On the (R,s,Q) Inventory Model when Demand is Modelled as a Compound Process

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Publication date:
1996

On the (R, s, Q) inventory model when demand is modelled as a compound Bernoulli process

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Abstract

In this paper we present an approximation method to compute the reorder point s in a (R, s, Q) inventory model with a service level restriction, where demand is modelled as a compound Bernoulli process, that is, with a fixed probability there is positive demand during a time unit, otherwise demand is zero. The demand size and replenishment leadtime are stochastic variables. It is shown that this kind of modelling is especially suitable for intermittent demand. Furthermore, an approximation for the expected average physical stock is derived. The quality of both the reorder point determination as well as the approximation for the expected average physical stock turn out to be excellent, as is verified by discrete event simulation.

1. Introduction

The (R, s, Q) inventory model has been studied exhaustively during the last decades. Under the regime of this inventory policy, every R time units the inventory position is monitored in order to make a replenishment decision. When the inventory position is below s, an integral multiple of Q is ordered such that the inventory position is raised to a value between s and $s+Q$. Many heuristic and optimal methods are developed to determine the values of the control parameters: R, s and Q. Among these methods, mainly two directions can be distinguished: methods that minimize total relevant costs (see e.g. Hadley and Whitin (1963), Das (1976), Johansen and Thorstenson (1993)), and methods that are based on achieving a pre-specified customer service level (see e.g. Schneider (1981,1990), de Kok (1991a,1991b), Tijms and Groenevelt (1984), and Tijms (1994)). As costs of shortages are difficult to quantify in practice, we focus on a service level approach. To be more precise, we use as service criterion the fraction of demand delivered directly from shelf, which is often denoted as the fill rate or P_2 service level (see Silver and Peterson (1985)). In this approach we assume that customer orders which can not be satisfied directly from shelf are backordered.

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Dunsmuir and Snyder (1989) developed a model where intermittent demand is modelled as a compound Bernoulli process, that is, with a fixed probability ($0 < \pi_D \leq 1$) there is positive demand during a time unit, otherwise demand is zero. The main intention of this paper is to go into the details of the compound Bernoulli modelling concept.

The increasing importance of intermittent demand modelling can be argued as follows. Firstly, the increasing information technology makes more detailed information available for all kind of processes. Hence, it is easy to collect daily or weekly demand information instead of monthly data. This, however, has consequences for the modeling of the demand process. The probability that demand is positive during a month is often one, whereas the probability that demand is positive during a day may be considerably less than one. Shortening the time unit over which demand data are analysed, has the advantage that the reaction time is short when sudden changes occur. This may be very important in a dynamic world. The importance of intermittent demand modelling is also obvious in spare parts environments, even with monthly demand. Further, we also observed intermittent demand in inventory management of medicines in a medical center with many departments: as management wants to keep inventories of certain medicines a low level (nursing departments), it has the consequence that demand processes were intermittent on that level.

Another important motivation for using the compound Bernoulli modelling, is forcasting. When intermittent demand is forecasted, it appears (see e.g. Willemain et al. (1994)) that the separation idea (called 'Croston's forecast procedure' in the forecast literature) is better than the single exponential smoothing procedure, applied to the non-separated demand data. These conclusions remain to hold for all kind of data scenarios: interarrivals and demand occurrences cross-correlated or not, or interarrivals autocorrelated.

Basically, we will adapt the method presented by Dunsmuir and Snyder, such that the compound Bernoulli modelling is applicable for a more general class of situations. It is assumed that information is available on a daily basis. The review period, however, may be once a week or once a month, hence we consider situations with a review period R is larger than one. The reorder level is not required to be positive and the shortages at the beginning of a replenishment cycle are not neglected. Finally, we include the undershoot, which is evidently important in situations where demand is lumpy and not of unit size.

The paper is organized as follows. In section 2 a formal model description is given for the demand process modelled as a compound Bernoulli process. In section 3 an approximation method is presented for computing the reorder level s when the demand process is a compound Bernoulli process. This method is called the compound Bernoulli method (CBM), furthermore an approximation is given for the expected average physical stock level. In section 4 the CBM as well as the approximation for the expected average physical stock level are validated by simulation, and the results will be compared with the results of the method presented by Dunsmuir and Snyder. Furthermore, examples are given how these result can be used by the management in practical situations. Finally, in section 5 some conclusions are given.

2. The Model

Let us assume that daily demand information is available. The demand size of the n-th day is denoted by D_n, and D_k^* denotes the demand size of the k-th day on which demand is positive. It is assumed that the D_n's as well as the D_k^*'s are independent and identically distributed random variables, with distribution functions $F_D(.)$ and $F_{D^*}(.)$, respectively.

When the demand process is modelled as a compound Bernoulli process and the probability that demand is positive is denoted by π_D, then the distribution functions $F_D(.)$ and $F_{D^*}(.)$ are related through

$$F_D(y) = \begin{cases} 1 - \pi_D & \text{if} \quad y = 0 \\ 1 - \pi_D + \pi_D F_{D^*}(y) & \text{if} \quad y > 0 \end{cases} \tag{1}$$

The relation between the moments of D and D^* can easily be derived:

$$\mathbb{E}D^k = \pi_D \mathbb{E}D^{*k} \quad k = 1, 2, \ldots \tag{2}$$

Remark: For small time units the compound Bernoulli process is approximately a compound Poisson process (e.g. see Feller(1970) pp. 153). Hence the compound Bernoulli process can be seen as the discrete time variant of the compound Poisson process.

We assume that the leadtimes do not cross in time, implying that the leadtimes of replenishment orders $L_1, L_2, \ldots$ are dependent random variables. For (s, Q)-modelling with interrelated leadtimes we refer to Heuts and de Klein (1995). Because the demand process is a discrete time process, we assume that the leadtimes are an integral number of days. Furthermore, it is assumed that customer orders are handled at the beginning of a day, whereas replenishment orders are handled at the end of the day. This aspect is important in case the customer service measure is actually determined in practice or in the simulation experiments.

Customer orders which cannot be delivered directly from stock will be backordered. As performance measure the P_2-service measure is used (see Silver and Peterson (1985)): the long-run fraction of demand delivered directly from shelf, which is denoted by $\beta(R, s, Q)$.

