Nonregular Graphs with Three Eigenvalues

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Abstract. We study nonregular graphs with three eigenvalues. We determine all the ones with least eigenvalue −2, and give new infinite families of examples.

1. Introduction

In this paper we have a look at the graphs that are generalizations of strongly regular graphs (cf. [3, 13]) by dropping the regularity condition. More specifically, we consider graphs of which the adjacency matrices have three distinct eigenvalues. Seidel (cf. [12]) did a similar thing for the Seidel matrix by introducing strong graphs, which turned out to have an easy combinatorial characterization. Similarly, a nice combinatorial characterization is found for graphs with three Laplace eigenvalues [7]. For more on graphs with *few* eigenvalues we refer to [6].

Regular graphs with three eigenvalues are well known to be strongly regular. Therefore we focus on the nonregular graphs with three (adjacency) eigenvalues. Here the combinatorial simplicity seems to disappear with the regularity. This all lies in the algebraic consequence that the all-one vector is no longer an eigenvector. The Perron-Frobenius eigenvector still contains a lot of information, but the problem is to derive this eigenvector from the spectrum.

Earlier results on nonregular graphs with three eigenvalues were found by Bridges and Mena [2] and Muzychuk and Klin [11]. In this paper we determine all the ones with least eigenvalue −2, and give new families of examples by using symmetric designs, affine designs, antipodal covers of the complete graph and systems of linked symmetric designs.

2. Examples

A large family of (in general) nonregular examples is given by the complete bipartite graphs $K_{m,n}$ with spectrum $\{[\sqrt{mn}]^1, [0]^{m+n-2}, [-\sqrt{mn}]^1\}$. Note that a connected graph with three eigenvalues has diameter two, hence a bipartite one must be complete bipartite. Other examples were found by Bridges and Mena [2] and Muzychuk and Klin [11], most of them being cones.

2.1. Cones

A *cone* over a graph G is obtained by adding a vertex to G that is adjacent to all vertices of G. If G is a strongly regular graph on v vertices, with degree k and smallest eigenvalue s, then the cone over G is a graph with three eigenvalues if and only if $s(k - s) = -v$ (cf. [11]). This condition is satisfied by infinitely many strongly regular graphs, which implies that there are infinitely many cones with three eigenvalues. A small example is given by the cone over the Petersen graph, with spectrum $\{[5]^1, [1]^5, [-2]^5\}$ (see Figure 1). Bridges and Mena [2] obtained results on cones with distinct eigenvalues λ_0, λ_1 and $-\lambda_1$. They proved that such graphs are cones over strongly regular (v, k, λ, λ) graphs with three possible exceptions (for the spectra): there is an example with spectrum $\{[21]^1, [3]^{19}, [-3]^{26}\}$, an example with spectrum $\{[56]^1, [4]^{41}, [-4]^{55}\}$, and the case with spectrum $\{[204]^1, [6]^{127}, [-6]^{161}\}$ is open.
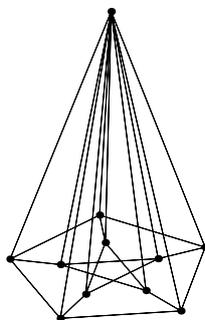

Figure 1. The cone over the Petersen graph

2.2. Switching in strongly regular graphs

Other examples can be constructed by switching in a strongly regular graph. Switching with respect to some subset of the vertices means that we interchange the edges and nonedges between the subset and its complement. Muzychuk and Klin [11] found parameter conditions for switching in a strongly regular graph to obtain a nonregular graph with three eigenvalues. Moreover, they proved that the only graph that can be obtained from switching in a triangular graph is the one obtained by switching in $T(9)$ with respect to an 8-clique. This gives a graph with spectrum $\{[21]^1, [5]^7, [-2]^{28}\}$.

We find new examples by switching with respect to an 8-clique in the strongly regular graph that is obtained from a polarity in Higman's symmetric 2-(176, 50, 14) design, and by switching with respect to three vertex-disjoint 6-cliques in the strongly regular Zara graph with parameters (126, 45, 12, 18), and by switching with respect to 8 vertex-disjoint 8-cliques in the point graph (with parameters (512, 196, 60, 84)) of the partial geometry $pg(7, 27, 3)$. Also, switching with respect to two vertex-disjoint 16-cliques in the Latin square graph $L_7(16)$, 5 vertex-disjoint 25-cliques in $L_{12}(25)$, 6 vertex-disjoint 27-cocliques

in $L_{15}(27)$, 27 vertex-disjoint 243-cliques in $L_{120}(243)$ and 76 vertex-disjoint 361-cliques in $L_{180}(361)$ give examples of nonregular graphs with three eigenvalues. A more complicated example is constructed by Martin [private communication]. Take the strongly regular $(105, 72, 51, 45)$ graph on the flags (incident point-line pairs) of $PG(2, 4)$, where two distinct flags (p_1, l_1) and (p_2, l_2) are adjacent if $p_1 = p_2$ or $l_1 = l_2$ or ($p_1 \notin l_2$ and $p_2 \notin l_1$). Now switching with respect to a set of 21 flags with the property that every point and every line is in precisely one flag (such a set exists by elementary combinatorial theory since it corresponds to a perfect matching in the incidence graph of $PG(2, 4)$) yields a nonregular graph with spectrum $\{[60]^1, [9]^{21}, [-3]^{83}\}$.

