Constructive Application of the Linear Tracing Procedure to Polymatrix Games

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Abstract

Polymatrix games concern a class of noncooperative multiperson games in normal form. Characteristic for these games is that the payoffs for any player are additive in the payoffs obtained against the individual other players. Harsanyi and Selten (1988) developed the tracing procedure for selecting an equilibrium in general noncooperative games. However, its practical application may be cumbersome because of nonlinearities involved. In this paper we show that an adapted version of the algorithm developed by van den Elzen and Talman (1991) may serve as a finite method for computerizing the linear tracing procedure when applied to polymatrix games. The method works via complementary pivoting and generates a piecewise linear path. This path constitutes a projection of the path generated by the linear tracing procedure which is in general nonlinear. Stated in game-theoretic terms, the path generated by the pivoting procedure describes the adjustment of the beliefs underlying the linear tracing procedure.

Keywords: Polymatrix game, linear tracing procedure, equilibrium selection, complementary pivoting, expanding set, computation.

1 Introduction

In this paper we are concerned with polymatrix games as formulated by Janovskaya (1968). These games are a generalization of the well-known bi-matrix games. Now we deal with more players, but the payoff that a player obtains when playing a certain strategy consists of the sum of the payoffs obtained from playing against each individual player apart. The canonical example for this type of games concerns two-player normal form games with incomplete information, i.e. one or both players are uncertain about the type of the other player.

For these games we want to consider the linear tracing procedure as originally developed by Harsanyi (1975). This procedure constitutes an important tool in the equilibrium selection theory of Harsanyi (1976). Later on this selection theory was elaborated in the standard work of Harsanyi and Selten (1988). The linear tracing procedure operates in so-called basic games. Roughly speaking, these are subgames having no inferior actions. Throughout the paper we will assume that we are dealing with games that are basic. The procedure starts from a prior, representing the initial guesses of the individual players about the behaviour of the other players. In their theory, Harsanyi and Selten assume that this prior is common over all players, i.e. all players have the same initial ideas about a certain player. The players now start reacting optimally against that prior. An equilibrium is reached by tracing a homotopy path along which more and more information about the whole game is revealed by increased interaction among the players, whereas simultaneously the weight of the prior decreases. Of course, the selected equilibrium depends in general on the specific prior. However, given a certain prior, the procedure is well-defined for a generic game. Harsanyi and Selten (1988) also developed a variant of the linear tracing procedure, namely the logaritmic tracing procedure. By adding a logaritmic term to the payoffs, the method is always well-defined. However, for practical computation the method seems to be of no use. In the sequel we will only deal with the linear tracing procedure.

Previously, van den Elzen and Talman (1995) presented a constructive method for mimicking the tracing procedure in case of bi-matrix games. That method was already presented in van den Elzen and Talman (1991) in a different context. Here, we present a generalization for polymatrix games. For that we use the more general method as presented in van den Elzen and Talman (1994), again in a very different setting. The algorithm presented is not only convenient as a computationally attractive and constructive method to handle the tracing procedure. Also as an algorithm for computing an equilibrium in a polymatrix game it seems to be superior to existing methods based on Howson (1972).

The contents of the paper are as follows. In Section 2 we introduce some notation and define the polymatrix game. Also some variants that can be handled are discussed. Next, we show in Section 3 the relation between the linear tracing procedure and our procedure in the context of polymatrix games. Section 4 provides a more detailed description of the working of our procedure. We illustrate this with an example as given by Harsanyi and Selten (1988). Finally, in Section 5 we consider the method in relation to the algorithm of Howson (1972).

2 The polymatrix game

Let us start with some preliminaries. For given integer $s > 0$, we denote by I_s the set $\{1, \ldots, s\}$. Furthermore, $\mathbb{R}^m_+$ stands for the nonnegative orthant of the m-dimensional Euclidean space, i.e. $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m | x_r \geq 0, \ \forall r \in I_m\}$. Related to a subset $X \subset \mathbb{R}^m$ we denote by $\mathrm{bd}(X)$ the boundary of X relative to its affine hull. Given two vectors x and y in $\mathbb{R}^m$, we denote by $[x, y]$ the line segment of vectors between x and y, i.e. $[x, y] = \{z \in \mathbb{R}^m | z = \lambda x + (1-\lambda)y, \ 0 \leq \lambda \leq 1\}$. Occasionally, $[x, y]$ denotes a curve connecting x and y, rather than a line segment, as will be clear from the context. Open and half-open segments are denoted by (x, y) and $(x, y]$ or $[x, y)$, respectively, with obvious meaning. By $e(r)$ we denote the r-th standard unit vector, whose dimension will be clear from the context. Similarly, we denote by $\underline{0}$ and e the vector of zeroes and the vector of ones of appropriate length, respectively. Furthermore, by $x^\top$ $(A^\top)$ we mean the transpose of a vector x (matrix A). Finally, given a matrix A, the k-th column of A is indicated by A_k.

