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# A NEW METHOD FOR ASSESSING JUDGMENTAL DISTRIBUTIONS

J.J.A. Moors*

M.H. Schuld*

A.C.A. Mathijssen*

**Summary**

For a number of statistical applications subjective estimates of some distributional parameters - or even complete densities are needed. The literature agrees that it is wise behaviour to ask only for some quantiles of the distribution; from these, the desired quantities are extracted. Quite a lot of methods have been suggested up to now; the number of quantiles they need varies from three to nine or more.

Still another method is proposed here. Individuals are asked the relatively simple task of presenting the seven values that divide the total probability mass into eight equal parts. From these so-called octiles four estimates for location, dispersion, skewness and 'peakedness' are derived. Moreover, these four values uniquely determine one distribution within either the Pearson or the Johnson system. Consequently, there is no need for 'optimal' approximating formulae.

**Key words:** Bayesian analysis, estimation, Johnson system, octiles, Pearson system, PERT, quantiles, subjective probability, systems of distributions.

* Tilburg University, P.O. Box 90153, 5000 LE Tilburg, Netherlands

# 1 Introduction

Eliciting from individuals judgemental probability distributions for a certain phenomenon is an important issue in a number of areas of (applied) statistics. In risk analysis and decision making in general, experts' opinions are needed on the probability of (damaging) events. PERT is based upon individual judgements on the probability distribution of random activity times, particularly on their means and variances. The whole field of Bayesian statistics roots in the concept of prior distributions for unknown parameters.

Whether the subjective probability statements are concerned with random events or with unknown parameters - there seems to be general agreement in the literature that the straight assessment of a probability distribution and/or its means and variance is a task too complicated for most individuals. In stead, judgements about direct probabilities and quantiles are elicited first; next, the statistician derives from these the probability distribution and/or it first moments. See MERKHOFER (1987) or SPETZLER & STAËL von HOLSTEIN (1975), e.g.

So in practice, in eliciting a subjective probability distribution or cumulative distribution function $F$, the individual is asked to specify a number of quantiles $x_\alpha$, defined by $F(x_\alpha) = \alpha$ (for continuous $F, 0 < \alpha < 1$). Since the assessment of quantiles $x_\alpha$ with $\alpha$ close to 0 or 1 is considered to be fairly difficult, a much-used set of quantiles in this setting is the triad

$$(x_{0.1}, \quad x_{0.5}, \quad x_{0.9}) \tag{1}$$

Now, two options arise.

In Bayesian statistics, a complete prior distribution is wanted. As a rule, such prior is chosen from a pre-determined class of distributions. If the prior has to be symmetric and unimodal for example, an obvious choice would be the class of normal distributions $\{N(\mu, \sigma^2) : \mu \in \mathbb{R}, \sigma^2 > 0\}$; of course, $x_{0.9} - x_{0.5} = x_{0.5} - x_{0.1}$ should hold. The class of beta distributions $\{Be(\alpha, \beta) : \alpha > 0, \beta > 0\}$ is richer in the sense that it contains distributions of many different shapes: unimodal, $U$-, $J$- and inverse $J$-shaped. Note that if the prior is restricted to the unit interval, the triad (1) corresponds with precisely one beta distribution.

In PERT, as in many other decision problems, only the mean $\mu$ and the variance $\sigma^2$ of the underlying distribution are needed. Approximating formulae to derive these moments from the given set of quantiles abound. One example, based on (1), are the extended Swanson-Megill approximations:

$$\begin{cases} \hat{\mu} &= 0.3x_{0.1} + 0.4x_{0.5} + 0.3x_{0.9} \\ \hat{\sigma}^2 &= 0.3(x_{0.1} - \hat{\mu})^2 + 0.4(x_{0.5} - \hat{\mu})^2 + 0.3(x_{0.9} - \hat{\mu})^2 \end{cases} \tag{2}$$

See KEEFER & BODILY (1983) for this and other approximating formulae.

Even if the assessed quantiles are exact, the true values of $\mu$ and $\sigma^2$ of course depend on the exact distribution. Therefore, it is of interest to investigate the quality of approximations like (2) over a set of underlying distributions. KEEFER & VERDINI (1993) chose for this set the class of beta distributions mentioned before. Recently, LAU et al. (1995) extended this research into three directions:

(i) *Systems* of distributions were taken into consideration, like Pearson's and Johnson's. (Note that the Pearson system contains the class of beta distributions - among many others.)

