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Abstract

In this paper we propose a model to describe the effectiveness of coordination in a continuously repeated two-player game. We study how the choice of a decision rule by a coordinator affects the strategic behavior of the players, resulting in more or less cooperation. Our model requires the analysis of an infinite-horizon nonlinear differential game with a one-dimensional state space, and we propose a method to obtain numerically the stationary feedback Nash equilibria for such games. This method is based on solving the associated Hamilton-Jacobi-Bellman-Isaacs equations directly.

1 Introduction

In this paper we provide a general model to study the role of a coordinator in reaching a cooperative equilibrium for a continuously repeated game. The model allows the individual players to react in a strategic fashion to the behavior of the coordinator, and so the natural question arises whether there is a way in which a coordinator can encourage cooperative play. In Klompstra (1992), in the context of linear-quadratic differential games, it is shown that if players are allowed to switch in time between cooperative and noncooperative behavior, such switches do indeed occur. With this idea in mind, we can conclude that the decision whether or not to cooperate has a dynamical flavor, i.e. willingness to cooperate should be modelled in a dynamic way. The theory of (strategic) bargaining (see for instance Houba (1994); Houba and de Zeeuw (1994); Osborne and Rubinstein (1991); de Zeeuw (1984)), shows that using threats to play noncooperatively, individual players can influence the final outcome of a game. This idea is used in our model, in the sense that individual players can influence the behavior of the coordinator by deviating from a cooperative strategy. The coordinator can then be interpreted as an institution appointed to promote a prespecified mode of play. In this way our model can be interpreted as a noncooperative hierarchical control model. In hierarchical control theory (see for instance Mesarovic et al. (1970); Jamshidi (1983); Singh

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(1980)) models have been developed in which a coordinator is introduced as a mechanism for finding a Pareto efficient equilibrium for a dynamic hierarchical control system. As already pointed out in Weeren (1993), it is necessary for all players to commit themselves to cooperate with the coordinator in order for the coordination to be successful. Therefore, the models as developed in Mesarovic et al. (1970); Jamshidi (1983); Singh (1980) can be viewed as cooperative hierarchical control models. The model as proposed in this paper is a first step towards the incorporation of strategic behavior into the hierarchical control framework.

The model in this paper is based upon a two-player static game, which is played repeatedly. We introduce the notion of coordination and arrive at a differential game with nonlinear dynamics. Unfortunately, it is impossible to handle such a game analytically, but we will show how such a game can be handled numerically. We consider this the second main contribution of our paper. We will show how stationary feedback Nash equilibria of a general nonlinear differential game over an infinite time horizon, with a scalar state, can be obtained numerically. As is well-known (see Başar and Olsder (1995); Feichtinger and Wirl (1993); Tsutsui and Mino (1990)) these feedback Nash equilibria can be described by the so-called Hamilton-Jacobi-Bellman-Isaacs (HJBI) equations. In this paper we propose to solve these equations directly using recently developed methods for solving differential-algebraic equations (see Brenan et al. (1989); Griepentrog and März (1986); Hairer et al. (1989); Hairer and Wanner (1991)).

The outline of the paper is as follows. In section 2 we will develop a general model describing the strategic interactions between players and coordinator. In section 3 we will discuss solution methods for differential-algebraic equations in general and for HJBI equations in particular. Then in section 4 and 5 we present two mechanisms fitting the general model of section 2. In section 4 we present a redistribution mechanism in which the coordinator is given direct control over the distribution of the payoffs between the individual players. We will discuss how one can numerically obtain all stationary feedback Nash equilibria of the resulting differential game using the methods developed in section 3, and illustrate this by a worked example. In section 5 we present another possible mechanism fitting the general model, which we refer to as the Pareto mechanism. In this case the coordinator influences the choice of Pareto efficient strategy, in such a way that the resulting differential game describes a movement along the Pareto frontier of the underlying static game. Finally, in section 6 we present some conclusions. Moreover an appendix in which the HJBI equations for infinite-horizon differential games are derived is included.

2 General model formulation

2.1 Introduction

Consider the following situation. Two players repeatedly play a nonzero-sum game G. Assume now that the game G depends in some way (through the payoffs that the players receive, or through the strategy spaces that are available to them) on a parameter $\alpha \in [0,1]$ that may vary in time. The value of α is determined by a 'coordinator' through some decision rule that takes the actions of the players into account. In this way the decisions of the players can influence their future payoffs, and a differential game arises which we shall refer to as the 'controlled game'. Working over an infinite horizon and comparing the asymptotic values of the equilibria of the controlled game to the possible modes of play in the original game G, we can see whether the decision rule chosen by the coordinator is effective in establishing cooperation between the players.

We formalize this idea as follows. Consider a two-player static game G in strategic form, with strategy spaces Γ_i and payoff functions π_i. In this game the objective for player i is to maximize his payoff π_i. From this game G we construct a new game, $G(\alpha)$, for every α in $[0,1]$, where α is the variable that is manipulated by the coordinator. Denote by $\Gamma_i(\alpha)$ the strategy spaces of $G(\alpha)$ and by $\nu_i(\alpha, \gamma_1(\alpha), \gamma_2(\alpha))$ the payoffs. Now assume that the coordinator can discriminate between the strategies $\gamma_i(\alpha)$ chosen by the individual players, and uses a decision rule

$$\dot{\alpha} = f(\alpha, \gamma_1(\alpha), \gamma_2(\alpha))$$

to determine the future values of α. Finally, by choosing as a criterion either

$$\mathcal{L}_i = \int_0^{t_f} \nu_i(\alpha(t), \gamma_1(\alpha(t)), \gamma_2(\alpha(t)))dt$$

or

$$\mathcal{L}_i = \int_0^{\infty} e^{-rt}\nu_i(\alpha(t), \gamma_1(\alpha(t)), \gamma_2(\alpha(t)))dt$$

for some $r > 0$, a differential game is specified, which we refer to as the controlled game. So the construction of a controlled game from a static game G is done in the following steps:

Step 1: construction of a coordination mechanism $G \rightarrowtail G(\alpha)$,

Step 2: specification of a decision rule

$$\dot{\alpha} = f(\alpha, \gamma_1(\alpha), \gamma_2(\alpha)),$$

for the coordinator,

Step 3: choice between a finite-horizon criterion and an infinite-horizon discounted criterion, and in the latter case specification of $r > 0$.

2.2 Construction of a controlled game

In this subsection we will construct a class of controlled games we will use in the remainder of this paper. First we make some assumptions on the underlying static game G.

Assumption 2.1
The strategy spaces $\Gamma_i \subseteq \mathbb{R}^k$ are convex.
The payoff functions $\pi_i : \Gamma_1 \times \Gamma_2 \to \mathbb{R}$, $i = 1, 2$, are twice differentiable and strictly concave, i.e. $\begin{pmatrix} \frac{\partial^2 \pi_i}{\partial \gamma_1^2} & \frac{\partial^2 \pi_i}{\partial \gamma_1 \partial \gamma_2} \\ \frac{\partial^2 \pi_i}{\partial \gamma_1 \partial \gamma_2} & \frac{\partial^2 \pi_i}{\partial \gamma_2^2} \end{pmatrix} < 0.$

By $\bar{\gamma} = (\bar{\gamma}_1, \bar{\gamma}_2) \in \Gamma_1 \times \Gamma_2$ we denote a Nash equilibrium of the game G.

Denote by $\gamma_i^*(\alpha)$ the cooperative strategy for player i, to be played when the coordinator announces α. Furthermore, denote by $\gamma_i^a(\alpha)$ an alternative strategy, that player i would play when playing noncooperatively. We have to make a choice for the alternative strategy γ^a. The issue on how to choose such an alternative strategy is closely related to the issue of choosing threatpoints or disagreement strategies in bargaining theory (see e.g. van Damme (1991); Houba (1994); Osborne and Rubinstein (1991)). A possible choice of alternative strategy is a Nash equilibrium for the underlying game G. Especially in the case that G has a unique Nash equilibrium this seems a good choice, for the Nash equilibrium is the standard equilibrium concept in noncooperative situations.

We introduce $c_i(t)$, which is a parameter reflecting the willingness of player i to play cooperatively at time instant t. If $c_i(t) = 0$ then player i chooses to play the alternative strategy $\gamma_i^a(\alpha(t))$ and if $c_i(t) = 1$ then player i chooses to play the strategy $\gamma_i^*(\alpha(t))$. We allow the players to hesitate between cooperative and noncooperative play by allowing the parameter $c_i(t)$ to take values between 0 and 1. For given c_i, the strategy played by player i is given by $u_i(c_i) := c_i\gamma_i^*(\alpha) + (1 - c_i)\gamma_i^a(\alpha)$.

Now we assume that the coordinator, by observing the actions of both players at time-instant t, can determine the values of $c_i(t)$. Using this information the coordinator adjusts the value of $\alpha(t)$. The process of coordination can be described by a decision rule

$$\dot{\alpha}(t) = f(\alpha(t), c_1(t), c_2(t)). \tag{1}$$

This decision rule has to satisfy some properties:

1. f has to be sufficiently smooth, i.e. f has to be at least twice differentiable w.r.t. c_i, and at least differentiable w.r.t. α,
2. $\forall_{c_1,c_2} f(0, c_1, c_2) \geq 0, f(1, c_1, c_2) \leq 0$,
3. $\frac{\partial^2 f}{\partial c_i \partial c_j} = 0$, $\frac{\partial f}{\partial c_i} \not\equiv 0$.

The smoothness condition is imposed in order to prevent some technical difficulties in the sequel of this paper. Clearly this condition might be weakened at the expense of some technical difficulties. The second condition is crucial, in the sense that it guarantees that $\alpha(t)$ remains in $[0, 1]$ for all t. Note that, due to this property, every nontrivial choice for f will be nonlinear. Finally, the third condition is sufficient to guarantee that the optimization problems we will encounter are strictly concave, and that the mechanism is not trivial. Obviously also this condition might be weakened, and in this case a more delicate analysis would be required. An example of a coordination rule satisfying properties 1 to 3 is

$$f(\alpha, c_1, c_2) = \beta\alpha(1 - \alpha)(c_2 - c_1),$$

where $\beta \in (0, \infty)$ is an arbitrary constant. This decision rule reflects the intuition that whenever one of the players shows less willingness to cooperate, the coordinator might try to convince this player to play more cooperatively in the future by choosing a new α, which is more favorable for that particular player. When β is chosen in $(-\infty, 0)$, the decision rule is such that the coordinator punishes any player who is not playing cooperatively.

