Strong Time-Consistency in the Cartel-versus-Fringe Model

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STRONG TIME-CONSISTENCY IN THE CARTEL-VERSUS-FRINGE MODEL

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Abstract

In the seventies and eighties, the cartel-versus-fringe model was introduced in the theory of exhaustible natural resources to characterize markets with one large coherent cartel and a big number of small suppliers named the fringe. It was considered appropriate to use the von Stackelberg solution concept but because solutions could only be derived in an open-loop framework time-inconsistency resulted. This paper solves time-inconsistency in the cartel-versus-fringe model by providing the feedback von Stackelberg equilibrium for all cost configurations.

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1 Introduction

The oil price shocks in the seventies increased the interest for the theory of the supply of raw materials from exhaustible natural resources, which dates back to the seminal work of Hotelling (1931).

Salant (1976) suggested to characterize the supply side of the oil market by one big cartel and a large number of small suppliers called the fringe. This market structure was thoroughly analysed in a number of papers: Pindyck (1978), Salant (1982), Lewis and Schmalensee (1980a, 1980b) and Ulph and Folie (1980). These studies all employ the Nash equilibrium concept. Gilbert (1978) put forward that in order to characterize the market power of the cartel the von Stackelberg equilibrium concept might be more appropriate. The idea is that the cartel can determine its extraction path first and the suppliers in the fringe can only react, which is actually beneficial for the cartel. However, this equilibrium has the unpleasant property that the cartel's strategy is time-inconsistent: the optimal extraction path ex ante ceases to be optimal ex post, when the analysis is done again at a future point in time, which means that, in the absence of binding commitments, the cartel has an incentive to renege on the announced extraction path (Ulph and Folie (1981), Newbery (1981), Ulph (1982), Groot, Withagen and de Zeeuw (1992)). This problem attracted a lot of attention (Karp and Newbery (1991,1992,1993)), but up to now the literature does not provide a time-consistent solution for the cartel-versus-fringe model.

Time-inconsistency in this context is a somewhat paradoxical concept. In the models employed so far it is assumed that the suppliers choose their extraction paths on the basis of information about all *initial* stocks of the resource only. In the check for time-consistency it is then assumed that the suppliers also have information about all stocks of the resource at that current point in time. However, this implies that the strategy space is changed because now it is assumed that the suppliers can condition their extractions on the current stocks of the resource. The problem is that in checking whether such an equilibrium is time-consistent or not, strategy spaces are used which normally yield different equilibria than the one that was considered in the first place. In the theory of differential games this problem is treated as follows (Başar (1989)). Equilibria in which the suppliers only know the initial resource stocks are called open-loop equilibria, whereas equilibria in which the suppliers condition their extractions on the current resource stocks are called closed-loop equilibria. Open-loop equilibria are called weakly time-consistent if reconsidering the extraction paths at any point in time, given the equilibrium resource stocks at that point in time, does not change the extraction paths. If this is not the case the equilibria are time-inconsistent. In the literature described above the conclusion was that the open-loop von Stackelberg equilibrium of the cartel-versus-fringe model is time-inconsistent for some economically relevant parameter values. In order to remove this undesirable property one can require that the extraction paths are in equilibrium at any point in time for any value of the resource stocks at that point in time. Note that by doing this the equilibrium becomes of the closed-loop type. Moreover, the resulting time-consistency is stronger in the sense that the suppliers not only will have no incentive to renege on their strategy if time passes by but also no incentive to renege when the value of the resource stocks for some reason differs from the expected one, for example as a consequence of a new oil deposit discovery. Therefore, this equilibrium is called strongly time-consistent. It is customary in the literature on differential games to use the name feedback equilibrium, but one also finds designations like credible, Markov perfect or rational expectations equilibrium.

This paper provides the feedback von Stackelberg equilibrium for the cartel-versus-fringe model with linear demand and constant marginal extraction costs, introduced by Ulph and Folie (1981) and Newbery (1981)[1]. The point of departure for the analysis is the open-loop equilibrium as given in Groot, Withagen and de Zeeuw (1990). The extraction paths depend on the relative position of the marginal extraction costs and the choke price. In some cases the open-loop von Stackelberg equilibrium is weakly time-consistent. It will be shown that these extraction paths can also be supported by a feedback equilibrium, so that the outcome is even strongly time-consistent. In cases where the open-loop outcome is time-inconsistent we will derive the extraction schedules that are supported by a feedback equilibrium. The result is, however, not unique in certain cases. It depends on the values of the parameters which schedule will be chosen by the dominant player, the cartel.

Section 2 presents the cartel-versus-fringe model and section 3 summarizes the open-loop equilibrium. In section 4 the Hamilton-Jacobi-Bellman equation is derived for the value function of the cartel which characterizes the feedback equilibrium. Section 5 presents the result of this paper: a strongly time-consistent equilibrium for the cartel-versus-fringe model. The appendices contain most of the mathematical calculations.

2 The cartel-versus-fringe model

The supply side of some markets for exhaustible natural resources, such as the oil market, can be characterized by a large coherent cartel and a big number of small suppliers called the fringe. The cartel has initial resource stock S_0^c and extracts $E^c(t)$ at time t with constant marginal extraction costs k^c. Similarly, each fringe member has initial resource stock S_{0i}^f and extracts $E_i^f(t)$ at time t with constant marginal extraction costs $k^f, i = 1, \cdots, N$, where N

[1] Karp and Tahvonen (1995) consider the case of extraction costs which depend linearly on remaining stocks. This facilitates the analysis considerably. We stick to the original model.

denotes the number of fringe members. The dynamics of the extraction of the resource stocks is described by

$$\dot{S}^c(t) = -E^c(t), S^c(0) = S_0^c \ , \tag{1}$$

$$\dot{S}_i^f(t) = -E_i^f(t), S_i^f(0) = S_{0i}^f, \quad i = 1, \cdots, N \ , \tag{2}$$

where the extraction rates and the stocks have to be nonnegative at any point in time.

The demand function is assumed to be linear with a so-called choke price $\overline{p}$ indicating that above this price there is no demand for this resource, for example because of the availability of a backstop technology. It follows that in market equilibrium the price p is also a linear function of total supply:

$$p(t) = \overline{p} - E^c(t) - E^f(t), \text{ with } E^f = \sum_{i=1}^{N} E_i^f \ . \tag{3}$$

The marginal extraction costs k^c and k^f are assumed to be smaller than the choke price $\overline{p}$, because otherwise either the cartel or the fringe or both will not exploit the resource and the problem is not interesting.

The objective of the producers is to choose extraction paths $E(\cdot)$ that maximize discounted profits, which are given by

$$\int_0^\infty e^{-rt}(\overline{p} - E^c(t) - E^f(t) - k^c)E^c(t)dt \ , \tag{4}$$

$$\int_0^\infty e^{-rt}(\overline{p} - E^c(t) - E^f(t) - k^f)E_i^f(t)dt, \ i = 1, \cdots, N \ , \tag{5}$$

for the cartel and the fringe members, respectively. The discount rate r is assumed to be constant and the same for all producers.

In order to characterize the market power of the cartel two steps are suggested. First, the von Stackelberg solution concept is employed for the game between cartel and fringe with the cartel as the leader and the aggregate fringe as the follower where the fringe members play a Nash game among themselves. Second, it can be shown that in the limit, when the number of fringe members N is taken to go to infinity, which implies that the aggregate fringe acts as a price-taker. For the open-loop case this is shown in Groot, Withagen and de Zeeuw (1990) and for the feedback case the reader is referred to section 4 of this paper.