A polymatrix game as defined by Janovskaya (1968) is a special type of a noncooperative multiperson game, with say N players. Let us index the players by i, $i \in I_N$. Player i, $i \in I_N$, has n_i pure actions, with action $k \in I_{n_i}$ denoted by (i, k). Together these actions constitute the action set $I(i)$ of player i, i.e. $I(i) = \{(i, 1), \ldots, (i, n_i)\}$. The total action set is denoted by I, i.e. $I = \cup_{i=1}^N I(i)$. Characteristic for polymatrix games is that the interaction between any pair of players is independent from the behaviour of the other players. By A^{ij}, $i \neq j$, we denote the partial payoff matrix of player i related to the interaction with player j. Of course, the dimension of this matrix is $n_i \times n_j$, and the element A^{ij}_{kl} denotes the payoff to player i playing (i, l) from the interaction with player j playing action (j, k). The partial payoff to player j in that situation is indicated by A^{ji}_{lk}. Summarizing, the polymatrix game Γ is denoted by $\Big(N, (n_1, \ldots, n_N), (A^{ij})_{i \neq j \in I_N}\Big)$. The case $N = 2$ corresponds to the standard bi-matrix game.

Allowing for mixed strategies we derive that the strategy space of player i equals the $(n_i - 1)$-dimensional unit simplex denoted by S^{n_i-1}. More precisely, $S^{n_i-1} := \{q_i \in \mathbb{R}_+^{n_i} \mid \sum_{k=1}^{n_i} q_{ik} = 1\}$. In general, a strategy vector in the game is denoted by $q \in S$, where $q = (q_1^\top, \ldots, q_N^\top)$, with q_i the strategy played by player $i \in I_N$, and $S := \Pi_{i=1}^N S^{n_i-1}$. The polytope S is called the strategy space of the game. Sometimes we denote a strategy vector q as (q_i, q_{-i}), where q_{-i} indicates the strategies of all players except i. Given a polymatrix game Γ, the expected payoff function $H_i : S \mapsto \mathbb{R}$ of player i is defined as

$$H_i(q) = q_i^\top \sum_{j \neq i} A^{ij} q_j. \tag{2.1}$$

A strategy vector $q^* \in S$ is a Nash equilibrium of Γ if

$$H_i(q^*) \geq H_i(q_i, q_{-i}^*), \quad \forall q_i \in S^{n_i-1}, \ \forall i \in I_N. \tag{2.2}$$

The set of Nash equilibria of the polymatrix game Γ is denoted by $NE(\Gamma)$.

The canonical situation that can be described by a polymatrix game concerns two-player games in normal form with incomplete information. Assume that the potential number of types of player i, $i \in I_2$, is $|T_i|$. Then the Bayesian-Nash equilibrium can be seen as a Nash equilibrium for the game with $|T_1| + |T_2|$ players, one for each type. The payoffs related to each pair of types for the different players are independent of the play by the other types. Another example of a polymatrix game concerns a model of a homogeneous oligopoly with linear costs and demands (see Quintas (1989)).

However, we can also look at polymatrix games from a totally different angle. In fact, polymatrix games can be seen as linearized noncooperative games in normal form. If we linearize the expected payoff functions of a normal form game around a certain strategy vector, i.e. we take the first-order Taylor expansion, then we obtain a polymatrix game (see van den Elzen and Talman (1994)).

Finally, we note that the algorithm discussed in this paper can also be applied to a more general class of polymatrix games as presented in Gowda and Sznajder (1996) for two players and in Mohan and Neogy (1994) for the general case. In these games the expected payoff $\hat{H}_i : S \mapsto \mathbb{R}$ of player i equals

$$\hat{H}_i(q) = q_i^\top \sum_{j \neq i} \max_{A^{ij} \in \mathcal{A}^{ij}} A^{ij} q_j, \tag{2.3}$$

where the set $\mathcal{A}^{ij}$ consists of a finite number of payoff matrices. Now, any player can select row by row. Of course, if $\mathcal{A}^{ij}$ consists of only one element then we are back in the standard polymatrix game.

3 Comparison of both procedures

Let be given an N-player polymatrix game Γ. As mentioned in the introduction the linear tracing procedure starts from a common prior $p = (p_1^\top, \ldots, p_N^\top) \in S$, where p_i denotes the idea of players $j \neq i$ about the strategy handled by player i. Thus, these opinions are taken to be equal among all players. Next, Harsanyi and Selten (1988) consider games Γ_p^t, $t \in [0,1]$, with expected payoff functions $H_i^{tp} : S \mapsto \mathbb{R}$, $i \in I_N$, given by

$$H_i^{tp}(q) = tH_i(q) + (1-t)H_i(q_i, p_{-i}). \tag{3.1}$$

Now, the linear tracing procedure generates a path $L(\Gamma, p)$ in the set

$$\mathcal{L}(\Gamma, p) := \{(q,t) \in S \times [0,1] \mid q \in NE(\Gamma_p^t)\}, \tag{3.2}$$

that connects $S \times \{0\}$ and $S \times \{1\}$. Thus, at $t = 0$ the players start by playing optimally, in fact optimizing, against the prior. Next, by increasing t gradually more information about the game is revealed and the players interact more and more while playing Nash. Finally, an equilibrium of the original game is obtained. The linear tracing procedure is generically well-defined, i.e. there exists a unique path $L(\Gamma, p)$ for almost all games given a certain prior p.

