A Note on the Kinks at the Mean Variance Frontier

Vörös, J.; Kriens, J.; Strijbosch, L.W.G.

Publication date:
1997

Citation for published version (APA):
Vörös, J., Kriens, J., & Strijbosch, L. W. G. (1997). *A Note on the Kinks at the Mean Variance Frontier*. (FEW Research Memorandum; Vol. 743). Department of Econometrics and Operations Research.

A Note on the Kinks at the Mean Variance Frontier

J. **Vörös**, J. **Kriens**[*], L.W.G. **Strijbosch**[*]

Janus Pannonius University, Faculty of Business and Economics, Pécs, Rákóczi 80, Hungary

**Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands*

ABSTRACT

In this paper the standard portfolio case with short sales restrictions is analyzed. Dybvig pointed out that if there is a kink at a risky portfolio on the efficient frontier, then the securities in this portfolio have equal expected return and the converse of this statement is false. For the existence of kinks at the efficient frontier the sufficient condition is given here and a new procedure is used to derive the efficient frontier, i.e. the characteristics of the mean variance frontier.

1. Introduction

In this paper the standard portfolio case with short sales restrictions is analyzed. We formulate this problem as follows: let **a** and **V** denote the mean vector and covariance matrix of asset returns, respectively. For the different values of the expected return, p, the problem can be formulated as

$$\mathbf{x} \geq \mathbf{0} \tag{1a}$$

$$\mathbf{1'x} = 1 \tag{1b}$$

$$\mathbf{a'x} = p \tag{1c}$$

$$z(p)/2 = \min \{\mathbf{x'Vx}\}/2 \tag{1d}$$

where x_i is the fraction of wealth invested in risky security i, i=1, ..., n, $\mathbf{x} = [x_1, ..., x_n]$, and z(p) indicates the minimum level of risk when we expect return p per unit capital invested. Let **V** be positive definite, for simplicity.

According to Dybvig (1984), if there is a kink at a risky portfolio (i.e. z(p) is not differentiable), then the securities in this portfolio have equal expected return and the converse of this statement is false.

The aim of this paper is to determine the sufficient conditions for the existence of kinks at the mean variance frontier. We provide these conditions through a special procedure developed for revealing the composition of the efficient frontier to reflect the differentiability nature of the problem.

2. The efficient frontier with a riskless asset

Let y be the portion of the unit capital invested in a riskless security at the interest rate r (the negative value of y shows lending). In this case (1) is modified as:

$$\mathbf{x} \geq \mathbf{0} \tag{2a}$$

$$\mathbf{1'x} + y = 1 \tag{2b}$$

$$\mathbf{a'x} + ry = p \tag{2c}$$

$$z^*(p)/2 = \min \{\mathbf{x'Vx}\}/2 \tag{2d}$$

where $z^*(p)$ indicates the minimum risk when the expected return level is p.

Obviously, when y = 0, we have the standard portfolio case with risky assets, and it is well known that $\sqrt{z^*(p)}$ is tangent of $\sqrt{z(p)}$. Thus, it is somewhat a straightforward idea to derive z(p) with the help of $z^*(p)$, by using r as a parameter on the interval $(-\infty, \max\{a_i\})$ and always setting y = 0. This way, we can have the full description of the efficient frontier for the standard portfolio case with risky assets. Thus when z(p) has a kink, i.e. it is not differentiable, there should be a set of interest rates r defining the same market portfolio.

To analyze (2), let us suppose that for a given p, with y=0, we know the optimal status of the variables satisfying the Kuhn-Tucker conditions, and p is in an open interval where the value of every variable belonging to a set, let us call this set M, is positive, while the value of the variables not in M is zero at this interval. The set of indices of the zero variables *at this open interval* is signed by N. For simplicity, we assume that the first k ($\leq$ n) variables are in the set M. From the definition of set M it follows, that M has at least two elements with $a_i \neq a_j$ for *this value of p.*

Partitioning matrix **V** and vector **a** according to this positioning:

$$\mathbf{V} = \begin{bmatrix} \mathbf{V}_{11} & \mathbf{V}_{12} \\ \mathbf{V}_{21} & \mathbf{V}_{22} \end{bmatrix}$$

and $\mathbf{a}' = [\mathbf{a}_1', \mathbf{a}_2']$, where $\mathbf{V}_{11}$ is of kxk, and similarly, $\mathbf{a}_1$ has k elements.