(ii) Approximating formulae for $\mu$ and $\sigma^2$ were considered, based on more quantiles than usual (up to 9).

(iii) Simulation and regression were used to find the optimal weights for the quantiles in the approximating formulae.

The present paper proposes a new method for the subjective assessment of a probability distribution and/or its first moments, based on the seven quantiles $x_{i/8}$, $i = 1, 2, \ldots, 7$. This septet is easily assessed; it leads to measures for location, dispersion, skewness and 'peakedness', that are uniquely determined within both the Pearson and the Johnson system. Hence, in Bayesian applications, once that one of these two systems has been selected, the unique prior distribution follows directly from the seven elicited quantiles. Mean and variance follow and suffice for PERT-type applications. A computer program in MATLAB for doing the necessary computations has been developed.

## 2 Octile-based measures and their properties

The quantiles

$$E_i = x_{i/8} \quad (i = 1, 2, \ldots, 7) \tag{3}$$

divide the area under a given density into eight equal parts. These *octiles* $E_i$ therefore generalize the concepts of median and quartile. This property, in combination with the fact that the extreme values of $\alpha$ ($\frac{1}{8}$ and $\frac{7}{8}$) are very moderate here, facilitate the assessment of the octiles: the median halves the total 'probability mass', the first quartile again halves the lower half, and so on.

Well-known characteristics of any probability distribution are location, dispersion, skewness and 'peakedness'. Familiar measures for these four characteristics, based on the central moments

$$\mu_i = \int_{\mathrm{R}} (x-\mu)^i dF(x), \; i = 2, 3, \ldots \tag{4}$$

are the mean $\mu$, the standard deviation $\sigma = \sqrt{\mu_2}$, the excess $\beta_1 = \mu_3/\sigma^3$ and the kurtosis $\beta_2 = \mu_4/\sigma^4$. Alternative measures, based on the octiles, are – for $E_6 \neq E_2$ – given by the foursome $(Q, R, S, T)$:

$$\begin{cases} Q &= E_4 \\ R &= (E_6 - E_2)/2 \\ S &= (E_6 - 2E_4 + E_2)/(E_6 - E_2) \\ T &= (E_7 - E_5 + E_3 - E_1)/(E_6 - E_2) \end{cases} \tag{5}$$

The median $Q$, the half interquartile range $R$ and Bowley's measure of skewness $S$ are very familiar again. In the same spirit, the quantity $T$ was developed as a quantile-based alternative to $\beta_2$. It is based on a new interpretation of kurtosis, see MOORS (1988); that is the reason for the quotation marks around 'peakedness'. Note that the shape parameters $S$ and $T$ are location-scale invariant, as are $\beta_1$ and $\beta_2$. Further, the quantile-based measures are more robust than their moment-based counterparts and exist even for distributions without finite moments; e.g., $T = 2$ for a Cauchy distribution.

If the population parameters $Q, R, S$ and $T$ are unknown, they can be estimated by their sample counterparts **q, r, s** and **t**. Limiting properties of these four random variables are discussed in MOORS et al. (1993) or MOORS et al. (1995).

Now, the Pearson system of distributions contains all densities satisfying a certain differential equation; Table 1 presents a survey of this extremely rich system.

**Table 1.** Outline of the Pearson system.

| Name | Type | Density* | Range | Parameters |
|---|---|---|---|---|
| Beta 1 | I | $x^{p-1}(1-x)^{q-1}$ | [0,1] | $p, q > 0$ |
| Student | VII | $(1+x^2/n)^{-(n+1)/2}$ | $\mathbb{R}$ | $n > 0$ |
| Arctan | IV | $(1+x^2)^{-m}\exp[v \text{ arctan } x]$ | $\mathbb{R}$ | $m > 1/2, v \in \mathbb{R}$ |
| Inverse gamma | V | $x^{-(\rho+1)}e^{-1/x}$ | $\mathbb{R}^+$ | $\rho > 0$ |
| Beta 2 | VI | $x^{p-1}/(x+1)^{p+q}$ | $\mathbb{R}^+$ | $p, q > 0$ |
| Gamma | III | $x^{\rho-1}e^{-x}$ | $\mathbb{R}^+$ | $\rho > 0$ |

* up to a normalizing constant.

Roman figures correspond with the original codes by Pearson, and with Figure 1. See STUART & ORD (1987) for full details.