From (3.1) and (3.2) we derive that the linear tracing procedure traces a homotopy path in the payoffs. Alternatively, the method of van den Elzen and Talman (1991) formulates a homotopy in the strategies. To become more precise, we need some additional notation. For given $b \in [0,1]$, let $S_p(b) := \{q \in S \mid q \geq bp\}$. Note that $S_p(0) = S$ whereas $S_p(1) = \{p\}$. The set of restricted Nash equilibria of the polymatrix game Γ with strategies restricted to $S_p(b)$ is denoted by $NE_{S_p(b)}(\Gamma)$. The pivoting procedure now generates a path $B(\Gamma, p)$ in the set

$$\mathcal{B}(\Gamma, p) := \{q \in S \mid q \in NE_{S_p(b)}(\Gamma),\ b \in [0,1]\}, \tag{3.3}$$

connecting $p \in NE_{S_p(1)}(\Gamma)$ and a $q^* \in NE_{S_p(b^*)}(\Gamma)$ for some $b^* \geq 0$, with q^* being a Nash equilibrium of the game. Thus, this procedure obtains a Nash equilibrium via a sequence of restricted Nash equilibria, where the restricted set of strategies expands from $\{p\}$ to S.

For polymatrix games it now holds that whenever the path generated by the linear tracing procedure exists it is equivalent, up to projection, to the path generated by the pivoting procedure. This will be shown in the theorem below. For the special case of bi-matrix games this was already shown in van den Elzen and Talman (1995).

Theorem 3.1. Let be given a polymatrix game Γ and a prior p. Furthermore, let q^* and b^* be as above. If the path $L(\Gamma, p)$ is unique then the part of $L(\Gamma, p)$ related to $t \in [0, 1-b^*]$ is in 1-1 correspondence to the path $B(\Gamma, p)$. More precisely, the pair $(q, t) \in L(\Gamma, p)$ relates to $tq + (1-t)p \in B(\Gamma, p)$. Furthermore, the path $\{(q, t) \in L(\Gamma, p) \mid t > 1 - b^*\}$ is nonlinear in t and corresponds to $q^* \in B(\Gamma, p)$, with $q^* = tq + (1-t)p$, $\forall t > 1 - b^*$.

Proof. Consider a pair $(q, t) \in L(\Gamma, p)$, $t \in [0, 1-b^*]$. From (3.1) and the linear structure of the payoffs, it is obvious that $\forall i \in I_N$, q_i is optimal against $tq_{-i} + (1-t)p_{-i}$. Thus, the vector $tq + (1-t)p \in NE_{S_p(1-t)}(\Gamma)$. But the latter vector is on the path $B(\Gamma, p)$. For $t > 1 - b^*$ we obtain that the vector $tq + (1-t)p$ is equal to q^*, i.e. q is nonlinear in t.

On the other hand, let $q \in B(\Gamma, p)$. In case $q \in NE_{S_p(b)}(\Gamma)$ with $b > b^*$ then q can be uniquely written as $q = bp + (1-b)\bar{q}$, where $\bar{q} \in S$. Furthermore, the actions corresponding to positive components of $\bar{q}$ are optimal against all strategy vectors in $S_p(b)$. Thus, $\forall i$, $\bar{q}_i$ is optimal against $(1-b)\bar{q}_{-i} + bp_{-i}$. But, again from (3.1) we then infer that $\bar{q} \in NE(\Gamma_p^{1-b})$. If $q \in NE_{S_p(b)}(\Gamma)$, $b \leq b^*$, then $q = q^* = bp + (1-b)\bar{q}(b)$, with $\bar{q}(b) \in NE(\Gamma_p^{1-b})$. □

From the first part of the proof we derive that the path generated by the pivoting procedure is indeed a path of adapted beliefs, i.e. given $(q, t) \in L(\Gamma, p)$, q_i is optimal against the beliefs $tq_{-i} + (1-t)p_{-i}$, $\forall i$. But $tq + (1-t)p \in B(\Gamma, p)$. From Theorem 3.1 we further conclude that t going from zero to one in the tracing procedure corresponds to b going from one to some $b^* \geq 0$ in the pivoting procedure. Crucial for the proof is the linear structure in the payoffs of polymatrix games, i.e. $\forall q \in S$, $\forall t \in [0, 1]$, $H_i(q_i, tq_{-i} + (1-t)p_{-i}) = tH_i(q) + (1-t)H_i(q_i, p_{-i})$. For general N-person noncooperative games Theorem 3.1 will not hold.