3 The open-loop von Stackelberg equilibrium

The general structure of the open-loop von Stackelberg equilibrium for this cartel-versus-fringe model can be found in Ulph and Folie (1981), Newbery (1981) and Ulph (1982). These results had to be slightly modified because the effect of a negative shadowprice for the stock of the fringe had been ignored in the decision problem of the cartel (see Groot, Withagen and de Zeeuw (1992)). The results depend on the cost configuration and can be nicely presented by the position of two price trajectories and one marginal cost trajectory. Define

$$P^f(t, \lambda^f) = k^f + \lambda^f e^{rt} \ , \tag{6}$$

$$P^m(t, \lambda^c) = \tfrac{1}{2}(\overline{p} + k^c + \lambda^c e^{rt}) \ , \tag{7}$$

$$MC(t, \lambda^c, \mu^c) = k^c + (\lambda^c - \mu^c)e^{rt} \ , \tag{8}$$

where λ^f is the (constant) shadow price for the fringe with respect to its own stock, and λ^c and μ^c are the (constant) shadow prices for the cartel with respect to its own stock and the stock of the fringe. These shadow prices are of course not exogenously given but are part of the solution. The price P^f is called the competitive price, because this price results whenever the price-taking fringe produces, and the price P^m is called the monopoly price, because this price results when the cartel is not constrained by the fringe. MC denotes the marginal costs of the cartel for producing at the competitive price and consists of the extraction cost plus the shadow price of loosing one unit of its own stock plus the shadow price of leaving one unit in the stock of the competitor (note that μ^c is negative).

The results are summarized in figures 1-6. A detailed derivation can be found in Groot, Withagen and de Zeeuw (1990). In these figures the symbols C^m, C and F refer to intervals of time where the cartel is the sole supplier at the monopoly price P^m, the cartel is the sole supplier at the competitive price P^f and the fringe is the sole supplier (of course at the competitive price), respectively.

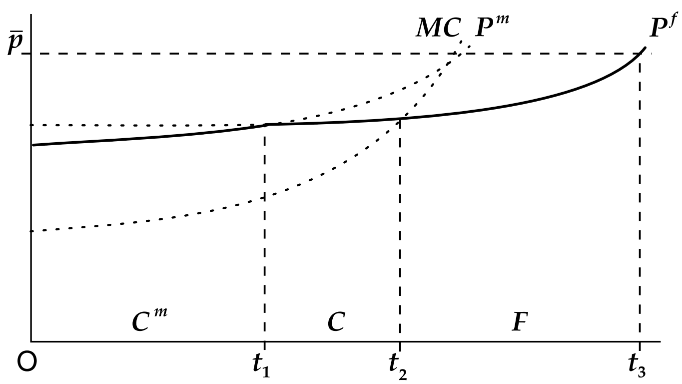

Figure 1

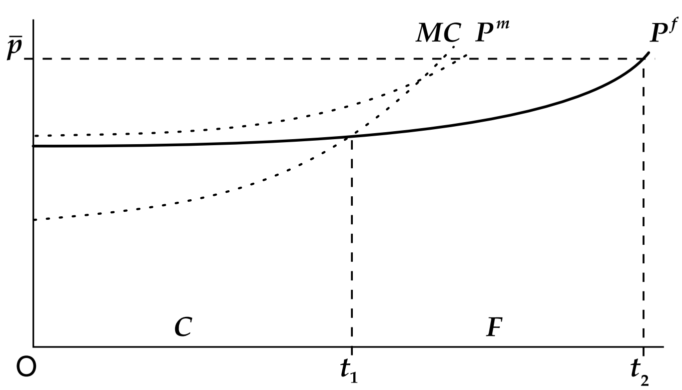

Figure 2

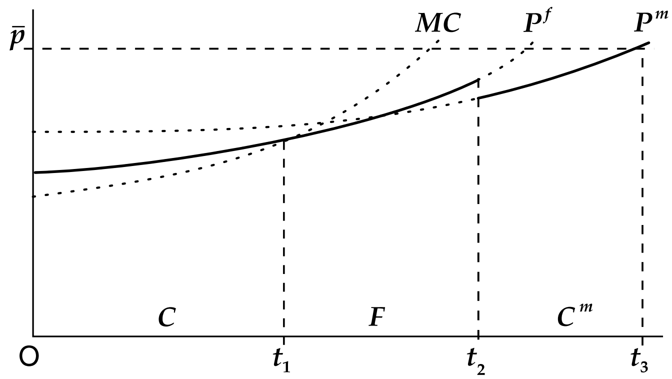

Figure 3

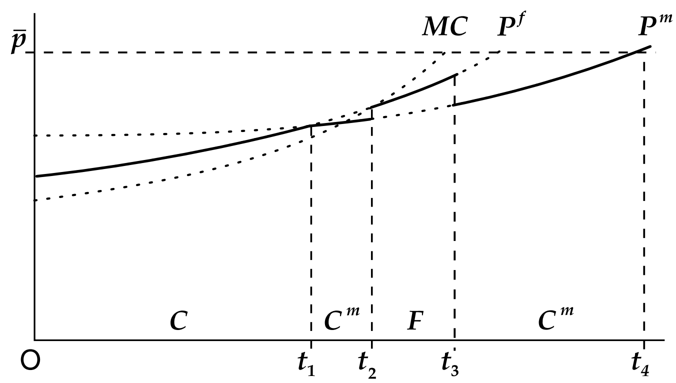

Figure 4

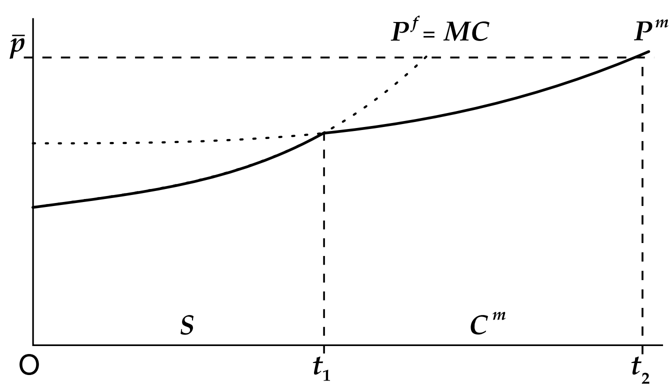

Figure 5

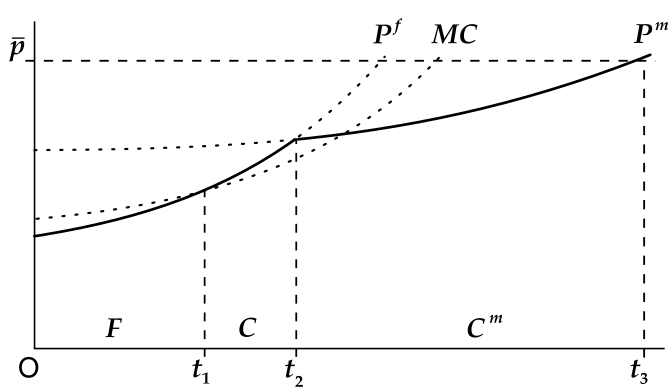

Figure 6

If $k^f > \frac{1}{2}(\overline{p} + k^c)$ either figure 1 or figure 2 applies. If the initial resource stock of the cartel is relatively large, the cartel produces first at the monopoly price and then at the competitive price after which the fringe produces. If the initial resource stock of the cartel is relatively low, only the last two stages occur. It is not difficult to see that the equilibrium of figure 2 is weakly time-consistent. If for some $t \in (0, t_1)$ the equilibrium is reconsidered the only difference with the original problem is that the stock of the cartel has decreased which implies that the equilibrium will not change. The equilibrium of figure 1 is also weakly time-consistent, because it can be shown that the length of the time intervals $[t_1, t_2]$ and $[t_2, t_3]$ does not depend on the initial stock of the cartel. Therefore, any reconsideration along the equilibrium path will not lead to a change in the extraction schedule.

