A Single-Stage Approach to Anscombe and Aumann's Expected Utility

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A Single-Stage Approach to Anscombe and Aumann's Expected Utility*

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ABSTRACT. Anscombe and Aumann showed that if one accepts the existence of a physical randomizing device such as a roulette wheel then Savage's derivation of subjective expected utility can be considerably simplified. They, however, invoked compound gambles to define their axioms. We demonstrate that the subjective expected utility derivation can be further simplified and need not invoke compound gambles. Our simplification is obtained by closely following the steps by which probabilities and utilities are revealed.

KEYWORDS: subjective expected utility, revealed preference, decision analysis, subjective probability, multistage gambles

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1. INTRODUCTION

The most well-known justification for subjective expected utility theory (SEU) was provided by Savage (1954). Savage's hallmark contribution was to derive both utilities and probabilities from preferences. His axioms are appealing but the analysis is complex and requires a rich set of available acts and states of the world. Soon after publication of Savage's work, several authors attempted to simplify the derivation of SEU (Rubin, 1949; Blackwell and Girshick, 1954; Chernoff, 1954; Anscombe and Aumann, 1963; Pratt, Raiffa, and Schlaifer, 1964). The simplifications were obtained by introducing an objective randomizing device such as a roulette wheel. These alternative derivations utilized a two-stage setup where an act yields a probability distribution under each state of the world. In contrast, Savage's act yields a prize (degenerate lottery) under each state of the world. This added complexity of the two-stage setup paid dividends in the simplification of the state space and of the axioms and proofs.

Anscombe and Aumann’s (1963) (AA) approach for deriving SEU has been found to be the most attractive. They adapted the independence axiom of von Neumann and Morgenstern (1944) (vNM) to the two-stage setup by means of a "reversal of order in compound lotteries" assumption. They permit finite state spaces, thus avoiding several mathematical complications (Stinchcombe, 1994). The AA approach attained a celebrity status because the axioms are highly transparent and the proofs are simple (Kreps, 1988). Subsequently, many authors have fruitfully employed the AA setup in refinements and modifications of the SEU model (Fishburn, 1970, 1982; Hazen, 1987, 1989; Schmeidler, 1989; Karni, 1993; Machina and Schmeidler, 1992, 1995; Nau, 1993; Eichberger and Kelsey, 1993).

The aim of this paper is to further simplify the AA approach. Like AA, we use a randomizing device for calibrating subjective probabilities and utilities. One distinction between our approach and that of AA is that, in AA's formulation, under each uncertain event the outcome can be any lottery, whereas in our approach we only need to assume that the outcome is a prize, i.e. a degenerate lottery. Thus we avoid the two-stage setup. This strategy of employing a one-stage approach rather than the traditional two-stage setup has been used in Sarin & Wakker (1992) to simplify Schmeidler's (1989) derivation of Choquet-expected utility.

Complications that are introduced by two-stage setups have been described by Loomes and Sugden (1986), Segal (1990), and Luce and von Winterfeldt (1994). A second distinction between our approach and that of AA is that in our setup we need not invoke all assignments of prizes to events. The two distinctions imply that we need fewer structural assumptions and obtain greater flexibility in modeling. Arguments for the need for structural simplicity have been given by Aumann (1962, 1971), Krantz, Luce, Suppes, and Tversky (1971), Suppes, Krantz, Luce, and Tversky (1989), Fishburn (1967, 1976), Karni (1985), Nau (1992), and Green and Osband (1991).

The strategy in our axiomatization is to closely follow the steps of elicitation commonly employed in decision analysis. Consistency conditions are imposed that serve to prevent contradictions in the elicitations. The resulting axioms are highly transparent and the proofs are elementary. We hope that the obtained simplicity is considered a virtue.

The consistency conditions are provided in Section 2. In Section 3, a formal presentation of our axioms and characterization theorem is given. Section 4 describes some examples that illustrate the simplicity of our model; Example 4.2 shows that our model is more general than the AA model. Section 5 presents an alternative characterization through a dominance axiom; the dominance axiom resembles stochastic dominance. Section 6 provides a conclusion.

2. CONSISTENCY CONDITIONS

The problem addressed here is the evaluation of an act $(H_1,x_1;\cdots;H_n,x_n)$ yielding a prize x_j if event H_j occurs, where probabilities of the events H_j are not given. The events H_j are disjoint and exhaustive. AA gave an example where the H events refer to the outcome of a horse race. Their terminology has been generally accepted, therefore we call these events "horse events." We do not impose restrictions on prizes; they could be quantitative variables such as money or could be qualitative such as good or bad health.

In the evaluation of a gamble two inputs are required. One is the probability of events and the other is the utility of prizes. Two elicitation assumptions, one dealing with the elicitation of utilities and the other with the elicitation of probabilities, and one valuation assumption give us the desired SEU of a gamble.

For the calibration of utilities and subjective probabilities we follow AA in using a randomizing device such as a roulette wheel. Physical ("known," "objective") probabilities are generated by using the roulette wheel. Thus any desired probability distribution over prizes can be specified in terms of a lottery contingent on the outcomes of the roulette wheel. Following AA, we assume:

ELICITATION ASSUMPTION 2.1. Preferences over roulette wheel lotteries are represented by vNM expected utility.