Up to positive linear transformations, distributions within the Pearson system are uniquely determined by the pair $(\beta_1, \beta_2)$. As a consequence, a one-one-correspondence exists between all Pearson distributions with finite fourth moment and the foursome $(\mu, \sigma, \beta_1, \beta_2)$. WAGEMAKERS et al. (1992) or MOORS et al. (1995) showed that a slightly more general property holds for (5): a one-one-relation exists between all distributions within the Pearson system and

(i) the quartet $(Q, R, S, T)$,

(ii) the pair $(S, T)$, up to positive linear transformations.

Figure 1a shows this last relation. Note that the restriction of a finite fourth moment can be dropped now.

Johnson's system is based on three transformations of the standard normal, leading to the three classes of distributions $S_B$, $S_L$ and $S_U$ (of which the lognormal distributions $S_L$ are best-known).

**Table 2.** Outline of the Johnson system.

| Name | Type | Density | Range | Parameters |
|---|---|---|---|---|
| Lognormal | $S_L$ | $\frac{\delta}{\sqrt{2\pi}}\frac{1}{y}\exp[-\delta \ln y)^2/2]$ | $\mathbb{R}^+$ | $\delta \in \mathbb{R}^+$ |
| Bounded range | $S_B$ | $\frac{\delta}{\sqrt{2\pi}}\frac{1}{y(1-y)}\exp[-\{\gamma + \delta \ln (\frac{y}{1-y})\}^2/2]$ | [0,1] | $\gamma \in \mathbb{R}, \delta \in \mathbb{R}^+$ |
| Unbounded range | $S_U$ | $\frac{\delta}{\sqrt{2\pi}}\frac{1}{\sqrt{1+y^2}}\exp[-\{\gamma + \delta \ln (y + \sqrt{1+y^2}\}^2/2]$ | $\mathbb{R}$ | $\gamma \in \mathbb{R}, \delta \in \mathbb{R}^+$ |

For the lognormal distributions the - location - parameter $\gamma$ has been deleted.

Again, distributions within the Johnson system are uniquely determined by either $(\mu, \sigma, \beta_1, \beta_2)$ or $(Q, R, S, T)$. See STUART & ORD (1987) and WAGEMAKERS et al. (1992) or MOORS et al. (1995), respectively. Figure 1b shows the one-one-relation between $(S, T)$ and location-scale-invariant Johnson distributions. Note that in Figure 1 the (symmetric) parts for $S < 0$ have been omitted.

**Figure 1.** $(S, T)$-plane.

a. Pearson system

b. Johnson system

More detailed versions of Figure 1 can be found in MOORS et al. (1995) as Figures 2 and 4.

# 3 Eliciting a subjective distribution

Now, the new method to assess a judgmental distribution and/or its mean and variance can be introduced. It consists of six steps.

(i) Decide whether the Pearson or the Johnson system is most appropriate to describe the subjective distribution.

(ii) The expert's opinion on the seven octiles $E_i$ is elicited.

(iii) From (ii) the four quantile-based measures $(Q, R, S, T)$ are calculated.

(iv) From $(S, T)$, the type and parameters of the unique location-scale invariant distribution are found within the chosen system.

(v) The linear transformation is found that leads to the one distribution with measures $(Q, R, S, T)$ within the chosen system.

(vi) The mean and variance of this unique distribution are calculated.

Of course, steps (iv)-(vi) can be carried out for both systems, leading to two subjective distributions. In that case, the final choice between these two replaces step (i).

The advantages of this new method can be summarized as follows.

1. Octiles $E_i$ are a simple generalization of the median : the first quartile $E_2$ is the median of the 'lower half' of the probability distribution, and so on. Besides, the octiles correspond to moderate probabilities. Their assessment therefore will be relatively easy.
2. The octile-based quantities $(Q, R, S, T)$ allow a meaningful interpretation as measures for location, dispersion, skewness and 'peakedness', respectively.
3. The foursome $(Q, R, S, T)$ determines exactly one probability distribution within the Pearson system and within the Johnson system.
4. Consequently, within each system a unique mean and variance correspond with the original seven octiles. So there is no need for approximating formulae like (2) and surely no need for assessing the quality of these approximations.
5. The Pearson system contains distributions with infinite moments; even for them the new method is appropriate. Therefore, our proposal seems particularly applicable when fat tails occur. Of course, if no moments exist, step (vi) will have to be omitted.

