markets on which this example is based can be found in Magill & Shafer (1991).

EXAMPLE 7 We consider a two period asset market model with dealers charging bid-ask spreads. For each portfolio there is a buying price and a selling price and the difference between these two prices is the bid-ask spread. There are two assets that can be traded in the first period (date zero), and payoff takes place in the second period (date one). There are two states of nature in this second period, i.e., $n = 2$. A consumption bundle of an agent therefore consists of a vector $(x_0, x_1, x_2) \in I\!R^3_+$, where x_0 denotes the amount of money the agent owns at date zero, x_1 denotes the amount of money he will own at date one if state one occurs, and x_2 denotes the amount of money he will own at date one if state two occurs. The endowment (before trading) of the agents is given by $w^1 = w^2 = (4, 3, 3)$. Since there are two states at date one, the payoff of an asset can be represented by a vector in $I\!R^2$. The payoff vectors for the two assets are given by $A^1 = (1, 0)$ and $A^2 = (0, 1)$, respectively. Hence, asset 1 pays off one if state 1 is true at date one and zero if state 2 is true, and the reverse holds for asset 2.

As stated before, the assets can only be traded through the intermediation of a dealer. For trading a portfolio $z = (z_1, z_2) \in I\!R^2$, i.e., buying z_1 units of asset 1 (or selling $-z_1$ if z_1 is negative), and buying z_2 units of asset 2, the dealer charges an amount $q(z) = \pi z + \gamma|z_2 - z_1|$, for given $\pi \in I\!R^2$ and $\gamma \geq 0$. Thus, $q(z)$ consists of a linear part πz, which is the "price," augmented by a subadditive, positive part $\gamma|z_2 - z_1|$ representing the "risk-premium" charged by the dealer for his intermediation. Since then $q(z) > -q(-z)$ for all portfolios with a risky payoff (i.e., $z_1 \neq z_2$), the dealer makes a profit by buying this portfolio from one agent and selling it to the other. In this example, the dealer's firm is owned by agent 2, hence the dealer's profit, denoted π^d, is returned to agent 2 after trade. Since prices for portfolios have to be paid at date zero, the trade of portfolio $z^i \in I\!R^2$ by agent i has the following effect on his resources:

$$\begin{aligned}
x_0^i &= w_0^i - q(z^i) + \xi^i \pi^d \\
x_1^i &= w_1^i + z_1^i \\
x_2^i &= w_2^i + z_2^i,
\end{aligned}$$

where $\xi^1 = 0$ and $\xi^2 = 1$ denote the shares of the respective agents in the dealer's profit. Now, using their initial resources w^i, the agents can trade asset portfolios $z = (z_1, z_2)$ in order to maximize their utility. Agent 1 has utility function $u^1(x_0, x_1, x_2) = \sqrt{x_0} + \sqrt{x_1} + 3\sqrt{5x_2}$. Agent 2 has utility function $u^2(x_0, x_1, x_2) = \sqrt{7x_0} + \sqrt{7x_1} + \sqrt{x_2}$.

In addition to the presence of the dealer, there is another restriction on the trading possibilities of the agents. Agent one is only allowed to buy portfolios $z = (z_1, z_2)$ satisfying $z_1 + z_2 \leq 1$ and agent two is only allowed to buy portfolios $z = (z_1, z_2)$ satisfying $z_1 \leq 1$. Now let $x^i \in I\!R^3_+$ and $z^i \in I\!R^2$ denote the consumption bundle and the asset portfolio that agent i will choose as a result of his utility maximizing problem for a given π^d, for $i = 1, 2$. It can easily be shown that, for

$$\pi^d = 0.87$$

$$\pi = (-0.855, 1.145)$$

$$\gamma = 0.145$$

$$x^1 = (1, 2, 5), \quad x^2 = (7, 4, 1)$$

$$z^1 = (-1, 2), \quad z^2 = (1, -2),$$

the market is in equilibrium. That is, both agents have maximized their utility, the market in assets and money clears (i.e., there is no excess demand or excess supply), and the dealer's profit equals $\pi^d = q((-1, 2)) + q((1, -2))$.