The assumption is utilized to elicite utilities over prizes by standard procedures (Mosteller and Nogee, 1951; Raiffa, 1968; Farquhar, 1984). In spite of the expected utility assumption for lotteries it is by no means clear that preferences over uncertain acts will satisfy SEU (Ellsberg, 1961; Schmeidler, 1989). Keynes (1921) and Knight (1921) emphasized the distinction between risk and uncertainty. Ramsey (1931), de Finetti (1937), and Savage (1954) argued that subjective probabilities can be assigned to uncertain events. The first step to obtain an SEU representation for acts is to show that beliefs about uncertain events (horse events) can be quantified by probabilities.

In the standard procedure to calibrate the probability of an event H, a favorable prize x^* and an unfavorable prize x_* are fixed. Then a roulette-wheel probability p is elicited such that the decision maker is indifferent between the act (H,x^*; *not*-H,x_*) and the lottery (p,x^*; 1-p,x_*). This indifference is displayed below.

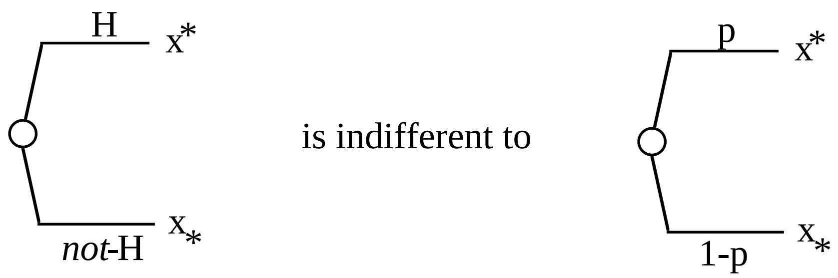

FIGURE 1

The method for eliciting probabilities described above is based on a comparison of likelihoods of different events through bets on events. Events E and E' are equally likely if one equally prefers betting on E to betting on E', i.e. (E,x^*; *not*-E,x_*) is indifferent to (E',x^*; *not*-E',x_*). In this case we write E $\sim_l$ E'. This method for calibrating probabilities has been well understood throughout history (Borel, 1939; Sections 39 and 48).

A consistency check for subjective probabilities will require that if the probability of H is assessed to be p, the probability of a disjoint event H' is assessed to be q, and in a cross-checking the probability of the union H∪H' is assessed to be r, then r must be equal to p+q. This additivity of probability is ensured by the following assumption on likelihood comparisons through bets on events:

ELICITATION ASSUMPTION 2.2. If H $\sim_l$ R and H' $\sim_l$ R' then H∪H' $\sim_l$ R∪R' must hold, for all disjoint horse events H and H' and disjoint roulette events R and R'.

While the above assumption may seem self-evident, we note that, as early as 1949, Shackle had argued for nonadditive probabilities. Ellsberg (1961) provided some ingenious examples where Assumption 2.2 is violated. Recently, many authors (Schmeidler, 1989; Gilboa, 1987; etc.) have developed alternative models for decision under uncertainty that relax Assumption 2.2. The above two assumptions guarantee that utilities are assigned to prizes and that beliefs can be quantified by additive probabilities. We now turn to the valuation of acts, for which it must be specified how probabilities and utilities are aggregated into an overall valuation. At this stage it is still possible that the decision maker uses methods of valuation that deviate from SEU. A way to elicit the value of an act $(H_1,x_1;\cdots;H_n,x_n)$ is as follows. First one elicits the probabilities $p_1,\cdots,p_n$ of the events $H_1,\cdots,H_n$ using the roulette wheel. By additivity, assured by the above assumption, $p_1,\cdots,p_n$ sum to one. Therefore we can construct a "matching" lottery $(R_1,x_1;\cdots;R_n,x_n)$, where the events $R_1,\cdots,R_n$ are roulette wheel events with probabilities $p_1,\cdots,p_n$ respectively. By Assumption 2.1, the roulette wheel lottery is valued by the *expected utility* $\sum_{j=1}^{n} p_jU(x_j)$. For consistency we would want the value of the act $(H_1,x_1;\cdots;H_n,x_n)$ to be the same. This consistency is ensured by the following condition.

VALUATION ASSUMPTION 2.3. An act $(H_1,x_1;\cdots;H_n,x_n)$ is indifferent to a lottery $(R_1,x_1;\cdots;R_n,x_n)$ whenever $H_1 \sim_l R_1,\cdots,H_n \sim_l R_n$.

The above assumption is based on the general principle that two gambles be judged indifferent if each prize is equally likely under the two. This method of evaluation of an uncertain act by matching it with a lottery with the same probability distribution has been widely used in decision analysis (Raiffa, 1968, Section 5.3 and page 109/110; Schlaifer, 1969, Section 4.4.5). Although the above conditions are guided by the elicitation process and serve to avoid contradictions in the assessments, they are easily reformulated as preference conditions. This is obtained mainly by replacing the equal-likelihood conditions in Assumptions 2.2 and 2.3 by their definitions in terms of bets on events. It is straightforward to observe that Assumption 2.2 implies additivity of probability and Assumption 2.3 implies an SEU valuation for all acts. In this manner the above assumptions, along with some common assumptions that are described in the next section, provide an elementary characterization of SEU.

3. A FORMAL PRESENTATION

Our setup involves two basic elements. One is the horse event H and the other is the roulette event R. The set of horse events, denoted by $\mathcal{H}$, may be viewed as containing subsets of an underlying horse-state space S (finite or infinite). Some richness conditions concerning $\mathcal{H}$ are specified in Assumption 3.1 below. The set of roulette events, $\mathcal{R}$, contains all the subintervals of [0,1);[1] to each interval R a probability is assigned that corresponds to the length of the interval. Our notation is as follows.