Because the two methods are generically equivalent, knowledge about one method can be applied to the other. In van den Elzen and Talman (1991) it was proved that the pivoting procedure ends up in a perfect equilibrium whenever the prior is completely mixed, i.e. all its components are positive. With the theorem above we derive that this goes through for the linear tracing procedure. This is especially relevant because Harsanyi and Selten are searching for perfect equilibria. Here we derive that pertubations of the game are not necessary for that in case the prior is completely mixed.

The crucial difference between both methods is that the path generated by the pivoting procedure is piecewise linear, whereas the path of the linear tracing procedure may have nonlinear segments. One such nonlinearity has already explicitly been discussed in the theorem. However, nonlinearities occur frequently. This can be verified from Theorem 3.1 where we derived that $q \in S_p(b)$ generated by the pivoting procedure is reflected in $(\frac{1}{1-b}(q - bp), 1 - b)$ generated by the linear tracing procedure.

4 The pivoting procedure

In this section we first show that the procedure of van den Elzen and Talman, when applied to polymatrix games, is in fact a complementary pivoting procedure that can easily be implemented on a computer. Next, we illustrate the procedure along with the example given in Harsanyi and Selten (1988), Section 4.19.

Again, let be given a polymatrix game $\Gamma = \Big(N, (n_1, \ldots, n_N), (A^{ij})_{i \neq j \in I_N}\Big)$, and a prior p. From (3.3) we derive that the pivoting procedure considers restricted Nash equilibria. More precisely, it considers Nash equilibria of Γ where the strategy space is restricted to $S_p(b)$, with $b \in [0, 1]$. To characterize these Nash equilibria we first introduce the marginal payoff functions. The marginal payoff function of player i, $i \in I_N$, is the function $\bar{H}_i : S \mapsto \mathbb{R}^{n_i}$ with

$$\bar{H}_i(q) = \sum_{j \neq i} A^{ij} q_j. \tag{4.1}$$

Concerning the interpretation of the marginal payoffs, observe that $\bar{H}_{ik}(q)$, $k \in I_{n_i}$, denotes the payoff for player i if he plays action (i, k) with probability one, whereas the other players play q_{-i}. Now each vector $q \in NE_{S_p(b)}(\Gamma)$, with $b \in [0, 1]$, is characterized as follows: $q \in S$ and $\forall (i, k) \in I$,

$$\begin{array}{lll} q_{ik} = bp_{ik} & \text{if} & \bar{H}_{ik}(q) < \max_\ell \bar{H}_{i\ell}(q) \\ q_{ik} \geq bp_{ik} & \text{if} & \bar{H}_{ik}(q) = \max_\ell \bar{H}_{i\ell}(q). \end{array} \tag{4.2}$$

Expression (4.2) is quite obvious. At unrestricted Nash equilibria the nonoptimal actions are played with probability zero. Restricted to $S_p(b)$ these are played with the minimally allowed probability. Next, we add slack variables in system (4.2) in order to get rid of the inequalities. By $T_i(q)$ we denote the set of optimal actions of player i at the strategy vector q. The set of all optimal actions at q is denoted by $T(q)$. Sometimes when it is not confusing, we will not indicate the dependency of T_i and T on q. Furthermore, $\forall i \in I_N$, $\max_\ell \bar{H}_{i\ell}(q)$ is denoted by β_i. With all of this and by substituting (4.1) into (4.2), we can rewrite the set $\mathcal{B}(\Gamma, p)$ as the set of strategy vectors q satisfying $\forall i \in I_N$

$$\begin{array}{l} q_i = bp_i + \sum_{(i,k)\in T_i(q)} \lambda_{ik} e(k) \\ \text{and} \\ \sum_{j\neq i} A^{ij} q_j + \sum_{(i,h)\notin T_i(q)} \mu_{ih} e(h) = \beta_i e, \end{array} \tag{4.3}$$

with $\sum_{(i,k)\in T_i(q)} \lambda_{ik} = 1-b$, $b \in [0,1]$, $\lambda_{ik} \geq 0$ for $(i,k) \in T_i(q)$, and $\mu_{ih} \geq 0,\ (i,h) \notin T_i(q)$.