The Kuhn-Tucker conditions for problem (2) are given by:

$$\mathbf{V}_{11}\mathbf{x}_1 - \mathbf{1}u_1 - \mathbf{a}_1 u_2 = \mathbf{0} \tag{3a}$$

$$\mathbf{V}_{21}\mathbf{x}_1 - \mathbf{1}u_1 - \mathbf{a}_2 u_2 - \mathbf{v} = \mathbf{0} \tag{3b}$$

$$-u_1 - ru_2 = 0 \tag{3c}$$

$$\mathbf{1'x_1} + y = 1 \qquad (3d)$$

$$\mathbf{a_1'x_1} + ry = p \qquad (3e)$$

$$\mathbf{x_1 \geq 0}, \quad \mathbf{v \geq 0}, \quad \mathbf{x_2} = \mathbf{0} \qquad (3f)$$

where u_1, u_2, and $\mathbf{v}$ are the Lagrange variables.

Using $\mathbf{C}$ as the inverse of $\mathbf{V_{11}}$, from (3a) we have that:

$$\mathbf{x_1} = \mathbf{C1}u_1 + \mathbf{Ca_1}u_2 \qquad (4)$$

from which

$$\mathbf{1'x} = \mathbf{1'C1}u_1 + \mathbf{1'Ca_1}u_2 = 1 - y \qquad (5a)$$

and

$$\mathbf{a_1'x_1} = \mathbf{a_1'C1}u_1 + \mathbf{a_1'Ca_1}u_2 = p - ry \qquad (5b)$$

follow.

Let $\mathbf{1'C1} = f$, $\mathbf{1'Ca_1} = d$, and $\mathbf{a_1'Ca_1} = e$. Then (5a-b) can be written as:

$$fu_1 + du_2 = 1 - y \qquad (5a)$$

and

$$du_1 + eu_2 = p - ry \qquad (5b).$$

Now let us consider the expression:

$$fr^2 - 2dr + e \qquad (6).$$

Property : The value of $fr^2 - 2dr + e$ is zero only when $a_i = a_j = a$ for every $i,j \in M$ and $r=a$, otherwise positive.

Proof: f is positive as $\mathbf{V}$ is positive definite. Thus (6) is a convex parabola in r and its discriminant is $4(d^2 - ef)$ which is always negative except when $a_i = a_j$ for every $i,j \in M$ and in this case its value is zero (see Vörös (1987)).

Excluding the case of $a_i = a_j = r$ for every $i,j \in M$, it can be assumed that (6) is always positive. The direct consequence of this statement is that (5) will have always a unique solution. Since multiplying (5a) by r and using (3c), from (5) we have:

$$u_2 (fr^2 - 2dr + e) = p-r ,$$

from which we gain that

$$u_2 = (p-r)/(fr^2 - 2dr + e) \qquad (7a)$$

and

$$u_1 = r(r-p)/(fr^2 - 2dr + e). \qquad (7b)$$

As in (4) the value of variables x depends on only the Lagrange variables u_1 and u_2, (7) gives the solution. In similar way, from (3a) - multiplying it by $\mathbf{x_1}$-, we have that

$$z^*(p) = (p - r)^2/(fr^2 - 2dr + e). \qquad (8)$$

Now, we have to determine the largest interval of r for which this status of the variables is appropriate. Assumptions in (3f) give the size of this interval: first, variables x can not be negative. Thus from (4), using here the values u_1 and u_2 given by (7):

$$\mathbf{Ca_1} - r\mathbf{C1} \geq \mathbf{0} \quad \text{for set } M \qquad (9a)$$

and on the other hand, again from (3f), as **v** can not be negative:

$$\mathbf{V_{21}Ca_1} - \mathbf{a_2} + r(\mathbf{1} - \mathbf{V_{21}C1}) \geq \mathbf{0} \quad \text{for set } N. \qquad (9b)$$

Here we suppose that (p-r) can be considered positive.

Based on these results, a procedure can be created for determining the efficient frontier of the risky assets (and z* too).

Procedure:

Step 1.: for p= max $\{a_i\}$ fill up sets M and N.

Step 2.: determine the smallest r for which assumptions (9a-b) still hold.