In our view, the main disadvantage consists of the necessary calculations in step (iv): no analytical expressions exist to find the parameters of the distribution with given pair $(S, T)$. Therefore, a MATLAB program was developed which executes the calculations for steps (ii)-(v). This program can be obtained from the authors. The following examples show its use.

**Example 1.**
Assume that the following vector of octiles is given:

$$(E_1, E_2, E_3, E_4, E_5, E_6, E_7) = (0, \;\; 2.5, \;\; ,5, \;\; 7, \;\; 9.5, \;\; 12.5, \;\; 16)$$

Then the four quantile-based measures equal

$$(Q, R, S, T) = (7, \;\; 5, \;\; 0.1, \;\; 1.15)$$

The unique Pearson distribution with these quantile measures is a Beta 1 distribution (type I in Table 1) with the density

$$f(z) = \frac{1}{B(p,q)} z^{p-1}(1-z)^{q-1}, z \in [0,1]$$

where $z = (x - \xi)/\lambda$ and $B$ denotes the beta-function. The parameter values are found to be

$$(\xi, \lambda, p, q) = (-3.1545, \;\; 34.3407, \;\; 1.5101, \;\; 3.1704)$$

Hence mean and variance of this distribution equal

$$(\mu, \sigma^2) = (7.925, \;\; 45.370)$$

Similarly, the unique Johnson distribution can be derived. It is of type $S_B$ (see Table 2) with density

$$f(z) = \frac{\delta}{z(1-z)\sqrt{2\pi}} \exp\left[-\left\{\gamma + \delta \ln\left(\frac{z}{1-z}\right)\right\}^2/2\right], z \in [0,1]$$

where

$$(\xi, \lambda, \gamma, \delta) = (-5.0109,\ \ 35.3460,\ \ 0.6936,\ \ 1.0444)$$

Now it follows

$$(\mu, \sigma^2) = (7.859,\ \ 45.088)$$

which values are very close to mean and variance of the unique Pearson distribution. Indeed, the two densities closely resemble each other, as Table 3 illustrates.

**Table 3.** Subjective densities with $(Q, R, S, T) = (7,\ \ 5,\ \ 0.1,\ \ 1.15)$.

| x | 1000 $f(x)$ | | x | 1000 $f(x)$ | | x | 1000 $f(x)$ | |
|---|---|---|---|---|---|---|---|---|
| | Pearson | Johnson | | Pearson | Johnson | | Pearson | Johnson |
| -5 | 0 | 0.00 | 7 | 52.56 | 52.55 | 19 | 17.67 | 18.16 |
| -3 | 13.17 | 17.91 | 9 | 47.76 | 47.70 | 21 | 12.52 | 12.80 |
| -1 | 44.31 | 40.75 | 11 | 42.05 | 42.05 | 23 | 8.11 | 8.02 |
| 1 | 53.89 | 52.58 | 13 | 35.87 | 36.04 | 25 | 4.58 | 4.07 |
| 3 | 56.75 | 56.57 | 15 | 29.56 | 29.94 | 27 | 2.03 | 1.30 |
| 5 | 55.84 | 55.89 | 17 | 23.42 | 23.93 | 29 | 0.51 | 0.08 |

Finally, for the two distributions the seven octiles were calculated directly.

**Table 4.** Octiles for subjective distributions.

| | $E_1$ | $E_2$ | $E_3$ | $E_4$ | $E_5$ | $E_6$ | $E_7$ |
|---|---|---|---|---|---|---|---|
| exact | 0 | 2.5 | 5 | 7 | 9.5 | 12.5 | 16 |
| Pearson | 0.2110 | 2.5075 | 4.7154 | 7.0068 | 9.5273 | 12.5050 | 16.5200 |
| Johnson | 0.1532 | 2.5000 | 4.7109 | 7.0002 | 9.5229 | 12.5005 | 16.4657 |

The three quartiles, $E_2, E_4$ and $E_6$ are uniquely determined by the triad $(Q, R, S)$; hence, their agreement with the exact values is very good. Although the recalculated $T$ is exact up to four decimal places, 'odd' octiles show larger discrepancies. □