The inventory position in the (R, s, Q) model is not a regenerative proces. Nevertheless, the following basic formula applies (see Tijms (1994) p.53)

$$1 - \beta(R, s, Q) = \frac{\text{the expected shortage during an arbitrary replenishment cycle}}{\text{the expected demand during an arbitrary replenishment cycle}} \tag{3}$$

Another important performance measure is the expected average physical stock needed to maintain the required service level, denoted by $\mu(R, s, Q)$. Denote $H(R, s, Q)$ as the expected area between the physical stock level and the time-axis during a replenishment cycle, and $C(R, s, Q)$ as the expected duration of a replenishment cycle. Based on the renewal reward theorem (see Tijms (1994) p. 33) a simple expression for $\mu(R, s, Q)$ is:

$$\mu(R, s, Q) = \frac{H(R, s, Q)}{C(R, s, Q)} \tag{4}$$

In order to derive expressions for $\beta(R,s,Q)$ and $\mu(R,s,Q)$ we define

$Z(n)$	$:=$	the total demand during n time periods;
$Z^*(n)$	$:=$	the total demand during n time periods, given that in at least one period the demand is positive;
$X(n)$	$:=$	the inventory position on time epoch n;
T_k	$:=$	the point in time at which the inventory position drops below s for the k-th time after 0;
U_k	$:=$	$s - X(T_k)$ (the k-th undershoot);
τ_k	$:=$	the first review moment after T_k;
W_k	$:=$	$\tau_k - T_k$;
$\hat{L}_k$	$:=$	$L_k + W_k$;
$U_{R,k}$	$:=$	$s - X(\tau_k)$;
Z_k	$:=$	$Z(\hat{L}_k) + U_k = Z(L_k) + U_{R,k}$;
Z_k^*	$:=$	$Z^*(\hat{L}_k) + U_k$;

When the reorder quantity (Q) is large relative to the expected demand per time unit,

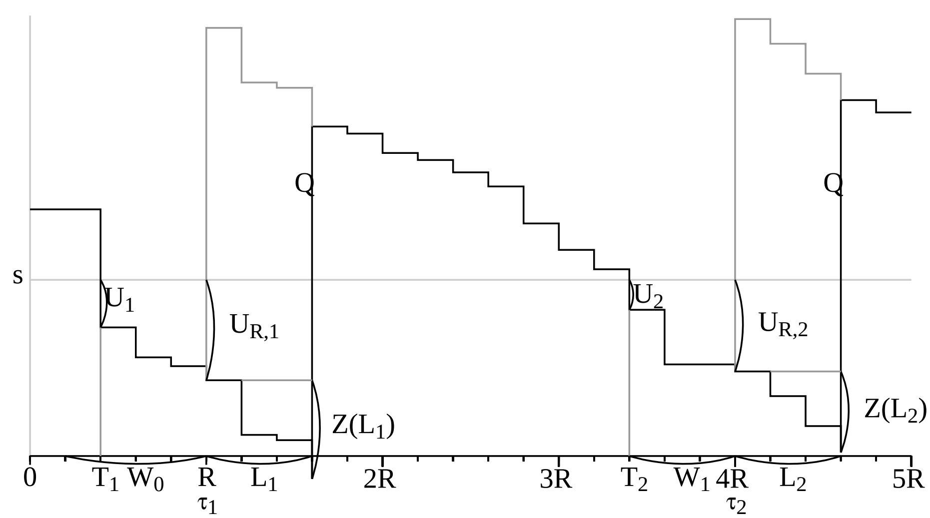

Figure 1: Evolution of the net stock and inventory position during the first replenishment cycle

or when the target service (β) is low, the target service can be realized, even when the reorder level (s) is negative. Therefore, we do not require the reorder point to be positive. Especially, in cases where Q and β are both small, the shortages at the beginning of a replenishment cycle are relevant. Thus we do not neglect the expected shortage at the beginning of a replenishment cycle. Consider the first replenishment cycle after zero. Given that the backlog at the end of the cycle equals the sum of backlog at the start of the cycle and the unsatisfied demand during the cycle, it can be derived (see e.g. de Kok(1991b)) that, for $Q \gg I\!\!ED$ (i.e. exactly one Q is ordered), s must satisfy the service equation (Figure 1 may be enlightening):

$$\beta(R,s,Q) = \begin{cases} 0 & s \leq -Q \\ 1 - \frac{EZ_2 - s - E(Z_1 - s - Q)^+}{Q} & -Q < s \leq 0 \\ 1 - \frac{E(Z_2 - s)^+ - E(Z_1 - s - Q)^+}{Q} & 0 < s \end{cases} \tag{5}$$

where $x^+ := \max\{0, x\}$. In the sequel of this paper we will use (5) as starting point to derive expressions for $\beta(R,s,Q)$. As in Tijms and Groenevelt (1984) to obtain values for (the incomplete moments) $I\!\!E(Z_2 - s)^+$ and $I\!\!E(Z_1 - s - Q)^+$ for given values for s and Q, we assume that the Z_1 has a generalized Erlang distribution, which is uniquely determined by its first two moments. The smallest value for s which satisfies (5) can then be find by using a local search procedure such as Golden Section search.

To obtain an expression for $\mu(R,s,Q)$ we consider the first replenishment cycle after zero. Based on results from renewal theory we derive (see Appendix 1):

$$\mu(R,s,Q) = \begin{cases} 0 & s \leq -Q \\ \int_0^{s+Q} \frac{(s+Q-x)^2}{2Q} dF_{Z(\hat{L}_1)}(x) & -Q < s \leq 0 \\ \int_0^{s+Q} \frac{(s+Q-x)^2}{2Q} dF_{Z(\hat{L}_1)}(x) - \int_0^{s} \frac{(s-x)^2}{2Q} dF_{Z(\hat{L}_2)}(x) & s > 0 \end{cases} \tag{6}$$