Next, we substitute the first set of equations in (4.3) into the second set and collect the resulting equations for all players, thus obtaining the final system

$$b \begin{bmatrix} \sum_{j\neq 1} A^{1j} p_j \\ \sum_{j\neq 2} A^{2j} p_j \\ \vdots \\ \sum_{j\neq N} A^{Nj} p_j \\ e \end{bmatrix} + \sum_{(1,k)\in T_1} \lambda_{1k} \begin{bmatrix} \underline{0} \\ A_k^{21} \\ \vdots \\ A_k^{N1} \\ e(1) \end{bmatrix} + \ldots + \sum_{(N,k)\in T_N} \lambda_{Nk} \begin{bmatrix} A_k^{1N} \\ A_k^{2N} \\ \vdots \\ \underline{0} \\ e(N) \end{bmatrix}$$

$$+ \sum_{(1,h)\notin T_1} \mu_{1h} \begin{bmatrix} e(h) \\ \underline{0} \\ \vdots \\ \underline{0} \\ \underline{0} \end{bmatrix} + \ldots + \sum_{(N,h)\notin T_N} \mu_{Nh} \begin{bmatrix} \underline{0} \\ \underline{0} \\ \vdots \\ e(h) \\ \underline{0} \end{bmatrix} - \beta_1 \begin{bmatrix} e \\ \underline{0} \\ \vdots \\ \underline{0} \\ \underline{0} \end{bmatrix} - \ldots \tag{4.4}$$

$$- \beta_N \begin{bmatrix} \underline{0} \\ \underline{0} \\ \vdots \\ e \\ \underline{0} \end{bmatrix} = \begin{bmatrix} \underline{0} \\ \underline{0} \\ \vdots \\ \underline{0} \\ e \end{bmatrix},$$

with $b \in [0,1]$, $\lambda_{ik} \geq 0$, and $\mu_{ih} \geq 0$.

The last N equations in the system above follow from summing up for each $i \in I_N$ the first set of equations in (4.3). These equations, together with $b \geq 0$ and $\lambda_{ik} \geq 0$, express that $q \in S$. Observe that system (4.4) contains $\sum_{i \in I_N}(n_i + 1)$ linear equations in $1 + \sum_{i \in I_N}(n_i + 1)$ unknowns. Assuming standard nondegeneracy, the system has a 1-dimensional piecewise linear manifold of solutions. The pivoting algorithm traces a specific path in that manifold. At the starting vector p we have that $b = 1$, $T_i(p) = (i, k_i)$, with $\bar{H}_{ik_i}(p) = \max_\ell \bar{H}_{i\ell}(p)$ and $\beta_i = \bar{H}_{ik_i}(p)$, $i \in I_N$. Furthermore, $\mu_{ik} = \beta_i - \bar{H}_{ik}(p)$, $(i,k) \neq (i,k_i)$. All other variables are zero. Next, the algorithm decreases b from 1. This implies by the last N equations that $\forall i \in I_N$, λ_{ik_i} is increased from zero. If the variable μ_{ih}, $(i,h) \notin T_i$, becomes zero then the complementary variable λ_{ih} is increased from zero, and vice versa. Finally, the algorithm stops if $b = 0$ or if $\mu_{ih} = 0$, $\forall (i,h) \notin T$ with $p_{ih} > 0$.

We know already that the algorithm generates a sequence of restricted Nash equilibria. However, in the sequel it will be handy to give an interpretation in terms of beliefs that directly follows from the algorithm. Firstly, at the start each player has a unique optimal reply against the other players' strategies, being action (i, k_i) for player $i \in I_N$. From the start the common beliefs related to these N actions are increased (λ_{ik_i} is increased from zero), whereas all other probabilities are, relatively to the starting probabilities, equally decreased (decrease in b). If along the path some action (i,h) becomes optimal ($\mu_{ih} = 0$), then the related relative probability is increased above the relative probabilities of the other nonoptimal actions. More precisely, λ_{ih} is increased from zero which induces that $\frac{q_{ih}}{p_{ih}} > b$ ($q_{ih} > 0$ if $p_{ih} = 0$). Reversely, if the probability (belief) attached to an action that was optimal previously becomes relatively minimal ($\lambda_{ik} = 0$), then the related probability relative to p is kept equal to that of all other nonoptimal actions ($\frac{q_{ik}}{p_{ik}} = b$, or $q_{ik} = 0$ if $p_{ik} = 0$), and μ_{ik} may increase from zero, i.e. the action becomes nonoptimal.

Example 4.1. Let us illustrate both procedures along with an example taken from Harsanyi and Selten (1988), Section 4.19. There are 3 players each having 2 actions. Concerning the payoffs it holds that

$$A^{13} = \begin{bmatrix} 240 & 0 \\ 0 & 272 \end{bmatrix} \quad A^{31} = \begin{bmatrix} 108 & 0 \\ 0 & 60 \end{bmatrix}$$

$$A^{23} = \begin{bmatrix} 40 & 0 \\ 0 & 72 \end{bmatrix} \qquad A^{32} = \begin{bmatrix} 108 & 0 \\ 0 & 60 \end{bmatrix}.$$

All other partial payoff matrices are zero matrices. This polymatrix game has three Nash equilibria of which one is mixed. The mixed equilibrium equals $\left((\frac{5}{7},\frac{2}{7})^\top,(0,1)^\top,(\frac{17}{32},\frac{15}{32})^\top\right)$. At the pure equilibria all the players play their first (second) action.