If $k^f = \frac{1}{2}(\overline{p} + k^c)$ the equilibrium is again as in figure 2.

If $k^c < k^f < \frac{1}{2}(\overline{p}+k^c)$ figures 2, 3 and 4 sketch the possible equilibria. There can be either no monopoly phase or only one monopoly phase at the end, or, in addition to that, a monopoly phase between the phases where the cartel and the fringe produce at the competitive price. The three figures represent the situations where the initial resource stock of the cartel, as compared to the initial resource stock of the fringe, is respectively low, higher and high. In the last two cases the equilibrium is time-inconsistent. This can be seen as follows. The fringe produces before the stock of the cartel is exhausted. If in this production stage the equilibrium is reconsidered it will again start with the common feature of figures 2, 3 and 4, namely that the cartel produces first at the competitive price. This is different from the continuation of the ex ante equilibrium where the fringe produces at the competitive price. Note also that the price path is discontinuous in these cases, which is another undesirable property of this equilibrium.

If $k^f = k^c$ figure 5 applies. In fact there are infinitely many equilibria because in the phase of simultaneous production supply is not uniquely divided between the cartel and the fringe. One of these equilibria is the open-loop Nash equilibrium, which is always weakly time-consistent. All the other equilibria are time-inconsistent.

Finally, if $k^f < k^c$ the equilibrium is depicted in figure 6. This extraction profile is clearly time-inconsistent because after the stock of the fringe is exhausted the cartel will ex post immediately produce at the monopoly price and not first at the competitive price.

In the absence of binding commitments time-inconsistency is an undesirable property of an equilibrium. In section 5 we will see what happens when time-consistency is required. In the next section the necessary methodology is derived first.

4 The Hamilton-Jacobi-Bellman equation

As was already stated in the Introduction, in order to remove time-inconsistency one requires that the extraction paths are in equilibrium at any point in time for any value of the resource stocks at that point in time. In fact one requires that Bellman's principle of optimality holds. This implies that dynamic programming yields the result. The central concept in dynamic programming is the value function which denotes at any point in time for any value of the resource stocks at that point in time the equilibrium profits-to-go for the supplier. The set of value functions for all suppliers have to satisfy the so-called Hamilton-Jacobi-Bellman equations (see Başar and Olsder (1982)). For the cartel-versus-fringe model these equations are

$$\frac{\partial V^c}{\partial t} + \max_{E^c}[e^{-rt}(p - k^c)E^c - \frac{\partial V^c}{\partial S^c}E^c - \sum_{j=1}^{N}\frac{\partial V^c}{\partial S_j^f}E_j^f] = 0 \ , \tag{9}$$

$$\frac{\partial V_i^f}{\partial t} + \max_{E_i^f}[e^{-rt}(p - k^f)E_i^f - \frac{\partial V_i^f}{\partial S^c}E^c - \sum_{j=1}^{N} \frac{\partial V_i^f}{\partial S_j^f}E_j^f] = 0, \quad i = 1, \cdots, N \;, \tag{10}$$

where p is given by equation (3) and where V denotes the value function, which is a function of time and all resource stocks.

The maximizations in the Hamilton-Jacobi-Bellman equations yield the extraction rates at time t as a function of the resource stocks. In the von Stackelberg equilibrium concept the fringe members take the extraction of the cartel as given. They are assumed to play Nash among themselves which, if $E_i^f > 0$, yields as first-order conditions

$$e^{-rt}(\overline{p} - E^c - \sum_{j=1}^{N} E_j^f - k^f - E_i^f) - \frac{\partial V_i^f}{\partial S_i^f} = 0, \quad i = 1, \cdots, N \;. \tag{11}$$

If we assume identical fringe members, summing up the equations in (11) and dividing the result by N and then taking the limit for N goes to infinity leads to an expression for the extraction of the aggregate fringe in case the fringe produces. It follows that

$$E^f = \sum_{j=1}^{N} E_j^f = \max(0, \overline{p} - E^c - k^f - e^{rt}\frac{\partial V^f}{\partial S^f}) \;. \tag{12}$$

If the problem had been set up as if the total fringe acts collectively but takes the price p as given, the same behaviour results. This means that the aggregate fringe acts as a price-taker. Therefore the original problem reduces to the following Hamilton-Jacobi-Bellman equation for the cartel

$$\frac{\partial V^c}{\partial t} + \max_{E^c}[e^{-rt}(p - k^c)E^c - \frac{\partial V^c}{\partial S^c}E^c - \frac{\partial V^c}{\partial S^f}E^f] = 0 \;, \tag{13}$$

where the cartel is restricted by a price-taking aggregate fringe (12) and p satisfies (3).

The difficulty in this type of problems is to find the value function. Note also that in this problem the reaction of the fringe is a correspondence and not a function. However, from any given extraction schedule the profits-to-go can be calculated and it can then be checked if this extraction schedule and these profits-to-go satisfy the Hamilton-Jacobi-Bellman equation. This procedure will be followed in the next section.

5 The feedback von Stackelberg equilibrium

Figures 1-6 sketch the extraction schedules for the cartel-versus-fringe model in case of an open-loop information structure, which means that the extractions are a function of time and the initial resource stocks. The von Stackelberg equilibria of figures 1, 2 and 5 (if the simultaneous production is divided as in the Nash equilibrium) are weakly time-consistent but the equilibria of figures 3, 4 and 6 are time-inconsistent. The concept of a feedback von Stackelberg equilibrium solves this but also leads to a stronger form of time-consistency. In a case of weak time-consistency the extractions will not change at any point in time if the equilibrium is recalculated with the equilibrium values of the resource stocks as initial stocks. In a feedback equilibrium the extraction rates are not only a function of time but also of the current resource stocks. Strong time-consistency means that these mappings remain in equilibrium at any point in time and for any value of the resource stocks at that point in time. The feedback equilibrium is in that sense robust against unexpected shocks to the resource stocks. Note that the strategies seen as functions do not change but that the actual rates of extraction will change in case of unexpected changes in the resource stocks. In this section it will be shown that the extraction schedules of the weakly time-consistent open-loop von Stackelberg equilibria are also supported by feedback equilibria and hence are also strongly time-consistent. Furthermore, the feedback equilibria will be derived for the cases where the open-loop equilibrium is time-inconsistent.

In section 4 it was shown that the feedback von Stackelberg equilibrium is characterized by the Hamilton-Jacobi-Bellman equation (13) for the cartel, restricted by a price-taking aggregate fringe. This equation consists of the optimization problem for the cartel at any point in time for any value of the resource stocks, and a partial differential equation in the value function of the cartel, where the value function denotes the profits-to-go. It is extremely difficult to solve this HJB equation and therefore we follow a verification procedure which means that we start from feasible extraction schedules and show whether or not the HJB equation is satisfied. The calculations are tedious but have a common structure.

