Incidence Matrix Games

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Abstract

We consider the two-person zero-sum game in which the strategy sets for Players I and II consist of the vertices and the edges of a directed graph respectively. If Player I chooses vertex v *and Player II chooses edge* e*, then the payoff is zero if* v *and* e *are not incident and otherwise it is* 1 *or* -1 *according as* e *originates or terminates at* v*. We obtain an explicit expression for the value of this game and describe the structure of optimal strategies. A similar problem is considered for undirected graphs and it is shown to be related to the theory of 2-matchings in graphs.*

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1 Introduction

A *two-person zero-sum game*, also known as a *matrix game,* consists of two players, each with finitely many pure strategies. We denote the players as Player I and Player II and let their strategy sets be $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$ respectively. If Player I chooses strategy i and Player II chooses strategy j then the payoff to Player I from Player II is a_{ij}. The $m \times n$ matrix $A = [a_{ij}]$ is called the *payoff matrix.* A mixed strategy consists of a probability distribution over the set of pure strategies. If $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$ are mixed strategies for Player I and II respectively, then $E_A(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j$ is the expected payoff to Player I.

We will assume that the reader is familiar with the fundamental aspects of matrix games, see, for example, [6, 7].

In particular, the well-known Minimax Theorem due to von Neumann asserts the following: Let A be an $m \times n$ payoff matrix. Then there exists a real number called the *value* of A, denoted $val(A)$, and mixed strategies $x^0 = (x_1^0, \ldots, x_m^0)$ and $y^0 = (y_1^0, \ldots, y_n^0)$ for Players I and II respectively such that

$$\min_j \sum_{i=1}^{m} a_{ij} x_i^0 = val(A) = \max_i \sum_{j=1}^{n} a_{ij} y_j^0.$$

The strategies x^0, y^0 are said to be *optimal* for Players I,II respectively.

We will also make use of the basic concepts from Graph Theory without defining them explicitly, see, for example, [1,4].

The main purpose of this paper is to solve (i.e., to determine the value and optimal strategies of) two very natural games related to graphs. In the first game the strategy sets consist of the vertices and the edges of a directed graph. If Player I chooses vertex v and Player II chooses edge e, then the payoff is zero if v and e are not incident and otherwise it is 1 or -1 according as e originates or terminates at v. Observe that if the graph is the directed cycle on three vertices, then this game reduces to the well known "Stone-Paper-Scissors" game [6]. For a different generalization of the Stone-Paper-Scissors game see [3].

In Section 2 we obtain a simple, explicit expression for the value of this game. We also determine the structure of optimal strategies.

In Section 3 we consider the same game but with an undirected graph. The payoff is thus zero if v and e are not incident and is 1 otherwise. The value of such games is related to some fundamental graph theoretic notions such as the matching number and the vertex covering number. The problem of determining the optimal strategies is shown to be intimately related to the theory of 2-matchings [5], which is a well-studied area of Graph Theory.

2 The directed incidence matrix game

We now introduce some terminology and notation. Let $G = (V, E)$ be a directed graph with at least one edge. We assume, unless stated otherwise, that $V = \{v_1, \ldots, v_m\}$ and $E = \{e_1, \ldots, e_n\}$. The (vertex-edge) incidence matrix of G is the $m \times n$ matrix $A = [a_{ij}]$ defined as follows: $a_{ij} = 0$ if v_i and e_j are not incident; otherwise, $a_{ij} = 1$ or -1 according as e_i originates or terminates at v_i.

The following result will be used in the sequel. The proof is easy and hence is omitted.

Lemma 1 *Let the matrix A be the direct sum of the matrices $A_1, A_2, \ldots, A_k$,* *where* $val(A_i) > 0$, *for* $i = 1, 2, \ldots, k$, *i.e.,*

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k \end{bmatrix}.$$

Then

$$val(A) = \left\{ \sum_{i=1}^{k} \frac{1}{val(A_i)} \right\}^{-1}.$$

A vertex $v \in V$ is called a *source* if it has indegree zero and a *sink* if it has outdegree zero. A *star* is a connected graph in which all vertices, except one which is called the *centre,* have degree 1.

Lemma 2 *Let* $G = (V, E)$ *be a directed graph with the* $m \times n$ *incidence matrix* A. *Then* $0 \leq val(A) \leq 1$. *Furthermore,* $val(A) = 0$ *if* G *has a directed cycle and* $val(A) = 1$ *if* G *is a star with the centre being a source.*

Proof: As usual, let $V = \{v_1, \ldots, v_m\}$ and $E = \{e_1, \ldots, e_n\}$. Let $\tilde{e_j}, j = 1, \ldots, n$ denote the pure strategy for Player II which chooses edge e_j with probability 1. The strategy $x = (\frac{1}{m}, \ldots, \frac{1}{m})^T$ for Player I satisfies $E_A(x, \tilde{e_j}) = 0, j = 1, \ldots, n$. Thus $val(A) \geq 0$. Since $a_{ij} \leq 1$ for all i, j, it is clear that $val(A) \leq 1$.