Now define the set function v on $2^{\{1,2\}}$ as follows: $v(\emptyset) = 0, v(\{1\}) = -0.71$, $v(\{2\}) = 1.29$, and $v(\{1, 2\}) = 0.29$. Then it immediately follows that $q(z) = \int z dv$ for all $z \in I\!R^2$, i.e., the equilibrium price of a portfolio equals the signed Choquet integral of its payoff. Moreover, it is clear that since π_1 is negative, these equilibrium asset prices cannot be represented by a Choquet integral with respect to a monotone set function (e.g. a capacity as in Chateauneuf et al., 1996). Because of the risk premium, prices are nonlinear and cannot be represented by a linear integral either. So, in this case, only the extension to signed Choquet integrals yields the desired result. This example shows that traditional no-arbitrage pricing is not always suitable to represent equilibrium asset prices. The extension from discounted expected values to signed Choquet integrals does yield a solution. $\square$

5 An application to multi-period decisions

Gilboa (1989) initiated the application of Choquet integrals to multiperiod decisions. He was motivated by the finding that preferences do not satisfy separability over time (Kreps & Porteus, 1978; Loewenstein & Elster, 1992), a phenomenon that underlies habit formation in economics (Constantinides, 1990). For example, people are sensitive to changes in income (Loewenstein & Prelec, 1991; Loewenstein & Elster, 1992). Decreases in income are valued highly negative. The Choquet integral can model dependency on orderings of outcomes, and thus sensitivity towards increases and decreases of income. Gilboa derived a special, "Markovian," version of the Choquet integral where the utility of current income can depend on the income of the previous period (Kahneman & Thaler, 1991). He retained the classical monotonicity condition of Choquet integrals and did not permit actual decreases of utility in income. There is, however, evidence that sensitivity towards the pattern of income can be so strong as to overrule even monotonicity (Hsee & Abelson, 1991). A person may prefer an income pattern (10, 10) to the income pattern (11, 10) so as to avoid the decrease in the second period.

Shalev (1994) developed a model that permits such violations of monotonicity. We will demonstrate that his model is in fact a special case of signed Choquet integration and we use that observation to present a simplified analysis. Our functional thus provides a common generalization of the ones of Gilboa (1989) and Shalev (1994). One difference between our model and theirs is that we assume linearity directly in outcomes, whereas they assume that outcomes are probability distributions over prizes and then use expected utility over those, i.e., they use linearity with respect to probabilistic mixtures rather than with respect to quantitative outcomes.

The set $\{1, \ldots, n\}$ now designates a set of periods, and an n-tuple is an *income profile*. Shalev introduced the following, basic, condition. Two income profiles x, y are *sequentially comonotonic* if there are no two adjacent time periods $s, s+1$ such that $x_s > y_s$ and $y_{s+1} > x_{s+1}$. A set of income profiles is *sequentially comonotonic* if every pair in it is so. The preference relation $\succeq$ satisfies *sequential additivity* if $x \succeq y$ implies $x + z \succeq y + z$ whenever

$\{x, y, z\}$ is sequentially comonotonic. Obviously, if income profiles are comonotonic then they are also sequentially comonotonic. Hence, sequential additivity implies comonotonic additivity.

The functional used in this section to value income profiles is of the form

$$\lambda_1 x_1 + \sum_{j=2}^{n} [\lambda_j x_j - \tau_j (x_{j-1} - x_j)^+] \tag{3}$$

where the λ_js sum to one, and for any real number β we define $\beta^+ = \max\{0, \beta\}$. Such a functional is called a *sequential Choquet integral*.