2.3. Symmetric and affine designs

A new family of nonregular graphs with three eigenvalues is constructed from symmetric 2-$(q^3 - q + 1, q^2, q)$ designs. Such designs exist if q is a prime power and $q - 1$ is the order of a projective plane (cf. [1]). Take the bipartite incidence graph of such a design and add edges between all blocks. The resulting graph has spectrum $\{[q^3]^1, [q - 1]^{(q-1)q(q+1)}, [-q]^{(q-1)q(q+1)+1}\}$. The smallest example is derived from the complement of the Fano plane, and we find a graph with spectrum $\{[8]^1, [1]^6, [-2]^7\}$. The next case comes from 2-$(25, 9, 3)$ designs. Denniston [8] found that there are exactly 78 such designs, and so there are at least 78 graphs with spectrum $\{[27]^1, [2]^{24}, [-3]^{25}\}$. By using the same methods as in the proof of Theorem 7, we were able to show that these are all the graphs with this spectrum.
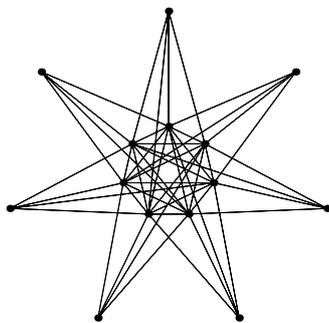
Figure 2. The graph derived from the complement of the Fano plane

Next, consider the design on the points and planes of the affine geometry $AG(3, q)$ of dimension 3 over the field with q elements. Take the bipartite incidence graph of this design, and add an edge between two planes if they intersect (in q points). This graph is nonregular with three eigenvalues. In fact, for any affine 2-$(q^3, q^2, q + 1)$ design this construction gives a graph with spectrum $\{[q^3 + q^2 + q]^1, [q]^{q^3 - 1}, [-q]^{q^3 + q^2 + q}\}$. The smallest example $(q = 2)$ of this infinite family is on 22 vertices with spectrum $\{[14]^1, [2]^7, [-2]^{14}\}$, and was already constructed by Bridges and Mena [2]. Note that the

graph derived from the complement of the Fano plane is an induced subgraph of this graph.

3. Nonintegral eigenvalues

For all known nonregular examples, except for the complete bipartite ones, all eigenvalues are integral. We shall prove that the only graphs with three eigenvalues for which the largest eigenvalue is not integral, are the complete bipartite graphs. Without proof we give the following useful lemma [folklore].

LEMMA 1. *Let G be a connected graph on v vertices, with eigenvalues λ_i and corresponding multiplicities m_i, and with largest eigenvalue λ_0. Denote by k_{max} and k_{ave} the largest and average vertex degree, respectively. Then $vk_{ave} = \sum_i m_i \lambda_i^2 \leq v\lambda_0$ with equality if and only if G is regular. Moreover, $\lambda_0 \leq k_{max}$, also with equality if and only if G is regular.*

PROPOSITION 2. *Let G be a connected graph with three distinct eigenvalues of which the largest is not an integer. Then G is a complete bipartite graph.*

Proof. Suppose G has v vertices, adjacency matrix A with largest eigenvalue λ_0 and remaining eigenvalues λ_1 and λ_2. Since λ_0 is simple and not integral, it follows that at least one of these remaining eigenvalues is also simple and nonintegral. If we have only three vertices, then there is only one connected, noncomplete graph: $K_{1,2}$. So we may assume to have more than three vertices. In this case the remaining eigenvalue is of course not simple, and it follows that λ_0 and say λ_2 are of the form $\frac{1}{2}(a \pm \sqrt{b})$, with a, b integral, and then λ_1 is integral. Moreover, since $\lambda_2 \geq -\lambda_0$, we must have $a \geq 0$. Since the adjacency matrix of G has zero trace, it follows that $a + (v - 2)\lambda_1 = 0$, so a is a multiple of $v - 2$, and $\lambda_1 \leq 0$.

If $a = \lambda_1 = 0$, then G is bipartite, hence complete bipartite. If $\lambda_1 = -1$, then $a = v - 2$, and it follows that -1 is the smallest eigenvalue of G, otherwise we would have $\lambda_0 > v - 1$, which is a contradiction. But then $A + I$ is a positive semidefinite matrix of rank two, and we would have that G is the disjoint union of two cliques, which is again a contradiction. If $\lambda_1 = -2$, then $a = 2(v - 2)$, and it follows that -2 is the smallest eigenvalue of G. Now $A + 2I$ is positive semidefinite of rank two, which cannot be the case. For the remaining case we have $\lambda_1 \leq -3$, and then $a \geq 3(v - 2)$. From $\lambda_0 \leq v - 1$, we now find that we can have at most three vertices, and so the proof is finished. $\square$

So if we have a graph with three eigenvalues, which is not a complete bipartite graph, then we know that its largest eigenvalue is integral. The remaining two eigenvalues, however, can still be nonintegral, with many of the strongly regular conference graphs as examples. The following proposition reflects what is known as the "half-case" for strongly regular graphs.

PROPOSITION 3. *Let G be a connected graph on v vertices with three eigenvalues* $\lambda_0 > \lambda_1 > \lambda_2$, *which is not a complete bipartite graph. If not all eigenvalues are integral, then v is odd and* $\lambda_0 = \frac{1}{2}(v - 1)$, $\lambda_1, \lambda_2 = -\frac{1}{2} \pm \frac{1}{2}\sqrt{b}$, *for some* $b \equiv 1 \pmod 4$, $b \leq v$, *with equality if and only if G is strongly regular. Moreover, if* $v \equiv 1 \pmod 4$ *then all vertex degrees are even, and if* $v \equiv 3 \pmod 4$ *then* $b \equiv 1 \pmod 8$.

Proof. According to the previous proposition λ_0 is integral, so λ_1 and λ_2 must be of the form $\frac{1}{2}(a \pm \sqrt{b})$, with a, b integral, with the same multiplicity $\frac{1}{2}(v - 1)$. Since the adjacency matrix has zero trace, we have $\lambda_0 + \frac{1}{2}a(v - 1) = 0$. Since $0 < \lambda_0 < v - 1$, and a is integral, we must have $\lambda_0 = \frac{1}{2}(v - 1)$, $a = -1$. Now $\lambda_1\lambda_2$ is integral, and it follows that $b \equiv 1 \pmod 4$. Moreover, by Lemma 1 we have that the average vertex degree $k_{\text{ave}} = (\lambda_0^2 + \frac{1}{2}(v - 1)(\lambda_1^2 + \lambda_2^2))/v = \frac{1}{2}\lambda_0(v + b)/v$ is at most λ_0, with equality if and only if G is strongly regular. This inequality reduces to $b \leq v$.

