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Cooperative games with stochastic payoffs: deterministic equivalents

JEROEN SUJJS ¹  PETER BORM

Abstract

This paper focuses on the subclass of cooperative games with stochastic payoffs in which the preferences of the agents are such that a stochastic payoff can be represented by a deterministic equivalent. To each game within this class one can associate a game with deterministic payoffs. It is shown that the core of such a cooperative game with stochastic payoffs is nonempty if and only if the core of the associated game is nonempty.

Keywords: cooperative games, stochastic variables, deterministic equivalents, preferences.

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1 Introduction

In general, the payoff of a coalition in a cooperative game is assumed to be known with certainty. In many cases, however, payoffs to coalitions are uncertain. This would not raise a problem, if the agents can await the realizations of the payoffs before deciding which coalitions to form and which allocations to settle on. But if the formation of coalitions and allocations has to take place before the payoffs are realized, standard cooperative game theory can not longer be applied.

Charnes and Granot (1973) considered cooperative games in stochastic characteristic function form. For these games the value $V(S)$ of a coalition $S$ is allowed to be a stochastic variable. It was suggested to allocate the stochastic payoff of the grand coalition in two stages. In the first stage, so called prior payoffs are promised to the agents. These prior payoffs are determined in such a way that there is a fair chance that this promise will be realized. In the second stage the realization of the stochastic payoff is awaited and, subsequently, a possibly nonfeasible prior payoff vector has to be adjusted to this realization in some way. This approach was elaborated in Charnes and Granot (1976), Charnes and Granot (1977), and Granot (1977). Most of the adjustment processes use a specific choice of objections as an adjustment base.

Suijs, Borm, De Waegenaere and Tijs (1995) also considered cooperative games with stochastic payoffs, but in a slightly different way than the authors above. The most significant differences between the model introduced by Charnes et al. (1973) and the model introduced by Suijs et al. (1995) is that the latter explicitly incorporates preferences on stochastic payoffs for each agent and allows each coalition to choose from several actions. In Suijs et al. (1995) the reader is provided with some applications of the model and suggestions for possible choices of preferences. Moreover, it is shown that for a special class of cooperative games with stochastic payoffs the core of the game is nonempty if and only if the game balanced.

In this paper we continue on the model introduced by Suijs et al. (1995). We consider a subclass of games in which the preferences of the agents are such that a
stochastic payoff can be represented by its deterministic equivalent, i.e., the amount of money for which an agent is indifferent between receiving the stochastic payoff and this amount. Using these deterministic equivalents we can associate to each cooperative game with stochastic payoffs within this class a cooperative game with deterministic payoffs. Subsequently, we show that the core of the cooperative game with stochastic payoffs is nonempty if and only if the core of the associated game with deterministic payoffs is nonempty.

The paper is organized as follows. In Section 2 we recall the definitions concerning cooperative games with stochastic payoffs and introduce deterministic equivalents. Moreover, we provide the reader with some examples of preferences for which the deterministic equivalent is well defined. Finally, we introduce the game with deterministic payoffs that can be associated with a game with stochastic payoffs. Section 3 proves the main result concerning the core of these two games.

2 Cooperative games with stochastic payoffs: deterministic equivalents

In this section we focus on a special class of cooperative games with stochastic payoffs to which one can associate a cooperative game with deterministic payoffs. First we recall some of the definitions concerning cooperative games with stochastic payoffs as introduced by Suijs et al. (1995). A cooperative game with stochastic payoffs is described by a tuple \( \Gamma = (N, (A_S)_{SCN}, (X_S)_{SCN}, (\succ_i)_{i \in N}) \), where \( N \) is the set of agents, \( A_S \) the finite set of actions a coalition \( S \) can take, \( X_S : A_S \to L^1(\mathbb{R}) \) the payoff function of coalition \( S \), assigning to each action \( a \in A_S \) a stochastic payoff \( X_S(a) \in L^1(\mathbb{R}) \) with finite expectation, and \( \succ_i \) the preference relation of agent \( i \) over the set \( L^1(\mathbb{R}) \) of stochastic payoffs with finite expectation. The class of all cooperative games with stochastic payoffs with agent set \( N \) is denoted by \( SG(N) \).