Formula (3) is based on the idea that a regular weighted sum applies if the sequence is increasing, i.e., if $x_j > x_{j-1}$ for all j. However, for decreases there is a "decision weight penalty," that is, if $x_{j-1} > x_j$, then decision weight τ_j is shifted from x_{j-1} to the lower outcome x_j, leading to subtraction of a term $\tau_j(x_{j-1} - x_j)$. In extreme cases, the τ_js may be so large that violations of monotonicity result. The terminology in this interpretation is adapted from the example of income valuation, where preference is monotonic if income is increasing ($\lambda_j > 0$ for all j), but for decreases in salary a "penalty" is subtracted, i.e., $\tau_j > 0$ for all j. The following analysis also considers the general case of negative λ_js and τ_js.

Let us next explain that a sequential Choquet integral is a signed Choquet integral indeed. The decision weights π_j, defined in Formula (1) in Section 2, are as follows, where we consider four cases (and $x_0 = -\infty$, $x_{n+1} = \infty$):

(1) $\pi_j = \lambda_j$ if $x_{j-1} < x_j < x_{j+1}$.

(2) $\pi_j = \lambda_j + \tau_j$ if $x_{j-1} > x_j$ and $x_j < x_{j+1}$.

(3) $\pi_j = \lambda_j - \tau_{j+1}$ if $x_{j-1} < x_j$ and $x_j > x_{j+1}$.

(4) $\pi_j = \lambda_j + \tau_j - \tau_{j+1}$ if $x_{j-1} > x_j > x_{j+1}$.

If there is an identity between x_j and x_{j-1} or x_{j+1}, then the rank-ordering of these outcomes, and the belonging case above, can be chosen arbitrarily. The belonging signed capacity satisfies $v(i) = \lambda_i - \tau_{i+1}$ ($\tau_{n+1} = 0$) for all i. For a "connected" set $E = \{i, i+1, \cdots, i+k\}$, $v(E) = \lambda_i + \cdots + \lambda_{i+k} - \tau_{i+k+1}$. Note that v assigns value 0 to the empty set, and value 1 to the entire state space, hence it is a signed capacity indeed. For a general set E, $v(E)$ can be seen to be the sum of the capacities of the separate components of E. Next we give the general formula for the signed capacity.

$$v(E) = \sum_{j \in E} \lambda_j - \sum_{j \in E : j+1 \notin E} \tau_{j+1}.$$

We conclude:

LEMMA 8 *The functional in Equation (3) is a signed Choquet integral.* □

An alternative interpretation can be given, where a sequential Choquet integral is a weighted sum, adjusted for variations between consecutive terms. The following formula is equivalent to a signed Choquet integral after appropriate substitutions, described next.

$$p(s_1)x_1 + \sum_{j=2}^{n} (p(s_j)x_j + \delta_j |x_j - x_{j-1}|). \quad (4)$$

To see that this formula is equivalent to Formula (3), rewrite the latter as

$$\lambda_1 x_1 + \sum_{j=2}^{n} [\lambda_j x_j - \frac{\tau_j}{2}|x_j - x_{j-1}| + \frac{\tau_j}{2}(x_j - x_{j-1})].$$

Then use the following substitutions: $\delta_j = -\tau_j/2$ for $j = 2, \ldots, n$, $p(s_1) = \lambda_1 - \tau_2/2$, $p(s_j) = \lambda_j + \tau_j/2 - \tau_{j+1}/2$ for $j = 2, \ldots, n-1$, and $p(s_n) = \lambda_n + \tau_n/2$. Formula (4) was used by Gilboa (1989) for sequential "nonsigned" Choquet integrals. Yet another, equivalent, formula was used by Shalev (1994, Theorem 1).

THEOREM 9 *Let $\succeq$ be a binary relation on $I\!R^n$. It can be represented by a sequential Choquet integral with respect to a signed capacity if and only if*

(1) $\succeq$ is a weak order;

(2) $\succeq$ is continuous;

(3) $\succeq$ is constant-monotonic;

(4) $\succeq$ satisfies sequential additivity.