First we concentrate on the extraction schedule of figure 2 where the cartel starts producing at the competitive price followed by the fringe. We have to show that at each time t between 0 and t_1 for each set of resource stocks S^c and S^f it is optimal for the cartel to fully supply the market at the competitive price, given by equation (6). This leads to the optimality conditions

$$k^f + \lambda^f e^{rt} - k^c - \frac{\partial V^c}{\partial S^c} e^{rt} + \frac{\partial V^c}{\partial S^f} e^{rt} > 0 \qquad (0 \leq t < t_1) \tag{14}$$

$$\tfrac{1}{2}(\overline{p} + k^c + \frac{\partial V^c}{\partial S^c} e^{rt}) > k^f + \lambda^f e^{rt} \qquad (0 \leq t < t_1) \tag{15}$$

and the HJB equation becomes

$$\frac{\partial V^c}{\partial t} + e^{-rt}(k^f + \lambda^f e^{rt} - k^c - \frac{\partial V^c}{\partial S^c} e^{rt})(\overline{p} - k^f - \lambda^f e^{rt}) = 0 \ . \tag{16}$$

The second optimality condition states that the cartel cannot act as a monopolist, because the corresponding price would be above the competitive price and the fringe would start to produce. The first optimality condition states that it is optimal for the cartel to have the fringe refrain from producing at the competitive price, because the price minus marginal extraction costs minus the shadow value of one unit less of its own resource exceeds the shadow value of lowering the resource stock of the fringe. Finally, the third condition is the resulting HJB equation that guarantees strong time-consistency.

Because the competitive price P^f given by equation (6) equals the choke price $\overline{p}$ at t_2, the variable λ^f is a function of t_2 and the parameters $\overline{p}$, k^f and r. Because the extraction path is known, the profits-to-go of the cartel at time t can be expressed in t, t_1 and t_2, and the parameters $\overline{p}, k^c, k^f$ and r. Furthermore, t_1 and t_2 are implicitly given as functions of t, S^c and S^f by the conditions that both the cartel and the fringe exhaust their resource stocks. It follows that the partial derivatives of the value function V^c of the cartel with respect to t, S^c and S^f can be determined (see Appendix A). Tedious but straightforward calculations then show that condition (14) is satisfied for $k^f > k^c$, condition (15) is satisfied for $k^f < \widehat{k}$ for some $\widehat{k}$ with $\frac{1}{2}(\overline{p} + k^c) < \widehat{k} < \overline{p}$, and condition (16) is always satisfied. We can conclude that

Proposition 1

If $k^c < k^f < \widehat{k}$ with $\frac{1}{2}(\overline{p} + k^c) < \widehat{k} < \overline{p}$, the extraction schedule of figure 2, where first the cartel produces at the competitive price and then the fringe, is a candidate for a feedback von Stackelberg equilibrium.

We do *not* find that the extraction schedule of figure 2 satisfies the HJB-equation for the whole parameter range $k^f > \frac{1}{2}(\overline{p} + k^c)$, which is not surprising since the extraction schedule of figure 1 can also be the equilibrium here. We can, however, prove strong time-consistency in the upper range directly from the open-loop equilibrium. The open-loop equilibrium conditions (see Groot, Withagen and de Zeeuw (1990))

$$P^f(t_1) = MC(t_1), \int_t^{t_1} 2(P^m(s) - P^f(s))ds + \int_{t_1}^{t_2} \mu^c e^{rs} ds = 0 \ , \tag{17}$$

where P^f, P^m and MC are given in equations (6)-(8), yield expressions for the shadow prices λ^c and μ^c which then prove to be equal to the partial derivatives of the value function of the cartel with respect to its own stock and the stock of the fringe, respectively (see Appendix A). It follows that the optimality conditions (14) and (15) hold, which can be seen from the position of P^f, P^m and MC, and the HJB equation (16) is also satisfied for $k^f \geq \frac{1}{2}(\overline{p} + k^c)$. We can conclude that

Proposition 2

The weakly time-consistent open-loop von Stackelberg equilibria of the form of figure 2, where first the cartel produces at the competitive price followed by the fringe, are also strongly time-consistent.

A corollary of proposition 2 is that the last two stages of the extraction schedule of figure 1, where the cartel produces first at the monopoly price and then at the competitive price followed by the fringe, are strongly time-consistent. It remains to be shown that this also holds for the first stage. The optimality conditions are

$$k^f + \lambda^f e^{rt} - k^c - \frac{\partial V^c}{\partial S^c} e^{rt} + \frac{\partial V^c}{\partial S^f} e^{rt} \geq 0 \quad (0 \leq t < t_1) \tag{18}$$

$$\tfrac{1}{2}(\overline{p} + k^c + \frac{\partial V^c}{\partial S^c} e^{rt}) < k^f + \lambda^f e^{rt} \qquad (0 \leq t < t_1) \ , \tag{19}$$

and the HJB equation becomes

$$\frac{\partial V^c}{\partial t} + \tfrac{1}{4} e^{-rt} (\overline{p} - k^c - \frac{\partial V^c}{\partial S^c} e^{rt})^2 = 0 \tag{20}$$

The second optimality condition states that the cartel can act as a monopolist and the first optimality condition assures that simultaneous production at the competitive price cannot be better for the cartel than producing alone at that price.

The verification procedure again involves lots of calculations but is similar in structure to the analyses above (see Appendix B). The intersections of P^f with $\overline{p}$, P^m and MC, given by equations (6)-(8), yield λ^f, λ^c and μ^c as explicit functions of t_1, t_2, t_3 and the parameters $\overline{p}$, k^c, k^f and r. The profits-to-go of the cartel can then also be written as a function of these variables and t. Exhaustion of the two resource stocks plus the equilibrium condition (see Groot, Withagen and de Zeeuw (1990))

$$\int_{t_1}^{t_2} 2(P^m(s) - P^f(s))ds + \int_{t_2}^{t_3} \mu^c e^{rs} ds = 0 \tag{21}$$

yield t_1, t_2 and t_3 as implicit functions of t, S^c and S^f. It follows that the partial derivatives of the value function V^c for the cartel with respect to t, S^c and S^f can be determined. The last two partial derivatives prove to be equal to λ^c and μ^c respectively, so that optimality conditions (18) and (19) hold, which can be seen immediately from the positions of P^f, P^m and MC. Furthermore, the HJB equation (20) proves to be satisfied. We can conclude that

Proposition 3

The weakly time-consistent open-loop von Stackelberg equilibrium of the form of figure 1, where the cartel produces first at the monopoly price and then at the competitive price followed by the fringe, is also strongly time-consistent.

It may seem that we have solved the problem for the cost configuration $k^f > k^c$. We have verified that the time-consistent open-loop equilibria satisfy the HJB equation and we have found a candidate for the feedback equilibrium in the form of figure 2 in the case of time-inconsistency. However, another extraction schedule can also satisfy the HJB equation for $k^f < \frac{1}{2}(\overline{p} + k^c)$ and is therefore also a candidate for the feedback equilibrium. It is the schedule of figure 5, where cartel and fringe first produce simultaneously at the competitive price after which the cartel is a monopolist. The optimality conditions are

$$k^f + \lambda^f e^{rt} - k^c - \frac{\partial V^c}{\partial S^c} e^{rt} + \frac{\partial V^c}{\partial S^f} e^{rt} = 0 \quad (0 \leq t < t_1) \tag{22}$$

and (15). The HJB equation becomes (using (22))

$$\frac{\partial V^c}{\partial t} - \frac{\partial V^c}{\partial S^f}(\overline{p} - k^f - \lambda^f e^{rt}) = 0 \; . \tag{23}$$

The first optimality condition now states that the cartel is indifferent on how to divide the production along the simultaneous stage.