We first apply the pivoting algorithm. We start at the prior p given in the example of Harsanyi and Selten (1988), i.e. $p=\left((\frac{1}{3},\frac{2}{3})^\top,(\frac{1}{3},\frac{2}{3})^\top,(\frac{5}{8},\frac{3}{8})^\top\right)$. The path generated by the pivoting algorithm is depicted in Figure 4.1. The bar represents the strategy space S. The numbers in the tuples correspond to actions of players. For example, the vertex $(2,1,1)$ indicates the strategy vector at which both players 2 and 3 play their first action, whereas the first player plays his second action. Further, the top (bottom) of the strategy space consists of strategy vectors q at which $q_{11}=1$ ($q_{11}=0$) etc. Also, some crucial information concerning the marginal payoffs is denoted in the figure. For example, at $q_{31}=\frac{17}{32}$ it holds that $\bar{H}_{11}(q)=\bar{H}_{12}(q)$, i.e. player 1 is indifferent concerning his actions. The same holds for player 2 if $q_{31}=\frac{9}{14}$ and for player 3 if $q_{11}+q_{21}=\frac{5}{7}$.

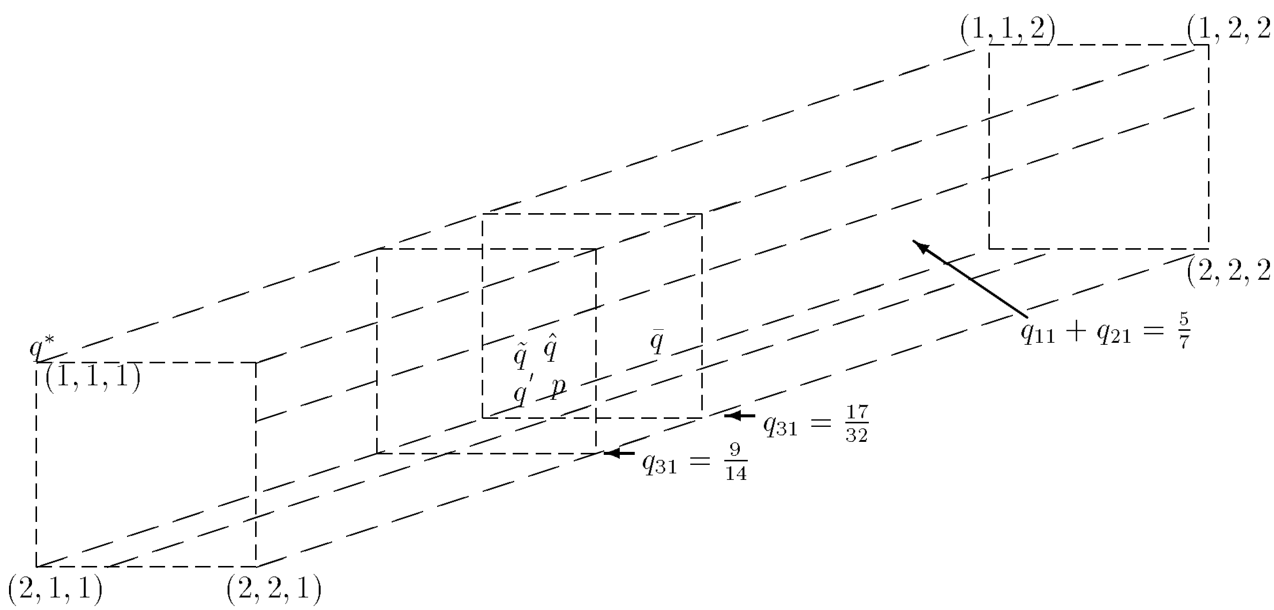

Figure 4.1. The path of strategy vectors generated in $S=S^1\times S^1\times S^1$ by the pivoting algorithm starting at p.

At p we derive for the marginal payoff vectors that $\bar{H}_1(p)=(150,102)^\top$, $\bar{H}_2(p)=(25,27)^\top$, and $\bar{H}_3(p)=(72,80)^\top$. Thus, from p the probabilities related to the actions $(1,1)$, $(2,2)$, and $(3,2)$ are increased, i.e. in Figure 4.1 the algorithm moves into the direction of $(1,2,2)$. Further information about the path of the algorithm can be summarized as follows,