Step 3.: if r = - ∞, stop. Otherwise, move indices giving smallest r from M to N if this is determined by (9a) or (*and*) from N to M, if it is given by (9b) (*as well*). Go to *Step 2*.

3. Kinks on the efficient frontier

The nature of *Step 3* has a strong connection with the differentiability of the efficient frontier. We emphasize that even if the fraction of a security in a particular portfolio is zero for a given r, the security still can be in set M. These cases appear when variables are just entering M.

Theorem: Consider the sets M that are determined by the *Procedure*. If there is a set M for which $a_i=a_j$ for all $i,j \in M$, then the efficient frontier of the risky assets is not differentiable.

Proof: Considering the case $a_i=a_j = a$ for all $i,j \in M$, we can write that

$$e = a^2 f \text{ and } d = af.$$

Using these values and u_1 and u_2 in (4), we have:

$$\mathbf{x_1} = (1/f)\, \mathbf{C1},$$

which is independent from r. So, subgradiants can be drawn from a whole interval of r's to the same market portfolio at the efficient frontier.

Example:

Let us consider Dybvig's (1984) example, where

$$\mathbf{a'} = [1, 3, 4] \text{ and } \quad \mathbf{V} = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & 1+\varepsilon & 2 \\ 0 & 2 & 4+\varepsilon \end{bmatrix}$$

and $0 < \varepsilon \leq 1/3$. (**V** is positive definite at this interval.) As Dybvig pointed out, for $0 < \varepsilon \leq 1/3$ the optimal portfolio is: $\mathbf{x'} = [0, 1, 0]$ for the expected return level $p = 3$. However, when $0 < \varepsilon < 1/3$ there is a kink at $p=3$ on the efficient frontier, while for $\varepsilon = 1/3$ there is no kink.

Taking the crucial second iteration when $M_2 = \{2, 3\}$ ($M_1 = \{3\}$), the problem can be structured in the following way:

$$N_2 = \{1\}, \ \mathbf{a_2} = [1], \text{ and}$$

$$\mathbf{V_{11}} = \begin{bmatrix} 1+\varepsilon & 2 \\ 2 & 4+\varepsilon \end{bmatrix}, \qquad \mathbf{a_1} = [3, 4], \ V_{21} = [0, 0].$$

From these, assumptions (9a-b) are:

$$r \leq (4 + 3\varepsilon) / (2+\varepsilon) \qquad (10a)$$

$$r \geq (2-4\varepsilon) / (1-\varepsilon) \qquad (10b)$$

$$r \geq 1, \qquad (10c)$$

respectively. Representing these functions (see Figure 1), we can conclude that:

- at the interval $0 < \varepsilon < 1/3$ the lower bound for r is given by assumption (10b) exclusively. Thus at the third iteration step: $M_3 = \{2\}$, consequently, the efficient frontier is not differentiable, because $a_i = a_j$ for every $i,j \in M$.

- when $\varepsilon = 1/3$, the value of the right-hand side of (10b) is exactly one, thus the lower bound is given by both (10b) and (10c). This means that variable 3 is leaving the set M, while variable 1 is entering the set M at the same time. Thus the new M set is: $M_3 = \{1, 2\}$, for which $a_i \neq a_j$ for every $i,j \in M$. Consequently, the efficient frontier has no kink.

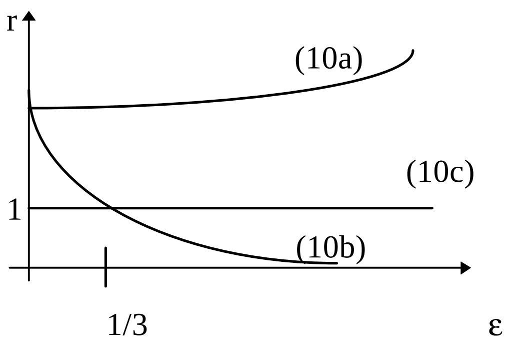

Figure 1.: The assumptions in (10)

References

Dybvig, P. H.(1984)**:** Short Sales Restrictions and Kinks on the Mean Variance Frontier, The Journal of Finance, 39(1), March 1984, 239-244

Vörös, J. (1987): The Explicit Derivation of the Efficient Portfolio Frontier in the Case of Degeneracy and General Singularity, European Journal of Operational Research, 32(2), 302-310.