3. The compound Bernoulli method

The motivation behind the compound Bernoulli model is the distinction between the situations that the demand during $\hat{L}_k$ is zero or positive. Due to this distinction, the service equation (5) has to be adjusted. We denote the $\hat{L}_k$'s as the pseudo leadtimes and the probability of positive demand during the pseudo leadtime as $\pi_{\hat{L}}$, i.e. $\pi_{\hat{L}} = I\!\!P(Z(\hat{L}_k) > 0), k \geq 1$. In the situation that the demand during the pseudo leadtime is zero, which occurs with probability $1 - \pi_{\hat{L}}$, backlogs only occurs when the undershoot is larger than s, the value of the reorder point. However, for the situation that the demand during the pseudo leadtime is positive, backlog occurs when the demand during the pseudo leadtime (given it is positive) plus the undershoot is larger than s. Combining both possible situations, analogously to (5) the following relation can be derived, considering the first replenishment cycle after zero:

$$\beta(R,s,Q) = \begin{cases} 0 & s \leq -Q \\ 1 - \frac{\pi_{\hat{L}}\left(EZ_2^* - s - E(Z_1^* - s - Q)^+\right) + (1-\pi_{\hat{L}})\left(EU_2 - s - E(U_1 - s - Q)^+\right)}{Q} & -Q < s \leq 0 \\ 1 - \frac{\pi_{\hat{L}}\left(E(Z_2^* - s)^+ - E(Z_1^* - s - Q)^+\right) + (1-\pi_{\hat{L}})\left(E(U_2 - s)^+ - E(U_1 - s - Q)^+\right)}{Q} & 0 < s \end{cases} \tag{7}$$

To compute (7) we again approximate the distribution functions of Z_1^* and U_1 by that of a generalized Erlang distribution. Hence, the first two moments of Z_1^* and U_1, and the probability $\pi_{\hat{L}}$ are sufficient to calculate the $\beta(R, s, Q)$ for given R, s and Q. To obtain values for $\mu(R, s, Q)$ also approximation of relevant distribution functions by a generalized Erlang distribution is used to compute the integrals of expression (6). Hence, only the first two moments of $Z(\hat{L}_1)$ are required. The expressions for $\pi_{\hat{L}}$, the first two moments of U_1, $Z(\hat{L}_1)$ and Z_1^* are derived below.

For the first two moments of U_1 we use the asymptotic results for the first two moments of the residual lifetime distribution, which yields according to (2)

$$\mathbb{E}U_1 \simeq \frac{\mathbb{E}(D^*)^2}{2\mathbb{E}D^*} = \frac{\mathbb{E}D^2}{2\mathbb{E}D} \tag{8}$$

$$\mathbb{E}U_1^2 \simeq \frac{\mathbb{E}(D^*)^3}{3\mathbb{E}D^*} = \frac{\mathbb{E}D^3}{3\mathbb{E}D} \tag{9}$$

We calculate $\pi_{\hat{L}}$ from the generating function of $\hat{L}_1$, denoted by $P_{\hat{L}_1}(.)$, via

$$\pi_{\hat{L}} = 1 - \mathbb{E}(1-\pi_D)^{\hat{L}_1} \tag{10}$$

$$= 1 - P_{\hat{L}_1}(1-\pi_D) \tag{11}$$

Because $\hat{L}_1$ is a convolution of two discrete random variables, a closed form expression for $P_{\hat{L}_1}(.)$ will not be available in general. Therefore we use the method of Adan et al.(1994) to fit a discrete distribution using the first two moments. This method is based on the approach used by Tijms (1994) for continuous random variables, and results in a distribution from the classes: Poisson, mixture of binomial, mixture of negative-binomial and mixture of geometric distributions (see Appendix 2). For any of these distributions the generating function can be determined. To obtain the first two moments of $\hat{L}_1$ we need the first two moments of L_1, which are assumed to be given, and the first two moments of W_1. It can be shown that (see Appendix 3) W_1 is uniformly distributed over $\{0, 1, \ldots, R-1\}$. Hence

$$\mathbb{E}W_1 = \tfrac{1}{2}(R-1) \tag{12}$$

$$\mathbb{E}W_1^2 = \tfrac{1}{6}(R-1)(2R-1) \tag{13}$$

Because L_1 and W_1 are independent, one obtains

$$\mathbb{E}\hat{L}_1 = \mathbb{E}L_1 + \mathbb{E}W_1 \tag{14}$$

$$\mathbb{E}\hat{L}_1^2 = \mathbb{E}L_1^2 + 2\mathbb{E}L_1\mathbb{E}W_1 + \mathbb{E}W_1^2 \tag{15}$$

Since $Z(\hat{L}_1)$ is a stochastic sum of i.i.d. random variables, we have

$$\mathbb{E}Z(\hat{L}_1) = \mathbb{E}\hat{L}_1\mathbb{E}D \tag{16}$$

$$\sigma^2(Z(\hat{L}_1)) = \mathbb{E}\hat{L}_1\sigma^2(D) + \sigma^2(\hat{L}_1)(\mathbb{E}D)^2 \tag{17}$$

What remains is expressions for the first two moments of Z_1^*, and therefore expressions for the first two moments of $Z^*(\hat{L}_1)$. For this purpose we use that the probability of positive demand during the pseudo leadtime is a Bernoulli experiment (with probabilty $\pi_{\hat{L}}$). From $\sigma^2(D^*) \geq 0$ it follows by using (2) with $k = 1$ and $k = 2$ that in case the compound

Bernoulli modelling is applied $c_D^2 \geq (1-\pi_D)/\pi_D$. Then using the appropriate analogy of this condition we conclude that only under special conditions moments for $Z^*(\hat{L}_1)$ can be derived, namely when $c^2_{Z(\hat{L}_1)} \geq (1-\pi_{\hat{L}})/\pi_{\hat{L}}$:

$$\mathbb{E}Z^*(\hat{L}_1) = \frac{\mathbb{E}Z(\hat{L}_1)}{\pi_{\hat{L}}} \tag{18}$$

$$\sigma^2(Z^*(\hat{L}_1)) = \frac{\sigma^2(Z(\hat{L}))}{\pi_{\hat{L}}} - \frac{(1-\pi_{\hat{L}})\mathbb{E}Z(\hat{L}_1)^2}{\pi_{\hat{L}}^2} \tag{19}$$

Hence when $c^2_{Z(\hat{L}_1)} < (1-\pi_{\hat{L}})/\pi_{\hat{L}}$ the compound Bernoulli model can not be applied. In this situation we propose to use expressions (8),(9),(16) and (17) to obtain values for the first two moments of Z_1, and use service equation (5) to calculate the reorder point s.