From the equation $(A - \lambda_0 I)(A - \lambda_1 I)(A - \lambda_2 I) = O$, we find that

$$A^3 = \frac{1}{2}(v - 3)A^2 + (\frac{1}{2}(v - 1) + \frac{1}{4}(b - 1))A - \frac{1}{8}(v - 1)(b - 1)I.$$

The diagonal element of this matrix corresponding to vertex x counts twice the number of triangles Δ_x through x. Thus we find that

$$\Delta_x = \frac{1}{4}(v - 3)d_x - \frac{1}{16}(v - 1)(b - 1),$$

where d_x is the vertex degree of x. Since Δ_x is integral we find that if $v \equiv 1 \pmod 4$, then d_x must be even, for every vertex x. If $v \equiv 3 \pmod 4$, then we must have $b \equiv 1 \pmod 8$. □

Of course, also if the eigenvalues are integral, we find restrictions for the degrees from the expression for Δ_x.

PROPOSITION 4. *Let G be a graph with three integral eigenvalues* λ_0, λ_1 *and* λ_2. *If all three eigenvalues are odd, then all vertex degrees are odd. If one of them is odd and two are even, then all vertex degrees are even. Moreover, if a vertex x has degree d_x, then the number of triangles through x is given by* $\Delta_x = \frac{1}{2}(\lambda_0 + \lambda_1 + \lambda_2)d_x + \frac{1}{2}\lambda_0\lambda_1\lambda_2$.

Returning to the graphs of Proposition 3, we should mention that although we do not know any nonregular example, we might consider the cone over the Petersen graph (with $v = 11$, $b = 9$) as one. However, we should be careful here. The nonbipartite graphs with nonintegral eigenvalues must have at least three distinct vertex degrees, as we shall see in the next section.

4. The Perron-Frobenius eigenvector

An important property of connected graphs with three eigenvalues is that $(A - \lambda_1 I)(A - \lambda_2 I)$ is a rank one matrix. It follows that we can write

$$(A - \lambda_1 I)(A - \lambda_2 I) = \alpha\alpha^T, \text{ with } A\alpha = \lambda_0\alpha.$$

Moreover, from the Perron-Frobenius theorem it follows that the Perron-Frobenius eigenvector α is a positive eigenvector, that is, all its components are positive. From the quadratic equation we derive that

$d_x = -\lambda_1\lambda_2 + \alpha_x^2$ is the degree of vertex x,

$\lambda_{x,y} = \lambda_1 + \lambda_2 + \alpha_x\alpha_y$ is the number of common neighbours of x and y, if they are adjacent,

$\mu_{x,y} = \alpha_x\alpha_y$ is the number of common neighbours of x and y, if they are not adjacent.

If we assume G not to be complete bipartite, so that $\lambda_1 + \lambda_2$ and $\lambda_1\lambda_2$ are integral, it follows that $\alpha_x\alpha_y$ is an integer for all x and y. We immediately see that this imposes strong restrictions for the possible degrees that can occur. We also see that if the graph is regular, then we have a strongly regular graph.

4.1. Two vertex degrees

Now suppose that G has only two vertex degrees (which is the case in most known nonregular examples), say k_1 and k_2, with respective α_1 and α_2. Now fix a vertex x of degree k_1. Let $k_{1,1}$ and $k_{1,2}$ be the numbers of vertices of degree k_1 and k_2, respectively, that are adjacent to x. Then it follows that $k_{1,1} + k_{1,2} = k_1$ and since $A\alpha = \lambda_0\alpha$, it follows that $k_{1,1}\alpha_1 + k_{1,2}\alpha_2 = \lambda_0\alpha_1$. These two equations uniquely determine $k_{1,1}$ and $k_{1,2}$, and the solution is independent of the chosen vertex x of degree k_1. Similarly we find $k_{2,1}$ and $k_{2,2}$, and it follows that the partition of the vertices according to their degrees is regular. This imposes the restriction $n_1k_{1,2} = n_2k_{2,1}$, where n_i is the number of vertices of degree k_i. Note that the equations $n_1 + n_2 = v$, and $n_1k_1 + n_2k_2 = \sum_i m_i\lambda_i^2$, determine n_1 and n_2 from the spectrum and k_1 and k_2. Note also that the quotient matrix with respect to the regular partition has an eigenvalue λ_0, and hence also another integral eigenvalue ($k_{1,1} + k_{2,2} - \lambda_0$). This implies that also the graph has this other integral eigenvalue (with eigenvector $(k_{1,2}\alpha_2,..., k_{1,2}\alpha_2, -k_{2,1}\alpha_1,..., -k_{2,1}\alpha_1)^T$, the first part corresponding to the vertices of degree k_1). Thus we find that if G is nonbipartite with nonintegral eigenvalues, then it has at least three distinct vertex degrees.

4.2. Cones

If we assume that G is a cone, say over H, then we can prove that we have at most three distinct degrees. Indeed, take a vertex with degree $k_1 = v - 1$, with corresponding α_1, and suppose we have another vertex of degree k_i (with corresponding α_i). Now the common neighbours of these vertices are all neighbours of the latter except the first vertex. So $k_i - 1 = \lambda_{1,i} = \alpha_1\alpha_i + \lambda_1 + \lambda_2$. But also $k_i = -\lambda_1\lambda_2 + \alpha_i^2$, and so we get a quadratic equation for k_i, and so k_i can take at most two values. If k_i takes only one value, then we easily see that H must be strongly regular. If we have precisely two other degrees, say k_2, k_3, with respective α_2, α_3, then it follows from the quadratic equation that $\alpha_1 = \alpha_2 + \alpha_3$. Here it also follows quite easily that the partition of the vertices according to their degrees is regular. Bridges and Mena [2] used this to show that there are only three parameter sets for cones with eigenvalues λ_0, $\pm\lambda_1$ over a nonregular graph.

4.3. Three vertex degrees

When we are trying to construct, or prove nonexistence of a graph G with given spectrum, and with possible vertex degrees k_i (and corresponding α_i), we have the following restrictions. If n_i is the number of vertices of degree k_i, then first of all we have the restrictions $\sum_i n_i = v$, and $\sum_i n_i k_i = \sum_i m_i \lambda_i^2$, as we have seen before. Another restriction is obtained by adding the entries in the vector equation $A\alpha = \lambda_0\alpha$. Thus we obtain $\sum_i n_i k_i \alpha_i = \underline{1}^T A\alpha = \underline{1}^T \lambda_0 \alpha = \lambda_0 \sum_i n_i \alpha_i$, hence $\sum_i n_i(k_i - \lambda_0)\alpha_i = 0$.