A further assumption we make is that both players exactly know the mechanism f the coordinator is using. This creates a possibility for strategic behavior by both players. By choosing c_1 and c_2 the players can influence the behavior of the coordinator. A nonlinear differential game emerges, where α is the state variable, c_1 and c_2 are the controls, and with the payoff functionals

$$L_i = \int_0^{t_f} \nu_i(\alpha(t), u_1(c_1(t)), u_2(c_2(t)))dt, \tag{2}$$

in which $u_i(c_i(t)) = c_i(t)\gamma_i^*(\alpha(t)) + \big(1 - c_i(t)\big)\gamma_i^a(\alpha(t))$. We refer to this newly defined differential game as the controlled game.

Note that by introducing $u_i(c_i) = c_i\gamma_i^*(\alpha) + \big(1 - c_i\big)\gamma_i^a(\alpha)$ the payoff for player i at time instant t is given by $\nu_i(\alpha(t), u_1(c_1(t)), u_2(c_2(t)))$, which we will sometimes write with some abuse of notation as $\nu_i(\alpha(t), c_1(t), c_2(t))$. In the sequel of this paper we will assume that ν_1 and ν_2 are strictly concave in $\big(c_1, c_2\big)$.

2.3 Equilibria of the controlled game

A natural solution concept to consider for the controlled game is the feedback Nash equilibrium (see Başar and Olsder (1995)). As is well-known (see Başar and Olsder (1995)), feedback Nash equilibria for this differential game correspond to solutions of the Hamilton-Jacobi-Bellman-Isaacs (HJBI) equations

$$\begin{aligned} -\frac{\partial V_1}{\partial t}(\alpha, t) &= \max_{c_1 \in [0,1]} \left\{ \frac{\partial V_1}{\partial \alpha}(\alpha, t) f(\alpha, c_1, c_2) + \nu_1(\alpha, u_1(c_1), u_2(c_2)) \right\}, & (3) \\ -\frac{\partial V_2}{\partial t}(\alpha, t) &= \max_{c_2 \in [0,1]} \left\{ \frac{\partial V_2}{\partial \alpha}(\alpha, t) f(\alpha, c_1, c_2) + \nu_2(\alpha, u_1(c_1), u_2(c_2)) \right\}, & (4) \end{aligned}$$

where V_1 and V_2 denote value functions.

In order to facilitate our analysis, we will consider the controlled game over an infinite time horizon, with discounted payoffs. This produces the payoff functionals

$$L_i = \int_0^{\infty} e^{-rt} \nu_i(\alpha(t), u_1(c_1(t)), u_2(c_2(t)))dt, \tag{5}$$

where $u_i(c_i(t)) = c_i(t)\gamma_i^*(\alpha(t)) + \big(1 - c_i(t)\big)\gamma_i^a(\alpha(t))$. Moreover we shall restrict attention to stationary feedback Nash equilibria[1] corresponding to continuously differentiable value functions.

[1] In Feichtinger and Wirl (1993); Maskin and Tirole (1994); Tsutsui and Mino (1990) these are called Markov perfect Nash equilibria.

Now the HJBI equations describing stationary feedback Nash equilibria (see appendix and e.g. Feichtinger and Wirl (1993); Tsutsui and Mino (1990)) reduce to the system

$$rV_1(\alpha) = \max_{c_1 \in [0,1]} \left\{ V_1'(\alpha) f(\alpha, c_1, c_2) + \nu_1(\alpha, u_1(c_1), u_2(c_2)) \right\}, \tag{6}$$

$$rV_2(\alpha) = \max_{c_2 \in [0,1]} \left\{ V_2'(\alpha) f(\alpha, c_1, c_2) + \nu_2(\alpha, u_1(c_1), u_2(c_2)) \right\}. \tag{7}$$

Remark 2.2 The characterization of stationary feedback Nash equilibria by the HJBI equations (6–7) must be understood in the following way. It can be shown (see appendix) that if $(\bar{V}_1, \bar{V}_2, \bar{c}_1, \bar{c}_2)$ are continuously differentiable solutions of (6–7) such that $\bar{V}_1$ and $\bar{V}_2$ are bounded, then the pair of strategies $(\bar{c}_1, \bar{c}_2)$ is a stationary feedback Nash equilibrium. (See also Tsutsui and Mino (1990)).

Remark 2.3 Note that by requiring stationary feedback Nash equilibria, Folk-theorem-like results do not immediately hold, for trigger strategies are not admissible (see Maskin and Tirole (1994)). Nevertheless, stationary feedback Nash equilibria are in general not unique (see Feichtinger and Wirl (1993); Tsutsui and Mino (1990); Weeren et al. (1994)).

Equivalent (by the concavity assumptions) to (6–7) is the system

$$\frac{\partial f}{\partial c_1}(\alpha, c_1, c_2) V_1'(\alpha) + \frac{\partial \nu_1}{\partial c_1}(\alpha, c_1, c_2) = \eta_1, \tag{8}$$

$$\frac{\partial f}{\partial c_2}(\alpha, c_1, c_2) V_2'(\alpha) + \frac{\partial \nu_2}{\partial c_2}(\alpha, c_1, c_2) = \eta_2, \tag{9}$$

$$f(\alpha, c_1, c_2) V_1'(\alpha) + \nu_1(\alpha, c_1, c_2) - rV_1(\alpha) = 0, \tag{10}$$

$$f(\alpha, c_1, c_2) V_2'(\alpha) + \nu_2(\alpha, c_1, c_2) - rV_2(\alpha) = 0, \tag{11}$$

$$0 \le c_1 \le 1, \ \big(1 - c_1\big)\eta_1 \le 0, \ c_1\eta_1 \ge 0,$$

$$0 \le c_2 \le 1, \ \big(1 - c_2\big)\eta_2 \le 0, \ c_2\eta_2 \ge 0.$$

In case no constraint on c_1 and c_2 is active, this results in the system of differential-algebraic equations

$$\frac{\partial f}{\partial c_1}(\alpha, c_1, c_2) V_1'(\alpha) + \frac{\partial \nu_1}{\partial c_1}(\alpha, c_1, c_2) = 0, \tag{12}$$

$$\frac{\partial f}{\partial c_2}(\alpha, c_1, c_2) V_2'(\alpha) + \frac{\partial \nu_2}{\partial c_2}(\alpha, c_1, c_2) = 0, \tag{13}$$

$$f(\alpha, c_1, c_2) V_1'(\alpha) + \nu_1(\alpha, c_1, c_2) - rV_1(\alpha) = 0, \tag{14}$$

$$f(\alpha, c_1, c_2) V_2'(\alpha) + \nu_2(\alpha, c_1, c_2) - rV_2(\alpha) = 0, \tag{15}$$

3 Treatment of HJBI-DAEs

3.1 General DAEs

In the current section we will discuss the (numerical) treatment of DAEs in general and the HJBI-DAEs in particular. For a more extensive treatment of general DAEs the interested reader is referred to Brasey and Hairer (1993); Brenan et al. (1989); Gear (1988); Griepentrog and März (1986); Hairer et al. (1989); Hairer and Wanner (1991).

By a differential-algebraic equation (DAE) is meant an equation of the form

$$F(t, y, y') = 0, \tag{16}$$

in which y is a function of t, and y' is the first derivative of y with respect to t. Regarding this DAE we can consider the system of equations

$$\begin{aligned} F(t, y, y') &= 0 \\ \frac{d}{dt} F(t, y, y') &= 0 \\ &\vdots \\ \frac{d^{j-1}}{dt^{j-1}} F(t, y, y') &= 0 \end{aligned} \tag{17}$$

which can be written as

$$\mathbf{F}_j(t, y, \mathbf{y}_j) = 0, \tag{18}$$

where

$$\mathbf{y}_j = \begin{pmatrix} y' \\ \vdots \\ y^{(j)} \end{pmatrix}. \tag{19}$$

Then the (differential) index of (16) is defined in the following way (see Brenan et al. (1989); Gear (1988)).

Definition 3.1 The index of (16) is the smallest ν such that $\mathbf{F}_{\nu+1}(t, y, \mathbf{y}_j) = 0$ uniquely determines the variable y' as a continuous function of y, t.

The index is of crucial importance in selecting a numerical solution method for a given DAE. Backward differentiation formulas (BDF) have emerged as the most popular and best understood class of linear multistep methods for DAEs (see Brenan et al. (1989); Hairer and Wanner (1991)). In general, multistep methods and Runge-Kutta methods are not stable for higher-index DAE systems. In the case of a system of DAEs of index 0 or 1 it is always possible to use these methods. A well-known implementation of the BDF technique is provided in the Fortran package DASSL (as described in Brenan et al. (1989)). For the treatment of higher-index systems the reader is referred to Brasey and Hairer (1993); Brenan et al. (1989); Gear (1988); Hairer et al. (1989); Hairer and Wanner (1991).