We are now able to calculate the first two moments of Z_1^*, $Z(\hat{L}_1)$ and U_1, which enables us to calculate the reorder level and the associated average physical stock level. However, it should be noted that the moments are based on asymptotic relations obtained from renewal theory, which makes this method an approximation, even in case the distribution functions of Z_1^*, $Z(\hat{L}_1)$ and U_1 truely are generalized Erlang distributions. To validate the quality of this approximation we use simulation (see section 4).

4. Numerical results

To show the impact of the extensions of the CBM with respect to the method presented by Dunsmuir and Snyder, we use all cases that are considered in their paper. The reorder levels calculated according to Dunsmuir and Snyder as well as the reorder level calculated by the CBM are both validated by simulation. Putting it more precisely, the actual resulting service level is computed via simulation, given a value for the reorder point, and this level is compared to the required service level. The closer these two levels are to each other the better the method performs. The number of sub-runs is fixed at 10 (exclusive the initialisation run), and the sub-run length is chosen such that 100.000 customers are evaluated. The results are tabulated in table 4.1., in which s_1 denotes the reorder point calculated by Dunsmuir and Snyder with the associated actual service level β_1 (between brackets the 95% confidence interval is given), and s_2 denotes the reorder point calculated by the CBM with the associated actual service level β_2.

Table 4.1.: Simulation results for $\beta = 0.95$ and $(\mathbb{E}L_1\,;\,\sigma(L_1)) = (2\,;\,0)$

				Dunsmuir & Snyder		CBM	
π_D	$\mathbb{E}D^*$	$\sigma(D^*)$	Q	s_1	β_1	s_2	β_2
0.36	3.00	1.41	2	6.00	0.8521 ($\pm$ 0.0010)	8.14	0.9480 ($\pm$ 0.0011)
			3	5.00	0.8115 ($\pm$ 0.0010)	7.74	0.9481 ($\pm$ 0.0008)
			4	4.30	0.7891 ($\pm$ 0.0016)	7.38	0.9485 ($\pm$ 0.0013)
0.28	10.30	3.51	5	18.30	0.8586 ($\pm$ 0.0022)	24.15	0.9477 ($\pm$ 0.0016)
			7	17.00	0.8492 ($\pm$ 0.0023)	23.32	0.9479 ($\pm$ 0.0015)
			10	15.00	0.8324 ($\pm$ 0.0026)	22.17	0.9480 ($\pm$ 0.0015)
0.45	201.60	212.40	200	730.00	0.8956 ($\pm$ 0.0037)	942.24	0.9492 ($\pm$ 0.0031)
			300	640.00	0.8782 ($\pm$ 0.0038)	898.73	0.9490 ($\pm$ 0.0028)
			400	570.00	0.8670 ($\pm$ 0.0037)	858.56	0.9493 ($\pm$ 0.0029)
0.64	846.60	384.60	1100	1975.00	0.8517 ($\pm$ 0.0022)	2575.06	0.9509 ($\pm$ 0.0008)
			1700	1725.00	0.8444 ($\pm$ 0.0018)	2384.73	0.9502 ($\pm$ 0.0008)
			2200	1600.00	0.8519 ($\pm$ 0.0018)	2251.34	0.9499 ($\pm$ 0.0009)

The cases considered by Dunsmuir and Snyder are such that the undershoot appears to be very important, which is shown by the bad service performance in case the undershoot is neglected ($\beta_1 < \beta = 0.95$). On the contrary, the CBM has an excellent performance for all cases considered by Dunsmuir and Snyder ($\beta_2 \approx \beta = 0.95$), this in spite of the small values of Q when compared with $\mathbb{E}D^*$.

Next we use simulation to validate the quality of the CBM in terms of service performance and expected average physical stock for a wide range of parameter values. The results are given in Table 4.2., in which s_1 denotes the reorder point calculated by the CBM with the associated actual service level β_1 (between brackets the 95% confidence interval is given), and $\mu(R, s, Q)$ the expected physical stock calculated by expression (6) with the associated actual physical stock μ_1. From these simulation results it can be concluded that the performance of the CBM and the quality of expression (6) for the expected average physical stock, are excellent for most situations considered. However, for the situation that $\pi_D = 0.9$ and $(\mathbb{E}L_1\,;\,\sigma(L_1)) = (10\,;\,4)$ (see bold printed results in Table 4.2.) the actual service is too large. A possible explanation for this is, that in these situations the replenishment orders might cross in time, because of the relative long leadtimes and high demand rates. This violates the previous made assumption that the replenishment orders do not cross. Moreover, notice that for small values of Q (Q=10) the method is still valid in some cases. Furthermore, it can be concluded that the approximation for the expected average physical stock is excellent for all situations considered.