An allocation of a stochastic payoff \( X_S(a) \) to the coalition \( S \) is described by a tuple
\((d, r|a) \in \mathbb{R}^S \times \mathbb{R}^S\) such that \(\sum_{i \in S} d_i = E(X_S(a))\) and \(\sum_{i \in S} r_i = 1\) and \(r_i \geq 0\) for all \(i \in S\). The payoff to agent \(i \in S\) according to the allocation \((d, r|a)\) equals \(d_i + r_i(X_S(a) - E(X_S(a)))\). This payoff will henceforth be denoted by \((d, r|a)_i\). Note that the payoff \((d, r|a)_i\) is stochastic. Summarizing, \(d \in \mathbb{R}^S\) is an allocation of the expectation \(E(X_S(a))\) of the stochastic payoff, and \(r \in \mathbb{R}^S\) is an allocation of the residual \(X_S(a) - E(X_S(a))\), also called the risk of the stochastic payoff \(X_S(a)\). The set of all allocations for coalition \(S\) is denoted by \(Z(S)\).

The core of a game with stochastic payoffs is defined as follows. Let \(\Gamma \in SG(N)\) and \((d, r|a) \in Z(S)\). Then the allocation \((d, r|a)\) is a core allocation for the game \(\Gamma\) if for each coalition \(S\) there is no allocation \((d, r|\tilde{a}) \in Z(S)\) such that \((d, r|\tilde{a}) \succ_i (d, r|a)\) for all \(i \in S\). The set of all core allocations for \(\Gamma\) is denoted by \(Core(\Gamma)\).

In the remainder of this paper we focus on a special class of cooperative games with stochastic payoffs. For the games in this subclass the preferences \((\succ_i)_{i \in N}\) are such that for each \(i \in N\) there exists a function \(m_i : L^1(\mathbb{R}) \to \mathbb{R}\) satisfying

(M1) for all \(X \in L^1(\mathbb{R}) : X \sim_i m_i(X)\),

(M2) for all \(X, Y \in L^1(\mathbb{R}) : X \succ_i Y\) if and only if \(m_i(X) \geq m_i(Y)\),

(M3) for all \(d \in \mathbb{R} : m_i(d) = d\),

(M4) for all \(X \in L^1(\mathbb{R}) : m_i(X - m_i(X)) = 0\),

(M5) for all \(X \in L^1(\mathbb{R})\) and all \(d, d' \in \mathbb{R}\) with \(d < d' : m_i(d + X) < m_i(d' + X)\).

The interpretation is that \(m_i(X)\) equals the amount of money \(m\) for which agent \(i\) is indifferent between receiving the amount \(m_i(X)\) with certainty and receiving the stochastic payoff \(X\). The amount \(m_i(X)\) is called the deterministic equivalent of \(X\). Condition (M2) states that agent \(i\) weakly prefers one stochastic payoff to another one if and only if the deterministic equivalent of the first is greater than or equal to the deterministic equivalent of the latter. Condition (M3) states that the deterministic equivalent of a deterministic payoff \(d\) equals \(d\) itself. From the conditions...
(M1), (M2) and (M3) it then follows that the preference relation \( \preceq_i \) can be represented by \( m_i \). Note, however, that \( m_i \) is not necessarily the utility function of agent \( i \). Condition (M4) states that an agent is indifferent between receiving the stochastic payoff \( X - m_i(X) \) and receiving the payoff zero. Finally, condition (M5) states that the preferences over stochastic payoffs of the form \( d + X \) are monotonically increasing in \( d \). Remark that condition (M4) is not implied by the other four conditions. Conditions (M4) and (M5), however, are equivalent with condition (M6) below:

\[ m_i(d + X) = d + m_i(X). \]

(M6) for all \( X \in L^1(\mathbb{R}) \) and all \( d \in \mathbb{R} \).

Obviously, condition (M6) implies conditions (M4) and (M5). For the converse, suppose that \( m_i(d + X) > d + m_i(X) \) for some \( d \in \mathbb{R} \) and some \( X \in L^1(\mathbb{R}) \). Then we get the following contradiction,

\[ 0 = m_i(d + X - (d + m_i(X))) < m_i(d + X - (d + m_i(X))) = m_i(X - m_i(X)) = 0. \]

Here the first and the last equality follow from condition (M4) and the inequality follows from condition (M5). Of course, a similar argument holds if one would suppose that \( m_i(d + X) < d + m_i(X) \).