If G has three possible vertex degrees k_1, k_2 and k_3, then the respective numbers of vertices n_i are determined by the three equations given. The integrality of the numbers n_i is a further obvious, but strong, restriction. Furthermore, we shall prove now that also here the partition of the vertices according to their degrees is regular. Let $k_{i,j}$ be the number of vertices of degree j adjacent to a vertex of degree i. Fix i, then obviously $\sum_j k_{i,j} = k_i$, and by inspecting (an entry corresponding to a vertex of degree k_i in) the equation $A\alpha = \lambda_0\alpha$ we find that $\sum_j \alpha_j k_{i,j} = \alpha_i \lambda_0$. The third equation, which together with these two will determine all $k_{i,j}$ (in case of three vertex degrees), is derived from inspecting the equation $A^2\underline{1} = (\lambda_1 + \lambda_2)A\underline{1} - \lambda_1\lambda_2 I\underline{1} + \alpha\alpha^T\underline{1}$. Now we find that

$$\sum_j k_j k_{i,j} = (\lambda_1 + \lambda_2)k_i - \lambda_1\lambda_2 + (\sum_j n_j \alpha_j)\alpha_i.$$

Hence if we have only three degrees then the partition is regular. The $k_{i,j}$ thus found must of course be integral, and furthermore $n_i k_{i,j} = n_j k_{j,i}$ and $0 \le k_{i,j} \le n_j - \delta_{ij}$.

4.4. Four vertex degrees

Another restriction (the fourth) on the numbers n_i is found by adding all entries in the matrix equation $(A - \lambda_1 I)(A - \lambda_2 I) = \alpha\alpha^T$. Thus we find that

$$(\sum_i n_i \alpha_i)^2 = \underline{1}^T \alpha \alpha^T \underline{1} = \underline{1}^T (A - \lambda_1 I)(A - \lambda_2 I)\underline{1} = \sum_i n_i (k_i - \lambda_1)(k_i - \lambda_2).$$

If G has (at most) four vertex degrees, then after substitution of the other three equations for n_i (from Section 4.3), this gives a quadratic equation for, say, n_4. Hence there are at most two possibilities for the numbers n_i. These observations will reduce the number of cases to be checked in the classification of graphs in the root system E_8 in the next section.

5. Graphs with least eigenvalue –2

The results from Section 3 imply that the only connected graphs with three eigenvalues, all greater than –2 are the complete bipartite graphs $K_{1,2}$ and $K_{1,3}$ and graphs from Proposition 3 with $b = 5$. However, here we can only have the strongly regular 5-cycle C_5, which is the unique graph with spectrum $\{[2]^1, [-\frac{1}{2} + \frac{1}{2}\sqrt{5}]^2, [-\frac{1}{2} - \frac{1}{2}\sqrt{5}]^2\}$.

PROPOSITION 5. *If G is a connected graph with three distinct eigenvalues, all greater than –2, then G is either $K_{1,2}$, $K_{1,3}$, or C_5.*

Proof. By the previous remarks, besides $K_{1,2}$ and $K_{1,3}$ we only have to check spectra from Proposition 3 with $b = 5$. First suppose $v > 9$ (Note that $v \equiv 1 \pmod 4$). Then for the average vertex degree we have $k_{ave} < \frac{1}{2}(v - 1)$. Since the vertex degrees must be even, there must be a vertex x of degree $d_x \leq \frac{1}{2}(v - 1) - 2$. If $d_x > 2$, then for the number of triangles Δ_x through x we have

$$\Delta_x = \frac{1}{4}(v - 3)(d_x - 1) - \frac{1}{2} > \frac{1}{4}(v - 5)(d_x - 1) \geq \binom{d_x}{2},$$

which is a contradiction. Also if $d_x = 2$, then $\Delta_x = \frac{1}{4}(v - 5) > 2$, a contradiction. The case $v = 9$ can be excluded by the following arguments, using the Perron-Frobenius eigenvector α. Here it follows that there must be a vertex x of degree 2, and so with $\alpha_x = 1$. Now α is an integral vector, implying that the vertex degrees can only take values 2, 5, 10,... But the vertex degrees must be even, and at most 8, so it follows that the graph is regular, which is a contradiction. □

Muzychuk and Klin [11] posed the problem of classifying all graphs with three eigenvalues, all of which are at least –2. By the characterization of Cameron, Goethals, Seidel and Shult [5], it follows that such a graph is a generalized line graph or can be

represented by roots in the lattice E_8. Using this, we find the following.

THEOREM 6. *If G is a connected graph with three distinct eigenvalues, all at least −2, then G is isomorphic to one of $K_{1,2}$, $K_{1,3}$, $K_{1,4}$, C_5, $L_2(n)$, $n \geq 2$, $T(n)$, $n \geq 4$, or $CP(n)$, $n \geq 2$, or G is represented by a subset of E_8.*

Proof. First, suppose that G is a connected line graph, not C_5 or $K_{1,2}$, of some graph H, and G has three eigenvalues, say $\lambda_0 > \lambda_1 > \lambda_2 = -2$. Here we may assume that H is connected. Then the adjacency matrix A of G can be written as $A = N^T N - 2I$, where N is the vertex-edge incidence matrix of H. Now $NN^T = D + B$, where D is the diagonal matrix of vertex degrees in H, and B is the adjacency matrix of H. It follows that $D + B$ has eigenvalues $\lambda_0 + 2$, $\lambda_1 + 2$, and possibly 0. Suppose 0 is an eigenvalue with eigenvector u. Then $N^T u = \underline{0}$. This implies that if i and j are adjacent in H, then $u_i = -u_j$. So H is bipartite. Moreover, since $D + B$ has three distinct eigenvalues, it follows that H has diameter at most two (the diameter is easily proven to be smaller than the number of distinct eigenvalues of $D + B$), so H must be a complete bipartite graph $K_{m,n}$. Since the line graph of a nonregular complete bipartite graph has four distinct eigenvalues (unless m or n equals one, then we get a complete graph, H must be the complete bipartite graph $K_{n,n}$, $n \geq 2$, with the lattice graph $L_2(n)$ as line graph. Now suppose that 0 is not an eigenvalue. Then $D + B$ has only two distinct eigenvalues, and it follows that H is a complete graph K_n, with the triangular graph $T(n)$ as line graph.