3.2 The index of HJBI-DAEs

Returning to the system of coupled DAEs (8–11), we note that whenever these systems are index 0 or 1, they can directly be solved with the use of DASSL. First we take a look at the system of HJBI-DAEs (12–15), i.e. the system of HJBI equations in case the constraints $c_1 \geq 0$, $c_1 \leq 1$, $c_2 \geq 0$ and $c_2 \leq 1$ are not active. For ease of notation, we will ignore the arguments of f,V_i and ν_i in the rest of this section. Define

$$y := \begin{pmatrix} V_1 \\ V_2 \\ c_1 \\ c_2 \end{pmatrix}. \tag{20}$$

To determine the index we first compute the Jacobian $F_{y'}$,

$$F_{y'} = \begin{pmatrix} \frac{\partial f}{\partial c_1} & 0 & 0 & 0 \\ 0 & \frac{\partial f}{\partial c_2} & 0 & 0 \\ f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \end{pmatrix}. \tag{21}$$

Clearly $F_{y'}$ is not invertible, hence the index of the system of HJBI-DAEs is at least 1. Differentiating the system of HJBI-DAEs once, using $\frac{\partial^2 f}{\partial c_i \partial c_j} = 0$ and (12–13), we obtain the additional equations

$$\begin{aligned}
\frac{\partial^2 f}{\partial \alpha \partial c_1} V_1' + \frac{\partial f}{\partial c_1} V_1'' + \frac{\partial^2 \nu_1}{\partial \alpha \partial c_1} + \frac{\partial^2 \nu_1}{\partial c_1^2} c_1' + \frac{\partial^2 \nu_1}{\partial c_1 \partial c_2} c_2' &= 0, && (22)\\
\frac{\partial^2 f}{\partial \alpha \partial c_2} V_2' + \frac{\partial f}{\partial c_2} V_2'' + \frac{\partial^2 \nu_2}{\partial \alpha \partial c_2} + \frac{\partial^2 \nu_2}{\partial c_1 \partial c_2} c_1' + \frac{\partial^2 \nu_2}{\partial c_2^2} c_2' &= 0, && (23)\\
f V_1'' - \left(\frac{\partial f}{\partial \alpha} - r\right)\left(\frac{\partial \nu_1}{\partial c_1} / \frac{\partial f}{\partial c_1}\right) + \frac{\partial \nu_1}{\partial \alpha} + \left(\frac{\partial \nu_1}{\partial c_2} - \frac{\partial f}{\partial c_2} \cdot \frac{\partial \nu_1}{\partial c_1} / \frac{\partial f}{\partial c_1}\right) c_2' &= 0, && (24)\\
f V_2'' - \left(\frac{\partial f}{\partial \alpha} - r\right)\left(\frac{\partial \nu_2}{\partial c_2} / \frac{\partial f}{\partial c_2}\right) + \frac{\partial \nu_2}{\partial \alpha} + \left(\frac{\partial \nu_2}{\partial c_1} - \frac{\partial f}{\partial c_1} \cdot \frac{\partial \nu_2}{\partial c_2} / \frac{\partial f}{\partial c_2}\right) c_1' &= 0. && (25)
\end{aligned}$$

In this way we find the following lemma:

Lemma 3.2 *The system of HJBI-DAEs (12–15) has index 1 if and only if the equations (12–15) together with the equations (22–25) determine y' uniquely as a continuous function of y and α.*

Note that we can eliminate V_1'' and V_2'' from (24–25) using equations (22–23). Using this elimination we can straightforwardly derive that (24–25) constitute an implicit ODE for c_1' and c_2' if and only if the matrix

$$\mathcal{J} := \begin{pmatrix} -f \cdot \frac{\partial^2 \nu_1}{\partial c_1^2} / \frac{\partial f}{\partial c_1} & \frac{\partial \nu_1}{\partial c_2} - \frac{\partial}{\partial c_2}\left(f \cdot \frac{\partial \nu_1}{\partial c_1}\right) / \frac{\partial f}{\partial c_1} \\ \frac{\partial \nu_2}{\partial c_1} - \frac{\partial}{\partial c_1}\left(f \cdot \frac{\partial \nu_2}{\partial c_2}\right) / \frac{\partial f}{\partial c_2} & -f \cdot \frac{\partial^2 \nu_2}{\partial c_2^2} / \frac{\partial f}{\partial c_2} \end{pmatrix} \tag{26}$$

is nonsingular. So we now have the following result:

Proposition 3.3 *The system of HJBI-DAEs (12–15) has index 1 if and only if the matrix $\mathcal{J}$ given by (26) is nonsingular.*

The systems of DAEs which emerge when one of the constraints $c_1 \geq 0$, $c_1 \leq 1$, $c_2 \geq 0$ or $c_2 \leq 1$ becomes active, can be shown to have index at least one in a similar fashion. Moreover, conditions as described in proposition 3.3 can be derived.

In case two constraints are active, the equations are either index 0 (i.e. implicit ODEs) or algebraic, depending on whether $f(\alpha, c_1, c_2) = 0$ or not.

Remark 3.4 Note that the method of studying HJBI equations via the so-called auxiliary equations as introduced in Tsutsui and Mino (1990) is closely related to the setup described in this section. The model considered in Tsutsui and Mino (1990) is of a more special form than the one considered here, which makes it possible to obtain explicit expressions for the equilibrium feedback strategies and to substitute these in the HJBI equations. Then, by differentiating the HJBI equations (implicit) ODEs are obtained. These ODEs have the property that they do not depend on V_1 and V_2. After deriving the ODEs, symmetry conditions are used to reduce the system of ODEs to a single first order ODE in $y = V_1' = V_2'$. This ODE is then solved analytically. In Feichtinger and Wirl (1993) a similar setup is used.

4 A redistribution mechanism

As already discussed in section 2, there are several ways in which the coordination parameter α may affect the underlying static game G. In this section we consider the case in which the payoffs depend on α and the strategy spaces do not.

4.1 A symmetric redistribution game

We make the following assumptions about the underlying static game G.

Assumption 4.1
(i) *The game G is symmetric, i.e. $\Gamma_1 = \Gamma_2$ and $\pi_1(\gamma_1, \gamma_2) = \pi_2(\gamma_2, \gamma_1)$,*

(ii) *G has a unique Nash equilibrium $(\bar{\gamma}_1, \bar{\gamma}_2)$, with equilibrium payoffs $(\bar{\pi}_1, \bar{\pi}_2)$,*

(iii) *the unique Nash equilibrium of G is not Pareto efficient.*

The symmetry suggests restricting our attention to Pareto efficient strategies $\hat{\gamma}(\frac{1}{2})$ corresponding to $\mu = \frac{1}{2}$ (see theorem 5.1). So for the cooperative strategy we choose $\gamma_i^*(\alpha) = \hat{\gamma}_i(\frac{1}{2})$. The second assumption, that G has a unique Nash equilibrium, justifies the choice of this Nash equilibrium as the alternative strategy, i.e. $\gamma_i^a(\alpha) = \bar{\gamma}_i$. Note that both the cooperative strategies γ^* and the alternative strategies γ^a do not depend on α in this case. The extra payoffs from playing $u_i(c_i) = c_i\hat{\gamma}_i(\frac{1}{2}) + (1 - c_i)\bar{\gamma}_i$ are given by

$$\pi^*(c_1, c_2) := \pi_1(u_1(c_1), u_2(c_2)) + \pi_2(u_1(c_1), u_2(c_2)) - \pi_1(\bar{\gamma}_1, \bar{\gamma}_2) - \pi_2(\bar{\gamma}_1, \bar{\gamma}_2). \tag{27}$$

Now suppose that these extra payoffs are redistributed over the players by the coordinator, according to the rule

$$\nu_1(\alpha, u_1, u_2) := \alpha\pi^*(c_1, c_2), \tag{28}$$
$$\nu_2(\alpha, u_1, u_2) := (1 - \alpha)\pi^*(c_1, c_2). \tag{29}$$

Then the HJBI equations describing the stationary feedback Nash equilibria of the controlled game are given by

$$rV_1(\alpha) = \max_{c_1 \in [0,1]} \left\{ V_1'(\alpha) f(\alpha, c_1, c_2) + \alpha\pi^*(c_1, c_2) \right\}, \tag{30}$$
$$rV_2(\alpha) = \max_{c_2 \in [0,1]} \left\{ V_2'(\alpha) f(\alpha, c_1, c_2) + (1 - \alpha)\pi^*(c_1, c_2) \right\}, \tag{31}$$

or, as long as the constraints $c_1 \geq 0$, $c_1 \leq 1$, $c_2 \geq 0$, $c_2 \leq 1$ are not active, in the form (12–15):

$$\frac{\partial f}{\partial c_1} V_1' + \alpha \frac{\partial \pi^*}{\partial c_1} = 0, \tag{32}$$
$$\frac{\partial f}{\partial c_2} V_2' + (1 - \alpha) \frac{\partial \pi^*}{\partial c_2} = 0, \tag{33}$$
$$f V_1' + \alpha\pi^* - rV_1 = 0, \tag{34}$$
$$f V_2' + (1 - \alpha)\pi^* - rV_2 = 0. \tag{35}$$

As the coordinator's decision rule, we shall take

$$f(\alpha, c_1, c_2) = \beta\alpha(1 - \alpha)(c_2 - c_1),$$

with $\beta \neq 0$.

Now, if we solve (32–33) for (V_1', V_2'), and then substitute the result in (34–35), we obtain

$$-\beta\alpha(1 - \alpha)V_1' + \alpha \frac{\partial \pi^*}{\partial c_1} = 0, \tag{36}$$
$$\beta\alpha(1 - \alpha)V_2' + (1 - \alpha) \frac{\partial \pi^*}{\partial c_2} = 0, \tag{37}$$
$$\alpha(c_2 - c_1) \frac{\partial \pi^*}{\partial c_1} + \alpha\pi^* - rV_1 = 0, \tag{38}$$
$$-(1 - \alpha)(c_2 - c_1) \frac{\partial \pi^*}{\partial c_2} + (1 - \alpha)\pi^* - rV_2 = 0. \tag{39}$$

The matrix $\mathcal{J}$ (see (26)) is given by

$$\mathcal{J} := \begin{pmatrix} \alpha\big(c_2 - c_1\big)\frac{\partial^2\pi^*}{\partial c_1^2} & \alpha\left(\frac{\partial\pi^*}{c_1} + \frac{\partial\pi^*}{\partial c_2} + \big(c_2 - c_1\big)\frac{\partial^2\pi^*}{\partial c_1\partial c_2}\right) \\ \big(1-\alpha\big)\left(\frac{\partial\pi^*}{c_1} + \frac{\partial\pi^*}{\partial c_2} - \big(c_2 - c_1\big)\frac{\partial^2\pi^*}{\partial c_1\partial c_2}\right) & -\big(1-\alpha\big)\big(c_2 - c_1\big)\frac{\partial^2\pi^*}{\partial c_2^2} \end{pmatrix}. \tag{40}$$

Note that $\mathcal{J}$ is nonsingular for all $\alpha \in \big(0,1\big)$ if and only if the matrix $\tilde{\mathcal{J}}$ given by

$$\tilde{\mathcal{J}} := \begin{pmatrix} 0 & \frac{\partial\pi^*}{\partial c_1} + \frac{\partial\pi^*}{\partial c_2} \\ -\left(\frac{\partial\pi^*}{\partial c_1} + \frac{\partial\pi^*}{\partial c_2}\right) & 0 \end{pmatrix} + \big(c_2 - c_1\big)\begin{pmatrix} \frac{\partial^2\pi^*}{\partial c_1^2} & \frac{\partial^2\pi^*}{\partial c_1\partial c_2} \\ \frac{\partial^2\pi^*}{\partial c_1\partial c_2} & \frac{\partial^2\pi^*}{\partial c_2^2} \end{pmatrix} \tag{41}$$

is nonsingular.