X:	An arbitrary set of *prizes*;[2]
$(H_1,x_1; \cdots; H_n,x_n)$:	An *act* ("pure horse lottery") yielding prize x_j in event H_j;
$\mathcal{A}$:	The set of all acts;
$(R_1,x_1; \cdots; R_n,x_n)$ $= (p_1,x_1; \cdots; p_n,x_n)$[3] :	A ("pure roulette") *lottery* yielding x_j in roulette event R_j with probability p_j ($1 \le n < \infty$, $\sum_{j=1}^{n} p_j = 1$);
$\mathcal{L}$:	The set of all (roulette) lotteries;

[1]The only reason for taking [0,1) instead of [0,1] is that the notation of partitions into intervals is now somewhat simpler; the intervals can all be left-closed and right-open.

[2]X can be finite as well as infinite.

[3]The identification of a lottery with the induced probability distribution over prizes will be justified in the sequel.

$G = (E_1,x_1; \cdots; E_n,x_n)$: A *gamble*, i.e. either an act or a lottery;

$\mathcal{G} = \mathcal{A} \cup \mathcal{L}$: The set of all gambles;

$\succcurlyeq$: The decision maker's preference relation on $\mathcal{G}$.

The notation $\succ$, $\sim$, $\preccurlyeq$, and $\prec$ is as usual. Acts are functions from the underlying horse state space S to X. Thus the events $H_1, \cdots, H_n$ partition the horse state space. Lotteries are functions from [0,1) to X, and the roulette events $R_1, \cdots, R_n$ partition the roulette wheel state space [0,1). Prizes are identified both with degenerate lotteries and with constant acts.[4] Thus preferences over prizes are generated by preferences over degenerate lotteries, and preferences over constant acts necessarily coincide with those preferences. To avoid triviality we assume throughout that two nonindifferent prizes $x^* \succ x_*$ are available. These two prizes are fixed throughout and are used below to obtain likelihood comparisons.

The set $\mathcal{L}$ of roulette lotteries consists of all roulette lotteries $(R_1,x_1; \cdots; R_n,x_n)$ for prizes $x_1, \cdots, x_n$ and subintervals $R_1, \cdots, R_n$ partitioning [0,1). Throughout we assume, without further mention, that all lotteries that generate the same probability distribution over prizes are indifferent. Therefore we often denote lotteries simply by the generated probability distributions. This is the common approach in decision under risk. Note that all simple probability distributions over prizes can be generated by the roulette wheel.

We do not assume that $\mathcal{A}$ contains all assignments of prizes to horse events. Below we describe our assumptions regarding horse events and acts. We will often use two-outcome gambles $(E,x^*;$ *not*-$E,x_*)$ in our analysis, hence we introduce the simplifying notation (E,x^*) for such gambles; similarly, (p,x^*) denotes $(p,x^*; 1-p,x_*)$.

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[4]It is sometimes convenient to assume one joint underlying state space set $S \times [0,1)$ that specifies both the uncertainties regarding horses and regarding the roulette wheel. Then a roulette wheel event R is related to the subset $S \times R$ of the joint state space and a horse event H to the subset $H \times [0,1)$ of the joint state space. In this case, degenerate lotteries and constant acts are formally identical.

ASSUMPTION 3.1 (Domain).[5]

H contains:

- $H_1, \cdots, H_n$ for each act $(H_1,x_1; \cdots; H_n,x_n)$;

- The union of all of its pairs of elements.

A contains the act (H,x^*) for each event $H \in$ H.

Equivalently, we could have first defined the collection H of events, and then define A as a subset of the set of functions from S to X that are "measurable" with respect to H. That is the most natural approach for Savage's (1954) model and other models where all functions from S to X are incorporated as acts. In our setup, where the set A can be fairly general, we preferred the setup where A is defined first and H is derived from A.

Now we turn to preference axioms that characterize SEU. Our axioms concern preferences over acts in A, preferences over lotteries in L, and preferences where an act is compared to a lottery. Of course both A and L are contained in G so, to avoid repetition, our axioms are formulated in terms of G.

AXIOM 3.2 (*Weak Ordering*). The preference relation $\succcurlyeq$ on G is complete and transitive.

The following axiom imposes a dominance condition. It implies that the likelihood of each event is between the likelihoods of the universal event and the impossible event. Alternatively, it can be interpreted as a monotonicity condition with respect to prizes, saying that replacing x_* by x^* is desirable.

AXIOM 3.3 (*Monotonicity*). $x^* \succcurlyeq (H,x^*) \succcurlyeq x_*$ for all $H \in$ H.

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[5]Observation A1 in the appendix demonstrates that H in the following assumption must be an "algebra."

For lotteries the classical independence axiom is imposed; we use here the version of Jensen (1967).

AXIOM 3.4 (*Independence axiom for lotteries*). For lotteries $L, L', Q \in \mathcal{L}$, if $L \succ L'$ and $0 < \lambda \leq 1$, then $\lambda L + (1-\lambda)Q \succ \lambda L' + (1-\lambda)Q$.[6]

The condition below extends the continuity condition of the vNM theorem to the present setup that also includes acts.

AXIOM 3.5 (*Continuity*). For lotteries L, L', and gamble G, if $L \succ G \succ L'$, then there exist $0 < \lambda < 1$ and $0 < \mu < 1$ such that $\lambda L + (1-\lambda)L' \succ G \succ \mu L + (1-\mu)L'$.