Next, we assume that G is a generalized line graph $L(H; a_1,..., a_m)$ (where m is the number of vertices of H), which is not a line graph, so at least one of the a_i is nonzero. Now G can be represented in $\mathbb{R}^n$, where $n = m + \sum_i a_i$, as follows. Take $\{e_{i,j} \mid i = 1,..., m, j = 0,..., a_i\}$ as orthonormal basis of $\mathbb{R}^n$, then we represent the vertices of G by the vectors $e_{i,0} + e_{j,0}$ for all edges $\{i, j\}$ in H, and the vectors $e_{i,0} + e_{i,j}$ and $e_{i,0} - e_{i,j}$ for all $i = 1,..., m$, $j = 1,..., a_i$, any two of them being adjacent if and only if they have inner product one. In matrix form, if N is the generalized $(0, \pm1)$-incidence matrix, that is, with rows representing the basis of $\mathbb{R}^n$, and columns representing the vertices of G, then $A = N^T N - 2I$ is the adjacency matrix of G. Now suppose that G has distinct eigenvalues $\lambda_0 > \lambda_1 > \lambda_2 = -2$, then NN^T has eigenvalues $\lambda_0 + 2$, $\lambda_1 + 2$, and possibly 0. Suppose we have an eigenvalue 0, say with eigenvector u. Then $N^T u = \underline{0}$, so if i and j are adjacent in H, then $u_{i,0} = -u_{j,0}$. For i with a_i nonzero, we have that $u_{i,0} = -u_{i,j}$, and $u_{i,0} = u_{i,j}$ for $j = 1,..., a_i$. So $u_{i,j} = 0$, and since we may assume H to be connected, it follows that $u = \underline{0}$, so 0 is not an eigenvalue of NN^T. Now let us have a closer look at this matrix. After rearrangement of the rows of N, it follows that

$$NN^T = \begin{pmatrix} D + B & O \\ O & 2I \end{pmatrix},$$

where D is the diagonal matrix with entries $D_{ii} = d_i + 2a_i$, where d_i is the vertex degree of

i in *H*, and *B* is the adjacency matrix of *H*. Thus it follows that $\lambda_1 = 0$ and that $D + B$ has distinct eigenvalues $\lambda_0 + 2$ (with multiplicity one) and (possibly) 2. If *H* is a graph on one vertex, then there are no further restrictions, and *G* then is a cocktail party graph $CP(n)$. Otherwise, *H* is a complete graph (since the diameter is one), and since $D + B - 2I$ is a rank one matrix, it follows that $D = 3I$. Since a_i is nonzero for some *i*, it now follows that $d_i = a_i = 1$ for all *i*. But then *H* is a single edge, and *G* is $K_{1,4}$. $\qquad\square$

The strongly regular graphs with all eigenvalues at least −2 have already been classified by Seidel (cf. [5]). Besides the 5-cycle, the lattice graphs, the triangular graphs and the cocktail party graphs, there are the Petersen graph, the complement of the Clebsch graph, the Shrikhande graph, the complement of the Schläfli graph, and the three Chang graphs.

In Section 2 we already saw some nonregular graphs with three eigenvalues, all of which are at least −2. Besides these, we also mention the cones over the lattice graph $L_2(4)$, the Shrikhande graph, the triangular graph $T(8)$ and the three Chang graphs. Moreover, we have the following graph with spectrum $\{[11]^1, [3]^7, [-2]^{16}\}$, which was found by Spence [private communication], and which is strongly related to the strongly regular lattice graph $L_2(5)$ (which has spectrum $\{[8]^1, [3]^8, [-2]^{16}\}$). Fix a vertex *x* in $L_2(5)$. Its set of neighbours can be partitioned into two 4-sets, each inducing a 4-clique. Now delete *x*, and (switch) interchange edges and nonedges between one of the 4-sets and the set of nonneighbours of *x*.

THEOREM 7. *The only connected graphs with three eigenvalues*, *all of which are at least −2*, *and which are not strongly regular or complete bipartite are the cone over the Petersen graph*, *the graph derived from the complement of the Fano plane* (*see Figure* 2), *the cone over the Shrikhande graph*, *the cone over the lattice graph* $L_2(4)$, *the graph on the points and planes of AG*(3, 2), *the graph related to the lattice graph* $L_2(5)$ (*see above*), *the cones over the Chang graphs*, *the cone over the triangular graph* $T(8)$, *and the graph obtained by switching in* $T(9)$ *with respect to an* 8-*clique*.

Proof. By Theorem 6 we only have to check the graphs that are represented in E_8. First we mention that a graph that is represented by a subset of E_8 has at most 36 vertices and vertex degrees at most 28, thus there are finitely many (see [4]). Note that both bounds are tight for the example obtained by switching in $T(9)$.

For graphs which are represented in the sublattice E_6, and which are not generalized line graphs, the multiplicity of the eigenvalue −2 is $v − 6$, where *v* is the number of vertices. Consequently such graphs have spectrum $\{[2(v-6)-5\lambda_1]^1, [\lambda_1]^5, [-2]^{v-6}\}$, and we also may assume that $\lambda_1 \geq 1$ (a connected graph with exactly one positive eigenvalue must be a complete multipartite graph). Using the inequality $\sum_i m_i \lambda_i^2 < v\lambda_0$ (Lemma 1), we find that $\lambda_1 = 1$ and $v = 11, 12, 13$.

For graphs which are represented in the sublattice E_7 (and which are not generalized line graphs or represented in E_6), the multiplicity of the eigenvalue −2 is $v − 7$. Consequently they have spectrum $\{[2(v-7)-6\lambda_1]^1, [\lambda_1]^6, [-2]^{v-7}\}$. Here we find that $\lambda_1 = 1, 2$ or 3. If $\lambda_1 = 1$, then $v = 12,..., 16$. In the case $\lambda_1 = 2$, we find $v = 17,..., 20$. If

$\lambda_1 = 3$, then $v = 22, 23, 24$.