The set of all games with stochastic payoffs with agent set \( N \) for which the preference relations \( (\preceq_i)_{i \in N} \) satisfy the conditions (M1) - (M5) is denoted by \( MG(N) \). Further the subclass of \( MG(N) \) where all payoffs are deterministic is denoted by \( DG(N) \). Note that for the games in \( DG(N) \) a coalition \( S \) can still choose between the different actions (if any) of \( A_S \). By the monotonicity condition (M5), however, it is optimal for coalition \( S \) to take the action \( a \in A_S \) which maximizes their joint payoff \( X_S(a) \). Hence, the number of actions of a coalition can be reduced to only one action without really changing the game.

Next, we give three examples of preferences for which the deterministic equivalent satisfies conditions (M1)-(M5) and one for which the deterministic equivalent does not exist.
Example 2.1 Consider von Neumann/Morgenstern preferences based on the utility function $u(t) = \alpha_1 + \alpha_2 \cdot a^{b(t+\beta)}$, $(t \in \mathbb{R})$, where $\alpha_1, \beta \in \mathbb{R}$, $\alpha_2, b \neq 0$ and $a > 0$. Then the conditions $(M1)$-$(M5)$ are satisfied. The deterministic equivalent of $X \in L^1(\mathbb{R})$ can be defined by $m(X) = u^{-1}(E(u(X)))$. It is easy to check that $m$ satisfies conditions $(M1)$-$(M3)$ and $(M5)$. For condition $(M4)$, let $X \in L^1(\mathbb{R})$. Then

$$u^{-1}(\tau) = \tau \cdot \log \left( \frac{\tau - \gamma}{\alpha_2} \right) - \beta$$

and

$$m(X - m(X)) = u^{-1}(E(u(X - m(X))))$$

$$= \frac{1}{\tau} \cdot \log \left( \frac{1}{\alpha_2} \left( \alpha_1 + \alpha_2 \cdot a^{b(t-m(X)+\beta)} dF_X(t) - \alpha_1 \right) \right) - \beta$$

$$= -m(X) + \frac{1}{\tau} \cdot \log \left( \frac{1}{\alpha_2} \left( \alpha_1 + \alpha_2 \cdot a^{b(t+\beta)} dF_X(t) - \alpha_1 \right) \right) - \beta$$

$$= -m(X) + m(X) = 0.$$

Finally, note that $u$ is a monotonically increasing and concave function if $a > 1$, $\alpha_2 < 0$ and $b < 0$. Consequently, an agent with such a utility function is risk averse.

Example 2.2 Let the preferences $\succsim_\alpha$ be such that for $X, Y \in L^1(\mathbb{R})$ it holds that $X \succsim_\alpha Y$ if $u^X_\alpha \geq u^Y_\alpha$, where $\alpha \in (0, 1)$ and $u^X_\alpha = \sup \{t \mid F_X(t) < \alpha \}$ denotes the $\alpha$-quantile of the distribution function $F_X$ of $X$. This type of preferences appear for example in insurance. They are used by insurance companies if the premium is determined on the basis of the percentile principle (also called chance constrained premium). With the deterministic equivalent of $X \in L^1(\mathbb{R})$ given by $m(X) = u^X_\alpha$ then conditions $(M1)$-$(M5)$ are satisfied.

Example 2.3 Let the preferences $\succsim^b$ be such that for $X, Y \in L^1(\mathbb{R})$ it holds that $X \succsim^b Y$ if $E(X) + b\sqrt{V(X)} \geq E(Y) + b\sqrt{V(Y)}$, where $V(X)$ denotes the variance of $X$. Let $X, Y \in L^1(\mathbb{R})$ and $u : \mathbb{R} \rightarrow \mathbb{R}$ a utility function. Then $X \succsim^b Y$ if $E(u(X)) \geq E(u(Y))$. 

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1 Let $X, Y \in L^1(\mathbb{R})$ and $u : \mathbb{R} \rightarrow \mathbb{R}$ a utility function. Then $X \succsim Y$ if $E(u(X)) \geq E(u(Y))$. 

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This type of preferences can be found for example in portfolio decision theory, where an agent’s evaluation of a portfolio depends on the expected revenue of the portfolio and the standard deviation of the revenue. With the deterministic equivalent of \( X \in L^1(\mathbb{R}) \) given by \( m(X) = E(X) + b\sqrt{V(X)} \), then conditions (M1)-(M5) are satisfied.