The sequential Choquet integral is uniquely determined. □

The functional, characterized in Theorem 9, is of the form characterized by Gilboa (1989) if we add monotonicity: $x \succeq y$ whenever $x_j \geq y_j$ for all j. The case in which all λ_js and τ_js are nonnegative is most interesting for the application to income evaluation. Positive λ_js are characterized by imposing the monotonicity condition only on income profiles x with $x_1 \leq \cdots \leq x_n$, and positive τ_js (for $j \geq 2$) are characterized by the preference condition

$$(x-a,\cdots,x-a,\mathbf{x}-\mathbf{a},\mathbf{x}+\mathbf{b},x+b,\cdots,x+b) \sim (x-a,\cdots,x-a,\mathbf{x},\mathbf{x},x+b,\cdots,x+b)$$

$$\Rightarrow$$

$$(x+b,\cdots,x+b) \succeq (x+b,\cdots,x+b,\mathbf{x}+\mathbf{a}+\mathbf{b},\mathbf{x},x+b,\cdots,x+b),$$

where the $(j-1)$th and jth coordinate have been boldprinted, and a and b are nonnegative. In this preference condition, the second preference has been generated by raising the first $j-1$ incomes by a term $a+b$. That implies a decrease in income in time period j, which generates a lower valuation of the right-hand income profile.

Appendix: Proofs

Proof of Theorem 1. First we demonstrate necessity of conditions (1) and (2). Continuity is obvious. Comonotonic additivity follows because the signed Choquet integral is linear on each comonotonic cone C^ρ and because every comonotonic triple x, y, z is contained in one such cone.

Henceforth, we assume conditions (1) and (2) and prove that V is a signed Choquet integral. We first restrict attention to the comonotonic cone of the form $\{x : x_1 \geq \cdots \geq x_n\}$, associated with the identity permutation ρ. Here V satisfies additivity (i.e., "Cauchy's equation"). Because V is continuous, it is linear on the cone, i.e., there exist real numbers $\pi_1, \ldots, \pi_n$ such that $V(x) = \pi_1 x_1 + \cdots + \pi_n x_n$ on this cone.

Similarly, for every permutation ρ, V satisfies Cauchy's equation on the belonging cone and there exist $\pi_{1,\rho}, \ldots, \pi_{n,\rho}$ such that $V(x) = \pi_{1,\rho} x_1 + \cdots + \pi_{n,\rho} x_n$ on this cone. For any subset E of $\{1, \ldots, n\}$, we define $v(E)$ as follows: First take a permutation ρ such that $E = \{\rho(1), \ldots, \rho(k)\}$, next define $v(E)$ as the sum of $\pi_{1,\rho}, \ldots, \pi_{k,\rho}$. The major conceptual point in this proof is to note that the definition of v is independent of ρ. That easily follows because $v(E) = V(1_E)$ where 1_E denotes the indicator function of E. Therefore, v is well-defined indeed. The π_ks are related to v by the formula

$$\pi_{k,\rho} = v(\pi_{1,\rho}, \ldots, \pi_{k,\rho}) - v(\pi_{1,\rho}, \ldots, \pi_{k-1,\rho}),$$

in agreement with Formula (1), and the theorem follows. □

Proof of Corollary 2. Necessity of conditions (1) – (4) is obvious; comonotonic additivity of $\succeq$ follows from comonotonic additivity of the signed Choquet integral. Hence we assume conditions (1) – (4) and derive the representation.

We identify real numbers (outcomes) and constant acts (constant n-tuples), and first prove that every act has a unique certainty equivalent. Suppose, for contradiction, that there exists an act x such that $x \succ \alpha$ for each constant act α. For any natural number n, $x/n \succ \alpha/n$ ($x/n \preceq \alpha/n$ would imply, because of comonotonicity and comonotonic additivity, $2x/n \preceq \alpha/n + x/n \preceq 2\alpha/n$, and then, by induction, $m \times x/n \preceq m \times \alpha/n$ for all natural numbers m; for $m = n$ a contradiction would result). As this holds for all real numbers α, we in fact have $x/n \succ \beta$ for all real numbers β. Limit taking for $n \to \infty$ and continuity of $\succeq$ then imply $0 \succeq \beta$ for each real number β, contradicting constant monotonicity. Contradiction similarly results if $x \prec \alpha$ for all real α. Hence, for each act x there exist real numbers α, β such that $\alpha \succeq x \succeq \beta$. By continuity, $\{\gamma \in I\!R : \gamma \succeq x\}$ and $\{\gamma \in I\!R : \gamma \preceq x\}$ are closed, we have already seen that both sets are nonempty, hence by connectedness of $I\!R$ they must intersect. Their intersection contains the certainty equivalent of x. It is unique because of constant monotonicity.