Before we show that the system of HJBI-DAEs (36–39) is an index 1 system, we first need the following lemma:

Lemma 4.2 *The function π^* defined in (27) is strictly concave, i.e. the matrix*

$$\begin{pmatrix} \frac{\partial^2\pi^*}{\partial c_1^2} & \frac{\partial^2\pi^*}{\partial c_1\partial c_2} \\ \frac{\partial^2\pi^*}{\partial c_1\partial c_2} & \frac{\partial^2\pi^*}{\partial c_2^2} \end{pmatrix}$$

is negative definite.

Proof: The symmetry of the game G implies $\bar{\gamma}_1 = \bar{\gamma}_2 =: \bar{\gamma}$ and $\hat{\gamma}_1(\frac{1}{2}) = \hat{\gamma}_2(\frac{1}{2}) =: \hat{\gamma}$. Then elementary calculus shows that

$$\begin{pmatrix} \frac{\partial^2\pi^*}{\partial c_1^2} & \frac{\partial^2\pi^*}{\partial c_1\partial c_2} \\ \frac{\partial^2\pi^*}{\partial c_1\partial c_2} & \frac{\partial^2\pi^*}{\partial c_2^2} \end{pmatrix} = (\hat{\gamma} - \bar{\gamma})^2 \left(\begin{pmatrix} \frac{\partial^2\pi_1}{\partial \gamma_1^2} & \frac{\partial^2\pi_1}{\partial \gamma_1\partial \gamma_2} \\ \frac{\partial^2\pi_1}{\partial \gamma_1\partial \gamma_2} & \frac{\partial^2\pi_1}{\partial \gamma_2^2} \end{pmatrix} + \begin{pmatrix} \frac{\partial^2\pi_2}{\partial \gamma_1^2} & \frac{\partial^2\pi_2}{\partial \gamma_1\partial \gamma_2} \\ \frac{\partial^2\pi_2}{\partial \gamma_1\partial \gamma_2} & \frac{\partial^2\pi_2}{\partial \gamma_2^2} \end{pmatrix}\right).$$

From the strict concavity of π_1 and π_2 it follows that

$$\begin{pmatrix} \frac{\partial^2\pi_i}{\partial \gamma_1^2} & \frac{\partial^2\pi_i}{\partial \gamma_1\partial \gamma_2} \\ \frac{\partial^2\pi_i}{\partial \gamma_1\partial \gamma_2} & \frac{\partial^2\pi_i}{\partial \gamma_2^2} \end{pmatrix} < 0,$$

for $i = 1, 2$, and hence

$$\begin{pmatrix} \frac{\partial^2\pi^*}{\partial c_1^2} & \frac{\partial^2\pi^*}{\partial c_1\partial c_2} \\ \frac{\partial^2\pi^*}{\partial c_1\partial c_2} & \frac{\partial^2\pi^*}{\partial c_2^2} \end{pmatrix} < 0$$

since $\hat{\gamma} \neq \bar{\gamma}$ by assumption 4.1 (iii). $\square$

Proposition 4.3 *The system of HJBI-DAEs (36–39) has index 1 on its domain of validity $\big(0,1\big) \times \big(0,1\big)$.*

Proof: We will show that the matrix $\tilde{\mathcal{J}}$ appearing in (41) is nonsingular for all $\alpha \in \big(0,1\big)$. We will consider two cases, first the case $c_1 \neq c_2$, and secondly the case $c_1 = c_2$. In the case $c_1 \neq c_2$ we note that $\tilde{\mathcal{J}}$ is the sum of a skew-symmetric matrix and a matrix that is, depending on the sign of $c_2 - c_1$, either positive or negative definite. Hence, for $c_1 \neq c_2$ $\tilde{\mathcal{J}}$ is nonsingular.

In the case $c_1 = c_2$, we see that

$$\tilde{\mathcal{J}} = \begin{pmatrix} 0 & \frac{\partial \pi^*}{\partial c_1} + \frac{\partial \pi^*}{\partial c_2} \\ -\left(\frac{\partial \pi^*}{\partial c_1} + \frac{\partial \pi^*}{\partial c_2}\right) & 0 \end{pmatrix}.$$

Note that $\pi^*(c_1, c_2) = \pi^*(c_2, c_1)$, and hence $\tilde{\mathcal{J}}$ is singular if and only if $\frac{\partial \pi^*}{\partial c_1} = \frac{\partial \pi^*}{\partial c_2} = 0$. Elementary calculus shows that $\frac{\partial \pi^*}{\partial c_1} = \frac{\partial \pi^*}{\partial c_2} = 0$ if and only if

$$\begin{aligned} \frac{\partial \pi_1}{\partial \gamma_1}(u_1(c_1), u_2(c_2)) + \frac{\partial \pi_2}{\partial \gamma_1}(u_1(c_1), u_2(c_2)) &= 0, \\ \frac{\partial \pi_1}{\partial \gamma_2}(u_1(c_1), u_2(c_2)) + \frac{\partial \pi_2}{\partial \gamma_2}(u_1(c_1), u_2(c_2)) &= 0. \end{aligned}$$

Note that these last equations are (see theorem 5.1) precisely the first order conditions characterizing $\hat{\gamma}(\frac{1}{2})$, and hence satisfied if and only if $c_1 = c_2 = 1$. However, $c_1 = c_2 = 1$ lies outside the domain of validity $\big(0,1\big) \times \big(0,1\big)$. $\square$

Because the system of HJBI-DAEs (36–39) has index 1, we can derive ODEs for c_1 and c_2 by differentiating (36–39) once. The resulting ODEs are given by

$$\begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \mathcal{J}^{-1} \begin{pmatrix} \left(-\big(c_2 - c_1\big) - \frac{r}{\beta\left(1-\alpha\right)}\right) \frac{\partial \pi^*}{\partial c_1} + \pi^* \\ \left(\big(c_2 - c_1\big) - \frac{r}{\beta\alpha}\right) \frac{\partial \pi^*}{\partial c_2} + \pi^* \end{pmatrix} \tag{42}$$

We shall be interested in particular in *symmetric* solutions, i.e. those for which $c_1(\alpha) = c_2(1-\alpha)$ and $V_1(\alpha) = V_2(1-\alpha)$. These solutions can be characterized as follows.

Lemma 4.4 *A solution $\big(V_1, V_2, c_1, c_2\big)$ of the HJBI-DAEs (36–39) is symmetric if and only if it satisfies $c_1(\frac{1}{2}) = c_2(\frac{1}{2})$.*

Proof: By writing the HJBI-DAEs (36–39) and its first derivatives, evaluated in $\alpha = \frac{1}{2}$, it is easily verified that there are only 2 degrees of freedom in specifying consistent initial[2] conditions, i.e. when 2 variables out of

$$\left\{V_1(\frac{1}{2}), V_2(\frac{1}{2}), c_1(\frac{1}{2}), c_2(\frac{1}{2}), V_1'(\frac{1}{2}), V_2'(\frac{1}{2}), c_1'(\frac{1}{2}), c_2'(\frac{1}{2})\right\}$$

are chosen, the other variables are fixed by the system of HJBI-DAEs (36–39) and its first derivatives, evaluated in $\alpha = \frac{1}{2}$.

[2]Note that in this case we consider $\alpha = \frac{1}{2}$ as the 'starting point', i.e. initial conditions are specified in $\alpha = \frac{1}{2}$.

Now let $\big(V_1(\alpha), V_2(\alpha), c_1(\alpha), c_2(\alpha)\big)$ be a solution of the HJBI-DAEs (36–39) corresponding to the initial conditions $\big(c_1(\frac{1}{2}), c_2(\frac{1}{2})\big)$. Then it can straightforwardly be shown that $\big(V_2(1-\alpha), V_1(1-\alpha), c_2(1-\alpha), c_1(1-\alpha)\big)$ is a solution of the HJBI-DAEs (36–39) corresponding to the initial conditions $\big(c_2(\frac{1}{2}), c_1(\frac{1}{2})\big)$. From (42) we see, because

$$g(\alpha, c_1, c_2) := \mathcal{J}^{-1} \begin{pmatrix} \left(-(c_2 - c_1) - \frac{r}{\beta(1-\alpha)}\right) \frac{\partial \pi^*}{\partial c_1} + \pi^* \\ \left((c_2 - c_1) - \frac{r}{\beta\alpha}\right) \frac{\partial \pi^*}{\partial c_2} + \pi^* \end{pmatrix}$$

is a C^1 function and hence satisfies a Lipschitz condition, that whenever $c_1(\frac{1}{2}) = c_2(\frac{1}{2})$, necessarily $c_1(\alpha) = c_2(1-\alpha)$ for all $\alpha \in (0,1)$. □

4.2 A worked example

As an example of a redistribution controlled game we choose for G a Cournot duopoly, in which the prices are determined by

$$p(y) = \begin{cases} 120 - y & \text{if} \quad y \leq 120 \\ 0 & \text{if} \quad y > 120 \end{cases} \tag{43}$$

and production costs are given by

$$c_i(y_i) = y_i^2. \tag{44}$$

Then the payoffs of G are given by (see e.g. Gibbons (1992); Takayama (1985)):

$$\pi_i(y_1, y_2) = y_i\,(120 - y_1 - y_2) - y_i^2. \tag{45}$$

The Nash equilibrium $\bar{\gamma}$ of G, with payoff $\bar{\pi}$, and the Pareto efficient strategy $\hat{\gamma}(\frac{1}{2})$, with payoff $\hat{\pi}$ are given by

$$\bar{\gamma} = 24, \tag{46}$$
$$\bar{\pi} = 1152, \tag{47}$$
$$\hat{\gamma}(\frac{1}{2}) = 20, \tag{48}$$
$$\hat{\pi} = 1200. \tag{49}$$

Hence, the additional payoffs after redistribution (27) are

$$\pi^*(c_1, c_2) = 96\,(c_1 + c_2) - 32\left(c_1^2 + c_1 c_2 + c_2^2\right). \tag{50}$$

Remark 4.5 The controlled game constructed in this way, can be interpreted as follows. Consider two firms who produce an identical product. Instead of selling the products themselves, the goods are sold on an instantaneously clearing market by a separate institution (the coordinator), who distributes the payoffs between the two firms using the decision rule f.