**Example 2.4** Let \( \preceq_F \) be the preferences based on the concept of stochastic domination, i.e., for \( X, Y \in L^1(\mathbb{R}) \) we have \( X \preceq_F Y \) if \( F_X(t) \leq F_Y(t) \) for all \( t \in \mathbb{R} \) and \( F_X(t) < F_Y(t) \) for at least one \( t \in \mathbb{R} \), where \( F_X \) and \( F_Y \) denotes the distribution function of \( X \) and \( Y \), respectively. Since this preference relation is not complete, there exists no function \( m : L^1(\mathbb{R}) \to \mathbb{R} \) satisfying conditions (M1)-(M5).

For each game in \( MG(N) \) we define an associated game in \( DG(N) \) in the following way. Let \( \Gamma \) be an element of \( MG(N) \). Consider a coalition \( S \) and an allocation \( (d, r|a) \) for this coalition. Since the stochastic payoff equals \( (d, r|a)_i = d_i + r_i(X_S(a) - E(X_S(a))) \) for each agent \( i \in S \), the deterministic equivalent for agent \( i \) equals \( m_i((d, r|a)_i) \). Consequently, the monetary equivalent of \( (d, r|a) \) for \( S \) equals \( \sum_{i \in S} m_i((d, r|a)_i) \). Moreover, the following property shows that an allocation \( (d, r|a) \in Z(S) \) is Pareto optimal for coalition \( S \) if and only if

\[
\sum_{i \in S} m_i((d, r|a)_i) = \max_{(d, r|\tilde{a}) \in Z(S)} \sum_{i \in S} m_i((d, r|\tilde{a})_i),
\]

**Proposition 2.5** Let \( S \subset N \). An allocation \( (d, r|a) \in Z(S) \) is Pareto optimal for \( S \) if and only if

\[
\sum_{i \in S} m_i((d, r|a)_i) = \max_{(d, r|\tilde{a}) \in Z(S)} \sum_{i \in S} m_i((d, r|\tilde{a})_i). \tag{1}
\]
**Proof:** We start with proving the 'if'-part of the statement. Let \((d, r|a) \in Z(S)\) satisfy expression (1). Suppose that \((d, r|a)\) is not a Pareto optimal allocation. Then there exists an allocation \((\hat{d}, \hat{r}|\hat{a}) \in Z(S)\) such that \((\hat{d}, \hat{r}|\hat{a}) \succ_i (d, r|a)\), for all \(i \in S\). From condition (M2) it then follows that

\[
\max_{(d, r|a) \in Z(S)} \sum_{i \in S} m_i((\hat{d}, \hat{r}|\hat{a})_i) = \sum_{i \in S} m_i((d, r|a)_i) < \sum_{i \in S} m_i((\hat{d}, \hat{r}|\hat{a})_i).
\]

This is a contradiction. Consequently, we must have that \((d, r|a)\) is a Pareto optimal allocation.

For the 'only if'-part, let \((d, r|a) \in Z(S)\) be Pareto optimal and suppose that \((d, r|a)\) does not satisfy expression (1). Then there exists an allocation \((\hat{d}, \hat{r}|\hat{a}) \in Z(S)\) such that \(\sum_{i \in S} m_i((d, r|a)_i) < \sum_{i \in S} m_i((\hat{d}, \hat{r}|\hat{a})_i)\). Next, define \((\hat{d}, \hat{r}|\hat{a}) \in Z(S)\) by

\[
\hat{d}_i = d_i - m_i((\hat{d}, \hat{r}|\hat{a})_i) + m_i((d, r|a)_i)
\]

\[
+ \frac{1}{|S|} \left( \sum_{j \in S} m_j((\hat{d}, \hat{r}|\hat{a})_j) - \sum_{j \in S} m_j((d, r|a)_i) \right),
\]

for all \(i \in S\), \(\hat{r}_i = \hat{r}_i\), for all \(i \in S\), and \(\hat{a} = \hat{a}\). Then for all \(i \in S\) it holds that

\[
m_i((\hat{d}, \hat{r}|\hat{a})_i) = m_i(d_i + \hat{r}_i(X_S(\hat{a}) - E(X_S(\hat{a}))))
\]