The verification procedure is much more difficult than in the previous cases because we do not know the division in the simultaneous stage. The extraction schedule of figure 5 is characterized by the variables λ^f and λ^c and therefore we can write the extraction of the fringe as a function of time s, λ^f and λ^c. The trick is to perform a transformation of these coordinates which finally leads to a partial differential equation in the value function that can be solved.

The transformation to the set of coordinates z, x and y is

$$s = t - r^{-1} \ln z \tag{24}$$

$$\lambda^f = \tfrac{1}{2} e^{-rt}(ax + by) \tag{25}$$

$$\lambda^c = e^{-rt} by \; , \tag{26}$$

with $a := (\overline{p} + k^c - 2k^f)$ and $b := (\overline{p} - k^c)$, where t still denotes the point in time at which we want to verify the HJB equation and s denotes the running time.

This transformation may look a bit strange at first sight but note that we also have

$$x = e^{rt - rt_1}, y = e^{rt - rt_2}, z = e^{rt - rs} \; . \tag{27}$$

The condition that the resource stocks at t will be exhausted yields the restrictions

$$\int_t^{t_1} E^f(s)ds = S^f, \int_t^{t_2} (E^f(s) + E^c(s))ds = S^f + S^c \ , \tag{28}$$

which leads to

$$\int_x^1 z^{-1} E^f(x,y,z)dz = rS^f, a(x - 1 - \ln x) + b(y - 1 - \ln y) = 2r(S^f + S^c) \ , \tag{29}$$

where $E^f(s) = E^f(s; \lambda^f, \lambda^c) = E^f(x, y, z)$.

The profits-to-go of the cartel can be written as a function of t, x and y. Restrictions (29) define x and y as implicit functions of S^c and S^f, from which the partial derivatives of x and y with respect to S^c and S^f can be derived. It is now possible, but rather complex, to derive a linear partial differential equation in the profits-to-go or the value function of the cartel (see Appendix C). In this derivation optimality condition (22) is used as well as the property that E^f as a function of x, y and z is homogeneous of degree zero. This partial differential equation can be solved so that we get an expression for the value function of the cartel. It is now straightforward, although again rather tedious, to derive the optimal extraction schedule of the cartel in the simultaneous phase. The optimality conditions (22) and the HJB equation (23) are always satisfied. The optimality condition (15) is satisfied for $k^f < \frac{1}{2}(\overline{p} + k^c)$ (see Appendix C). We can conclude that

Proposition 4

If $k^f < \frac{1}{2}(\overline{p} + k^c)$ the extraction schedule of figure 5, where cartel and fringe first produce simultaneously at the competitive price followed by a cartel monopoly, is a candidate for a feedback von Stackelberg equilibrium.

For the cost configuration $k^c < k^f < \frac{1}{2}(\overline{p} + k^c)$ propositions 1 and 4 thus yield two candidates for the feedback von Stackelberg equilibrium, Note, however, that in the derivation of the extraction schedule $S \rightarrow C^m$, where S denotes an interval of time with simultaneous supply, we have implicitly assumed that the extraction rate of the cartel is non-negative. In fact, the equilibrium does not always exist. If the cartel has a choice it will pick the extraction schedule with the highest profit. On the basis of numerical profit calculations figure 7 shows what will happen. In this figure the initial resource stock of the fringe S^f varies on one axis and its marginal extraction cost k^f on the other axis. The other parameters are fixed: the initial resource stock of the cartel S^c is equal to 200 and its marginal extraction cost k^c is equal to 50, the choke price $\overline{p}$ is equal to 100, and the interest rate is set at 10%. The dashed curve indicates the area where the equilibrium $S \rightarrow C^m$ exists (i.e. the corresponding extraction rates are positive). From proposition 1 we know that the extraction schedule $C \rightarrow F$ is a candidate for the feedback von Stackelberg equilibrium for all k^f between k^c and $\frac{1}{2}(\overline{p} + k^c)$. This implies that in the area between the dashed curve and k^c the cartel has a choice. The solid line indicates the points where the profits for the cartel are equal. Above this line the cartel will choose the extraction schedule $C \rightarrow F$ and below this line $S \rightarrow C^m$.

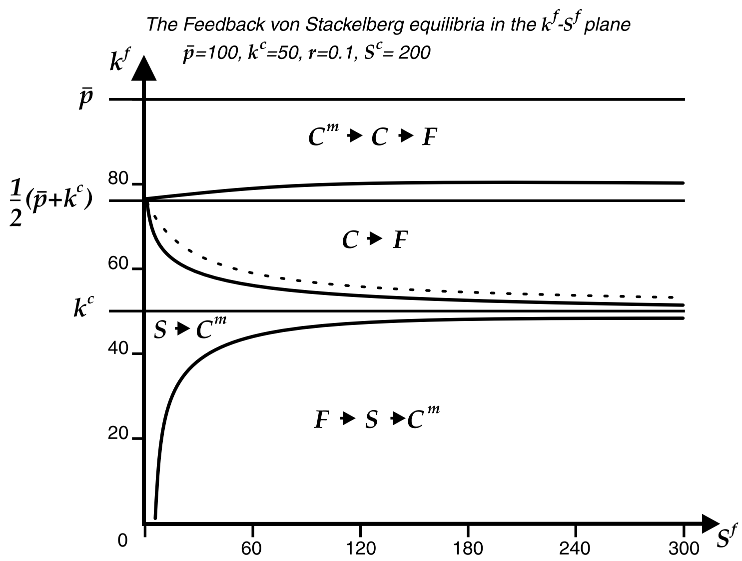

Figure 7

For the cost configuration $k^f \geq \frac{1}{2}(\overline{p} + k^c)$ propositions 2 and 3 show that the feedback von Stackelberg equilibrium coincides with the open-loop von Stackelberg equilibrium. Figure 7 again shows what will happen. For each k^f a level of S^f exists above which the monopoly phase disappears, and for each S^f a level of k^f exists below which the monopoly phase disappears.

If $k^f = k^c$ the situation is as follows. The extractions of the schedule $S \rightarrow C^m$ seen as a feedback equilibrium are equal to the extractions of the open-loop Nash equilibrium or the weakly time-consistent open-loop von Stackelberg equilibrium, which coincide here. Therefore this open-loop von Stackelberg equilibrium is also strongly time-consistent.

For the cost configuration $k^f < k^c$ the feedback von Stackelberg equilibrium is given by the extraction schedule of figure 5. If the initial resource stock of the fringe is relatively large and k^f is small these stages will be preceded by a stage where the fringe produces alone (see figure 7). In order not to complicate this paper any further a proof is omitted here.

The discovery of the extraction schedule $S \rightarrow C^m$ as a candidate feedback von Stackelberg equilibrium is the major achievement of this paper. Note that this result is in line with the conjecture of Newbery (1993) that the Nash equilibrium might be a good approximation for the feedback von Stackelberg equilibrium. It remains to be shown that $S \rightarrow C^m$ is the only candidate besides $C \rightarrow F$. The cartel will never choose a final stage S, because it is better to split this into $C \rightarrow F$. A final stage C is not time-consistent because C^m is better. C^m as final stage cannot be preceded by C for the same reason. It can be shown that $F \rightarrow C^m$ does not satisfy the HJB equation. The proof is very similar to the proofs of the propositions above and will be omitted here. To summarize, the last two stages of the extraction schedule in the feedback von Stackelberg equilibrium for the cartel-versus-fringe model are either $C \rightarrow F$ or $S \rightarrow C^m$. $C \rightarrow F$ always occurs if $k^f \geq \frac{1}{2}(\overline{p} + k^c)$ and $S \rightarrow C^m$ if $k^f \leq k^c$. $C \rightarrow F$ can be preceded by C^m if k^f is big and the initial stock of the cartel is relatively large, and $S \rightarrow C^m$ can be preceded by F if k^f is small and the initial stock of the fringe is relatively large. For the configuration $k^c < k^f < \frac{1}{2}(\overline{p} + k^c)$ there can be a choice and in that case the cartel will pick the one with the highest profit.