From (36–39) we find the HJBI-DAEs

$$V_1' = \frac{96 - 64c_1 - 32c_2}{\beta(1-\alpha)}, \tag{51}$$

$$V_2' = \frac{-96 + 32c_1 + 64c_2}{\beta\alpha}, \tag{52}$$

$$V_1 = \frac{\alpha}{r}\left((c_2 - c_1)(96 - 64c_1 - 32c_2) + \pi^*(c_1, c_2)\right), \tag{53}$$

$$V_2 = \frac{1-\alpha}{r}\left((c_1 - c_2)(96 - 32c_1 - 64c_2) + \pi^*(c_1, c_2)\right). \tag{54}$$

Similar HJBI-DAEs can be derived for the cases where one or more constraints on c_i become active.

By starting the integration at $\alpha = \frac{1}{2}$ and using symmetry, the systems of HJBI-DAEs are solved using DASSL (see Brenan et al. (1989)). In fact, we only calculate V_1, V_2, c_1 and c_2 for α from $\alpha = \frac{1}{2}$ to $\alpha = 0.999$, (thus avoiding the singularity at $\alpha = 1$), and then by using symmetry (i.e. $c_1(\alpha) = c_2(1-\alpha)$) we obtain the results for $\alpha = 0.001$ to $\alpha = 0.999$. The DASSL-output is then fed into Matlab, where we use spline interpolation and the built-in Runge-Kutta ODE solver to simulate the resulting closed-loop dynamics of the controlled game.

We have already seen in the proof of lemma 4.4 that in specifying consistent initial conditions the degree of freedom is 2, i.e. when one of the pairs of variables $\left(V_1(\frac{1}{2}), V_2(\frac{1}{2})\right)$ or equivalently $\left(c_1(\frac{1}{2}), c_2(\frac{1}{2})\right)$ is chosen, the others are fixed by the system of HJBI-DAEs. By requiring the extra symmetry condition $c_1(\alpha) = c_2(1-\alpha)$, $V_1(\alpha) = V_2(1-\alpha)$, i.e. $c_1(\frac{1}{2}) = c_2(\frac{1}{2})$, the degree of freedom is reduced to 1. In the experiments we have started by fixing the initial value of $c_1(\frac{1}{2}) = c_2(\frac{1}{2})$, which then fully determines the consistent initial conditions.

In the experiments we have fixed the parameters $\beta = \frac{1}{3}$ and $r = 1$. We have varied the initial conditions $c_1(\frac{1}{2}) = c_2(\frac{1}{2})$ (see table 1). Consistent values $V_1(\frac{1}{2}) = V_2(\frac{1}{2})$ are then calculated using the HJBI-DAEs (51-54) evaluated in $\alpha = \frac{1}{2}$.

$c_1(\frac{1}{2}) = c_2(\frac{1}{2})$	$V_1(\frac{1}{2}) = V_2(\frac{1}{2})$
0.75	45
0.79	45.8832
0.796	46.002432
0.8	46.08
0.9	47.52
0.99	47.9952

Table 1: Some consistent initial conditions for $\beta = \frac{1}{3}$, $r = 1$

The results of the experiments are shown in figures 1–6. In figure 1 and figure 2 we see that for the initial conditions corresponding to $c_1(\frac{1}{2}) = c_2(\frac{1}{2}) = 0.75$ and $c_1(\frac{1}{2}) = c_2(\frac{1}{2}) = 0.79$, the solutions V_1 and V_2 become unbounded, and hence do not correspond to a stationary feedback Nash equilibrium of the controlled game[3]. In the other cases (see figures 3,4,5,6), V_1

[3]Note that π^* is bounded and hence also $\alpha\pi^*$ and $(1-\alpha)\pi^*$. Using the fact that the discount factor r is

and V_2 are continuously differentiable and bounded, and hence by theorem A.3 correspond to stationary feedback Nash equilibria for the controlled game. In the figures 3,4,5,6 we have plotted $V_1(\alpha)$, $c_1(\alpha)$, the closed-loop mechanism $f(\alpha, c_1(\alpha), c_2(\alpha))$ and the simulated closed-loop dynamics $\dot{\alpha}(t) = f(\alpha(t), c_1(\alpha(t)), c_2(\alpha(t)))$ for $\alpha(0) = 0.35, 0.4, 0.45, 0.5, 0.55, 0.6$ and 0.65.

In figure 3 and 4 we see that the corresponding stationary feedback Nash equilibria support five different steady states; three of these are unstable ($\alpha = 0$, $\alpha = 0.5$ and $\alpha = 1$), and two are stable ($\alpha \approx 0.4$ and $\alpha \approx 0.6$). Finally, in figure 5 and figure 6, the stationary feedback Nash equilibria support only three steady states; two of these are unstable ($\alpha = 0$ and $\alpha = 1$) and one is stable ($\alpha = 0.5$). This suggests that somewhere between $c_1(\frac{1}{2}) = c_2(\frac{1}{2}) = 0.9$ and $c_1(\frac{1}{2}) = c_2(\frac{1}{2}) = 0.8$ a bifurcation takes place. To confirm this, we calculated the steady states, belonging to equilibria corresponding to initial conditions $c_1(\frac{1}{2}) = c_2(\frac{1}{2})$ ranging from 0.8 to 0.9. The outcomes are plotted in figure 7. In all the stable steady states we have also determined the corresponding values of the cooperation parameters $c_1 = c_2$. These values are plotted in figure 8. In figure 9 we have plotted the payoffs for player 1 and 2 in the stable steady states which are greater than or equal to $\frac{1}{2}$. The dotted line in this figure is the payoff in the steady state $\frac{1}{2}$.

In all stationary feedback Nash equilibria that we find, convergence takes place to a quite cooperative situation; a threshold value of approximately 0.84 is found for the cooperation coefficients. However, there are two ways in which this cooperation is achieved. If the players are already cooperative above the threshold value in a situation of equal distribution of profits ($\alpha = \frac{1}{2}$), then a symmetric solution is obtained. This symmetry breaks down however if the players are less cooperative at $\alpha = \frac{1}{2}$; the slightest deviation of the initial value $\alpha(0) = \frac{1}{2}$ will cause a process in which convergence takes place to a situation in which both players are equally cooperative but take unequal shares in the revenues of cooperation.

In a second experiment we have fixed the parameters at $r = 1$ and $\beta = -\frac{1}{3}$. In this case it is easily verified that the strategies $c_1 = c_2 \equiv 1$ give a stationary feedback Nash equilibrium, with corresponding value functions $V_1(\alpha) = 96\alpha$ and $V_2(\alpha) = 96(1-\alpha)$. Moreover, we calculated solutions of the HJBI equations corresponding to several initial conditions $c_1(\frac{1}{2}) = c_2(\frac{1}{2}) \in [0, 1)$. In all these calculated solutions V_1 and V_2 turn out to be unbounded. This suggests that the only symmetric stationary feedback Nash equilibrium is given by $c_1 = c_2 \equiv 1$. Apparently the mechanism in which the coordinator punishes any deviation from a joint cooperative strategy (i.e. the mechanism with $\beta < 0$) is more effective, in the sense that it supports full cooperation (i.e. $c_1 = c_2 \equiv 1$) as the only symmetric stationary feedback Nash equilibrium of the controlled game.

5 A Pareto mechanism

In this section we will consider a situation in which the coordination parameter α affects the underlying static game G not through the payoffs but rather through the strategy spaces of both players. We motivate the choice of coordination mechanism by the following result (see

positive, it immediately follows that any value function is necessarily bounded.