Table 4.2.: Simulation results to validate the CBM with $\mathbb{E}D^* = 5$

R	π_D	$\sigma(D^*)$	Q	$(\mathbb{E}L_1 ; \sigma(L_1))$	β	β_1	s_1	$X(R,s,Q)$	X_1
1	0.10	5	10	(1 ; 0)	0.99	0.9902 (± 0.0015)	20.81	25.32	25.30
1	0.10	5	10	(10 ; 4)	0.99	0.9909 (± 0.0015)	34.96	35.00	34.95
1	0.90	5	10	(1 ; 0)	0.99	0.9899 (± 0.0009)	28.37	28.88	28.86
1	0.90	5	10	(10 ; 4)	0.99	**0.9991** (± 0.0002)	118.18	78.30	78.05
1	0.10	10	10	(1 ; 0)	0.99	0.9914 (± 0.0022)	65.60	70.10	70.11
1	0.10	10	10	(10 ; 4)	0.99	0.9909 (± 0.0035)	80.13	80.18	80.12
1	0.90	10	10	(1 ; 0)	0.99	0.9901 (± 0.0014)	76.44	76.95	76.94
1	0.90	10	10	(10 ; 4)	0.99	**0.9945** (± 0.0013)	174.61	134.76	134.50
5	0.10	5	10	(1 ; 0)	0.95	0.9501 (± 0.0037)	14.75	18.33	18.30
5	0.10	5	10	(10 ; 4)	0.95	0.9518 (± 0.0032)	24.77	23.97	23.85
5	0.90	5	10	(1 ; 0)	0.95	0.9515 (± 0.0013)	36.53	28.24	28.18
5	0.90	5	10	(10 ; 4)	0.95	**0.9672** (± 0.0016)	102.79	54.43	53.83
5	0.10	10	10	(1 ; 0)	0.95	0.9520 (± 0.0047)	41.66	45.24	45.21
5	0.10	10	10	(10 ; 4)	0.95	0.9509 (± 0.0062)	52.44	51.68	51.58
5	0.90	10	10	(1 ; 0)	0.95	0.9492 (± 0.0026)	66.99	58.85	58.76
5	0.90	10	10	(10 ; 4)	0.95	**0.9555** (± 0.0035)	140.08	92.02	91.52
1	0.10	5	50	(1 ; 0)	0.95	0.9500 (± 0.0023)	4.32	28.84	28.77
1	0.10	5	50	(10 ; 4)	0.95	0.9486 (± 0.0038)	12.41	32.59	32.44
1	0.90	5	50	(1 ; 0)	0.95	0.9497 (± 0.0012)	10.01	30.57	30.57
1	0.90	5	50	(10 ; 4)	0.95	**0.9585** (± 0.0025)	75.65	56.28	55.80
1	0.10	10	50	(1 ; 0)	0.95	0.9521 (± 0.0042)	24.84	49.37	49.42
1	0.10	10	50	(10 ; 4)	0.95	0.9530 (± 0.0039)	35.13	55.34	55.21
1	0.90	10	50	(1 ; 0)	0.95	0.9489 (± 0.0026)	32.83	53.45	53.39
1	0.90	10	50	(10 ; 4)	0.95	**0.9571** (± 0.0021)	109.19	90.02	89.50
5	0.10	5	50	(1 ; 0)	0.99	0.9901 (± 0.0009)	16.03	39.55	39.49
5	0.10	5	50	(10 ; 4)	0.99	0.9898 (± 0.0017)	27.31	46.35	46.34
5	0.90	5	50	(1 ; 0)	0.99	0.9915 (± 0.0010)	40.20	51.74	51.71
5	0.90	5	50	(10 ; 4)	0.99	**0.9947** (± 0.0009)	116.61	87.72	87.31
5	0.10	10	50	(1 ; 0)	0.99	0.9911 (± 0.0023)	54.68	78.19	78.18
5	0.10	10	50	(10 ; 4)	0.99	0.9912 (± 0.0022)	67.95	87.00	86.88
5	0.90	10	50	(1 ; 0)	0.99	0.9898 (± 0.0017)	84.72	96.29	96.16
5	0.90	10	50	(10 ; 4)	0.99	0.9925 (± 0.0015)	173.68	144.86	144.50

To illustrate that negative values for the reorder level may be appropriate, we consider two situations. In the first situation a low service is required, whereas in the second situation the reorder quantity is large relatively to $\mathbb{E}D^*$, see tabel 4.3. Moreover, notice the excellent results in these situations of both the CBM as the approximations for the expected average physical stock.

Table 4.3.: Simulation results to illustrate negative reorder levels

R	π_D	$\sigma(D^*)$	Q	$(\mathbb{E}L_1;\sigma(L_1))$	β	β_1	s_1	$X(R,s,Q)$	X_1
1	0.10	5	50	(1 ; 0)	0.50	0.5022 ($\pm$ 0.0045)	-19.51	9.04	9.06
1	0.10	5	50	(10 ; 4)	0.50	0.5028 ($\pm$ 0.0048)	-15.13	9.42	9.47
1	0.90	5	50	(1 ; 0)	0.50	0.5004 ($\pm$ 0.0016)	-15.54	9.22	9.22
1	0.90	5	50	(10 ; 4)	0.50	0.4844 ($\pm$ 0.0041)	22.46	12.35	11.99
1	0.10	10	50	(1 ; 0)	0.50	0.4983 ($\pm$ 0.0069)	-13.28	13.23	13.24
1	0.10	10	50	(10 ; 4)	0.50	0.4995 ($\pm$ 0.0079)	-9.52	13.68	13.70
1	0.90	10	50	(1 ; 0)	0.50	0.4989 ($\pm$ 0.0031)	-9.76	13.61	13.61
1	0.90	10	50	(10 ; 4)	0.50	0.4845 ($\pm$ 0.0044)	25.81	18.14	17.43
1	0.10	5	500	(1 ; 0)	0.90	0.9003 ($\pm$ 0.0017)	-44.49	207.04	206.18
1	0.10	5	500	(10 ; 4)	0.90	0.8995 ($\pm$ 0.0021)	-40.02	207.06	206.32
1	0.90	5	500	(1 ; 0)	0.90	0.8999 ($\pm$ 0.0004)	-40.50	207.05	206.81
1	0.90	5	500	(10 ; 4)	0.90	0.8996 ($\pm$ 0.0025)	-0.00	207.60	207.27
1	0.10	10	500	(1 ; 0)	0.90	0.8994 ($\pm$ 0.0041)	-37.00	213.92	212.88
1	0.10	10	500	(10 ; 4)	0.90	0.8990 ($\pm$ 0.0048)	-32.51	214.03	212.98
1	0.90	10	500	(1 ; 0)	0.90	0.8995 ($\pm$ 0.0014)	-33.01	213.99	213.99
1	0.90	10	500	(10 ; 4)	0.90	0.8993 ($\pm$ 0.0025)	7.57	215.21	215.04

To conclude we indicate some restrictions to the application of the compound Bernoulli model. Note, that when the demand process is a compound renewal process, it is in general not true that the D_n's are independent identically distributed. Denote A as the actual interarrival time of customers when monitored continously. For situations with $\mathbb{E}A = 1/\pi_D$ and $\sigma^2(A) < (1-\pi_D)/\pi_D^2$ or $\sigma^2(A) > (1-\pi_D)/\pi_D^2$ the compound Bernoulli modelling is not appropriate. In the latter situations the independency assumption of the D_n's and D_k^*'s is violated. For a more rigorous treatment of these situations we refer to Janssen et al. (1996), who give an alternative modelling approach for the demand process, namely the compound renewal modelling; see also Sahin (1983).