6 Conclusion

The developments on the oil market in the seventies increased the interest for the so-called cartel-versus-fringe model for the supply side of markets for exhaustible natural resources. For certain cost configurations the more realistic models proved to be time-inconsistent. For a long time it remained an open problem how to solve time-inconsistency. This paper finally provides an answer.

Acknowledgement

The authors are very grateful to Jos Jansen for pointing out the way to solve the partial differential equation that arises from the HJB-equations for the $S \rightarrow C^m$ case

Appendix A

Consider figure 2, with extraction schedule $C \to F$. The corresponding extraction rates are given as follows. There exist a positive constant λ^f and points in time t_1 and t_2 with $t_2 > t_1 > 0$ such that

$$\left.\begin{aligned} E^c(s) &= \overline{p} - k^f - \lambda^f e^{rs} \\ \\ E^f(s) &= 0 \end{aligned}\right\} (0 \leq s < t_1) \tag{A1}$$

$$\left.\begin{aligned} E^c(s) &= 0 \\ \\ E^f(s) &= \overline{p} - k^f - \lambda^f e^{rs} \end{aligned}\right\} (t_1 \leq s \leq t_2) \tag{A2}$$

Fix some t with $0 \leq t < t_1$.

At t_2 the price must reach the choke price $\overline{p}$, so that

$$k^f + \lambda^f e^{rt_2} = \overline{p} \tag{A3}$$

We also have

$$\int_t^{t_2} E^f(s) ds = S^f(t)$$

Hence, using (A3),

$$(\overline{p} - k^f)[e^{rt_1 - rt_2} - 1 + rt_2 - rt_1] = rS^f(t) \tag{A4}$$

Similarly

$$\int_t^{t_2} E^c(s) ds = S^c(t)$$

Hence, using (A3),

$$(\overline{p}-k^f)[e^{rt-rt_2}-e^{rt_1-rt_2}+rt_1-rt]=rS^c(t) \tag{A5}$$

The profits of the cartel, from t onwards with $0 \leq t < t_1$, are

$$\Pi^c(t,t_1,t_2)=\int_t^{t_2} e^{-rs}[p(s)-k^c]E^c(s)ds$$

$$=\frac{e^{-rt}}{r}(\overline{p}-k^f)[(k^f-k^c)(1-e^{rt-rt_1})$$

$$+(\overline{p}+k^c-2k^f)e^{rt-rt_2}(rt_1-rt)$$

$$-(\overline{p}-k^f)(e^{rt_1-rt_2}-e^{rt-rt_2})e^{rt-rt_2}] \tag{A6}$$

Using the implicit function theorem we can find t_1 and t_2 as functions of t, S^c and S^f from (A4) and (A5), where S^c and S^f denote the stocks at time t. The value function of the cartel is then implicitly defined by

$$V^c(t,S^c,S^f)=\Pi^c(t,t_1(t,S^c,S^f),t_2(t,S^c,S^f))$$

The partial derivatives needed in the HJB-equation are found as:

$$\frac{\partial V^c}{\partial t} = \frac{\partial \Pi^c}{\partial t}+\frac{\partial \Pi^c}{\partial t_1}\frac{\partial t_1}{\partial t}+\frac{\partial \Pi^c}{\partial t_2}\frac{\partial t_2}{\partial t}=-r\Pi^c(t,t_1,t_2)\ ,$$

$$\frac{\partial V^c}{\partial S^c} = \frac{\partial \Pi^c}{\partial t_1}\frac{\partial t_1}{\partial S^c}+\frac{\partial \Pi^c}{\partial t_2}\frac{\partial t_2}{\partial S^c}$$

$$=\frac{(\overline{p}-k^f)(e^{rt_1-rt_2}+1-2e^{rt-rt_2})-(k^f-k^c)(1-e^{rt_2-rt_1})-(\overline{p}+k^c-2k^f)(rt_1-rt)}{(e^{rt_2}-e^{rt})}\ ,$$

$$\frac{\partial V^c}{\partial S^f} = \frac{\partial \Pi^c}{\partial t_1}\frac{\partial t_1}{\partial S^f} + \frac{\partial \Pi^c}{\partial t_2}\frac{\partial t_2}{\partial S^f}$$

$$= \frac{(\overline{p} - k^f)(e^{rt_1 - rt_2} - e^{rt - rt_2}) - (k^f - k^c)(1 - e^{rt - rt_1}) - (\overline{p} + k^c - 2k^f)(rt_1 - rt)}{(e^{rt_2} - e^{rt})}$$

We can now verify equations (14)–(16).

It is easy to check that equation (16) is satisfied.

Equation (14) reduces to

$$(k^f - k^c)(1 - e^{rt - rt_1}) > 0 \ ,$$

which is satisfied for $k^f > k^c$.

Equation (15) can be seen as a linear expression in k^f which has to be positive. It can be shown that this expression is decreasing, positive for $k^f = \frac{1}{2}(\overline{p} + k^c)$ and negative for $k^f = \overline{p}$. It follows that equation (15) is satisfied for $k^f < \widehat{k}$ with $\frac{1}{2}(\overline{p} + k^c) < \widehat{k} < \overline{p}$.

Finally, the partial derivatives $\dfrac{\partial V^c}{\partial S^c}$ and $\dfrac{\partial V^c}{\partial S^f}$ are equal to the expressions for λ^c and μ^c, respectively, that follow from the conditions (17).

Appendix B

Consider figure 1, with extraction schedule $C^m \rightarrow C \rightarrow F$.

The corresponding extraction rates are given as follows. There exist positive constants λ^f and λ^c, and points in time t_1, t_2 and t_3 with $t_3 > t_2 > t_1 > 0$ such that

$$\left.\begin{array}{l} E^c(s) = \frac{1}{2}(\overline{p} - k^c) - \frac{1}{2}\lambda^c e^{rs} \\ \\ E^f(s) = 0 \end{array}\right\} (0 \leq s < t_1) \tag{B1}$$

$$\left.\begin{aligned} E^c(s) &= \overline{p} - k^f - \lambda^f e^{rs} \\ \\ E^f(s) &= 0 \end{aligned}\right\} (t_1 \leq s < t_2) \tag{B2}$$

$$\left.\begin{aligned} E^c(s) &= 0 \\ \\ E^f(s) &= \overline{p} - k^f - \lambda^f e^{rs} \end{aligned}\right\} (t_2 \leq s \leq t_3) \tag{B3}$$

Fix some t with $0 \leq t < t_1$.