We define, for each act x, $V(x)$ as its certainty equivalent. By constant-monotonicity, this function represents $\succeq$. Consider $V(x)$ and $V(y)$, for comonotonic x, y. Then $x \sim V(x)$ implies $x+y \sim V(x)+y$. (Note here that each constant act is comonotonic with each other act.) $y \sim V(y)$ implies $V(x)+y \sim V(x)+V(y)$. Transitivity implies $x+y \sim V(x)+V(y)$. Hence $V(x+y) = V(x)+V(y)$. V is continuous because $\{x : V(x) \geq \alpha\} = \{x : x \succeq (\alpha, \ldots, \alpha)\}$ and $\{x : V(x) \leq \alpha\} = \{x : x \preceq (\alpha, \ldots, \alpha)\}$ are closed for all α, because of continuity of $\succeq$. By Theorem 1, V is a signed Choquet integral.

For uniqueness of the representation, note that the signed capacity v assigns value 1 to the state space, hence the signed Choquet integral assigns value α to each constant act α. That uniquely defines the representing signed Choquet integral as the certainty equivalent of each act. $\square$

Proof of Theorem 3. First we prove equivalence of (1) and (2). Because V is continuous, convexity holds if and only if midpoint convexity holds, therefore we consider midpoint convexity. We have

$$V(x+y) \leq V(x)+V(y) \Leftrightarrow \frac{V(x+y)}{2} \leq \frac{V(x)+V(y)}{2} \Leftrightarrow V(\frac{x+y}{2}) \leq \frac{V(x)+V(y)}{2}$$

where the last step applies positive homogeneity to the left-hand side.

Equivalence of (1) and (3) is well-known, even holding for general functions V; it will not be proved here. We finally turn to the equivalence of (1) and (4). We first prove that convexity of V implies concavity of v. Consider any two subsets A, B of $\{1, \ldots, n\}$. Then $v(A \cup B) + v(A \cap B) = V(1_{A \cup B}) + V(1_{A \cap B}) =$ (because $1_{A \cup B}$ and $1_{A \cap B}$ are comonotonic) $V(1_{A \cup B} + 1_{A \cap B}) =$ (because $1_{A \cup B} + 1_{A \cap B} = 1_A + 1_B$) $V(1_A + 1_B) \leq$ (because V is convex and hence, as shown before, subadditive) $V(1_A) + V(1_B) = v(A) + v(B)$.

We finally turn to the hardest part of the proof, i.e., the demonstration that (4) implies (1). Consider an act $(x_1, \ldots, x_n)$ and any rank-ordering ρ. Let $\pi_1, \ldots, \pi_n$ be the decision weights corresponding to ρ. It is not assumed that the rank-ordering is compatible with $x_1, \ldots, x_n$, it is just any arbitrary rank-ordering. Consider the sum

$$\pi_1 x_1 + \cdots + \pi_n x_n. \tag{5}$$

This sum need not be the V value of x because ρ need not be compatible with x. We next show:

LEMMA 10 *For any given x, (5) attains its maximum over ρ at any ρ compatible with x.*

The proof of the lemma is based on the following idea. Under concavity of v, more decision weight is assigned to a state as it moves up in ranking. Then the sum in (5) gets higher as more decision weight is shifted to the higher outcomes, i.e., as the permutation to generate the decision weights agrees more with the rank-ordering of x.