q	T	b
$p = \big((\frac{1}{3}, \frac{2}{3})^\top, (\frac{1}{3}, \frac{2}{3})^\top, (\frac{5}{8}, \frac{3}{8})^\top\big)$	$\{(1,1),(2,2),(3,2)\}$	1
$\bar{q} = \big((\frac{3}{7}, \frac{4}{7})^\top, (\frac{2}{7}, \frac{5}{7})^\top, (\frac{15}{28}, \frac{13}{28})^\top\big)$	$\{(1,1),(2,2),(3,1),(3,2)\}$	$\frac{6}{7}$
$\hat{q} = \big((\frac{3}{7}, \frac{4}{7})^\top, (\frac{2}{7}, \frac{5}{7})^\top, (\frac{9}{14}, \frac{5}{14})^\top\big)$	$\{(1,1),(2,1),(2,2),(3,1),(3,2)\}$	$\frac{6}{7}$
$\tilde{q} = \big((\frac{23}{63}, \frac{40}{63})^\top, (\frac{22}{63}, \frac{41}{63})^\top, (\frac{9}{14}, \frac{5}{14})^\top\big)$	$\{(1,1),(2,1),(2,2),(3,1)\}$	$\frac{20}{21}$
$q' = \big((\frac{23}{63}, \frac{40}{63})^\top, (\frac{23}{63}, \frac{40}{63})^\top, (\frac{9}{14}, \frac{5}{14})^\top\big)$	$\{(1,1),(2,1),(3,1)\}$	$\frac{20}{21}$
$q^* = \big((1,0)^\top, (1,0)^\top, (1,0)^\top\big)$	$\{(1,1),(2,1),(3,1)\}$	0

The pivoting procedure traces a piecewise linear path of belief vectors linking the prior p and the Nash equilibrium q^* of the polymatrix game. Computationally, the pivoting procedure obtains the equilibrium in five steps. Remarkable in this example is the increase in b along part of the path. More specifically, b increases from $\frac{6}{7}$ to $\frac{20}{21}$ when the path moves from $\hat{q}$ to $\tilde{q}$. Thus, the restricted set of actions shrinks along this piece of the path.

Let us now consider the linear tracing procedure operating on this example. Starting from the same prior p, it turns out that player 1 plays his first pure strategy along the whole path from $t = 0$ to $t = 1$. This enables us to depict the path generated by the tracing procedure simply by only considering the relevant part of the strategy space. For most priors this would be impossible because the introduction of the homotopy parameter leads in principle to 4-dimensional space $S \times [0,1]$. The parameter b in the pivoting procedure gives no such problems because this parameter is depicted within S. The path is represented in Figure 4.2 which is taken from Harsanyi and Selten (1988).

Observe that the increase of b during the pivoting procedure is reflected in a decrease in t from $\frac{1}{7}$ at M towards $\frac{1}{21}$ at N in the tracing procedure. This is of course in accordance with Theorem 3.1. Also the remainder of that theorem can be verified for this example. For example, when going from $t = 0$ to $t = \frac{1}{7}$ the strategy vector $\big((1,0)^\top, (0,1)^\top, (0,1)^\top\big)$ is an equilibrium of Γ_p^t. In the pivoting procedure this line segment corresponds to the segment $[p, \bar{q}]$.

In this example we see the occurrence of a nonlinear segment in the linear tracing procedure, as the parameter t goes backwards. However, the related piece of the path generated by the pivoting procedure is piecewise linear. These nonlinearities make it difficult to generate the path of the linear tracing procedure directly and with precision. The pivoting procedure may serve as an adequate computational technique for generating a projected copy of that path within a finite number of pivoting steps.