Secondly, when the probability distribution of the demand size is concentrated in a small number of points, it is in general false to assume that the distribution function of Z_1, $Z(\hat{L}_1)$ and U_1 are generalized Erlang distributions. Also, the expressions for the undershoot ((8) and (9)) as well as the result that W_1 is distributed uniformly on $\{0, 1, .., R-1\}$, are based on asymptotic results from renewal theory. Hence, for values of Q small as compare to $\mathbb{E}D$, these relations do not hold.

From a managerial point of view it is interesting to represent graphically various service levels versus the expected average physical stock level; see figures 2 and 3. In figures 2 and 3 we consider the situations with $R = 1$, $\mathbb{E}D^* = 5$, $\pi_D = 0.5$, $\mathbb{E}L_1 = 5$, and $\sigma(L_1) = 2$. In order to make a trade off between the customer service and the associated required average physical stock, the graph can be used as an aid for determining the target service level.

For the determination of the replenishment quantity Q often the economic order quantity is used (e.g see Silver and Peterson (1985)), which is also known as the Wilson lot size formula. A more sophisticated approach would be to minimize the ordering cost plus the holding cost subject to the service level constraint (see Moon and Choi (1994)). Following

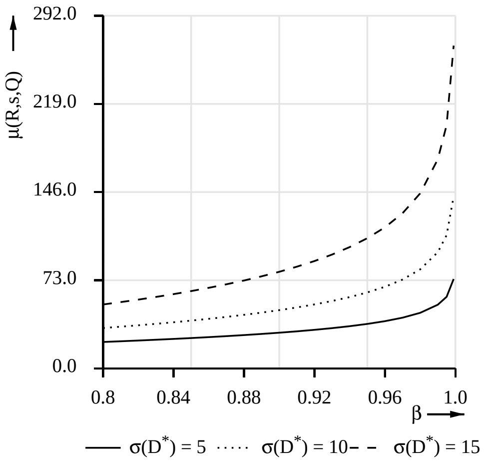

Figure 2: Expected average physical stock level versus $\beta \in (0.80, ..., 0.99))$.

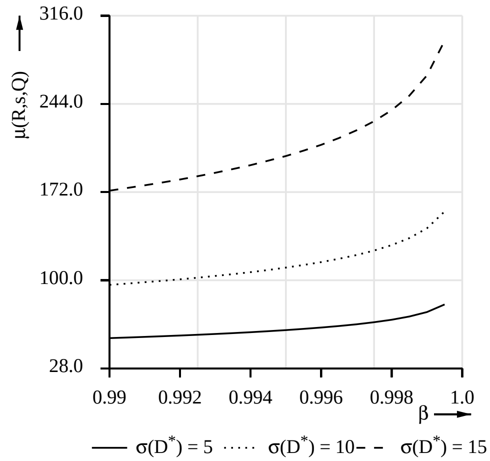

Figure 3: Expected average physical stock level versus $\beta \in (0.99, ..., 0.999)$.

the same approach as Moon and Choi yields

$$\begin{aligned} \text{MIN}\Big\{ \quad TRC(R,s,Q) &= \frac{A\mathbb{E}D}{Q} + h\mu(R,s,Q)\Big\} \\ \text{s.t.} \quad \beta(R,s,Q) &\geq \beta \\ Q &\geq 0 \end{aligned}$$

where h denotes the stock keeping costs per time unit and A the fixed ordering costs. This optimization problem can be solved by using the CBM to determine s for given Q such that $\beta(R,s,Q) = \beta$ using (6), to compute the value of the object function. Golden section search enables us to compute the optimal solution (s^*, Q^*).

Consider the following example, $R = 1$, $(\mathbb{E}D^*; \sigma(D^*) = (5;5)$, $\pi_D = 0.50$, $(\mathbb{E}L; \sigma(L)) = (10;2)$, $\beta = 0.95$, $A = 50$ \$ and h varies between $1, 5$ and 10 \$ / year (= 200 days). Notice that in this case the EOQ is given by $\sqrt{10000/h}$. In table 4.4 the results are given for the optimal replenishment quantity Q^* of the optimization problem described above.

Table 4.4.: The optimal replenishment quantities as function of the holding costs

h	EOQ	Q^*
10	100	114
5	141	157
1	316	335

Figure 4 shows that the solution for Q^* is robust for small values of h. Hence, in these situations the EOQ is nearly optimal.

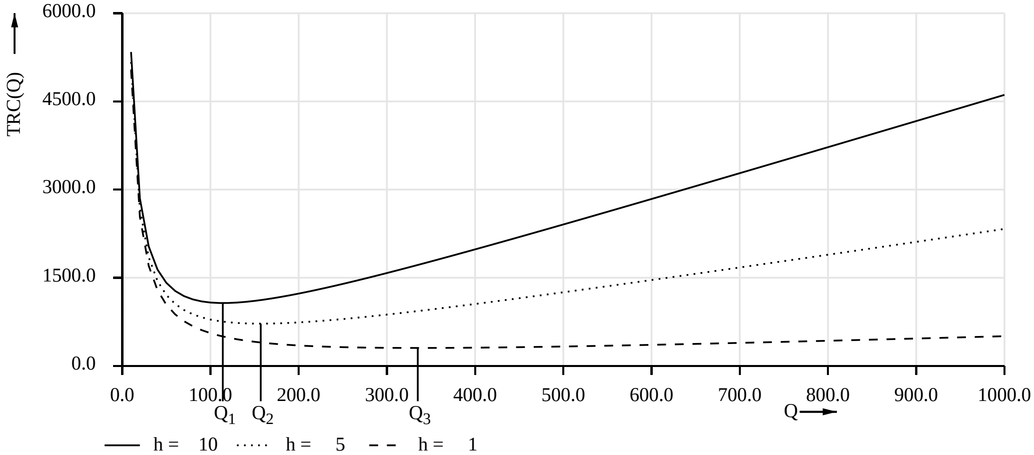

Figure 4: The expected ordering and holding costs.