For the remaining nonregular graphs represented by roots in E_8, we find possible spectra $\{[2(v-8) - 7\lambda_1]^1, [\lambda_1]^7, [-2]^{v-8}\}$ with $\lambda_1 = 1$ and $v = 13,..., 19$, $\lambda_1 = 2$ and $v = 18,..., 25$, $\lambda_1 = 3$ and $v = 23,..., 30$, $\lambda_1 = 4$ and $v = 29,..., 35$, or $\lambda_1 = 5$ and $v = 34,..., 36$.

By computer we checked these spectra for possible vertex degrees, using the property that the product of any two entries of the Perron-Frobenius eigenvector α is integral. We checked that in a particular sequence of degrees, we have $k_{min} < \lambda_0 < k_{max} \leq 28$, and we checked for a possible degree d_x that the corresponding number of triangles Δ_x obtained from Proposition 4 satisfies the condition $\Delta_x \leq \frac{1}{2}d_x(d_x - 1)$. Of course, we also checked that all vertex degrees were even in the cases with two even eigenvalues (by Proposition 4). If in a particular sequence only two vertex degrees remained, then these were checked for the conditions which follow from the fact that in that case the vertices of distinct degrees induce a regular partition. Similarly we computed the numbers n_i of degree k_i from the equations in Sections 4.3 and 4.4, and checked their integrality, in case a sequence of degrees contained three or four degrees. If only three degrees remained, then these were also checked for the further conditions of Section 4.3. In none of the possible sequences of degrees more than four degrees appeared. What remains are the following possibilities. By a $*$ we denote that a graph exists.

v	spectrum	degrees	
* 11	$\{[5]^1, [1]^5, [-2]^5\}$	4 10	($n_4 = 10$, $n_{10} = 1$, $k_{4,10} = 1$)
* 14	$\{[8]^1, [1]^6, [-2]^7\}$	4 10	($n_4 = 7$, $n_{10} = 7$, $k_{4,10} = 4$)
* 17	$\{[8]^1, [2]^6, [-2]^{10}\}$	7 16	($n_7 = 16$, $n_{16} = 1$, $k_{7,16} = 1$)
* 22	$\{[14]^1, [2]^7, [-2]^{14}\}$	7 16	($n_7 = 8$, $n_{16} = 14$, $k_{7,16} = 7$)
* 24	$\{[11]^1, [3]^7, [-2]^{16}\}$	7 10 15 22	($n_7 = 4$, $n_{10} = 16$, $n_{15} = 4$, $n_{22} = 0$, $k_{7,10} = 4$, $k_{7,15} = 0$, $k_{10,15} = 3$)
* 29	$\{[14]^1, [4]^7, [-2]^{21}\}$	13 28	($n_{13} = 28$, $n_{28} = 1$, $k_{13,28} = 1$)
30	$\{[16]^1, [4]^7, [-2]^{22}\}$	9 12 17 24	($n_9 = 2$, $n_{12} = 9$, $n_{17} = 18$, $n_{24} = 1$)
* 36	$\{[21]^1, [5]^7, [-2]^{28}\}$	12 18 28	($n_{12} = 0$, $n_{18} = 28$, $n_{28} = 8$, $k_{18,28} = 6$)

In E_6 there is one case, with $v = 11$. Here we must have one vertex of degree 10 and ten vertices of degree 4. It is obvious that this must be the cone over the Petersen graph. In the first case in E_7, with $v = 14$, we find seven vertices of degree 4 and seven vertices of degree 10. Moreover, the vertices of degree 4 induce a coclique, and the vertices of degree 10 induce a clique. Since any two vertices of degree 4 have two common neighbours (this follows from the Perron-Frobenius eigenvector), the edges between the vertices of degree 4 and the vertices of degree 10 form the incidence of the complement of the Fano plane, the unique 2-(7, 4, 2) design. So here we find the graph of Figure 2, which is now proven to be the unique graph with spectrum $\{[8]^1, [1]^6, [-2]^7\}$. The other case in E_7, with $v = 17$, must be a cone over a strongly regular graph with eigenvalues 6, 2 and -2, hence we must

have the cone over the lattice graph $L_2(4)$ or the cone over the Shrikhande graph.

Now consider the case in E_8 with $v = 22$. Here we have a regular partition, with 8 vertices of degree 7 and 14 vertices of degree 16. Furthermore, the vertices of degree 7 induce a coclique, and the vertices of degree 16 induce a (unique) 14-regular graph. Now the fact that any two vertices of degree 16 have 12 common neighbours uniquely determines this graph, and we find the graph found by Bridges and Mena, and which can be described with the points and planes of $AG(3, 2)$.

Next, consider the case on 24 vertices, with $n_7 = 4$, $n_{10} = 16$ and $n_{15} = 4$ (and $n_{22} = 0$). Here we have that $k_{7,7} = 3$, $k_{7,10} = 4$, $k_{7,15} = 0$, $k_{10,7} = 1$, $k_{10,10} = 6$, $k_{10,15} = 3$, $k_{15,7} = 0$, $k_{15,10} = 12$ and $k_{15,15} = 3$. First of all, the vertices of degree 7 induce a clique, and each vertex of degree 10 is adjacent to a unique vertex of degree 7. By using the fact that a vertex of degree 7 and a vertex of degree 10 that are not adjacent have two common neighbours, and a vertex of degree 10 is adjacent to 6 other vertices of degree 10, we find that the 4 neighbours of degree 10 of a vertex of degree 7 induce a clique. Also the vertices of degree 15 induce a clique. Furthermore, a vertex x of degree 7 and a vertex y of degree 15 have three common neighbours (which must be of degree 10). This implies that the unique neighbour z of degree 10 of x, which is not adjacent to y, has 6 common neighbours with y. But these are the remaining three vertices of degree 15 ($k_{10,15} = 3$), and the other three neighbours of degree 10 of x. Hence z is adjacent to the remaining three nonneighbours of y. Hence the nonneighbours of a vertex of degree 15 also induce a clique. Now it is clear that the induced graph on the vertices of degree 10 is the lattice graph $L_2(4)$, and that the graph we uniquely determined is the one found by Spence, and which is strongly related to the lattice graph $L_2(5)$.