To prove the lemma in detail, assume that ρ is an arbitrary rank-ordering, and consider a second rank-ordering ρ' that is almost identical to ρ. The only difference is that ρ' reversed two consecutively ordered states, hence $a = \rho(i) = \rho'(i+1)$ and $b = \rho(i+1) = \rho'(i)$ for some i. Let $\pi'_1, \ldots, \pi'_n$ denote the decision weights generated by ρ'. We compare the sum

$$\pi'_1 x_1 + \cdots + \pi'_n x_n \tag{6}$$

with the sum in (5). Obviously, $\pi'_k = \pi_k$ for k different than a or b, and hence

$$\sum_{k \neq a,b} \pi'_k = \sum_{k \neq a,b} \pi_k.$$

Because the total sum of decision weights is always $V(1, \cdots, 1) = v(\{1, \cdots, n\}) - v(\emptyset)$, it follows that

$$\pi'_a + \pi'_b = \pi_a + \pi_b.$$

Consequently, the only difference between the decision weights π_k and π'_k is that some decision weight has been reshifted between states a and b. If $x_a = x_b$, then (5) and (6) yield the same result as it does not matter how the total decision weight for states a and b is distributed between x_a and x_b. Next assume that $x_b > x_a$. Then ρ' agrees better with the rank-ordering of x than ρ. Now

$$\pi'_b \geq \pi_b$$

follows because of concavity of v. This is seen by letting D denote the set of states rank-ordered before a and b, substituting $A = D \cup \{a\}$ and $B = D \cup \{b\}$ in Equation (2), and reshifting terms. Moving state b one up in ranking increases its decision weight, so that under ρ' more of the common decision weight of a and b is assigned to the higher outcome x_b. Therefore, concavity of v implies that (6) $\geq$ (5).

This inequality is the central step in the proof of Lemma 10, and the implication (4)$\Rightarrow$(1). We conclude that making a rank-ordering agree more with the rank-ordering of an act (in the sense of a basic permutation of interchanging two consecutive states), increases the weighted sum assigned to the act which proves Lemma 10. *QED*

Now consider a convex combination $rx + (1-r)y$ of x and y. Then the V value of $rx + (1-r)y$ is

$$\pi_1(rx_1 + (1-r)y_1) + \cdots + \pi_n(rx_n + (1-r)y_n)$$

where the decision weights are derived from the rank-ordering of $rx + (1-r)y$. The V value can be rewritten as

$$r[\pi_1 x_1 + \cdots + \pi_n x_n] + (1-r)[\pi_1 y_1 + \cdots + \pi_n y_n]. \tag{7}$$

By Lemma 10, (7) becomes larger when the decision weights for the coordinates of x are replaced by the decision weights compatible with x, and similar for y, as we saw before. Hence, (7) $\leq rV(x) + (1-r)V(y)$.

We conclude that

$$V(rx + (1-r)y) \leq rV(x) + (1-r)V(y),$$

i.e., V is convex. □

Proof of Lemma 4. That $V \geq V'$ implies $v \geq v'$ follows immediately by restriction to indicator functions. Next assume that $v \geq v'$. The signed Choquet integral $V(x)$ can be rewritten, for ρ compatible with x, as

$$\sum_{j=1}^{n-1} v(\{\rho(1), \ldots, \rho(j)\})(x_{\rho(j)} - x_{\rho(j+1)}) + v(\{\rho(1), \ldots, \rho(n)\})x_{\rho(n)}.$$

This formula shows that $V(x) \geq V'(x)$ if $v \geq v'$.

Note that the equality $v(\{1,\ldots,n\}) = v'(\{1,\ldots,n\})$, is essential in the proof to ensure that $x_{\rho(n)}$ has the same weight for both V and V' in the displayed formula (both if it is positive and if it is negative). □

PROOF OF THEOREM 5. By Lemma 4, $V(x) \geq \sup\{\int x\,d\mu | \mu \in \mathcal{P}(v)\}$. Define μ as the additive set function that assigns to each single state j the value equal to the decision weight of j for a rank-ordering ρ that is compatible with x. By the definition of the signed Choquet integral, $V(x) = \int x\,d\mu$, and $\mu(\{1,\ldots,n\}) = v(\{1,\ldots,n\})$. Hence it remains to be demonstrated that $v \geq \mu$. This follows from Lemma 10, when applied to indicator functions. The supremum is taken at $\mu \in \mathcal{P}(v)$, hence it is a maximum. □

PROOF OF THEOREM 9. Throughout this proof we use the notational convention that, for all $x \in I\!R^n$, $x_0 = -\infty$ and $x_{n+1} = \infty$.