**Example 2.**
Now assume the following vector of octiles:

$$(E_1, E_2, E_3, E_4, E_5, E_6, E_7) = (0, \;\; 0.3, \;\; ,0.6, \;\; 1.15, \;\; 1.6, \;\; 2.3, \;\; 4)$$

so that

$$(Q, R, S, T) = (1.15, \;\; 1, \;\; 0.15, \;\; 1.5)$$

The unique Pearson distribution is an Arctan distribution (Type IV in Table 1) with the density

$$f(z) = c(1 + z^2)^{-m} \exp[-v \text{ arctan } z], \quad z \in \mathbb{R}$$

where $c$ is a normalizing constant. The parameter values are

$$(\xi, \lambda, m, v) = (0.2869, \;\; 1.6327, \;\; 1.5919, \;\; -1.0138)$$

leading to mean and variance

$$(\mu, \sigma^2) = (1.685, \;\; 25.140)$$

The unique Johnson distribution is of type $S_U$ with density

$$f(z) = \delta[2\pi(1 + z^2)]^{-0.5} \exp[-\{\gamma + \delta \sinh^{-1}(z)\}^2/2], \quad z \in \mathbb{R}$$

where here

$$(\xi, \lambda, \gamma, \delta) = (0.4913, \;\; 1.2399, \;\; -0.5181, \;\; 1.0179)$$

$$(\mu, \sigma^2) = (1.559, \;\; 6.381)$$

Values of the two densities are given in the Tabel 5.

**Table 5.** Subjective densities with $(Q, R, S, T) = (1.15,\ \ 1,\ \ 0.15,\ \ 1.5)$.

| $x$ | 1000 $f(x)$ | | $x$ | 1000 $f(x)$ | | $x$ | 1000 $f(x)$ | |
|---|---|---|---|---|---|---|---|---|
| | Pearson | Johnson | | Pearson | Johnson | | Pearson | Johnson |
| -5 | 1.47 | 1.64 | 1 | 299.27 | 301.11 | 7 | 10.17 | 10.38 |
| -4 | 2.86 | 3.35 | 2 | 181.39 | 180.98 | 8 | 6.89 | 6.83 |
| -3 | 6.41 | 7.64 | 3 | 89.67 | 91.36 | 9 | 4.85 | 4.65 |
| -2 | 17.63 | 20.08 | 4 | 46.39 | 48.31 | 10 | 3.54 | 3.25 |
| -1 | 61.30 | 62.60 | 5 | 26.07 | 27.35 | 11 | 2.65 | 2.33 |
| 0 | 207.89 | 200.97 | 6 | 15.79 | 16.44 | 12 | 2.02 | 1.71 |

At first sight the two densities bear a close resemblance, seemingly contradicting the large difference between the two variances. The explanation can be found in the far tails of the distributions, as Table 6 illustrates.

**Table 6.** Tail probablities of the two subjective densities.

| $x$ | 1000 | $P(X < x)$ | | $x$ | 1000 | $P(X > x)$ | |
|---|---|---|---|---|---|---|---|
| | Pearson | Johnson | ratio | | Pearson | Johnson | ratio |
| -5 | 3.479 | 2.969 | 1.17 | 10 | 16.963 | 11.744 | 1.44 |
| -10 | 0.777 | 0.336 | 2.31 | 15 | 7.211 | 3.540 | 2.04 |
| -15 | 0.319 | 0.074 | 4.34 | 20 | 3.898 | 1.378 | 2.83 |
| -20 | 0.170 | 0.023 | 7.53 | 25 | 2.412 | 0.629 | 3.84 |

The much fatter tails of the Pearson distribution correspond with the fact that the variance of Pearson type IV distributions only exists for $m > 1.5$. Our value $m = 1.6327$ is only slightly higher.

**Table 7.** Octiles for subjective distributions.

| | $E_1$ | $E_2$ | $E_3$ | $E_4$ | $E_5$ | $E_6$ | $E_7$ |
|---|---|---|---|---|---|---|---|
| exact | 0 | 0.3 | 0.6 | 1.15 | 1.6 | 2.3 | 4 |
| Pearson | -0.3038 | 0.2996 | 0.7333 | 1.1498 | 1.6297 | 2.2990 | 3.5919 |
| Johnson | -0.3293 | 0.3000 | 0.7359 | 1.1500 | 1.6292 | 2.2999 | 3.5639 |

Again, the agreement with the exact values is much better for the quartiles than for the other octiles. Nevertheless, $S$ and $T$ are accurate up to four decimal places. □

Note the striking difference between the two examples: in Example 1 the variances (and means) of the unique Pearson and Johnson distribution practically coincide, while in Example 2 there are large differences. This will be a consequence of the fact that the two distributions in Example 1 have a finite support, while the support is $\mathbb{R}$ in the second example. Example 2 in particular shows the importance of step 1: the choice of the system.

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