\[
= m_i(d_i + \hat{r}_i(X_S(\hat{a}) - E(X_S(\hat{a}))) - m_i((\hat{d}, \hat{r}|\hat{a})_i))
\]

\[
+m_i((d, r|a)_i) + \frac{1}{|S|} \left( \sum_{j \in S} m_j((\hat{d}, \hat{r}|\hat{a})_j) - \sum_{j \in S} m_j((d, r|a)_i) \right)
\]

\[
= m_i((d, r|a)_i) + \frac{1}{|S|} \left( \sum_{j \in S} m_j((\hat{d}, \hat{r}|\hat{a})_j) - \sum_{j \in S} m_j((d, r|a)_i) \right)
\]

\[
> m_i((d, r|a)_i),
\]

where the second equality follows from linearity condition (M6) and the third equality follows from condition (M4). Finally, condition (M2) implies that \((\hat{d}, \hat{r}|\hat{a})_i \succ_i (d, r|a)_i\) for all \(i \in S\). This contradicts the Pareto optimality of \((d, r|a)\). Hence, expression (1) is satisfied. \(\Box\)
Consequently, it is optimal for a coalition $S$ to maximize the expression $\sum_{i \in S} m_i((d, r|a)_i)$ over all possible allocations. Therefore we define the deterministic payoff of a coalition $S$ as the maximum of $\sum_{i \in S} m_i((d, r|a)_i)$ over all allocations $(d, r|a) \in Z(S)$. Note that if $\sum_{i \in S} m_i((d, r|a)_i)$ is continuous in $r$ this maximum always exists. Indeed, from the linearity condition (M6) it follows that
\[
\sum_{i \in S} m_i((d, r|a)_i) = \sum_{i \in S} (d_i + m_i(r_i(X_S(a) - E(X_S(a))) \rangle 
= E(X_S(a)) + \sum_{i \in S} m_i(r_i(X_S(a) - E(X_S(a))) \rangle.
\]
Consequently we have that $\max_{(d, r|a) \in Z(S)} \sum_{i \in S} m_i((d, r|a)_i)$ is independent of the choice of $d$. Hence, the maximum is actually taken over the compact sets \( \{ r \in [0,1]^S \mid \sum_{i \in S} r_i = 1 \} \) and $A_S$. For the preferences discussed in Examples 2.2 and 2.3 continuity of $m_i$ in $r$ is guaranteed. For the preferences discussed in Example 2.1, however, this need not be the case, as we show in the following example.

**Example 2.6** Consider a one person coalition $S = \{i\}$ with the utility function of agent $i$ equal to $u_i(t) = 1 - e^{-\mu t}$. Define the payoff of agent $i$ by $X_{\{i\}} = -Y$, where $Y$ is exponentially distributed with parameter $\lambda$ such that $0 < \lambda < \mu$. Since the coalition consists of only one person, the only possible payoff to agent $i$ is $X_{\{i\}}$. Then
\[
\max_{(d, r|a) \in Z(\{i\})} m_i((d, r|a)_i) = m_i(X_{\{i\}}) = \int_0^\infty (1 - e^{-\mu t} \lambda) e^{-\lambda t} dt = -\infty.
\]

We define for each cooperative game $\Gamma$ with stochastic payoffs given by $\Gamma = (N, (A_S)_{S \subseteq N}, (X_S)_{S \subseteq N}, (\kappa_i)_{i \in N}) \in MG(N)$ the associated cooperative game $\Delta_{\Gamma}$ with deterministic payoffs by $\Delta_{\Gamma} = (N, \{a_S\}_{S \subseteq N}, (x_S)_{S \subseteq N}, (\kappa_i)_{i \in N}) \in DG(N)$ with
\[
x_S = \max_{(d, r|a) \in Z(S)} \sum_{i \in S} m_i((d, r|a)_i),
\]
for all $S \subseteq N$ and $a_S$ an action in $A_S$ for which there exists an allocation that attains this maximum. Henceforth, we will call this game the deterministic equivalent of $\Gamma$. 
If for all $i \in N$ the function $m_i$ is differentiable with respect to $r_i$ and concave in $r_i$ the following properties hold for the allocation that maximizes expression (2).