Now suppose G has a directed cycle $\mathcal{C}$ with k vertices. Consider the strategy $y = (y_1, \ldots, y_n)^T$ for Player II given by

$$y_i = \begin{cases} \frac{1}{k}, & \text{if } e_i \text{ is an edge of } \mathcal{C} \\ 0, & \text{otherwise.} \end{cases}$$

Then $E_A(\tilde{v}_i, y) = 0, i = 1, \ldots, m$, where $\tilde{v}_i$ is the pure strategy for Player I which chooses vertex v_i with probability 1, and thus $val(A) \leq 0$. It follows that $val(A) = 0$ in this case.

Now suppose G is a star with v as the centre and suppose v has indegree zero. We assume, without loss of generality, that $v = v_1$. Then A has a saddle point at the position $(1, 1)$ and therefore $val(A) = a_{11} = 1$. ■

We say that G is *acyclic* if it has no directed cycles. Assume now that G is acyclic and for each $v \in V$, let $P(v)$ denote a directed path, originating at v and having maximum possible length. Clearly such a path must terminate at a sink. Note that there may be more than one such path but we choose and fix one. Let $\rho(v)$ denote the length of $P(v)$. If v is a sink then we set $\rho(v) = 0$. For each arc $e \in E$, let $\eta(e)$ denote the number of vertices v such that e is an arc of the path $P(v)$.

Lemma 3 $\sum_{v \in V} \rho(v) = \sum_{e \in E} \eta(e)$.

Proof: Let B be the $m \times n$ matrix defined as follows. The rows of B are indexed by V and the columns of B by E. If $v \in V, e \in E$, then the (v,e)-entry of B is 1 if $e \in P(v)$ and 0 otherwise. Then observe that $\{\rho(v) : v \in V\}$ are the row sums of B and $\{\eta(e) : e \in E\}$ are the column sums of B. Since the sum of the row sums must equal that of the column sums, the result is proved. ∎

The following is the main result of this section.

Theorem 4 *Let $G = (V, E)$ be an acyclic directed graph with the $m \times n$ incidence matrix A. Then*

$$val(A) = \frac{1}{\sum_{v \in V} \rho(v)}$$

is the value of the matrix game A with the optimal strategies $\{\frac{1}{\sum_{v \in V} \rho(v)} \rho(v) : v \in V\}$ for Player I and $\{\frac{1}{\sum_{v \in V} \rho(v)} \eta(e) : e \in E\}$ for Player II.

Proof: As usual, let $V = \{v_1, \ldots, v_m\}$ and $E = \{e_1, \ldots, e_n\}$. We assume that the graph is connected, since otherwise, we may prove the result for each connected component and then apply Lemma 1.

Fix $j \in \{1, \ldots, n\}$ and suppose that the arc e_j is from v_ℓ to v_k. We have

$$\frac{1}{\sum_{v \in V} \rho(v)} \sum_{i=1}^{m} \rho(v_i) a_{ij} = \frac{1}{\sum_{v \in V} \rho(v)} \{\rho(v_\ell) - \rho(v_k)\}. \tag{1}$$

Note that $\rho(v_\ell) \geq \rho(v_k) + 1$ and therefore it follows from (1) that

$$\frac{1}{\sum_{v \in V} \rho(v)} \sum_{i=1}^{m} \rho(v_i) a_{ij} \geq \frac{1}{\sum_{v \in V} \rho(v)}. \tag{2}$$

Now fix $i \in \{1, \ldots, m\}$ and let

$$U = \{j : e_j \text{ originates at } v_i\}, W = \{j : e_j \text{ terminates at } v_i\}.$$

We have

$$\frac{1}{\sum_{v \in V} \rho(v)} \sum_{j=1}^{n} a_{ij} \eta(e_j) = \frac{1}{\sum_{v \in V} \rho(v)} \left\{ \sum_{j \in U} \eta(e_j) - \sum_{j \in W} \eta(e_j) \right\}. \tag{3}$$