At t_3 the price must reach the choke price $\overline{p}$, so that

$$k^f + \lambda^f e^{rt_3} = \overline{p} \tag{B4}$$

The price path is continuous at t_1, so that

$$k^f + \lambda^f e^{rt_1} = \tfrac{1}{2}(\overline{p} + k^c) + \tfrac{1}{2}\lambda^c e^{rt_1} \tag{B5}$$

There also exists a constant μ^c such that

$$k^f + \lambda^f e^{rt_2} = k^c + (\lambda^c - \mu^c)e^{rt_2} \tag{B6}$$

We also have

$$\int_t^{t_3} E^f(s)ds = S^f(t)$$

Hence, using (B4),

$$(\overline{p} - k^f)[e^{rt_2 - rt_3} - 1 + rt_3 - rt_2] = rS^f(t) \tag{B7}$$

Similarly

$$\int_t^{t_3} (E^c(s) + E^f(s))ds = S^c(t) + S^f(t)$$

Hence, using (B4), (B5) and (B6),

$$2(\overline{p} - k^f)[e^{rt-rt_3} - 1 + rt_3 - rt] - (\overline{p} + k^c - 2k^f)*$$

$$[e^{rt-rt_1} - 1 + rt_1 - rt] = 2r(S^c(t) + S^f(t)) \tag{B8}$$

We must also have (see Groot et al. (1992))

$$\int_{t_1}^{t_2} 2(P^m(s) - P^f(s))ds + \int_{t_2}^{t_3} \mu^c e^{rs} ds = 0$$

This implies

$$\begin{aligned}&[(\overline{p} - k^f)e^{rt-rt_3} - (k^f - k^c)e^{rt-rt_2} - (\overline{p} + k^c - 2k^f)e^{rt-rt_1}]* \\ &(e^{rt_3-rt} - e^{rt_2-rt}) - (\overline{p} + k^c - 2k^f)(e^{rt_2-rt_1} - 1 + rt_1 - rt_2) = 0\end{aligned} \tag{B9}$$

Finally, the profits of the cartel, from t onwards with $0 \leq t < t_1$, are

$$\Pi^c(t, t_1, t_2, t_3) = \int_t^{t_3} e^{-rs}[p(s) - k^c]E^c(s)ds$$

$$= \frac{e^{-rt}}{4r}[(\overline{p} - k^c) - 2(\overline{p} - k^f)e^{rt-rt_3} + (\overline{p} + k^c - 2k^f)e^{rt-rt_1}]^2$$

Using the implicit function theorem we can find t_1, t_2 and t_3 as functions of t, S^c and S^f from (B7), (B8) and (B9), where S^c and S^f denote the stocks at time t. The value function of the cartel is then implicitly defined by

$$V^c(t, S^c, S^f) = \Pi^c(t, t_1(t, S^c, S^f), t_2(t, S^c, S^f), t_3(t, S^c, S^f))$$

The partial derivatives needed in the HJB-equation are found as:

$$\frac{\partial V^c}{\partial t} = \frac{\partial \Pi^c}{\partial t} + \frac{\partial \Pi^c}{\partial t_1}\frac{\partial t_1}{\partial t} + \frac{\partial \Pi^c}{\partial t_2}\frac{\partial t_2}{\partial t} + \frac{\partial \Pi^c}{\partial t_3}\frac{\partial t_3}{\partial t} = -r\Pi^c(t, t_1, t_2, t_3) \ ,$$

$$\frac{\partial V^c}{\partial S^c} = \frac{\partial \Pi^c}{\partial t_1}\frac{\partial t_1}{\partial S^c} + \frac{\partial \Pi^c}{\partial t_2}\frac{\partial t_2}{\partial S^c} + \frac{\partial \Pi^c}{\partial t_3}\frac{\partial t_3}{\partial S^c}$$

$$= 2(\overline{p} - k^f)e^{-rt_3} - (\overline{p} + k^c - 2k^f)e^{-rt_1} \ ,$$

$$\frac{\partial V^c}{\partial S^f} = \frac{\partial \Pi^c}{\partial t_1}\frac{\partial t_1}{\partial S^f} + \frac{\partial \Pi^c}{\partial t_2}\frac{\partial t_2}{\partial S^f} + \frac{\partial \Pi}{\partial t_3}\frac{\partial t_3}{\partial S^f}$$

$$= (\overline{p} - k^f)e^{-rt_3} - (\overline{p} + k^c - 2k^f)e^{-rt_1} - (k^f - k^c)e^{-rt_2}$$

Finally, it is easy to check that the partial derivative $\frac{\partial V^c}{\partial S^c}$ is equal to λ^c, using (B5) and (B4), that the partial derivative $\frac{\partial V^c}{\partial S^f}$ is equal to μ^c, using (B6), (B5) and (B4), and that the HJB equation (20) is satisfied.

Appendix C

Consider figure 5, with extraction schedule $S \to C^m$. There exist positive constants λ^f and λ^c and points in time t_1 and t_2 with $t_2 > t_1 > 0$ such that

$$E^c(s) + E^f(s) = \overline{p} - k^f - \lambda^f e^{rs} \quad (0 \le s < t_1) \tag{C1}$$

$$\left.\begin{aligned} E^c(s) &= \tfrac{1}{2}(\overline{p}-k^c) - \tfrac{1}{2}\lambda^c e^{rs} \\ \\ E^f(s) &= 0 \end{aligned}\right\} \ (t_1 \leq s \leq t_2) \tag{C2}$$

Fix some t with $0 \leq t < t_1$.

At t_2 the price must reach the choke price $\overline{p}$, so that

$$\tfrac{1}{2}(\overline{p}+k^c) + \tfrac{1}{2}\lambda^c e^{rt_2} = \overline{p} \tag{C3}$$

The price path is continuous at t_1, so that

$$k^f + \lambda^f e^{rt_1} = \tfrac{1}{2}(\overline{p}+k^c) + \tfrac{1}{2}\lambda^c e^{rt_1} \tag{C4}$$

Given λ^f and λ^c it can easily be calculated how much the cartel and the fringe will supply in total over the interval $[0, t_1)$. Central in our approach is the claim that, given λ^f and λ^c, there is also only one possible outcome along this first interval if $k^f \neq k^c$. This follows from the fact that the cartel will simply choose the best one for itself if there exist multiple candidates. Note that at any point in time between 0 and t_1 it is optimal to have the same λ's. We write

$$E^c(s;\lambda^f,\lambda^c) = \widehat{E}^c(\lambda^f e^{rs}, \lambda^c e^{rs}) \ (0 \leq s < t_1) \tag{C5}$$

$$E^f(s;\lambda^f,\lambda^c) = \overline{p} - k^f - \lambda^f e^{rs} - E^c(s;\lambda^f,\lambda^c) = \widehat{E}^f(\lambda^f e^{rs}, \lambda^c e^{rs}) \ (0 \leq s < t_1) \tag{C6}$$

In view of the transformation (24)–(26) we have

$$\widehat{E}^f(\lambda^f e^{rs}, \lambda^c e^{rs}) = \widehat{E}^f\left(\frac{ax+by}{2z}, \frac{by}{z}\right) =: \overline{E}^f(x,y,z)$$