**Proposition 2.7** Let $\Gamma \in MG(N)$, $S \subset N$ and $(d, r|a) \in Z(S)$. If $\sum_{i \in S} m_i((d, r|a)_i) = x_S$, then

$$\frac{\partial m_i((d, r|a)_i)}{\partial r_i} = \frac{\partial m_j((d, r|a)_j)}{\partial r_j}, \text{ if } r_i > 0 \text{ and } r_j > 0,$$

and

$$\frac{\partial m_i((d, r|a)_i)}{\partial r_i} \geq \frac{\partial m_j((d, r|a)_j)}{\partial r_j}, \text{ if } r_i > 0 \text{ and } r_j = 0.$$

**Proof:** See Appendix.

For an interpretation of these conditions, suppose that there are two agents which both bear a part of the risk and for which the marginal valuations of risk are not equal. Then reallocating (a part of) the risk from the agent with the lower marginal valuation to the agent with the higher marginal valuation increases $\sum_{k \in S} m_k((d, r|a)_k)$. Obviously, this is not possible if the agent with the lower marginal valuation of risk already bears no risk. Hence the distinction between two cases in the expression above is necessary. Finally, note that since $m_i(d_i + X) = d_i + m_i(X)$ holds for all agents $i$, a reallocation of $d$ leaves $\sum_{k \in S} m_k((d, r|a)_k)$ unchanged.

### 3 The core

In this section we will relate the core of a cooperative game with stochastic payoffs in $MG(N)$ to the core of its deterministic equivalent. We will show that the core of a game $\Gamma$ in $MG(N)$ is nonempty if and only if the core of its deterministic equivalent $\Delta_{\Gamma}$ is nonempty.
Theorem 3.1 If $\Gamma \in MG(N)$, then

$$Core(\Gamma) \neq \emptyset$$

if and only if $Core(\Delta_{\Gamma}) \neq \emptyset$. \hfill (3)

**Proof:** Let $\Gamma \in MG(N)$ such that $Core(\Delta_{\Gamma}) = \emptyset$. Suppose $(d, r|a) \in Core(\Gamma)$. Since $Core(\Delta_{\Gamma}) = \emptyset$ there exists a coalition $S \subseteq N$ such that $\sum_{i \in S} m_i((d, r|a); i) < x_S$. From Proposition 2.5 it then follows that coalition $S$ can improve the allocation $(d, r|a)$. Hence, $(d, r|a) \notin Core(\Gamma)$. Consequently, we must have $Core(\Gamma) = \emptyset$.

Next, let $Core(\Gamma) = \emptyset$ and suppose $y \in Core(\Delta_{\Gamma})$. Let $(d, r|a) \in Z(N)$ be such that $\sum_{i \in N} m_i((d, r|a); i) = x_N$. Define $(\tilde{d}, \tilde{r}|\tilde{a}) \in Z(N)$ by $\tilde{d}_i = y_i - m_i((d, r|a); i) + d_i$, for all $i \in N$, $\tilde{r}_i = r_i$, for all $i \in N$, and $\tilde{a} = a$. Then we have

$$m_i((\tilde{d}, \tilde{r}|\tilde{a}); i) = \tilde{d}_i + m_i(\tilde{r}_i(X_N(\tilde{a}) - E(X_N(\tilde{a}))))$$

$$= y_i - m_i((d, r|a); i) + m_i(d_i + r_i(X_N(a) - E(X_N(a))))$$

$$= y_i - m_i((d, r|a); i) + m_i((d, r|a); i) = y_i,$$

for all $i \in N$. Since $Core(\Gamma) = \emptyset$ there exists a coalition $S \subseteq N$ with an allocation $(\tilde{d}, \tilde{r}|\tilde{a}) \in Z(S)$ such that $(\tilde{d}, \tilde{r}|\tilde{a}); \succ (\tilde{d}, \tilde{r}|\tilde{a});$ or, equivalently, $m_i((\tilde{d}, \tilde{r}|\tilde{a}); i) > m_i((\tilde{d}, \tilde{r}|\tilde{a}); i)$ holds for all $i \in S$. But this leads to the following contradiction

$$x_S = \sum_{i \in S} y_i = \sum_{i \in S} m_i((\tilde{d}, \tilde{r}|\tilde{a}); i) < \sum_{i \in N} m_i((\tilde{d}, \tilde{r}|\tilde{a}); i) \leq x_S.$$

Hence, $y \notin Core(\Delta_{\Gamma})$, thus $Core(\Delta_{\Gamma}) = \emptyset$. \hfill $\square$

Note that expression (3) in Theorem 3.1 can be replaced by a similar statement in terms of allocations, i.e., if $(d, R) \in Z(N)$ and $y \in \mathbb{R}^N$ are such that $m_i((d, R); i) = y_i$ for all $i \in N$ then

$$(d, R) \in Core(\Gamma)$$

if and only if $y \in Core(\Delta_{\Gamma})$.