5. Conclusions

In this paper we developed a method (the CBM) for the determination of the reorder point s in a (R, s, Q) inventory model subject to a service level constraint, especially when the demand process is intermittent. Therefore, we modelled the demand process as a compound Bernoulli process. The motivation behind the compound Bernoulli model is the distinction between the situations that the demand during the pseudo leadtime is zero or positive. The presented method is an extention of the method introduced by Dunsmuir and Snyder (1989). The CBM as well as the approximation for the expected physical stock level turned out to perform excellent, in almost all situations considered. Finally, we point out that due to the speed of the computations for both the CBM as the approximation for the expected average physical stock, these result are extremely useful for scenario analysis, in order to support the management.

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Appendix 1: Proof of relation (6)

Given a random variable X with distribution function $F(.)$ and a random variable Y with distribution function $G(.)$, then the distibution function of the convolution of X and Y will be denoted by $F * G(.)$. The $n-$fold distribution of X with itself is denoted by $F^{n*}(.)$.

Lemma A.1.1.
Let $M(.)$ be the renewal function (i.e. $M(x) := \sum_{n=0}^{\infty} F_X^{n*}(x)$) generated by the random process X_n, $n = 1, 2, \ldots$ (with $0 < \mathbb{E}X_1 < \infty$), and let $U(.)$ the equilibrium excess distribution of X, then

$$M * U(x) = \frac{x}{\mathbb{E}X} \tag{A.1.1}$$

Proof:
Let $\tilde{X}(s)$ be the Laplace transform of X $(0 < \mathbb{E}X < \infty)$, thus $\tilde{X}(s) = \int_0^{\infty} e^{-sx} dF_X(x)$. As $\tilde{U}(s) = (1 - \tilde{X}(s))/(s\mathbb{E}X)$ and $\tilde{M}(s) = 1/(1 - \tilde{X}(s))$, it follows that the Laplace transform of the convolution equals $1/(s\mathbb{E}X)$. Hence, taking the inverse Laplace transform of $1/(s\mathbb{E}X)$ yields $M * U(x) = x/\mathbb{E}X$.
$\square$

Lemma A.1.2.
Let $M(.)$ be the renewal function generated by the random process X_n, $n = 1, 2, \ldots$ (with $0 < \mathbb{E}X < \infty$), and let $U(.)$ the equilibrium excess distribution of X. Further, let Y be a positive random variable with distribution function $F_Y(.)$, then for $s > 0$ it holds that

$$\int_0^s \int_0^{s-x} (s - x - y) dM(y) dF_Y * U(x) = \int_0^s \frac{(s - x)^2}{2\mathbb{E}X} dF_Y(x) \tag{A.1.2}$$

Proof:
Using lemma A.1.1. it easily follows

$$\begin{aligned}
& \int_0^s \int_0^{s-x} (s - x - y) dM(y) dF_Y * U(x) \\
= & \int_0^s \int_0^{s-x} (s - x - y) dM * U(y) dF_Y(x) \\
= & \frac{1}{\mathbb{E}X} \int_0^s \int_0^{s-x} (s - x - y) dy dF_Y(x) \\
= & \int_0^s \frac{(s - x)^2}{2\mathbb{E}X} dF_Y(x)
\end{aligned}$$

□

<u>P</u>roof of relation (6).
Define $H(x)$ as the expected area between the physical inventory level and the zero level, in case the physical stock level on epoch 0 equals x ($x \geq 0$), and there are no replenishments. Then conditioning with respect to the demand in the next period using relation (1), results in

$$H(x) = x + \int_0^x \pi_D H(x-y)dF_D^*(y) + (1-\pi_D)H(x) \tag{A.1.3}$$

hence

$$H(x) = \frac{x}{\pi_D} + \int_0^x H(x-y)dF_D^*(y) \tag{A.1.4}$$

then we find

$$H(x) = \int_0^x \frac{x-y}{\pi_D} dM(y) \tag{A.1.5}$$

which can be verified by substitution in (A.1.5), where $M(.)$ is the renewal function with respect to the D^* process.
Consider the first replenishment cycle after 0 (see Figure 1). In the sequel of the proof we will assume that $s > 0$, as for $s < 0$ the same approach can be applied. The expected physical stock at the beginning of the replenishment cycle (just after the replenishment arrived) at epoch $\tau_1 + L_1$, denoted by I_1, is equal to $s+Q-U_1+Z(\hat{L}_1)$, whereas the expected physical stock at the end of the replenishment cycle (just before the replenishment arrives) , denoted by I_2, is equal to $s - U_2 + Z(\hat{L}_2)$. Then it is easy to see that $H(R,s,Q)$ is given by $H(I_1) - H(I_2)$. Conditioning with respect to I_1 and I_2, using (A.1.5) and lemma A.1.2, we find

$$\begin{aligned}
H(R,s,Q) &= \int_0^{s+Q} H(s+Q-x)dF_{Z(\hat{L}_1)} * F_{U_1}(x) - \int_0^{s} H(s-x)dF_{Z(\hat{L}_2)} * F_{U_2}(x) \\
&= \int_0^{s+Q} \int_0^{s+Q-x} \frac{s+Q-x-y}{\pi_D} dM(y) dF_{Z(\hat{L}_1)} * F_{U_1}(x) \\
&\quad - \int_0^{s} \int_0^{s-x} \frac{s-x-y}{\pi_D} dM(y) dF_{Z(\hat{L}_2)} * F_{U_2}(x) \\
&= \int_0^{s+Q} \frac{(s+Q-x)^2}{2\pi_D \mathbb{E} D^*} dF_{Z(\hat{L}_1)}(x) - \int_0^{s} \frac{(s-x)^2}{2\pi_D \mathbb{E} D^*} dF_{Z(\hat{L}_2)}(x)
\end{aligned}$$

Notice that the expected duration of a replenishment cycle is given by $\frac{Q}{ED}$, hence for $s > 0$

$$\mu(R,s,Q) = \int_{0}^{s+Q} \frac{(s+Q-x)^2}{2Q} dF_{Z(\hat{L}_1)}(x) - \int_{0}^{s} \frac{(s-x)^2}{2Q} dF_{Z(\hat{L}_2)}(x) \tag{A.1.6}$$

□

Appendix 2: Two moment fit for the distribution function of a discrete random variable

This appendix is based on Adan et al. (1994).