As a tool in the proof, we define a change vector c as an $n+1$-tuple with $c_1 = + = c_{n+1}$, and for each j either c_j is $+$ or c_j is $-$. For every change vector c, C^c contains all x such that for all $j \geq 2$, $x_j \geq x_{j-1}$ if $c_j = +$ and $x_{j-1} \geq x_j$ if $c_j = -$. C^c is sequentially comonotonic. Note that x belongs to several C^cs if $x_j = x_{j-1}$ for some j. The constant income profiles belong to all sets C^c. C^c is the union of all rank-ordered cones whose change vector agrees with c. C^c is a convex cone because weak inequalities are kept under convex combinations.

LEMMA 11 *A set $E \subset I\!R^n$ is sequentially comonotonic if and only if it is contained in one set C^c.*

PROOF. C^c, and thus any of its subsets, is sequentially comonotonic. Next assume that E is any sequentially comonotonic set. We define the change vector c. Consider three, exclusive and exhaustive, cases as follows.

(1) There is $x \in E$ with $x_i > x_{i-1}$. Then, by sequential consistency, $y_i \geq y_{i-1}$ for all $y \in E$. We define $c_i = +$.

(2) There is $x \in E$ with $x_i < x_{i-1}$. Then, by sequential consistency, $y_i \leq y_{i-1}$ for all $y \in E$. We define $c_i = -$ in this case.

(3) $x_i = x_{i-1}$ for all i. Then c_i can be chosen arbitrarily.

We now have $E \subset C^c$. *QED*

LEMMA 12 *A sequential Choquet integral V is additive on any set C^c.*

PROOF. Within one set C^c, we can use the same decision weights π_j, defined before Lemma 8, for all Choquet integral calculations of V. Hence V is additive there. *QED*

After these preparations, we turn to the proof of the theorem. First assume the representation. Necessity of preference conditions (1), (2), and (3) is obvious, and (4) follows from Lemmas 11 and 12.

Next we assume Conditions (1) to (4), and derive the sequential Choquet integral representation. By the proof of Corollary 2, there exists a certainty equivalent $V(x)$ for each income profile x, and V represents preference. In fact, by the Corollary, V is a signed Choquet integral and satisfies additivity within each comonotonic cone. We prove, in a number of steps, that the decision weights of V are as for a sequential Choquet integral.

STEP 1. The decision weights depend only on the change vector c.

We demonstrate that V satisfies additivity on larger domains than comonotonic subsets, i.e., on whole sets C^c. By sequential additivity, for $x, y \in C^c$, $x \sim (\alpha, \cdots, \alpha)$ and $y \sim (\beta, \cdots, \beta)$ implies $x+y \sim (\alpha, \cdots, \alpha)+y \sim (\alpha, \cdots, \alpha)+(\beta, \cdots, \beta)$, which implies $V(x+y) = \alpha + \beta = V(x) + V(y)$. In other words, the continuous functional V satisfies additivity on the convex cone C^c. Hence it is linear there. That implies that there are "decision weights" π_j^c such that $V(x) = \sum_{j=1}^{n} \pi_j^c x_j$ on C^c. By uniqueness, the decision weights of the signed Choquet integral on all comonotonic cones contained within C^c must coincide with the π^cs. Therefore, the term decision weight is justified for these numbers. The reasoning shows that the decision weights in the signed Choquet integral are fully determined by the change vector of an income profile.

STEP 2. The jth decision weight depends only on c_j and c_{j+1}.