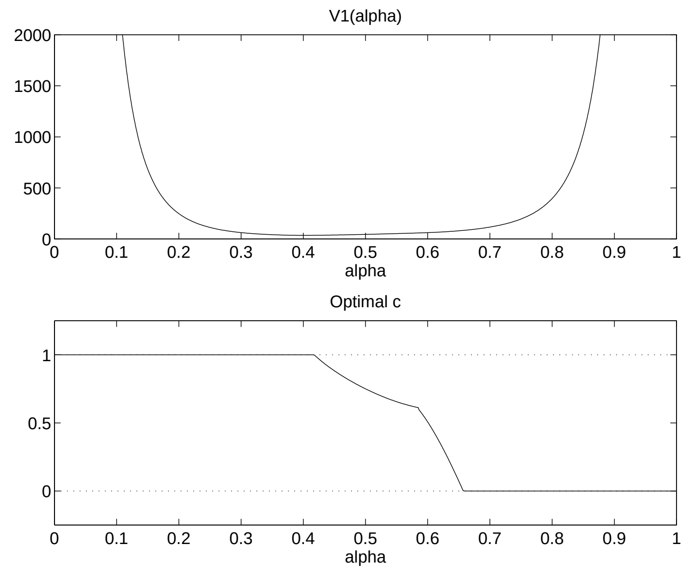

Figure 1: $c_1(\frac{1}{2}) = c_2(\frac{1}{2}) = 0.75$

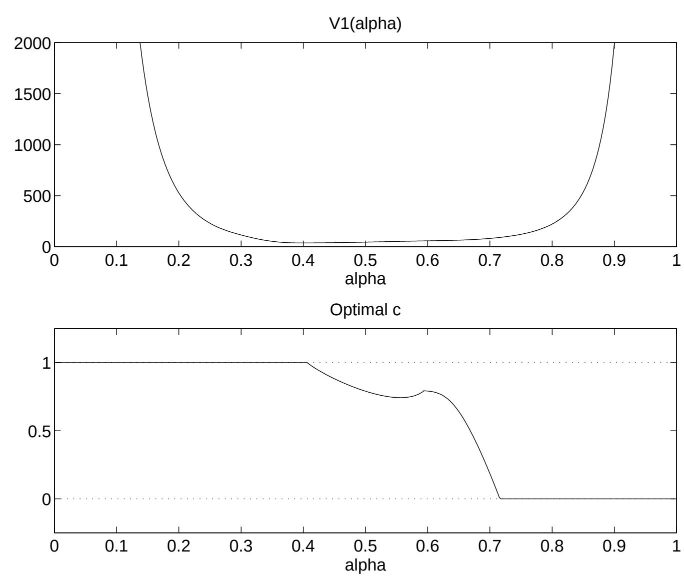

Figure 2: $c_1(\frac{1}{2}) = c_2(\frac{1}{2}) = 0.79$

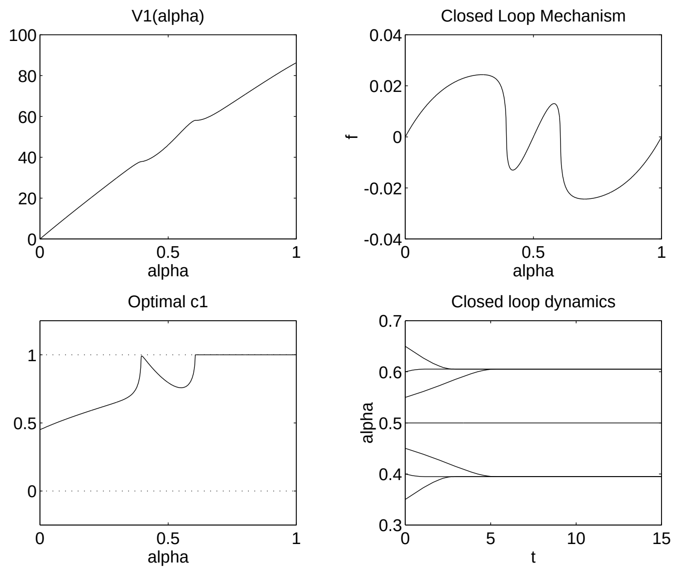

Figure 3: $c_1(\frac{1}{2}) = c_2(\frac{1}{2}) = 0.796$

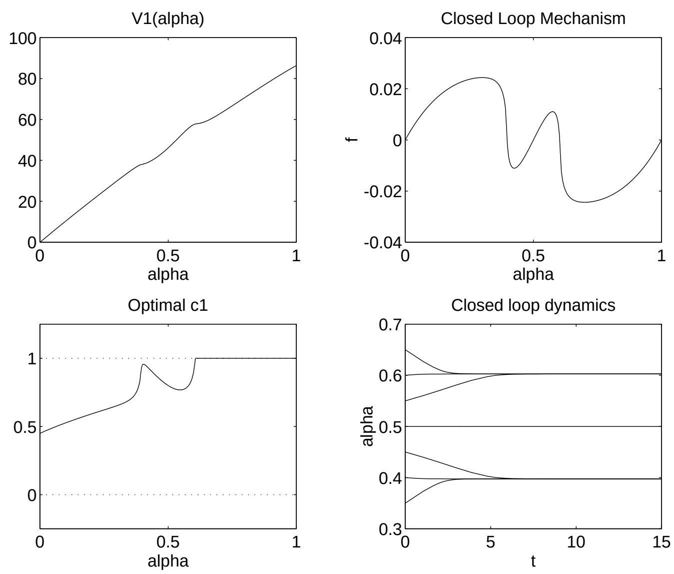

Figure 4: $c_1(\frac{1}{2}) = c_2(\frac{1}{2}) = 0.8$

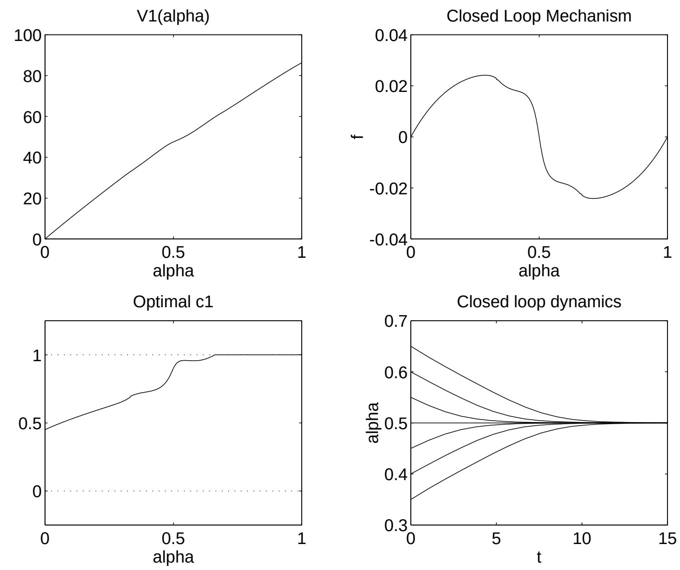

Figure 5: $c_1(\frac{1}{2}) = c_2(\frac{1}{2}) = 0.9$

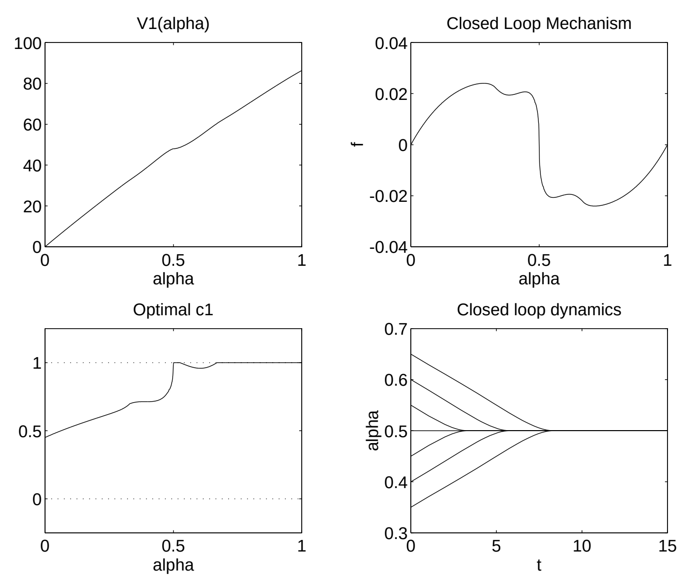

Figure 6: $c_1(\frac{1}{2}) = c_2(\frac{1}{2}) = 0.99$

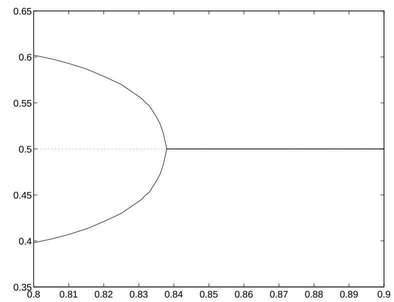

Figure 7: Bifurcation of steady states

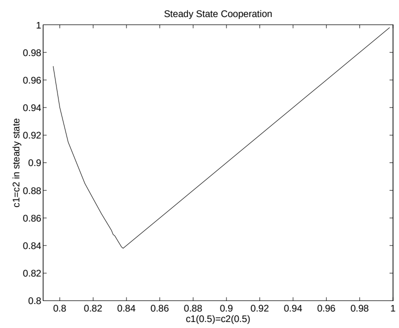

Figure 8: Steady state cooperation

e.g. Takayama (1985); Verkama (1994))

Theorem 5.1 *For all $\mu \in (0,1)$ holds that if $(\hat{\gamma}_1, \hat{\gamma}_2) \in \Gamma_1 \times \Gamma_2$ satisfies*

$$(\hat{\gamma}_1, \hat{\gamma}_2) = \arg \max_{(\gamma_1,\gamma_2)\in\Gamma_1\times\Gamma_2} \{\mu\pi_1(\gamma_1,\gamma_2) + (1-\mu)\pi_2(\gamma_1,\gamma_2)\},$$

then $(\hat{\gamma}_1, \hat{\gamma}_2)$ is Pareto efficient.
Moreover, if Γ_1, Γ_2 are convex, and π_1, π_2 are concave, then for all Pareto efficient $(\hat{\gamma}_1, \hat{\gamma}_2)$ there exists a $\mu \in [0,1]$, such that

$$(\hat{\gamma}_1, \hat{\gamma}_2) = \arg \max_{(\gamma_1,\gamma_2)\in\Gamma_1\times\Gamma_2} \{\mu\pi_1(\gamma_1,\gamma_2) + (1-\mu)\pi_2(\gamma_1,\gamma_2)\}.$$