If $U = \emptyset$, i.e., if v_i is a sink, then the right hand side of (3) is clearly non-positive. Now suppose that $U \neq \emptyset$. Observe that for any vertex $v \neq v_i$, the path $P(v)$ either contains exactly one edge from U and one edge from W or has no intersection with either U or W. Thus for any $v \neq v_i$, the path $P(v)$ either makes a contribution of 1 both to $\sum_{j \in U} \eta(e_j)$ and $\sum_{j \in W} \eta(e_j)$ or does not contribute to either of these terms. Also the path $P(v_i)$ makes a contribution of 1 to $\sum_{j \in U} \eta(e_j)$ but none to $\sum_{j \in W} \eta(e_j)$. Thus if v_i is not a sink, then

$$\sum_{j \in U} \eta(e_j) - \sum_{j \in W} \eta(e_j) = 1.$$

Therefore we conclude that for $i \in \{1, \ldots, m\}$,

$$\frac{1}{\sum_{v \in V} \rho(v)} \sum_{j=1}^{n} a_{ij} \eta(e_j) \leq \frac{1}{\sum_{v \in V} \rho(v)}. \tag{4}$$

The result is proved in view of (2) and (4). ∎

As a corollary of Theorem 4 we now obtain a converse of (the second part of) Lemma 2. The *if* parts in the next result were proved in Lemma 2, while the *only if* parts follow from Theorem 4.

Lemma 5 *Let $G = (V, E)$ be a directed graph with the $m \times n$ incidence matrix A. Then $val(A) = 0$ if and only if G has a directed cycle and $val(A) = 1$ if and only if G is a star with the centre being a source.*

The next result gives bounds on the value of the incidence matrix of a directed, acyclic, connected graph.

Theorem 6 *Let $G = (V, E)$ be a connected, acyclic, directed graph with the $m \times n$ incidence matrix A. Then*

$$\frac{1}{\binom{m}{2}} \leq val(A) \leq 1.$$

Proof: It was shown in Lemma 2 that $val(A) \leq 1$. The lower bound will be proved by induction on the number of vertices. As before, for each $v \in V$, let $P(v)$ denote a directed path, originating at v and having maximum possible length, and let $\rho(v)$ denote the length of $P(v)$. In view of Theorem 4, we must show that

$$\sum_{v \in V} \rho(v) \leq \frac{m(m-1)}{2}. \tag{5}$$

Clearly, (5) holds for $m = 2, n = 1$. We assume that (5) holds for graphs with fewer than $m(\geq 3)$ vertices and proceed by induction.

Let v_0 be a source and let e_0 be an arc originating at v_0 which is contained in $P(v_0)$. For each vertex v of G other than v_0, $P(v)$ is the path of maximum possible length in $G \setminus \{v_o\}$, the graph obtained from G by removing v_0. Also, we clearly have $\rho(v_0) \leq m - 1$. By the induction assumption,

$$\sum_{v \in V \setminus v_0} \rho(v) \leq \binom{m-1}{2}.$$

Thus

$$\sum_{v \in V} \rho(v) \leq m - 1 + \frac{(m-1)(m-2)}{2} = \frac{m(m-1)}{2}$$

and the proof is complete. ■

We remark that the lower bound in Theorem 6 is attained precisely when G is a directed path on m vertices.

We now consider the structure of optimal strategies.

Theorem 7 *Let $G = (V, E)$ be a directed, acyclic graph. Then the optimal strategy for Player I is unique.*

Proof: As before we assume that $V = \{v_1, \ldots, v_m\}$ and $E = \{e_1, \ldots, e_n\}$, and let A be the incidence matrix of G. Suppose $\{\phi(v), v \in V\}$ is optimal for Player I. Let $v_i \in V$ be a sink. We claim that $\phi(v_i) = 0$. To see this, let $\{y_1, \ldots, y_n\}$ be optimal for Player II. If $\phi(v_i) > 0$, then we must have

$$\sum_{j=1}^{n} a_{ij} y_j = val(A). \tag{6}$$

Since v_i is a sink, $a_{ij} \leq 0, j = 1, \ldots, n$, whereas, by Theorem 4, $val(A) > 0$. This contradicts (6) and the claim is proved.

Now let $v \in V$ be a vertex which is not a sink and let $v = u_0, u_1, \ldots, u_{k-1}, u_k = w$ be a path of maximum length originating at v. Since ϕ is optimal,

$$\phi(u_i) - \phi(u_{i+1}) \geq val(A), i = 0, 1, \ldots, k-1.$$

Thus

$$\sum_{i=0}^{k-1} \{\phi(u_i) - \phi(u_{i+1})\} \geq k val(A),$$

and hence

$$\phi(v) - \phi(w) \geq \rho(v) val(A).$$

Since w must necessarily be a sink, $\phi(w) = 0$ by our earlier claim and hence

$$\phi(v) \geq k val(A) = \rho(v) val(A). \tag{7}$$

Thus

$$1 = \sum_{v \in V} \phi(v) \geq val(A) \sum_{v \in V} \rho(v) = 1,$$

where the last equality follows by Theorem 4. Thus equality must hold in (7) and the result is proved. ∎

The assumption that G be acyclic is necessary in Theorem 7 as can be seen from the next example.

