Graphs with constant and

van Dam, E.R.; Haemers, W.H.

Publication date:
1995

Graphs with constant μ and $\bar{\mu}$

Edwin R. van Dam   Willem H. Haemers

Tilburg University, Department of Econometrics,
P.O. Box 90153, 5000 *LE Tilburg, The Netherlands*

Abstract. A graph G has constant $\mu = \mu(G)$ if any two vertices that are not adjacent have μ common neighbours. G has constant μ and $\bar{\mu}$ if G has constant $\mu = \mu(G)$, and its complement $\bar{G}$ has constant $\bar{\mu} = \mu(\bar{G})$. If such a graph is regular, then it is strongly regular, otherwise precisely two vertex degrees occur. We shall prove that a graph has constant μ and $\bar{\mu}$ if and only if it has two distinct restricted Laplace eigenvalues. Bruck-Ryser type conditions are found. Several constructions are given and characterized. A list of feasible parameter sets for graphs with at most 40 vertices is generated.

1. Introduction

We say that a noncomplete graph G has constant $\mu = \mu(G)$ if any two vertices that are not adjacent have μ common neighbours. A graph G has constant μ and $\bar{\mu}$ if G has constant $\mu = \mu(G)$, and its complement $\bar{G}$ has constant $\bar{\mu} = \mu(\bar{G})$. It turns out that only two vertex degrees can occur. Moreover, we shall prove that a graph has constant μ and $\bar{\mu}$ if and only if it has two distinct restricted Laplace eigenvalues. The Laplace eigenvalues of a graph are the eigenvalues of its Laplace matrix. This is a square matrix Q indexed by the vertices, with $Q_{xx} = k_x$, the degree of x, $Q_{xy} = -1$ if x and y are adjacent, and $Q_{xy} = 0$ if x and y are not adjacent. Note that if G has v vertices and Laplace matrix Q, then its complement $\bar{G}$ has Laplace matrix $vI - J - Q$. Since the Laplace matrix has row sums zero, it has an eigenvalue 0 with the all-one vector as eigenvector. The eigenvalues with eigenvectors orthogonal to the all-one vector are called restricted. The restricted multiplicity of an eigenvalue is the dimension of the eigenspace orthogonal to the all-one vector. Note that the graphs with one restricted Laplace eigenvalue are the complete and the empty graphs.

Graphs with constant μ and $\bar{\mu}$ form a common generalization of two known families of graphs. The regular ones are precisely the strongly regular graphs and for $\mu = 1$ we have the (nontrivial) geodetic graphs of diameter two.

Some similarities with so-called neighbourhood-regular or ΓΔ-regular graphs (see [4, 7]) occur. These graphs can be defined as graphs G with constant λ and $\bar{\lambda}$, that is, in G any two adjacent vertices have λ common neighbours, and in $\bar{G}$ any two adjacent vertices have $\bar{\lambda}$ common neighbours. Here also only two vertex degrees can occur, but there is no easy algebraic characterization.

2. Laplace eigenvalues and vertex degrees

In this section we shall derive some basic properties of graphs with constant μ and $\overline{\mu}$. We start with an algebraic characterization.

THEOREM 2.1. *Let G be a graph on v vertices. Then G has constant* μ *and* $\overline{\mu}$ *if and only if G has two distinct restricted Laplace eigenvalues* θ_1 *and* θ_2. *If so then only two vertex degrees* k_1 *and* k_2 *can occur, and* $\theta_1 + \theta_2 = k_1 + k_2 + 1 = \mu + v - \overline{\mu}$ *and* $\theta_1\theta_2 = k_1k_2 + \mu = \mu v$.

Proof. Let G have Laplace matrix Q. Suppose that G has two distinct restricted Laplace eigenvalues θ_1 and θ_2. Then $(Q - \theta_1 I)(Q - \theta_2 I)$ has spectrum $\{[\theta_1\theta_2]^1, [0]^{v-1}\}$ and row sums $\theta_1\theta_2$, so it follows that $(Q - \theta_1 I)(Q - \theta_2 I) = (\theta_1\theta_2/v)J$. If x is not adjacent to y, so $Q_{xy} = 0$ then $Q^2_{xy} = \theta_1\theta_2/v$, and so $\mu = \theta_1\theta_2/v$ is constant. Since the complement of G has distinct restricted Laplace eigenvalues $v - \theta_1$ and $v - \theta_2$, it follows that $\overline{\mu} = (v - \theta_1)(v - \theta_2)/v$ is also constant.