Note that if G is a lottery then this axiom is simply the continuity axiom of Jensen (1967). The axiom represents a generalization because G can also be an act.

AXIOM 3.6 (*Additivity*). For all horse events $H, H', H \cup H'$ and roulette events R, R', $R \cup R'$ for which $H \cap H' = R \cap R' = \varnothing$,

$(H,x^*) \sim (R,x^*)$ and $(H',x^*) \sim (R',x^*)$

imply

$(H \cup H',x^*) \sim (R \cup R',x^*)$.

This axiom states that if one is indifferent between betting on H or on R, and on H' or on R', then one must be indifferent between betting on $H \cup H'$ or on $R \cup R'$. In other words, if H and R are equally likely and so are H' and R', then $H \cup H'$ and $R \cup R'$ are also equally likely.

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[6] Here $\lambda L + (1-\lambda)L'$ assigns to each prize x the probability $\lambda L(x) + (1-\lambda)L'(x)$, where $L(x)$ and $L'(x)$ are the probabilities assigned to x by L and L', respectively.

AXIOM 3.7 (*Probabilistic Beliefs*). Act $(H_1,x_1; \cdots; H_n,x_n)$ is indifferent to lottery $(R_1,x_1; \cdots; R_n,x_n)$ whenever $(H_i,x^*) \sim (R_i,x^*)$ for all i.

The above axiom implies that an act be judged indifferent to a lottery if each prize is as likely under the act as it is under the lottery. We now state our main result.

THEOREM 3.8. Under the Domain Assumption 3.1, the following two statements are equivalent:

(i) SEU holds; i.e., there exists a utility function $U:X \rightarrow \mathbb{R}$ and a probability function P on the events, such that preferences over gambles are represented by the function

$$(E_1,x_1; \cdots; E_n,x_n) \mapsto \sum_{j=1}^{n} P(E_j)U(x_j).$$

(ii) $\succcurlyeq$ satisfies Axioms 3.2 (weak ordering), 3.3 (monotonicity), 3.4 (independence axiom for lotteries), 3.5 (continuity), 3.6 (additivity), and 3.7 (probabilistic beliefs).

PROOF. Necessity of the conditions in Statement (ii) is obvious, so we show sufficiency. For each horse event H we define the "probability" P(H) as the number p such that $(H,x^*) \sim (p,x^*)$. Existence of such a number p follows from continuity (see Lemma A2 in the appendix) and uniqueness follows from stochastic dominance on lotteries as implied by expected utility there.

To show additivity of probability, assume that H and H' are disjoint events and assume that P(H) = p, P(H') = q. Take for R the interval [0,p) and for R' the interval [p,p+q). Axiom 3.6 (additivity) implies that $(H \cup H',x^*) \sim (R \cup R',x^*)$; i.e., $P(H \cup H') = p+q$.

Next we turn to the evaluation of an act $(H_1,x_1; \cdots; H_n,x_n)$. We construct a "matching" lottery $(R_1,x_1; \cdots; R_n,x_n)$ as follows. Write $p_j = P(H_j)$. By additivity of probability as established above, the p_j's sum to one. Hence we can define intervals $R_1 = [0,p_1)$, $R_2 = [p_1, p_1+p_2)$, and $R_j = [p_1+\cdots+p_{j-1}, p_1+\cdots+p_j)$ for all $j>2$, and get $H_j \sim_j R_j$ for all j. By Axiom 3.7 (probabilistic beliefs), the lottery is indifferent to the act. Hence the value of the act is $\sum_{i=1}^{n} p_j U(x_j)$; i.e., it is the SEU value of the act. □

4. EXAMPLES

This section presents some examples to illustrate the flexibility of our approach. In all the examples, horses participate in a race, exactly one horse will win, and it is unknown which horse will win. The first example describes a simple single-stage approach that is compatible with our assumptions. The second example shows that our approach can accommodate preferences over compound lotteries. We show that the two-stage AA setup satisfies all of the assumptions of our approach, so that it can be considered a special case thereof. The third example further illustrates the increased flexibility that is possible through our approach.

EXAMPLE 4.1. There are two horses s,t. Acts are mappings from {s,t} to X, i.e. the prize resulting from an act depends on the horse that will win the race. Horse events are subsets of {s,t}. Roulette lotteries are generated by the random drawing of a number from [0,1), as in the previous sections. Obviously, a subjective probability p for s means that an act assigning a prize x to s and a prize y to t is indifferent to a roulette lottery assigning x to the interval [0,p) and y to [p,1). Note that we did not make independence assumptions, or other assumptions, about joint probability distributions of horses and numbers from [0,1). Simply, such assumptions are not needed in our setup. □

EXAMPLE 4.2 (the two-stage AA setup). Assume again that there are two horses s,t. Decision alternatives are *strategies*, i.e. they assign to each horse a probability distribution over prizes. In this setup, uncertainty is resolved in two stages. If the decision maker has chosen a strategy, then in the first stage it is decided which horse wins the race, resulting in the belonging probability distribution over prizes. Next, in the second stage, the probability distribution is played out by a choice of a random number from [0,1), to decide what prize finally results. The setup can be considered a special case of our one-stage setup.