Next, consider the cases with $\lambda_1 = 4$. The case on 29 vertices gives a cone over a strongly regular graph, hence over the triangular graph $T(8)$ or one of the three Chang graphs. The case on 30 vertices gives $n_9 = 2$, $n_{12} = 9$, $n_{17} = 18$, $n_{24} = 1$. Now consider the unique vertex x of degree 24. A nonneighbour of x of degree respectively 9, 12, 17, has respectively 4, 8 and 12 common neighbours with x. Hence such a nonneighbour is adjacent to respectively 5, 4 and 5 other nonneighbours of x. But there are only 5 nonneighbours of x, so all nonneighbours of x have degree 12. Now $k_{24,9} = n_9 = 2$, $k_{24,12} = n_{12} - 5 = 4$, and $k_{24,17} = n_{17} = 18$. But now we have a contradiction with the equation $9k_{24,9} + 12k_{24,12} + 17k_{24,17} = 368$ from Section 4.3, hence there can be no such graph.

Finally, consider the case on 36 vertices, which has $n_{18} = 28$ and $n_{28} = 8$ (and $n_{12} = 0$). Now each vertex of degree 28 is adjacent to all 7 other vertices of degree 28, and to 21 vertices of degree 18. Each vertex of degree 18 is adjacent to 6 vertices of degree 28, and to 12 vertices of degree 18. Now it follows that switching with respect to the vertices of degree 28 gives a 14-regular graph with spectrum $\{[14]^1, [5]^8, [-2]^{27}\}$, which must be the triangular graph $T(9)$. Hence a graph with spectrum $\{[21]^1, [5]^7, [-2]^{28}\}$ is obtained from switching with respect to an 8-clique in $T(9)$. $\qquad\square$

6. More examples

After investigating possible spectra, with the possibility of precisely two vertex degrees (and a regular partition) we constructed the following examples. From the strongly regular (40, 12, 2, 4) graphs we found several graphs on 39 vertices with spectra $\{[14]^1, [2]^{23}, [-4]^{15}\}$ and $\{[20]^1, [2]^{22}, [-4]^{16}\}$. Take such a strongly regular graph, and fix a vertex for which the neighbours induce a graph that is the union of 3-cycles and 6-cycles. This implies that we can partition the neighbours into two sets of size 6 such that any two vertices in different sets are not adjacent. Now delete the fixed vertex, and add edges in between the two sets of size 6 (so adding the edges of a $K_{6,6}$). It is not hard to prove that these graphs have spectrum $\{[14]^1, [2]^{23}, [-4]^{15}\}$. According to Spence [private communication], who recently classified all (28) strongly regular (40, 12, 2, 4) graphs, 117 graphs are found in the way described. By computer Spence also found 3 graphs with the same spectrum which are not constructed this way, but which also have vertex degrees 12 and 17. From the graphs thus found, we also obtain 120 graphs with spectrum $\{[20]^1, [2]^{22}, [-4]^{16}\}$ by switching with respect to the set of vertices of degree 12 (that is, by interchanging the edges and nonedges between the set of vertices of degree 12 and the set of vertices of degree 17).

Consider a 2-(45, 12, 3) design with a polarity with 36 absolute points. This is equivalent to the property that the design has a symmetric incidence matrix with 36 ones on the diagonal. Now take this incidence matrix minus its diagonal as adjacency matrix of a graph. If the graph has the further property that the 9 nonabsolute points induce a coclique in this graph, then after replacing this coclique by a clique, we find a graph with spectrum $\{[14]^1, [2]^{27}, [-4]^{17}\}$ with 36 vertices of degree 11 and 9 vertices of degree 20. According to Spence [private communication], 2-(45, 12, 3) designs with the required properties exist, and in fact he found that there are at least 9 corresponding graphs. Also here switching (with respect to the vertices of degree 11) gives other graphs with three eigenvalues, that is, with spectrum $\{[20]^1, [2]^{26}, [-4]^{18}\}$.

6.1. A mixture of antipodal covers and symmetric designs

Suppose we have an antipodal 4-cover of the complete graph K_{4t^2} with $c_2 = t^2$ (where t is some integer), that is, a graph on $16t^2$ vertices, such that the vertices can be partitioned into $4t^2$ sets of size 4, so-called antipodal classes, with the property that vertices in the same class are at distance three, each vertex is adjacent to precisely one vertex from each besides its own class, two adjacent vertices have $t^2 - 2$ common neighbours, and two nonadjacent vertices from distinct classes have t^2 common neighbours. Such covers are known to exist if $t = 2^i$, $i \geq 1$ (cf. [9]). Furthermore, suppose we have a symmetric 2-($4t^2$, $2t^2 - t$, $t^2 - t$) design D (a so-called Menon design), which is also known to exist if $t = 2^i$, $i \geq 1$ (cf. [1]).

The vertices of the graph G we are going to construct are the vertices and the antipodal classes of the antipodal cover. Two vertices of the cover are adjacent in the new graph if

and only if they are adjacent in the cover. All antipodal classes are mutually adjacent. Now in order to define the adjacency between a vertex and an antipodal class of the cover, we represent the points of the design D by the antipodal classes, and we do the same for the blocks. Now an antipodal class is adjacent to a vertex of the cover if the antipodal class is incident in D with the antipodal class containing the vertex. In this graph the vertices of the cover have degree $k_1 = 2t^2 - t + 4t^2 - 1$, and the antipodal classes have degree $k_2 = 4t^2 - 1 + 4(2t^2 - t)$. We claim that G has eigenvalues $\lambda_0 = 8t^2 - 2t - 1$, $\lambda_1 = 2t - 1$ and $\lambda_2 = -2t - 1$. Setting $\alpha_1 = \sqrt{2t^2 - t}$ and $\alpha_2 = 2\sqrt{2t^2 - t}$ gives the Perron-Frobenius eigenvector, from which we find λ_0. By counting common neighbours we find indeed that G has three eigenvalues: any two antipodal classes are adjacent, and in the design D they are incident with $t^2 - t$ common blocks, hence they have $4(t^2 - t)$ vertices of the cover as common neighbours. Moreover, they have $4t^2 - 2$ antipodal classes as common neighbours, so that $\lambda_{2,2} = 4(t^2 - t) + 4t^2 - 2 = \alpha_2^2 - 2$. Two adjacent vertices of the cover have $t^2 - 2$ vertices of the cover as common neighbours, and as the two vertices are in two distinct antipodal classes, which thus are incident in D with $t^2 - t$ common classes, they also have these $t^2 - t$ classes as common neighbours, so that $\lambda_{1,1} = t^2 - t + t^2 - 2 = \alpha_1^2 - 2$. Two nonadjacent vertices of the cover, which are from distinct classes have t^2 vertices of the cover as common neighbours, and $t^2 - t$ classes as common neighbours. Two nonadjacent vertices of the cover, which are in the same antipodal class have the same $(2t^2 - t)$ antipodal classes as neighbours (and no vertices of the cover as common neighbours), hence we find in both cases that $\mu_{1,1} = 2t^2 - t = \alpha_1^2$. If a vertex of the cover is adjacent to an antipodal class, then they have $2t^2 - t - 1$ antipodal classes as common neighbours. Since the antipodal class is incident with $2t^2 - t - 1$ classes not containing the vertex (and so adjacent to all vertices of these classes), and the vertex is adjacent to precisely one vertex from each such class, they have $2t^2 - t - 1$ vertices of the cover as common neighbours, showing that $\lambda_{1,2} = 2(2t^2 - t - 1) = \alpha_1\alpha_2 - 2$. Similarly, if a vertex of the cover is not adjacent to an antipodal class, then they have $2t^2 - t$ antipodal classes as common neighbours, and $2t^2 - t$ vertices of the cover as common neighbours, hence $\mu_{1,2} = 2(2t^2 - t) = \alpha_1\alpha_2$. It now follows from the equations in Section 4 that G has three eigenvalues, with $\lambda_1 + \lambda_2 = -2$ and $-\lambda_1\lambda_2 = 4t^2 - 1$.