Lemma A.2.1.

For a pair of non-negative, real numbers (μ_x, c_X). there exists a random variable X on the non-negative integers with mean μ_x and coefficient of variation c_X if and only if

$$c_X^2 \geq \frac{2k+1}{\mu_x} - \frac{k(k+1)}{(\mu_x)^2} - 1 \tag{A.1.7}$$

where k is the unique integer satisfying $k \leq \mu_x < k+1$. For the proof of Lemma A.2.1. we refer to Adan et al. (1994). Let X a random variable on the non-negative integers. Define $a := c_X^2 - 1/\mathbb{E}X$, then it follows from lemma A.2.1. that $a \geq -1$. The method is based on a selection out of four classes of distributions: Poisson, mixture of binomials, mixture of negative-binomials, and a mixture of geometric distributions. Define

Poisson distribution	$P(\mu, x) := \sum_{i=0}^{x} \frac{\mu^i}{i!} e^{-\mu}$	$x = 0, 1, \ldots$
Negative-binomial distribution	$NB(k, p, x) := \sum_{i=0}^{x} \binom{k+i-1}{k-1} p^k (1-p)^i$	$x = 0, 1, \ldots$
Binomial distribution	$BIN(k, p, x) := \sum_{i=0}^{x} \binom{k}{i} p^i (1-p)^{k-i}$	$x = 0, 1, \ldots, k$
Geometric distribution	$G(p, x) := \sum_{i=0}^{x} p(1-p)^i$	$x = 0, 1, \ldots$

Then there exists a random variable Y which matches the first two moments of X, if the distribution function of Y is chosen such that:

if $\frac{-1}{k} \leq a \leq \frac{-1}{k+1}$	$F_Y(x) = qBIN(k, p, x) + (1-q)BIN(k+1, p, x)$ where $q = \frac{1+a(1+k)+\sqrt{-ak(1+k)-k}}{1+a}$ and $p = 1 - \frac{\mu_x}{k+1-q}$	$x = 0, 1, \ldots, k+1$
if $a = 0$	$F_Y(x) = P(\mu, x)$ where $\mu = \mu_x$	$x = 0, 1, \ldots$
if $\frac{1}{k+1} \leq a \leq \frac{1}{k}$	$F_Y(x) = qNB(k, p, x) + (1-q)NB(k+1, p, x)$ where $q = \frac{a(1+k)-\sqrt{(1+k)(1-ak}}{1+a}$ and $p = 1 - \frac{\mu_x}{k+1-q+\mu_x}$	$x = 0, 1, \ldots, k+1$
if $a > 1$	$F_Y(x) = qG(p_1, x) + (1-q)G(p_2, x)$ where $q = \frac{1}{1+a+\sqrt{a^2-1}}$ and $p_1 = \frac{2}{2+\mu_x(1+a+\sqrt{a^2-1})}$ $p_1 = \frac{2}{2+\mu_x(1+a-\sqrt{a^2-1})}$	$x = 0, 1, \ldots$

Appendix 3:
Proof W_1 is uniformly distributed over $\{0, 1, \ldots, R-1\}$.

Define $W(x)$ as the time between the moment of undershoot of the reorderlevel s for the first time after zero and the next review epoch, given that the inventory position minus the reorder level equals x at time epoch 0, where $x > 0$.
Then for $0 \leq k < R$

$$\begin{aligned}
I\!P(W(x) = k) &= \sum_{m=1}^{\infty} I\!P\Big(\sum_{n=1}^{mR-k} Z_{T,n} > x, \sum_{n=1}^{mR-k-1} Z_{T,n} \leq x\Big) \\
&= \sum_{m=1}^{\infty} \Big(I\!P\Big(\sum_{n=1}^{mR-k-1} Z_{T,n} \leq x\Big) - I\!P\Big(\sum_{n=1}^{mR-k-1} Z_{T,n} \leq x, \sum_{n=1}^{mR-k} Z_{T,n} \leq x\Big)\Big) \\
&= \sum_{m=1}^{\infty} \Big(F_{Z_{T,1}}^{(mR-k-1)*}(x) - F_{Z_{T,1}}^{(mR-k)*}(x)\Big) \qquad \text{(A.3.1)}
\end{aligned}$$

Taking the Laplace transforms at both sides yields

$$\begin{aligned}
\tilde{W}_k(s) &:= \int_0^{\infty} e^{-sx} I\!P(W(x) = k)dx \\
&= \frac{1}{s}\int_0^{\infty} e^{-sx} d_x I\!P(W(x) = k) \\
&= \frac{1}{s}\sum_{m=1}^{\infty} \Big(\int_0^{\infty} e^{-sx} dF_{Z_{T,1}}^{(mR-k-1)*}(x) - \int_0^{\infty} e^{-sx} dF_{Z_{T,1}}^{(mR-k)*}(x)\Big) \\
&= \tilde{F}_{Z_{T,1}}^{R-k}(s)\frac{1-\tilde{F}_{Z_{T,1}}(s)}{s(1-\tilde{F}_{Z_{T,1}}^{R}(s))} \qquad \text{(A.3.2)}
\end{aligned}$$

where $\tilde{F}_{Z_{T,1}}(s) := \int_0^{\infty} e^{-sx} dF_{Z_{T,1}}(x)$
Since

$$\lim_{x\to\infty} I\!P(W(x) = k) = \lim_{s\downarrow 0} s\tilde{W}_k(s) \qquad \text{(A.3.3)}$$

we conclude from (A.3.2) that

$$\lim_{x\to\infty} I\!P(W(x) = k) = \lim_{s\downarrow 0} \tilde{F}_{Z_{T,1}}^{R-k}(s)\frac{1-\tilde{F}_{Z_{T,1}}(s)}{1-\tilde{F}_{Z_{T,1}}^{R}(s)} \qquad \text{(A.3.4)}$$

which implies, using l'Hopital's rule

$$\lim_{x\to\infty} I\!P(W(x) = k) = \frac{1}{R} \qquad \text{(A.3.5)}$$

which completes the proof.