Let $G = (V, E)$ where $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{v_1v_2, v_2v_3, v_3v_1, v_2v_4\}$. Then it can be verified that the strategies $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$ and $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ are both optimal for Player I.

Let us concentrate now on the dimension of the optimal strategy space of Player II, for an $m \times n$ incidence matrix. A pure strategy is called *essential* if it is used with positive probability in some optimal strategy, otherwise it is *inessential*. Let d_1, d_2 be the dimensions of the optimal strategy spaces $\mathcal{O}_1$ and $\mathcal{O}_2$ of Players I, II respectively, and let f_1, f_2 be the number of essential strategies for Players I,II respectively. Then according to a well-known result of Bohnenblust, Karlin and Shapley [2], $f_1 - d_1 = f_2 - d_2$.

Now let $G = (V, E)$ be a directed, acyclic graph with the $m \times n$ incidence matrix A. By Theorem 7, the optimal strategy for Player I is unique. Also any vertex which is not a sink is essential for Player I. Let s be the number of sinks in G and let t be the number of inessential strategies (i.e., arcs) for Player II. Then we conclude by the result of Bohnenblust, Karlin and Shapley that the dimension of the optimal strategy space of Player II is

$$n - m + s - t.$$

Example 8 *Let $G = (V, E)$ be the directed graph with vertices v_1, v_2, v_3, v_4 and edges $e_1 = (v_1, v_2), e_2 = (v_2, v_3), e_3 = (v_2, v_4)$, and $e_4 = (v_1, v_3)$. Let A be the corresponding incidence matrix, so*

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Then $val(A) = \frac{1}{3}$ and $\mathcal{O}_1 = \{(\frac{2}{3}, \frac{1}{3}, 0, 0)\}$. Thus $m = n = 4, s = 2, t = 1$, and $d_1 = 0$. Therefore $\mathcal{O}_2$ must be one-dimensional. In fact, it can be verified that $\mathcal{O}_2$ is precisely the line segment joining $\{(\frac{1}{3}, 0, \frac{2}{3}, 0)\}$ and $\{(\frac{1}{3}, \frac{2}{3}, 0, 0\})$.

3 The undirected incidence matrix game

In this section we consider undirected graphs. Let $G = (V, E)$ be a graph with m vertices and n edges. The incidence matrix A of G is an $m \times n$ matrix with a_{ij} being zero if the i-th vertex is not incident with the j-th edge and one otherwise. Our main interest in this section is in obtaining the value and the optimal strategies for the matrix game with the payoff matrix A.

We recall some graph-theoretic terminology and notation, see, for example, Lovász and Plummer [5]. A set of edges constitute a *matching* if no two edges in the set are incident with a common vertex. The maximum cardinality of a matching in G is the *matching number* of G, denoted by $\nu(G)$. A set of vertices of G is a *vertex cover* if they are collectively incident with all the edges in G. The minimum cardinality of a vertex cover is the *vertex covering number* of G denoted by $\tau(G)$.

We first prove a simple preliminary result.

Lemma 9 *Let $G = (V, E)$ be a graph with the $m \times n$ incidence matrix A. Then*

$$\frac{1}{\tau(G)} \leq val(A) \leq \frac{1}{\nu(G)}. \tag{8}$$

Proof: Let $\tau(G) = k, \nu(G) = \ell$, and suppose that vertices numbered $i_1, \ldots, i_k$ form a cover and that the edges numbered $j_1, \ldots, j_\ell$ form a matching. If Player I chooses vertices $i_1, \ldots, i_k$ uniformly with probability $\frac{1}{k}$, then against any pure strategy of Player II he is guaranteed a payoff of at least $\frac{1}{k}$. Similarly, if Player II chooses edges $j_1, \ldots, j_\ell$ uniformly with probability $\frac{1}{\ell}$, then against any pure strategy of Player I he loses at most $\frac{1}{\ell}$. These two observations together give (8). ■

Recall that a graph is said to have a *perfect matching* if it has a matching in which the edges are collectively incident with all the vertices. A graph is *Hamiltonian* if it has a cycle, called a *Hamilton cycle*, containing every vertex exactly once.