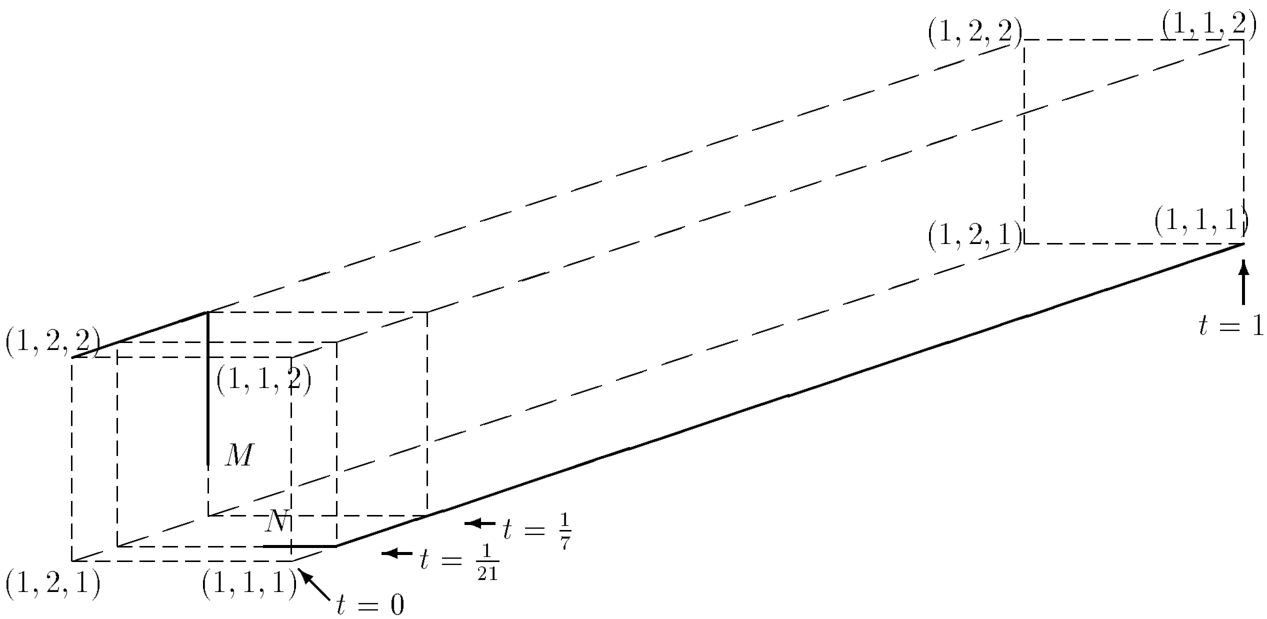

Figure 4.2. The path in $S \times [0,1]$ generated by the linear tracing procedure starting from prior p. Only strategy vectors with player 1 playing $e(1)$ are represented.

Remark. For the game Γ as presented in the example, the set $\mathcal{B}(\Gamma, p)$ contains exactly one other piecewise linear path connecting the other two equilibria. More precisely, the path connects the following strategy vectors

q	T	b
$\left((0,1)^\top, (0,1)^\top, (0,1)^\top\right)$	$\{(1,2),(2,2),(3,2)\}$	0
$\left((\frac{17}{60}, \frac{43}{60})^\top, (\frac{17}{60}, \frac{43}{60})^\top, (\frac{17}{32}, \frac{15}{32})^\top\right)$	$\{(1,1),(1,2),(2,2),(3,2)\}$	$\frac{17}{20}$
$\left((\frac{181}{420}, \frac{239}{420})^\top, (\frac{17}{60}, \frac{43}{60})^\top, (\frac{17}{32}, \frac{15}{32})^\top\right)$	$\{(1,1),(1,2),(2,2),(3,1),(3,2)\}$	$\frac{17}{20}$
$\left((\frac{5}{7}, \frac{2}{7})^\top, (0,1)^\top, (\frac{17}{32}, \frac{15}{32})^\top\right)$	$\{(1,1),(1,2),(2,2),(3,1),(3,2)\}$	0

The path above corresponds to a partly nonlinear path in $\mathcal{L}(\Gamma, p)$ that connects the same two Nash equilibria in $S \times \{1\}$. More precisely, the path connects $\left((0,1)^\top, (0,1)^\top, (0,1)^\top\right) \times \{1\}$, $\left((0,1)^\top, (0,1)^\top, (0,1)^\top\right) \times \{\frac{3}{20}\}$, $\left((\frac{62}{63}, \frac{1}{63})^\top, (0,1)^\top, (0,1)^\top\right) \times \{\frac{3}{20}\}$ and $\left((\frac{5}{7}, \frac{2}{7})^\top, (0,1)^\top, (\frac{17}{32}, \frac{15}{32})^\top\right) \times \{1\}$.

5 Relation to the Howson algorithm

In the introduction we already indicated that the pivoting procedure can also be viewed upon as an algorithm for solving polymatrix games. As such it may be compared with other algorithms related to this class of games. The standard algorithm for solving polymatrix games was presented by Howson (1972). This procedure first rewrites the equilibrium problem into the form of a Linear Complementarity Problem (LCP). Next, this LCP is solved by a complementary pivoting procedure. For the moment we confine ourselves to an interpretation in game-theoretic terms of the path generated by the Howson-procedure. The procedure starts from an arbitrary pure strategy vector, i.e. each player uses a pure strategy. Next, the procedure searches for an equilibrium for the first player given the initial stategies of the other players. In fact, a decision problem for player 1 is solved. Next, the same is done for player 2 while player 1 is kept in equilibrium and players $i > 2$ keep playing their initial strategy. The procedure continues till all players are in equilibrium. However, it may happen that along the path no equilibrium for say player $j > 1$ can be found. Then the equilibrium for player $j - 1$ is distorted and another equilibrium for that player is searched for. Thus, in theory it may happen that the algorithm goes back from a situation at which all but one player are in equilibrium to the one-person equilibrium problem. However, Howson (1972) shows that the algorithm always converges to an equilibrium. The Howson algorithm can be seen as a special case of the algorithm by Wilson (1971) that is designed for solving general noncooperative games in normal form.

Our algorithm seems to have some advantages above the method of Howson. First of all the game-theoretic interpretation of the latter algorithm is not attractive. The players are considered one by one, whereas we consider them all at once thereby using more information. Moreover, our procedure is allowed to start at an arbitrary strategy vector, whereas the Howson method has to start from a pure strategy vector. As a result our procedure may be able to find more equilibria than the Howson procedure. In fact, the differences between both methods correspond to the differences between the procedures of van den Elzen and Talman and Lemke-Howson for the case of bi-matrix games (see van den Elzen (1990)).

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