@Op 15 april heb ik, zie # tekens, wat veranderd. {#15 apr. 96#{#The basic idea is simple. General strategies are considered to be acts, where the state space describes both the uncertainty regarding the horse race and the random number drawn from [0,1). Our one-stage approach also needs lotteries; these are delivered by the strategies that assign the same probability distribution to all horses, i.e. for which the horse race is irrelevant.#}

{#Formally#}, we define a state space $\{s,t\}\times[0,1)$ that describes both the uncertainty regarding the horse race and the drawing of a random number from [0,1). Thus (s,α) designates a victory for horse s and the drawing of the number α from [0,1). An act is a mapping from that state space to the prize set. {#Before presenting a formal elaboration, we give an example of a strategy and a related act.#} Suppose a strategy yields the lottery $(p_1,x_1; p_2,x_2) = (R_1,x_1; R_2,x_2)$ if s wins, and the lottery $(q_1,y_1; q_2,y_2) = (R_1',y_1; R_2',y_2)$ if t wins. To describe the strategy in terms of our one-stage approach, we partition the state space into four events $\{s\}\times R_1$, $\{s\}\times R_2$, $\{t\}\times R_1'$, and $\{t\}\times R_2'$. The above strategy can then be described by the act given by:

$\{s\}\times R_1$	$\{s\}\times R_2$	$\{t\}\times R_1'$	$\{t\}\times R_2'$
x_1	x_2	y_1	y_2

That is, it yields x_1 for each (s,α) from $\{s\}\times R_1$, etc. Note that the conditional probability distribution over prizes, given {s}, generated by the act is precisely $(p_1,x_1; p_2,x_2)$ (=

$(R_1,x_1; R_2,x_2))$; similarly, the generated conditional probability distribution given {t} is $(q_1,y_1; q_2,y_2)$. A strategy is therefore identified with an act that, for each horse, generates a conditional probability distribution over prizes that is identical to the probability distribution assigned to the horse by the strategy. A more general elaboration is provided next.#15 apr. 96#}

Roulette lotteries are defined (as in Section 3) as probability distributions over prizes, and are identified with mappings from [0,1) to the prize set X. The associated objects from the AA two-stage setup are strategies that assign the same probability distribution to every horse. For example, the roulette lottery $(p_1,x_1; \cdots; p_n,x_n)$ is associated with the strategy f such that $f(s) = f(t) = (p_1,x_1; \cdots; p_n,x_n)$. For such a strategy it obviously does not matter which horse will win the race.

Horse events are subsets of $\{s,t\} \times [0,1)$. Consider now an act $(s{\times}R_1,x_1; \cdots; s{\times}R_n,x_n; t{\times}R_1',y_1; \cdots; t{\times}R_m',y_m)$. The act designates the map from $\{s,t\} \times [0,1)$ to X that assigns, for $i = 1,\cdots,n$, prize x_i to each (s,α) in R_i, and, for $j = 1,\cdots,m$, prize y_j to each (t,α) in R_j'.[7] Every act generates a conditional probability distribution over prizes conditional on s, and also conditional on t. For example, the above act generates the lottery $(p_1,x_1; \cdots; p_n,x_n)$ conditional on s, and the lottery $(q_1,y_1; \cdots; q_m,y_m)$ conditional on t, where p_i is the length of R_i and q_j is the length of R_j'.

The above notation also shows how strategies and acts can be identified. This is done by letting probability distributions, assigned to horses by a strategy, correspond to conditional probability distributions generated over prizes by acts. Thus Figure 2 illustrates an act as described above, but can also be taken to illustrate a strategy that assigns probability distribution $(p_1,x_1; \cdots; p_n,x_n)$ to horse s and a probability distribution $(q_1,y_1; \cdots; q_m,y_m)$ to horse t.

[7]Every act can be written in the above manner, because for a general act $(H_1,z_1;\cdots; H_k,z_k)$, with $H_1,\cdots, H_k$ an arbitrary partition of $\{s,t\} \times [0,1)$, we can define $n = m = k$, $x_j = y_j = z_j$, and split up each H_j into $s{\times}R_j$ and $t{\times}R_j'$.

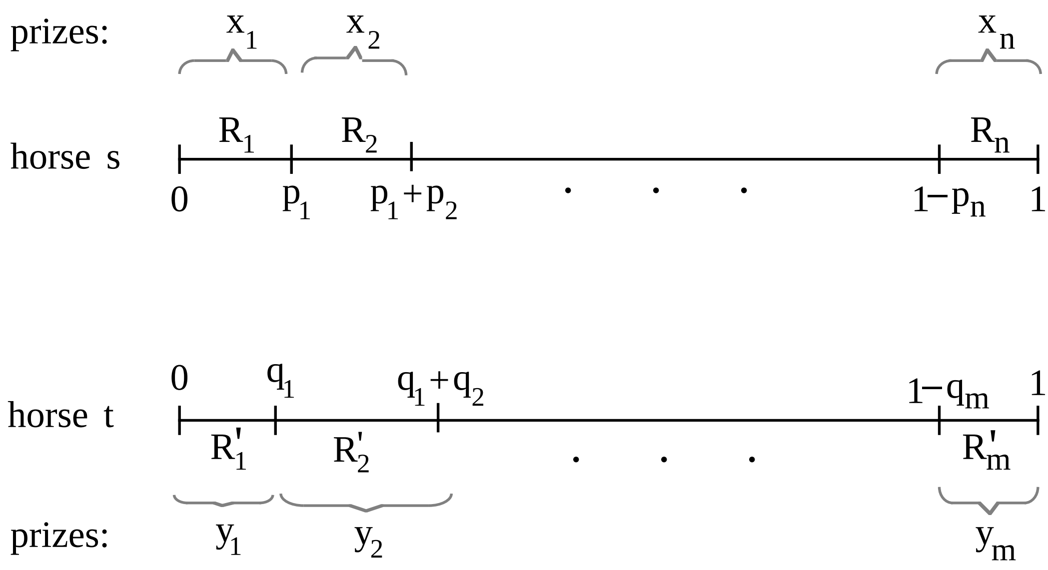

FIGURE 2. A strategy assigns to each horse a probability distribution over prizes. The associated act determines, for each horse and number from [0,1), a prize.