We will no longer assume that G is symmetric. Now the task of the coordinator is to choose the Pareto efficient strategy to be considered by the individual players, i.e. the coordinator determines the choice of μ according to theorem 5.1 at time instant t. The cooperative strategies to be considered are $\gamma_i^*(\alpha) = \hat{\gamma}_i(\alpha)$. In this section we will exclude the possibility of sidepayments or redistribution, by taking

$$\nu_i(\alpha, \gamma_1, \gamma_2) = \pi_i(\gamma_1, \gamma_2). \tag{55}$$

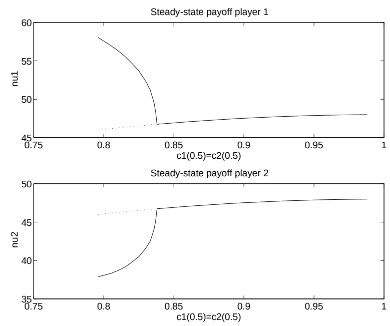

Figure 9: Steady-state payoffs

The HJBI equations describing the stationary feedback Nash equilibria of the controlled game, are given by

$$rV_1(\alpha) = \max_{c_1 \in [0,1]} \left\{ V_1'(\alpha) f(\alpha, c_1, c_2) + \pi_1(u_1(c_1), u_2(c_2)) \right\}, \tag{56}$$

$$rV_2(\alpha) = \max_{c_2 \in [0,1]} \left\{ V_2'(\alpha) f(\alpha, c_1, c_2) + \pi_2(u_1(c_1), u_2(c_2)) \right\}. \tag{57}$$

We find the following proposition:

Proposition 5.2 *Suppose G has a unique Nash equilibrium $\big(\bar\gamma_1, \bar\gamma_2\big)$. Furthermore, suppose the alternative strategy γ^a is such that for all $\alpha \in \big(0,1\big)$ the system of equations*

$$\begin{aligned} c_1(\alpha)\gamma_1^*(\alpha) + \big(1 - c_1(\alpha)\big)\gamma_1^a(\alpha) &= \bar\gamma_1, \\ c_2(\alpha)\gamma_2^*(\alpha) + \big(1 - c_2(\alpha)\big)\gamma_2^a(\alpha) &= \bar\gamma_2, \end{aligned}$$

has a unique solution $\big(\bar c_1(\alpha), \bar c_2(\alpha)\big)$, with $0 \le \bar c_i(\alpha) \le 1$. Then a stationary feedback Nash equilibrium of the controlled game is given by

$$\big(c_1(\alpha), c_2(\alpha)\big) = \big(\bar c_1(\alpha), \bar c_2(\alpha)\big).$$

The actions $\big(u_1(\bar c_1(\alpha(t))), u_2(\bar c_2(\alpha(t)))\big)$ played at every time instant t are equal to the Nash equilibrium $\big(\bar\gamma_1, \bar\gamma_2\big)$ of G.

Proof: For all $\alpha \in \big(0,1\big)$ the Nash equilibrium of G is recovered for $\big(c_1, c_2\big) = \big(\bar c_1(\alpha), \bar c_2(\alpha)\big)$, more precisely

$$\begin{aligned} \bar c_1(\alpha) &\in \arg\max_{c_1} \left\{ \pi_i(u_1(c_1), u_2(\bar c_2(\alpha))) \right\}, \\ \bar c_2(\alpha) &\in \arg\max_{c_2} \left\{ \pi_i(u_1(\bar c_1(\alpha)), u_2(c_2)) \right\}. \end{aligned}$$

Note that, for all $\alpha \in \big(0,1\big)$, $\pi_i(u_1(\bar c_1(\alpha)), u_2(\bar c_2(\alpha))) = \pi_i(\bar\gamma_1, \bar\gamma_2)$ does not depend on α. The HJBI equations are given by

$$\begin{aligned} rV_i(\alpha) &= \pi_i(u_1(\bar c_1(\alpha)), u_2(\bar c_2(\alpha))), \\ V_i'(\alpha) &= 0. \end{aligned}$$

$\square$

Remark 5.3 Note that although the actions at every time instant t equal the actions corresponding to the unique Nash equilibrium of G, they emerge from a different strategy. Moreover, these equilibrium strategies can give rise to a nontrivial dynamic behavior of α. So in this sense a coordination process does take place, but is never successful, because both players attain the same payoff as in the case of noncooperative play.

Remark 5.4 In general the conditions of proposition 5.2 will not be satisfied for all $\alpha \in \big(0,1\big)$. In that case a stationary feedback Nash equilibrium of the controlled game will allow for different actions to be played. Also in the case of multiple Nash equilibria for the game G different actions can be expected. Furthermore it is important to note that, even in the case that all the conditions of proposition 5.2 are fulfilled, this Nash equilibrium is not necessarily unique.

In the Cournot duopoly example that we introduced in the previous section, it can straightforwardly be shown that the stationary feedback Nash equilibrium described by proposition 5.2, in case the alternative strategies γ_i^a are chosen to be equal to the Nash strategies $\bar{\gamma}_i$, is in fact unique. This is a consequence of the fact that $c_1(\frac{1}{2}) = c_2(\frac{1}{2}) = 0$ provides the only consistent initial conditions for the HJBI-DAEs. In this case the determinant of the matrix $\mathcal{J}$ (see (26)) is given by

$$\det \mathcal{J} = -331776 \frac{16\alpha^4 - 32\alpha^3 + 3\alpha^2 + 13\alpha - 4}{\big(1 - 16\alpha + 16\alpha^2\big)^2}, \tag{58}$$

so that $\det \mathcal{J} = 0$ for $\alpha = -\frac{1}{8} + \frac{1}{8}\sqrt{17} \approx 0.390$ or for $\alpha = \frac{9}{8} - \frac{1}{8}\sqrt{17} \approx 0.610$, and moreover the denominator of $\det \mathcal{J}$ equals 0 for $\alpha = \frac{1}{2} \pm \frac{1}{4}\sqrt{3}$. Hence the system of HJBI-DAEs is locally of higher index. As a consequence the DASSL code can not be used to obtain the solution. However, using the code RADAU5 (see Hairer et al. (1989); Hairer and Wanner (1991)), which is also suited for semi-explicit index 2 and index 3 systems, it is possible to find the solution efficiently[4].

6 Conclusions

In this paper we have introduced a model for the process of coordination towards a cooperative equilibrium of a repeated game. An important aspect in this setup is that we have allowed for strategic behavior by the individual players, influencing the outcome of the coordination process. We have obtained a nonlinear differential game, with state variable α, which we called the controlled game. We have taken a closer look at two special cases of such a controlled game, namely a redistribution controlled game and a Pareto controlled game. Using recently developed methods for differential-algebraic equations (DAEs), we have described in what way for such differential games all stationary feedback Nash equilibria can be calculated. In a worked example of a repeated symmetric Cournot duopoly, we have illustrated the numerical method for the redistribution controlled game. We saw that for this example there are several qualitatively different stationary feedback Nash equilibria. In this example we saw that if the

[4] Of course in this case we did not need to use any numerical method to find the solution. However, we can use this case as a testcase for the different numerical methods for DAEs.

players are sufficiently willing to cooperate at the point where all extra payoffs are divided equally between the players, this point is supported by the stationary feedback Nash equilibria as the only stable steady state. However, in case the players are not sufficiently willing to cooperate at this point, the stability of the steady state is lost. Moreover, we have seen that if the coordinator's decision rule is changed in such a way that deviations from a cooperative strategy are punished by the coordinator, a symmetric stationary feedback Nash equilibrium exists that supports full cooperation of both players. In the case of the Pareto controlled game, we found for the same example that the unique stationary feedback Nash equilibrium does not support any cooperation at all; in this case the coordination mechanism is too weak to stimulate cooperation. We can conclude that the choice of coordination mechanism and the choice of decision rule for the coordinator can be viewed as a control problem; by choosing the appropriate mechanism and decision rule a global control objective can be pursued.

A Derivation of HJBI equations

In this appendix we derive the HJBI equations associated to stationary feedback Nash equilibria of general nonlinear differential games, where the state space is an open and bounded subinterval of $\mathbb{R}$. The results of this appendix can straightforwardly be generalized to more general state spaces. Although the results of this appendix may be considered essentially well known (see Feichtinger and Wirl (1993); Tsutsui and Mino (1990)), we have not been able to find suitable references in the existing literature. Therefore we have decided to provide the proofs in this appendix. Similar results for optimal control problems can be found in Fleming and Soner (1993).

First we consider the optimal control problem

$$\max_u \int_0^\infty e^{-r\tau}\pi(x(\tau), u(\tau))d\tau,$$

under the conditions

$$\begin{aligned} \dot{x} &= f(x, u), \\ x(0) &= x_0, \end{aligned}$$

in which $r > 0$.

Assume that for all x_0,

$$\max_u \int_0^\infty e^{-r\tau}\pi(x(\tau), u(\tau))d\tau < \infty.$$

Now define the value function

$$V(x_0) := \max_u \int_0^\infty e^{-r\tau}\pi(x(\tau), u(\tau))d\tau.$$

Then we find the following lemma:

Lemma A.1 *Assume that the value function V is continuously differentiable. Then V satisfies the Hamilton-Jacobi-Bellman equation*

$$rV(x_0) = \max_{u_0}\left\{\pi(x_0, u_0) + V'(x_0)f(x_0, u_0)\right\}.$$

Proof: Let $\Delta t > 0$ and let $x(\cdot)$ be any admissible trajectory satisfying $x(0) = x_0$. Then, because V is continuously differentiable

$$V(x(t)) = V(x_0) + V'(x_0)f(x_0, u(0))\Delta t + o(\Delta t),$$

for $\Delta t \to 0$.

Now we find

$$\begin{aligned}
V(x_0) &= \max_u \int_0^\infty e^{-r\tau}\pi(x(\tau),u(\tau))d\tau \\
&= \max_u \left\{ \int_0^{\Delta t} e^{-r\tau}\pi(x(\tau),u(\tau))d\tau + \int_{\Delta t}^\infty e^{-r\tau}\pi(x(\tau),u(\tau))d\tau \right\} \\
&= \max_u \left\{ \int_0^{\Delta t} e^{-r\tau}\pi(x(\tau),u(\tau))d\tau + e^{-r\Delta t}\int_0^\infty e^{-r\tau}\pi(x(\tau+\Delta t),u(\tau+\Delta t))d\tau \right\} \\
&= \max_u \left\{ \int_0^{\Delta t} e^{-r\tau}\pi(x(\tau),u(\tau))d\tau + e^{-r\Delta t}V(x(\Delta t)) \right\} \\
&= \max_{u_0} \left\{ \pi(x_0,u_0)\Delta t + e^{-r\Delta t}V(x_0) + e^{-r\Delta t}V'(x_0)f(x_0,u_0)\Delta t + o(\Delta t) \right\} \\
&= \max_{u_0} \left\{ \pi(x_0,u_0)\Delta t + (1-r\Delta t)V(x_0) + V'(x_0)f(x_0,u_0)\Delta t + o(\Delta t) \right\} \\
&= (1-r\Delta t)V(x_0) + \max_{u_0} \left\{ \left(\pi(x_0,u_0) + V'(x_0)f(x_0,u_0)\right)\Delta t + o(\Delta t) \right\},
\end{aligned}$$

for $\Delta t \to 0$.