In the next result we identify some classes of graphs for which the value of the corresponding game is easily determined.

Theorem 10 *Let $G = (V, E)$ be a graph with the $m \times n$ incidence matrix A.*

(i) If G is bipartite then $val(A) = \frac{1}{\nu(G)}$.

(ii) If G is the path on m vertices then $val(A) = \frac{m}{2}$ if m is even and $\frac{m-1}{2}$ if m is odd.

(iii If G has a perfect matching then $val(A) = \frac{2}{m}$.

(iv) If G is Hamiltonian then $val(A) = \frac{2}{m}$.

(v) If G is the complete graph on m vertices, then $val(A) = \frac{2}{m}$.

Proof: If G is bipartite, then by the well-known König Theorem (see, for example, [5]), $\nu(G) = \tau(G)$ and the result follows by Lemma 9. Therefore (i) is proved. Since a path is bipartite, (ii) follows in view of the observation that the matching number of a path on m vertices is $\frac{m}{2}$ if m is even and $\frac{m-1}{2}$ if m is odd. Similarly, if G has a perfect matching, then $\nu(G) = \tau(G) = \frac{m}{2}$ and (iii) is proved.

To prove (iv), first suppose that G is a cycle on m vertices. Then $n = m$ and the strategies for Players I, II which choose all pure strategies with equal probability, namely $\frac{1}{m}$, are easily seen to be optimal. Thus $val(A) = \frac{2}{m}$.

Now suppose G is Hamiltonian. Then the value of G is at least equal to the value of the game corresponding to a Hamilton cycle in G and thus, $val(A) \geq \frac{2}{m}$ by the preceding observation. Furthermore, if Player II chooses only the edges in the Hamilton cycle with equal probability then against any pure strategy of Player I he loses at most $\frac{2}{m}$. Therefore (iv) is proved.

Finally, (v) follows since a complete graph is clearly Hamiltonian. ■

We now turn to arbitrary graphs, not necessarily covered by Theorem 10. In this case the problem of determining the value and optimal strategies is closely related to the theory of 2-matchings and a theorem of Tutte 8. We now indicate how one can determine the structure of optimal strategies using Tutte's theorem.

We first need some definitions, see [5], p.214. Let $G = (V, E)$ be a graph. A 2-*matching* of G is an assignment of weights $0, 1$ or 2 to the edges of G such that the sum of the weights of edges incident with any vertex is at most 2. The sum of the weights in a 2-matching is called the *size* of the 2-matching. The maximum size of a 2-matching in G is denoted by $\nu_2(G)$.

A 2-*cover* of G is an assignment of weights $0, 1$ or 2 to the vertices of G such that the sum of the weights of the two endpoints of any edge is at least 2. The sum of the weights in a 2-cover is called the *size* of the 2-cover. The minimum size of a 2-cover in G is denoted by $\tau_2(G)$.

We now state a result due to Tutte [8], see [5].

Theorem 11 *If G is any graph, then $\nu_2(G) = \tau_2(G)$.*

Theorem 11 directly leads to the following result about the structure of optimal strategies in the game with the matrix being the incidence matrix of a graph. The proof is omitted.

Theorem 12 *Let $G = (V, E)$ be a graph with the $m \times n$ incidence matrix A. Then there exists a positive integer k such that $val(A) = \frac{2}{k}$. Furthermore, there* *exist partitions $V = V_1 \cup V_2 \cup V_3$, $E = E_1 \cup E_2 \cup E_3$ such that for Player I, choosing vertices in V_1 uniformly with probability $\frac{1}{k}$ and vertices in V_2 uniformly with probability $\frac{2}{k}$ is optimal. Similarly for Player II, choosing edges in E_1 uniformly with probability $\frac{1}{k}$ and edges in E_2 uniformly with probability $\frac{2}{k}$ is optimal.*

Example 13 *Consider the graph $G = (V, E)$ with the vertex set $V = \{v_1, v_2, \ldots, v_7\}$ and edges $E = \{v_1v_2, v_2v_3, v_1v_3, v_3v_4, v_4v_5, v_4v_6, v_5v_6, v_5v_7\}$. The incidence matrix of G is given by*

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The value of the game is $\frac{2}{7}$. An optimal strategy for Player I is to choose each of the vertices v_1, v_2, v_3 with probability $\frac{1}{7}$ and vertices v_4, v_5 each with probability $\frac{2}{7}$. An optimal strategy for Player II is to choose edges v_1v_2, v_2v_3, v_1v_3 each with probability $\frac{1}{7}$ and edges v_4v_6, v_5v_7 each with probability $\frac{2}{7}$.

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