Assume expected utility in the AA setup with utility function U and probabilities p that s wins and 1–p that t wins. That corresponds to expected utility in our setup with the same utility function U, and with a probability measure P on $\{s,t\}\times[0,1)$ that is naturally generated by the marginal probabilities $P(\{s\}\times[0,1)) = p$, $P(\{t\}\times[0,1)) = (1-p)$, and independence. That is, P assigns probability $p\bullet p_i$ to any set $\{s\}\times R_i$ where R_i is a subinterval of [0,1) with length p_i, probability $(1-p)\bullet q_j$ to any set $\{t\}\times R_j$ where R_j is a subinterval of [0,1) with length q_j, and is extended to other subsets of $\{s,t\}\times[0,1)$ by additivity. A strategy as in Figure 2 has expected utility

$$\sum_{i=1}^{n} p\cdot p_i U(x_i) + \sum_{j=1}^{m} (1-p)\cdot q_j U(y_j),$$

which coincides with the expected utility of the associated act in the one-stage setup. Further, all the conditions of Section 3 and Theorem 3.8 are satisfied. Thus every two-stage AA setup, with two horses, can be considered a special case of our one-stage approach.

Obviously, similar constructions are possible if there are more than two horses, where the lottery assigned to a horse by a strategy corresponds with the conditional probability distribution generated by a gamble conditional on the horse winning. □

The following example further illustrates the claim that in our setup fewer hypothetical assumptions must be made than in the two-stage AA setup.

EXAMPLE 4.3. Assume three horses, h_1,h_2,h_3. Suppose the decision maker considers three acts: Stake \$1 on horse 1, stake \$1 on horse 2, or not bet at all. Further suppose that a bet on horse 1 yields \$10 if horse 1 wins. Then the net profit, in dollars, of betting on horse 1 is 9 if it wins and -1 if some other horse wins. Similarly, a bet on horse 2 yields \$2 if horse 2 wins. Denoting acts by the corresponding net profit vectors, the three acts considered by the decision maker are:

$$(9,-1,-1), \quad (-1,1,-1), \quad \text{and} \quad (0,0,0).$$

In our model we assume all "pure" lotteries over the set of prizes $X = \{9,1,0,-1\}$, and from these the utilities of prizes result by standard procedures. Let us take $x^* = 1$ and $x_* = -1$ as the prizes for calibrating the probabilities of horse events. The three "imaginary" acts used for probability calibration are $(1,-1,-1)$, $(-1,1,-1)$, $(-1,-1,1)$. Our axiomatization, requiring availability of all acts $(H,x^*;H^c,x_*)$ for any horse event H, requires three more imaginary acts, i.e. the acts $(-1,1,1)$, $(1,-1,1)$, $(1,1,-1)$.[8] In elicitations, these acts can be

[8] Besides the trivial acts $(1,1,1)$ and $(-1,-1,-1)$.

used to verify additivity of probability and can thus provide cross-checkings for elicited probabilities.

With the pure lotteries over the prizes and the six imaginary acts just described, the domain of our theory is complete and the characterization Theorems 3.8 (and 5.2) can already be invoked. Obviously, if considered useful, one has the possibility of adding more imaginary acts. One may, for instance, add the three imaginary acts (9,0,0), (0,9,0), (0,0,9) to provide alternative calibrations of probability. Then still our characterization theorems apply and give a foundation to SEU. In short, there is considerable flexibility concerning domain.

AA invoke more imaginary choice alternatives in this example, with the displayed three actual acts. Like us, their model invokes the entire set L of lotteries over X = {9,1,0,–1}. Their model requires all 3^4 = 81 assignments of prizes to $\{h_1,h_2,h_3\}$, which adds 78 imaginary acts, including all, unrealistic, acts that assign net wins to all horses. But then (both in its original version and in its modern version) the model also requires all assignments of lotteries-over-X to $\{h_1,h_2,h_3\}$. In the original version, the domain was extended further by also including all lotteries over assignments as just described. □

5. A DOMINANCE-CHARACTERIZATION

Some may judge that the axiomatization given above is not satisfactory because the axioms are too close to the representation that they seek to characterize. Thus Axiom 3.6 may simply seem like a direct reformulation of additivity of probability. Note, however, that the axioms given above satisfy all the requirements of a characterization, i.e. they formulate conditions entirely in terms of the empirical primitive which is the preference relation.

In this section we provide an alternative characterization, that shares all the structural advantages of the analysis in the above sections. The present characterization resembles the one above but is based on an appeal to the intuition of stochastic dominance rather than additivity and probabilistic beliefs. A similar idea was used by Quiggin (1989) in a study of regret theory.