Moreover, it is not difficult to show that $y \in Core(\Delta_{\Gamma})$ if and only if $\sum_{i \in S} y_i \geq x_S$ for all $S \subseteq N$ and $\sum_{i \in N} y_i = x_N$. So, for the class of games $MG(N)$ the problem of
finding a core allocation of a cooperative game with stochastic payoffs is reduced to the problem of finding a core allocation of the corresponding deterministic equivalent, which is a similar problem as finding a core allocation of a transferable utility game.

Finally, it is not difficult to show that Proposition 2.5 and Theorem 3.1 still hold when the definition of an allocation is adjusted in the following way. Instead of $(d, r|a)$ one could define an allocation for coalition $S$ as a tuple $(d, Y) \in \mathbb{R}^S \times L^1(\mathbb{R})^S$ where $d \in \mathbb{R}^S$ is such that $\sum_{i \in S} d_i = E(X_S(a))$ and $Y \in L^1(\mathbb{R})^S$ is such that $\sum_{i \in S} Y_i = X_S(a) - E(X_S(a))$. The stochastic payoff of agent $i \in S$ then equals $d_i + Y_i$. Moreover, the results in this paper are not affected when, dependent on the type of problem, only a specific subclass of these allocations need to be considered.

Appendix

Proposition 2.7 Let $\Gamma \in MG(N)$, $S \subset N$ and $(d, r|a) \in Z(S)$. If $\sum_{i \in S} m_i((d, r|a); i) = x_S$ then

$$\frac{\partial m_i((d, r|a); i)}{\partial r_i} = \frac{\partial m_i((d, r|a); j)}{\partial r_j}, \text{ if } r_i > 0 \text{ and } r_j > 0$$

and

$$\frac{\partial m_i((d, r|a); i)}{\partial r_i} \geq \frac{\partial m_i((d, r|a); j)}{\partial r_j}, \text{ if } r_i > 0 \text{ and } r_j = 0.$$ 

Proof: Since

$$\sum_{i \in S} m_i((d, r|a); i) = E(X_S(a)) + \sum_{i \in S} m_i(r_i(X_S(a) - E(X_S(a)))) = x_S$$

we have that $r$ solves the following maximization problem

$$\max_r E(X_S(a)) + \sum_{i \in S} m_i(r_i(X_S(a) - E(X_S(a))))$$

s.t. $\sum_{i \in S} \bar{r}_i = 1$

$$\bar{r}_i \geq 0, \text{ for all } i \in S.$$
From the Karush-Kuhn-Tucker conditions\footnote{The Karush-Kuhn-Tucker conditions read as follows:} we know that there exist $\mu \in \mathbb{R}$ and $\nu_i \geq 0$ for all $i \in S$ such that

$$\frac{\partial m_i((d, r|a)_i)}{\partial r_i} = \mu - \nu_i, \text{ for all } i \in S$$

$$\nu_i \cdot r_i = 0, \text{ for all } i \in S.$$  

Since $\nu_i \geq 0$ and $\nu_i r_i = 0$ we have that $\frac{\partial m_i((d, r|a)_i)}{\partial r_i} \leq \mu$ if $r_i = 0$ and $\frac{\partial m_i((d, r|a)_i)}{\partial r_i} = \mu$ if $r_i > 0$. Hence,

$$\frac{\partial m_i((d, r|a)_i)}{\partial r_i} = \frac{\partial m_j((d, r|a)_j)}{\partial r_j}, \text{ if } r_i > 0 \text{ and } r_j > 0$$

and

$$\frac{\partial m_i((d, r|a)_i)}{\partial r_i} \geq \frac{\partial m_j((d, r|a)_j)}{\partial r_j}, \text{ if } r_i > 0 \text{ and } r_j = 0,$$

holds for all $i, j \in S$ with $i \neq j$.  

\hfill $\square$
References