Instead of the design D, we could also have taken its complement, a symmetric $2\text{-}(4t^2, 2t^2 + t, t^2 + t)$ design. This produces a graph with three eigenvalues $\lambda_0 = 8t^2 + 2t - 1$, $\lambda_1 = 2t - 1$ and $\lambda_2 = -2t - 1$. This is easily seen by noticing that this graph is obtained by switching in the graph described above, with respect to the subset of vertices consisting of the vertices of the cover.

6.2. Systems of linked symmetric designs

A system of l linked symmetric $2\text{-}(v, k, \lambda)$ designs is a collection of sets V_i, $i = 1, ..., l + 1$ and an incidence relation between each pair of sets forming a symmetric $2\text{-}(v, k, \lambda)$ design, such that for any i, j, h the number of $x \in V_i$ incident with both $y \in V_j$ and $z \in V_h$ depends only on whether y and z are incident. The incidence graph of such a system has

the union of all V_i as vertex set, where two vertices are adjacent if they are in distinct V_i and they are incident in the corresponding design of the system. This graph is one of the relations in a three-class association scheme (cf. [6, 10]).

Now suppose we have a system of $u + 2$ linked symmetric 2-($u^2(u + 2)$, $u(u + 1)$, u) designs. The incidence graph of this system is regular with degree $(u + 2)(u + 1)u$, any two adjacent vertices have $u + 1$ common neighbours, two nonadjacent vertices from distinct V_i have $(u + 1)^2$ common neighbours and two (nonadjacent) vertices from the same V_i have $(u + 2)u$ common neighbours (cf. [10]). The vertices of the graph G we shall construct are the vertices of the incidence graph and the sets V_i, $i = 1,..., l + 1$. The induced graph on the vertices of the incidence graph is this graph itself. A set V_i is adjacent to all vertices of the incidence graph except the vertices from V_i itself. In this graph the vertices of the incidence graph have degree $k_1 = (u + 2)(u + 1)u + u + 2$, and the sets V_i have degree $k_2 = (u + 2)^2u^2$. By taking $\alpha_1 = \sqrt{(u + 2)(u + 1)}$ and $\alpha_2 = u\sqrt{(u + 2)(u + 1)}$ we get the Perron-Frobenius eigenvector, from which we find $\lambda_0 = (u + 2)^2u$. By counting common neighbours we find indeed that G has three eigenvalues, with $\lambda_1 = u$, $\lambda_2 = -(u + 2)u$. Unfortunately, only for $u = 2$, the required system of linked designs is known to exist. In this case there is precisely one such system (cf. [10]).

7. Problems and remarks

Using the same methods as in the proof of Theorem 7 we were able to classify all graphs with less than 30 vertices and three eigenvalues, which are not strongly regular or complete bipartite (and all of them have least eigenvalue -2). Here only two (more) cases had to be checked. The following table consists of all cases with at most 30 vertices, besides the ones with least eigenvalue -2.

v	spectrum	degrees		existence
22	$\{[7]^1, [1]^{14}, [-3]^7\}$	5 21	($n_5 = 21$, $n_{21} = 1$, $k_{5,21} = 1$)	No
23	$\{[11]^1, [1.562]^{11}, [-2.562]^{11}\}$	6 12 22	($n_6 = 11$, $n_{12} = 11$, $n_{22} = 1$, $k_{6,12} = 3$, $k_{6,22} = 1$, $k_{12,22} = 1$)	No
30	$\{[11]^1, [1]^{21}, [-4]^8\}$	5 13	($n_5 = 15$, $n_{13} = 15$, $k_{5,13} = 3$)	No
30	$\{[16]^1, [1]^{20}, [-4]^9\}$	8 20	($n_8 = 15$, $n_{20} = 15$, $k_{8,20} = 8$)	No
30	$\{[12]^1, [2]^{15}, [-3]^{14}\}$	8 14	($n_8 = 15$, $n_{14} = 15$, $k_{8,14} = 4$)	?

It is interesting to note that all known examples have two or three distinct degrees, and hence the partition of the vertices according to their degrees is regular. A nice problem would be to construct examples with more than three degrees, and if possible, one for which the partition is not regular.

There are no examples known of two connected graphs with three eigenvalues, which

have the same spectrum, but have distinct sequences of degrees. Note that the connectivity is essential here: the complete bipartite graph $K_{1,4}$ and the disjoint union of a 4-cycle and a single vertex both have spectrum $\{[2]^1, [0]^3, [-2]^1\}$. We expect there will also be connected examples of graphs with the same spectrum, and with distinct (sequences of) degrees.

Finally we mention the problem of finding an example with nonintegral eigenvalues, which is not strongly regular or complete bipartite.

Acknowledgements

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