Minimal prizes of gambles play a special role here, hence we use subscripts 0 to denote them. We rank-order prizes of gambles; i.e., a typical gamble is now denoted by $(E_0,x_0; E_1,x_1; \cdots; E_n,x_n)$ where it is assumed that $x_0 \preccurlyeq x_1 \preccurlyeq \cdots \preccurlyeq x_n$. For two-outcome gambles (*not*-$E,x_*; E,x^*$) and $(1-p,x_*; p,x^*)$ we maintain the abbreviated notation (E,x^*) and (p,x^*). Now we turn to our new condition that extends the well-known idea of stochastic dominance. This condition is an alternative to the extension of stochastic dominance used in Sarin and Wakker (1992) to characterize a nonexpected utility model, "Choquet-expected utility." They use a one-stage approach similar to the present paper, and describe the axioms that should be added to their Choquet-expected utility model to characterize SEU. Their axioms are, however, more complicated than the ones presented in this paper.

Consider two lotteries $(p_0,x_0; p_1,x_1; \cdots; p_n,x_n)$ and $(q_0,x_0; q_1,x_1; \cdots; q_n,x_n)$, and assume that $p_1 \geq q_1, \cdots, p_n \geq q_n$. In other words, each nonminimal prize is more likely in the first lottery. This implies that the first lottery stochastically dominates the second. A generally accepted condition for rational choice is that the stochastically dominating lottery be preferred. The expected utility model as implied by the vNM axioms satisfies stochastic dominance. Strict stochastic dominance adds the requirement that the preference between the above two lotteries be strict if in addition, for some j, $x_j \succ x_0$ and $p_j > q_j$. We extend the principle of stochastic dominance now to general gambles. To do so, we define $H \succcurlyeq_l R$ if $(H,x^*) \succcurlyeq (R,x^*)$. Obviously, $\sim_l$ as defined above is the symmetric part, and the asymmetric part $\succ_l$ is defined as usual. Now we propose the following extension of stochastic dominance to horse events:

AXIOM 5.1 (Event-Dominance). $(H_0,x_0; H_1,x_1; \cdots; H_n,x_n) \succcurlyeq (R_0,x_0; R_1,x_1; \cdots; R_n,x_n)$ whenever $(H_1,x^*) \succcurlyeq (R_1,x^*), \cdots, (H_n,x^*) \succcurlyeq (R_n,x^*)$; further the former preference is strict if, in addition, $x_j \succ x_0$ and $(H_j,x^*) \succ (R_j,x^*)$ for some j. Similar conditions hold when preferences are reversed ($\preccurlyeq$ instead of $\succcurlyeq$ and $\prec$ instead of $\succ$) except in $x_j \succ x_0$.

The above axiom says that, if for each nonminimal prize the associated horse event in an act is more likely than the corresponding event in the lottery, then the act must be preferred to the lottery. Note that in the above axiom, it cannot be assumed a priori that from $H_1 \succcurlyeq_l R_1, \cdots, H_n \succcurlyeq_l R_n$ it follows that $R_0 \succcurlyeq_l H_0$. That conclusion does follow as an implication of the theorem below.

THEOREM 5.2. Under the Domain Assumption 3.1, the following two statements are equivalent:

(i) SEU holds.

(ii) $\succcurlyeq$ satisfies Axioms 3.2 (weak ordering), 3.3 (monotonicity), 3.4 (independence for lotteries), 3.5 (continuity), and 5.1 (event dominance).

□

In Theorem 5.2, the event dominance axiom replaces Axiom 3.6 (additivity) and Axiom 3.7 (probabilistic beliefs) of Theorem 3.8 in the previous section.

6. CONCLUSION

Our approach shares some similarities as well as some important differences with that of AA. We adopt their strategy of using a randomizing device (roulette wheel) to simplify the axiomatization. The important difference between our approach and that of AA is that we do not use compound gambles in our analysis. In AA's setup a horse event yields an outcome that is not a determinate prize but a probability distribution over prizes. Thus AA invoke all assignments of lotteries to horse events. In our single-stage approach, the outcomes of either a horse race or a roulette wheel spin are always final prizes (degenerate lotteries). Examples illustrating the simplicity of our approach have been given in Section 4.

The simplification of the measurement and application of expected utility obtained in this paper may be viewed as a modest contribution to a large literature that exists for deriving and justifying expected utility (for another recent justification, see Hammond, 1988). It is remarkable, however, that in spite of a long history of the study of the AA approach, no one has noticed the simpler single-stage approach. Several authors (Hazen, 1987; Schmeidler, 1989; Machina and Schmeidler, 1995; Nau, 1993) have used the two-stage setup of AA to derive new models for decision under uncertainty. We believe that the single-stage setup can lead to simplifications of these derivations. Finally, we hope that the single-stage approach will facilitate the presentation and teaching of expected utility, and the development of alternative models.

APPENDIX. PROOFS AND ADDITIONAL RESULTS

The first observation demonstrates that $\mathcal{C}$, the collection of horse events, is an "algebra."

OBSERVATION A1. Under Assumption 3.1, $\mathcal{H}$ is nonempty, it is closed under finite unions, intersections, and complementation; i.e., it is an algebra.