This immediately implies

$$\forall_{\Delta t>0}\; rV(x_0) = \max_{u_0} \left\{ \pi(x_0,u_0) + V'(x_0)f(x_0,u_0) + o(\Delta t)/\Delta t \right\},$$

and hence necessarily

$$rV(x_0) = \max_{u_0} \left\{ \pi(x_0,u_0) + V'(x_0)f(x_0,u_0) \right\}.$$

□

Now consider the differential game

$$\dot{x} = f(x,u_1,u_2),$$

with payoff functionals

$$\begin{aligned}
L_1(u_1,u_2) &:= \int_0^\infty e^{-r\tau}\pi_1(x(\tau),u_1(\tau),u_2(\tau))d\tau, \\
L_2(u_1,u_2) &:= \int_0^\infty e^{-r\tau}\pi_2(x(\tau),u_1(\tau),u_2(\tau))d\tau.
\end{aligned}$$

Because we are interested in stationary feedback Nash equilibria, we limit the admissible strategies to the class of stationary feedback strategies. As in (Başar and Olsder, 1995, section 5.3), we demand that the strategies satisfy a Lipschitz continuity condition to ensure well-posedness.

Now define the value functions V_1, V_2 by

$$\begin{aligned}
V_1(x_0,\gamma_2) &:= \max_{u_1} \int_0^\infty e^{-r\tau}\pi_1(x(\tau),u_1(\tau),\gamma_2(x(\tau)))d\tau, \\
V_2(x_0,\gamma_1) &:= \max_{u_2} \int_0^\infty e^{-r\tau}\pi_2(x(\tau),\gamma_1(x(\tau)),u_2(\tau))d\tau.
\end{aligned}$$

We assume that for all $\gamma_1 \in \Gamma_1$ and for all $\gamma_2 \in \Gamma_2$:

$$\begin{aligned}
\max_{u_1} \int_0^\infty e^{-r\tau}\pi_1(x(\tau), u_1(\tau), \gamma_2(x(\tau)))d\tau &< \infty, \\
\max_{u_2} \int_0^\infty e^{-r\tau}\pi_2(x(\tau), \gamma_1(x(\tau)), u_2(\tau))d\tau &< \infty.
\end{aligned}$$

Then we immediately find, using lemma A.1, provided V_1 and V_2 are continuously differentiable, that V_1 and V_2 satisfy the stationary Hamilton-Jacobi-Bellman-Isaacs equations

$$rV_1(x_0, \gamma_2) = \max_{u_0}\left\{\pi_1(x_0, u_0, \gamma_2(x_0)) + \frac{\partial V_1}{\partial x}(x_0, \gamma_2) f(x_0, u_0, \gamma_2(x_0))\right\},$$
$$rV_2(x_0, \gamma_1) = \max_{u_0}\left\{\pi_2(x_0, \gamma_1(x_0), u_0) + \frac{\partial V_2}{\partial x}(x_0, \gamma_1) f(x_0, \gamma_1(x_0), u_0)\right\}.$$

This leads to the following lemma:

Lemma A.2 *If $(\bar{\gamma}_1, \bar{\gamma}_2)$ is a stationary feedback Nash equilibrium, with continuously differentiable value functions V_1 and V_2, then V_1 and V_2 satisfy the HJBI equations*

$$\begin{aligned}
rV_1(x_0, \bar{\gamma}_2) &= \max_{u_0}\left\{\pi_1(x_0, u_0, \bar{\gamma}_2(x_0)) + \frac{\partial V_1}{\partial x}(x_0, \bar{\gamma}_2) f(x_0, u_0, \bar{\gamma}_2(x_0))\right\}, \\
rV_2(x_0, \bar{\gamma}_1) &= \max_{u_0}\left\{\pi_2(x_0, \bar{\gamma}_1(x_0), u_0) + \frac{\partial V_2}{\partial x}(x_0, \bar{\gamma}_1) f(x_0, \bar{\gamma}_1(x_0), u_0)\right\}.
\end{aligned}$$

Moreover

$$\begin{aligned}
\bar{\gamma}_1(x_0) &\in \arg\max_{u_0}\left\{\pi_1(x_0, u_0, \bar{\gamma}_2(x_0)) + \frac{\partial V_1}{\partial x}(x_0, \bar{\gamma}_2) f(x_0, u_0, \bar{\gamma}_2(x_0))\right\}, \\
\bar{\gamma}_2(x_0) &\in \arg\max_{u_0}\left\{\pi_2(x_0, \bar{\gamma}_1(x_0), u_0) + \frac{\partial V_2}{\partial x}(x_0, \bar{\gamma}_1) f(x_0, \bar{\gamma}_1(x_0), u_0)\right\}.
\end{aligned}$$

With a slight abuse of notation we write $V_i(x_0)$ instead of $V_i(x_0, \bar{\gamma}_j)$.

We now find the following theorem:

Theorem A.3 (Verification theorem) *Suppose $\left(\bar{V}_1, \bar{V}_2, \bar{\gamma}_1, \bar{\gamma}_2\right)$ are solutions of the HJBI equations*

$$\begin{aligned}
rV_1(x_0) &= \max_{u_0}\left\{\pi_1(x_0, u_0, \bar{\gamma}_2(x_0)) + V_1'(x_0) f(x_0, u_0, \bar{\gamma}_2(x_0))\right\}, \\
rV_2(x_0) &= \max_{u_0}\left\{\pi_2(x_0, \bar{\gamma}_1(x_0), u_0) + V_2'(x_0) f(x_0, \bar{\gamma}_1(x_0), u_0)\right\},
\end{aligned}$$

with

$$\begin{aligned}
\bar{\gamma}_1(x_0) &\in \arg\max_{u_0}\left\{\pi_1(x_0, u_0, \bar{\gamma}_2(x_0)) + V_1'(x_0) f(x_0, u_0, \bar{\gamma}_2(x_0))\right\}, \\
\bar{\gamma}_2(x_0) &\in \arg\max_{u_0}\left\{\pi_2(x_0, \bar{\gamma}_1(x_0), u_0) + V_2'(x_0) f(x_0, \bar{\gamma}_1(x_0), u_0)\right\},
\end{aligned}$$

and $\bar{V}_1, \bar{V}_2$ continuously differentiable and bounded.

Then, $\bar{V}_1$ and $\bar{V}_2$ are value functions, i.e.

$$\begin{aligned}
\bar{V}_1(x_0) &= \max_{\gamma_1} \int_0^\infty e^{-rt}\pi_1(x(t), \gamma_1(x(t)), \bar{\gamma}_2(x(t)))dt \\
\bar{V}_2(x_0) &= \max_{\gamma_2} \int_0^\infty e^{-rt}\pi_2(x(t), \bar{\gamma}_1(x(t)), \gamma_2(x(t)))dt
\end{aligned}$$

Proof: We find

$$\begin{aligned}
&\int_0^\infty e^{-rt}\pi_1(x(t), \bar{\gamma}_1(x(t)), \bar{\gamma}_2(x(t)))dt \\
=& \int_0^\infty e^{-rt}\left[r\bar{V}_1(x(t)) - f(x(t), \bar{\gamma}_1(x(t)), \bar{\gamma}_2(x(t)))\bar{V}_1'(x(t))\right]dt \\
=& \int_0^\infty \left[-\frac{d}{dt}\left[e^{-rt}\right]\bar{V}_1(x(t)) - e^{-rt}\dot{x}(t)\bar{V}_1'(x(t))\right]dt \\
=& \int_0^\infty -\frac{d}{dt}\left[e^{-rt}\bar{V}_1(x(t))\right]dt \\
=& \bar{V}_1(x_0).
\end{aligned}$$

Similarly, we can derive $\bar{V}_2(x_0) = \int_0^\infty e^{-rt}\pi_2(x(t), \bar{\gamma}_1(x(t)), \bar{\gamma}_2(x(t)))dt$.

Now suppose γ_1 is any admissible stationary feedback strategy. Then we find for all x:

$$\pi_1(x, \gamma_1(x), \bar{\gamma}_2(x)) + f(x, \gamma_1(x), \bar{\gamma}_2(x))\bar{V}_1'(x) \leqslant r\bar{V}_1(x),$$

and hence:

$$\begin{aligned}
&\int_0^\infty e^{-rt}\pi_1(x(t), \gamma_1(x(t)), \bar{\gamma}_2(x(t)))dt \\
\leqslant& \int_0^\infty e^{-rt}\left[r\bar{V}_1(x(t)) - f(x(t), \gamma_1(x(t)), \bar{\gamma}_2(x(t)))\bar{V}_1'(x(t))\right]dt \\
=& \int_0^\infty \left[-\frac{d}{dt}\left[e^{-rt}\right]\bar{V}_1(x(t)) - e^{-rt}\dot{x}(t)\bar{V}_1'(x(t))\right]dt \\
=& \int_0^\infty -\frac{d}{dt}\left[e^{-rt}\bar{V}_1(x(t))\right]dt \\
=& \bar{V}_1(x_0).
\end{aligned}$$

This implies

$$\bar{V}_1(x_0) = \max_{\gamma_1} \int_0^\infty e^{-rt}\pi_1(x(t), \gamma_1(x(t)), \bar{\gamma}_2(x(t)))dt,$$

and similarly we find

$$\bar{V}_2(x_0) = \max_{\gamma_2} \int_0^\infty e^{-rt}\pi_2(x(t), \bar{\gamma}_1(x(t)), \gamma_2(x(t)))dt.$$

□

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