We have seen, in signed Choquet integrals, that a decision weight π_j does not depend on all of the rank-ordering, but only on the "dominating set" of states i that are rank-ordered before j. Step 2 in this proof is similar. For j, and a change vector c', consider the convex cone of income profiles x such that

- $x_1 = \cdots = x_{j-1}$;
- x_{j-1} and x_j are ordered in agreement with c'_j (hence $x_j \geq x_{j-1}$ if $c'_j = +$, $x_j \leq x_{j-1}$ if $c'_j = -$);
- x_j and x_{j+1} are ordered in agreement with c'_{j+1};
- $x_{j+1} = \cdots = x_n$.

This cone is at least two-dimensional (unless $n = 1$, but for that case the theorem is trivial). It is the intersection of all cones C^c that have $c_j = c'_j$ and $c_{j+1} = c'_{j+1}$. Hence $V(x) = ax_{j-1} + bx_j + dx_{j+1}$ for uniquely determined weights a, b, d on this cone. It follows that $\pi_j^c = b$ for any change vector c with $c_j = c'_j$ and $c_{j+1} = c'_{j+1}$ (and $a = \pi_1^c + \ldots + \pi_{j-1}^c$, $c = \pi_{j+1}^c + \ldots + \pi_n^c$). Hence the decision weight π_j^c does not depend on all of the change pattern c, but only on c_j and c_{j+1}.

STEP 3. If c and c' differ only on the jth coordinate, say $c_j = +$ and $c'_j = -$, then $\pi_j^{c'} - \pi_j^c$ is independent of c and c'; this difference is called the jth *decision weight penalty*, and is denoted as τ_j.

We will now prove that τ_j is independent of c and c' indeed, for c and c' as just described. τ_j can be interpreted as the decision weight, taken off π_{j-1} when outcome x_j falls below x_{j-1}. Because each decision weight π_i depends only on the ith and $(i+1)$th coordinate of the change vector, $\pi_i^c = \pi_i^{c'}$ for all $i < j - 1$ and $i > j$. This implies that $\pi_{j-1}^c + \pi_j^c = \pi_{j-1}^{c'} + \pi_j^{c'}$, and hence $\pi_{j-1}^{c'} = \pi_{j-1}^c - \tau_j$. In other words, changing the change vector c only on its jth coordinate generates a decision weight shift from the $(j-1)$th coordinate to the jth coordinate and leaves all other decision weights unaffected.

Because the decision weights of the jth coordinate are independent of c's coordinates other than c_j and c_{j+1}, $\pi_j^{c'}$ and π_j^c, thus also their difference τ_j, depend only on c_j and c_{j+1}. In particular, τ_j is independent of $c_1, \cdots, c_{j-1}$. Similarly, $\pi_{j-1}^{c'}$ and π_{j-1}^c, thus also their difference τ_j, depend only on c_{j-1} and c_j. In particular, τ_j is independent of $c_{j+1}, \cdots, c_n$. We conclude that τ_j is independent of $c_1, \ldots, c_{j-1}, c_{j+1}, \ldots, c_n$, so it is independent of c and c'.

STEP 4. Definition of λ_js.

Define, for $c' = (+, \cdots, +)$, $\lambda_j = \pi_j^{c'}$ for all j. In the context of income evaluation, these are the decision weights for the empirically most favorable case where income always increases.

The decision weight π_j^c depends only on c_j and c_{j+1}, and is as follows.

(1) If $c_j = +$ and $c_{j+1} = +$, then $\pi_j^c = \lambda_j$, by the definition of λ_j.

(2) If $c_j = -$ and $c_{j+1} = +$, then $\pi_j^c = \lambda_j + \tau_j$, by (1) and Step 3.

(3) If $c_j = +$ and $c_{j+1} = -$, then $\pi_j^c = \lambda_j - \tau_{j+1}$, by (1) and Step 3.

(4) If $c_j = -$ and $c_{j+1} = -$, then $\pi_j^c = \lambda_j + \tau_j - \tau_{j+1}$, by (2) (or (3)) and Step 3.

By substitution (see also before Lemma 8) it now follows that V is as in Formula (3), i.e., it is a sequential Choquet integral. Uniqueness follows from Corollary 2. □

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