PROOF. It was assumed in the text that all simple lotteries are contained in $\mathcal{L}$, and that these can be identified with constant acts. Hence all constant acts are contained in $\mathcal{A}$, in particular the act $(S,x^*;\emptyset,x_*)$, where S denotes the universal event. By assumption 3.1, this implies that S and $\emptyset$ are contained in $\mathcal{H}$, and therefore $\mathcal{H}$ is nonempty. For every event H, (H,x^*) is contained in $\mathcal{H}$, i.e. $(H,x^*;H^c,x_*)$ is contained in $\mathcal{H}$. This implies that not only H, but also its complement H^c is contained in $\mathcal{H}$, so $\mathcal{H}$ is closed under complementation. As it is closed under union, it is an algebra, and is closed under finite unions and intersections. □

The following lemma elaborates a claim in the proof of Theorem 3.8.

LEMMA A2. For each event $H \in \mathcal{H}$ there exists a unique $p \in [0,1]$ such that $(H,x^*) \sim (p,x^*)$.

PROOF. By Axiom 3.3 (monotonicity), $(p,x^*) \succcurlyeq (H,x^*) \succcurlyeq (q,x^*)$ for p=1 and q=0. We may assume that both preferences are strict, as the other cases are trivial. Next consider any arbitrary p and q such that $(p,x^*) \succ (H,x^*) \succ (q,x^*)$.

By stochastic dominance on lotteries as implied by the expected utility representation there, $p > q$ and the same preferences hold for all $p' > p$ and $q' < q$. By Axiom 3.5 (continuity), $(p',x^*) \succ (H,x^*) \succ (q',x^*)$ for some $p'<p$ and $q'>q$: In the notation used in the definition of probabilistic continuity, $p' = \lambda p + (1-\lambda)q$ can be taken for some $0 < \lambda < 1$ and $q' = \mu p + (1-\mu)q$ for some $0 < \mu < 1$. Apparently, the set of p such that $(p,x^*) \succ (H,x^*)$ is an interval of the form $(\sigma,1]$ and the set of q such that $(H,x^*) \succ (q,x^*)$ is an interval of the form $[0,\tau)$. For all probabilities $\tau \le p \le \sigma$, $(p,x^*) \sim (H,x^*)$. By stochastic dominance on lotteries, $\sigma = \tau = p$ must hold; i.e., there exists a p as required and it is unique. □

Finally, we prove Theorem 5.2.

PROOF OF THEOREM 5.2. The proof of necessity of (ii) is immediate, so we assume (ii) and derive (i). As in the proof of Theorem 3.8, expected utility holds for lotteries and probability numbers $0 \le P(H) \le 1$ can be defined for all events H. The proof now proceeds in three stages. The first stage prepares for Stage 2, Stage 2 then derives Axiom 3.6, and Stage 3 derives Axiom 3.7. All events and acts used in this proof are available because of the Domain Assumption 3.1; this will not be made explicit anymore.

STAGE 1. For disjoint horse events H,H', $P(H) + P(H') \leq 1$.

To derive this stage let $P(H) = p$, $P(H') = q$ and assume, for contradiction, that $p+q > 1$; i.e., $q > 1-p$. Consider the lottery (p,x^*); assume that it is generated by the roulette wheel gamble (R,x^*) for $R = [0,p)$. We have $H \sim_l R$. Also $H' \succ_l$ *not*-R because $(H',x^*) \sim (q,x^*) \succ (1-p,x^*)$; the latter strict preference follows from $q > 1-p$ and expected utility for lotteries.

Consider the gamble $(H' \cup H,x^*) = ($*not*-$(H \cup H'),x_*; H',x^*; H,x^*)$, and compare it to the roulette wheel lottery $(\emptyset,x_*;$ *not*-$R,x^*; R,x^*)$. Because $H' \succ_l$ *not*-R and $H \succcurlyeq_l R$, by event dominance the former gamble is strictly preferred to the latter. That means, however, that $(H' \cup H,x^*) \succ x^*$, contradicting the monotonicity Axiom 3.3. Stage 1 has been established.

STAGE 2. For disjoint horse events H,H', $P(H \cup H') = P(H) + P(H')$.

Consider the act $(H \cup H',x^*)$. We rewrite the act as $($*not*-$(H \cup H'),x_*; H,x^*; H',x^*)$. Assume $P(H) = p$, $P(H') = q$. Compare the act just-described to the lottery $(1-p-q,x_*; p,x^*; q,x^*)$. By the above lemma, the latter lottery can be defined indeed. Assume that the lottery is generated by roulette wheel events $R = [0,p)$, $R' = [p,p+q)$, and *not*-$(R \cup R') = [p+q,1)$; i.e., the lottery corresponds to $($*not*-$(R \cup R'),x_*; R,x^*; R',x^*)$. Now $H \sim_l R$, $H' \sim_l R'$, and two-fold application of event dominance, once with $\succcurlyeq$ and once with $\preccurlyeq$, implies that the lottery is indifferent to the act $($*not*-$(H \cup H'),x_*; H,x^*; H',x^*)$. We can rewrite the lottery and act to obtain $(p+q,x^*) \sim (H \cup H',x^*)$. That implies that $P(H \cup H') = p+q$. Stage 2 has been established.

STAGE 3. Acts are valued by their SEU value.

Consider an act $(H_0,x_0;\cdots;H_n,x_n)$. Assume that $P(H_j) = p_j$ for all j. By additivity of probability the p_j's sum to 1, hence we can consider the lottery $(p_0,x_0;\cdots;p_n,x_n)$, corresponding to some $(R_0,x_0;\cdots;R_n,x_n)$. We have $H_j \sim_l R_j$ for all $j \geq 1$, therefore by two-fold application of event-dominance the gamble and the lottery are indifferent. This implies that the value of the gamble is its SEU value. □

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