The restrictions with respect to the stocks are

$$\int_t^{t_1} \widehat{E}^f(\lambda^f e^{rs}, \lambda^c e^{rs}) ds = S^f(t)$$

$$\int_t^{t_2} [\widehat{E}^c(\lambda^f e^{rs}, \lambda^c e^{rs}) + \widehat{E}^f(\lambda^f e^{rs}, \lambda^c e^{rs})] ds = S^c(t) + S^f(t)$$

which imply (omitting the time argument in S^c and S^f)

$$F(x, y) := \int_x^1 \tfrac{1}{z} \overline{E}^f(x, y, z) dz = r S^f \tag{C7}$$

$$a(x - 1 - \ln x) + b(y - 1 - \ln y) = 2r(S^c + S^f) \tag{C8}$$

The profits of the cartel, from time t onwards with $0 \leq t < t_1$, are

$$\Pi^c(t, x, y) = \int_t^{t_2} e^{-rs} [\overline{p} - E^f(s) - E^c(s) - k^c] E^c(s) ds$$

$$= \int_t^{t_1} e^{-rs} [k^f + \lambda^f e^{rs} - k^c][\overline{p} - k^f - \lambda^f e^{rs} - \widehat{E}^f(\lambda^f e^{rs}, \lambda^c e^{rs})] ds$$

$$+ \int_{t_1}^{t_2} e^{-rs} \tfrac{1}{2} [\overline{p} - k^c + \lambda^c e^{rs}] \tfrac{1}{2} [\overline{p} - k^c - \lambda^c e^{rs}] ds$$

$$= \frac{e^{-rt}}{4r} [b^2(1 - y)^2 - a^2(x - 1)^2 + 2(ax + by) a(x - 1 - \ln x)$$

$$-2(b - a) G(x, y) - 2(ax + by) F(x, y)] \tag{C9}$$

where

$$G(x,y) := \int_x^1 \overline{E}^f(x,y,z)dz\ . \tag{C10}$$

Using the implicit function theorem we can find x and y as functions of S^c and S^f from (C7) and (C8). Hence, the value function at time t is

$$V^c(t,S^c,S^f) = \Pi^c(t, x(S^c,S^f), y(S^c,S^f))$$

A necessary condition for the proposed exploitation patterns to be a feedback von Stackelberg equilibrium is (22). In terms of x and y the equation boils down to

$$\tfrac{1}{2}(b-a) + \tfrac{1}{2}(ax+by) = \Big[\overline{\Pi}_x^c(t,x,y)\left(\frac{\partial x}{\partial S^c} - \frac{\partial x}{\partial S^f}\right) +$$

$$+\overline{\Pi}_y^c(t,x,y)\left(\frac{\partial y}{\partial S^c} - \frac{\partial y}{\partial S^f}\right)\Big] \tag{C11}$$

where $\overline{\Pi}^c = e^{rt}\Pi^c$.

First we claim

$$G(x,y) = xG_x(x,y) + yG_y(x,y) - xF_x(x,y) - yF_y(x,y) \tag{C12}$$

This can be seen as follows. The definitions (C7) and (C10) imply

$$F_x(x,y) = -\tfrac{1}{x}\overline{E}^f(x,y,x) + \int_x^1 \tfrac{1}{z}\overline{E}_x^f(x,y,z)dz$$

$$F_y(x,y) = \int_x^1 \tfrac{1}{z}\overline{E}_y^f(x,y,z)dz$$

$$G_x(x,y) = -\overline{E}^f(x,y,x) + \int_x^1 \overline{E}_x^f(x,y,z)dz$$

$$G_y(x,y) = \int_x^1 \overline{E}_y^f(x,y,z)dz$$

Moreover, $\overline{E}^f$ is homogeneous of degree zero which is immediate from the definition of $\overline{E}^f$, so that

$$\int_x^1 (\frac{z-1}{z})[x\overline{E}_x^f(x,y,z) + y\overline{E}_y^f(x,y,z) + z\overline{E}_z^f(x,y,z)]dz = 0$$

It is easily seen that these equations yield (C12).

The next step is to solve G from (C9), to calculate G_x and G_y and to insert all three of them into (C12). This yields

$$xF_x + yF_y = \tfrac{1}{2}(a(x-1) + b(y-1)) + 2r(\overline{\Pi}^c - x\overline{\Pi}_x^c - y\overline{\Pi}_y^c)/(a(x-1) + b(y+1)) \quad \text{(C13)}$$

It follows from (C7) and (C8) that

$$\frac{\partial x}{\partial S^c} - \frac{\partial x}{\partial S^f} = -brx(y-1)/(b(y-1)xF_x - a(x-1)yF_y)$$

$$\frac{\partial y}{\partial S^c} - \frac{\partial y}{\partial S^f} = ary(x-1)/(b(y-1)xF_x - a(x-1)yF_y)$$

Hence, we find a second equation in xF_x and yF_y by inserting these results in (C11). We can then solve for xF_x and yF_y:

$$xF_x = \tfrac{1}{2}a(x-1) + \frac{2ra(x-1)\overline{\Pi}^c - (a(x-1) + b(y-1))2rx\overline{\Pi}_x^c}{(a(x-1) + b(y+1))(a(x-1) + b(y-1))} \quad \text{(C14)}$$

$$yF_y = \tfrac{1}{2}b(y-1) + \frac{2rb(y-1)\overline{\Pi}^c - (a(x-1) + b(y-1))2ry\overline{\Pi}_y^c}{(a(x-1) + b(y+1))(a(x-1) + b(y-1))} \quad \text{(C15)}$$

When (C14) is differentiated with respect to y and (C15) with respect to x, we find the following partial differential equation for Π^c, because $F_{xy} = F_{yx}$:

$$2ab(x-y)(a(x-1)+by)\Pi^c+$$

$$+b(a(x-1)+b(y-1))(b(y-1)-a(x-1))x\Pi^c_x$$

$$+a(a(x-1)+b(y-1))(b(y+1)+a(x-1)-2bx)y\Pi^c_y = 0 \tag{C16}$$

There are two restrictions that have to be taken into account when solving the partial differential equation. First, if $x=1$ then $t=t_1$ and $F(x,y)=G(x,y)=0$, so that

$$\Pi^c(t,1,y) = \frac{e^{-rt}}{4r}b^2(1-y)^2 \text{ for } 0<y<1$$

Second, $\Pi^c(t,1,1)=0$. We then find as the value function (The proof is available from the authors upon request)

$$V^c(t,S^c,S^f) = -\frac{e^{-rt}}{4r}[a(x-1)+b(y-1)]\{[a(x-1)+b(y+1)]^2-4b^2x^{\frac{a}{b}}y\}^{\frac{1}{2}} \tag{C17}$$

where it is to be understood that x and y are functions of S^c and S^f.

The last step is to find the functions $\overline{E}^f$ and $\overline{E}^c$. It follows from the earlier results that

$$xF_x + yF_y = -\overline{E}^f(x,y,x) - \int_x^1 \overline{E}^f_z(x,y,z)dz = -\overline{E}^f(x,y,1)$$

where we have used the homogeneity of $\overline{E}^f$ again. Since we know the function Π^c we thus find $\overline{E}^f$ from (C13). In terms of our original notation this yields

$$E^c(t) = \frac{(\overline{p}-k^f)(k^f-k^c)+(\lambda^f e^{rt})^2-(\overline{p}-k^f)\lambda^c e^{rt}\left[\dfrac{(2\lambda^f-\lambda^c)e^{rt}}{\overline{p}+k^c-2k^f}\right]^{\frac{\overline{p}+k^c-2k^f}{\overline{p}-k^c}}}{\left[(k^f-k^c+\lambda^f e^{rt})^2-(\overline{p}-k^c)\lambda^c e^{rt}\left[\dfrac{(2\lambda^f-\lambda^c)e^{rt}}{\overline{p}+k^c-2k^f}\right]^{\frac{\overline{p}+k^c-2k^f}{\overline{p}-k^c}}\right]^{\frac{1}{2}}}$$

We can now verify equation (22), (23) and (15).

It is easy to check that equations (22) and (23) are satisfied. Equation (15) reduces to

$$SQR < 2(\overline{p}-k^f-\lambda^f e^{rt}) \ ,$$

where SQR is the square root in the value function V^c given in (C17). It can be shown that this is precisely satisfied for $k^f < \frac{1}{2}(\overline{p}+k^c)$.

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