Now suppose that μ and $\overline{\mu}$ are constant. If x and y are adjacent then $(vI - J - Q)^2_{xy} = \overline{\mu}$, so $\overline{\mu} = (v^2I + vJ + Q^2 - 2vJ - 2vQ)_{xy} = Q^2_{xy} + v$, and if x and y are not adjacent, then $Q^2_{xy} = \mu$. Furthermore $Q^2_{xx} = k_x^2 + k_x$. Now

$$
\begin{aligned}
Q^2 &= (\overline{\mu} - v)(\mathrm{diag}(k_x) - Q) + \mu(J - I - \mathrm{diag}(k_x) + Q) + \mathrm{diag}(k_x^2 + k_x) \\
&= (\mu + v - \overline{\mu})Q + \mathrm{diag}(k_x^2 - k_x(\mu + v - \overline{\mu} - 1) - \mu) + \mu J.
\end{aligned}
$$

Since Q and Q^2 have row sums zero, it follows that $k_x^2 - k_x(\mu + v - \overline{\mu} - 1) - \mu + \mu v = 0$ for every vertex x. So $Q^2 - (\mu + v - \overline{\mu})Q + \mu vI = \mu J$. Now let θ_1 and θ_2 be such that $\theta_1 + \theta_2 = \mu + v - \overline{\mu}$ and $\theta_1\theta_2 = \mu v$, then $(Q - \theta_1 I)(Q - \theta_2 I) = (\theta_1\theta_2/v)J$, so G has distinct restricted Laplace eigenvalues θ_1 and θ_2. As a side result we obtained that all vertex degrees k_x satisfy the same quadratic equation, thus k_x can only take two values k_1 and k_2, and the formulas readily follow. □

Note that if the restricted Laplace eigenvalues are not integral, then they have multiplicities $m_1 = m_2 = (v - 1)/2$. If the Laplace eigenvalues are integral, then their multiplicities are not necessarily fixed by v, μ and $\overline{\mu}$. For example, there are graphs on 16 vertices with constant $\mu = 2$ and $\overline{\mu} = 6$ with Laplace spectrum $\{[8]^m, [4]^{15-m}, [0]^1\}$ for $m = 5, 6, 7, 8$ and 9.

The following lemma implies that the numbers of vertices of the respective degrees follow from the Laplace spectrum.

LEMMA 2.2. *Let G be a graph on v vertices with two distinct restricted Laplace eigenvalues θ_1 and θ_2 with restricted multiplicities m_1 and m_2, respectively. Suppose there are n_1 vertices of degree k_1 and n_2 vertices of degree k_2. Then $m_1 + m_2 + 1 = n_1 + n_2 = v$ and $m_1\theta_1 + m_2\theta_2 = n_1k_1 + n_2k_2$.*

Proof. The first equation is trivial, the second follows from the trace of the Laplace matrix. □

The number of common neighbours of two adjacent vertices is in general not constant, but depends on the degrees of the vertices.

LEMMA 2.3. *Let G be a graph with constant μ and $\overline{\mu}$, and vertex degrees k_1 and k_2. Suppose x and y are two adjacent vertices. Then the number of common neighbours λ_{xy} of x and y equals*

$$\lambda_{xy} = \begin{cases} \lambda_{11} = \mu - 1 + k_1 - k_2 & \text{if } x \text{ and } y \text{ both have degree } k_1, \\ \lambda_{12} = \mu - 1 & \text{if } x \text{ and } y \text{ have different degrees}, \\ \lambda_{22} = \mu - 1 + k_2 - k_1 & \text{if } x \text{ and } y \text{ both have degree } k_2. \end{cases}$$

Proof. Suppose x and y have vertex degrees k_x and k_y, respectively. The number of vertices that are not adjacent to both x and y equals $\overline{\mu}$. The number of vertices adjacent to x but not to y equals $k_x - 1 - \lambda_{xy}$, and the number of vertices adjacent to y but not to x equals $k_y - 1 - \lambda_{xy}$. Now we have that $v = 2 + \lambda_{xy} + \overline{\mu} + k_x - 1 - \lambda_{xy} + k_y - 1 - \lambda_{xy}$. Thus $\lambda_{xy} = \overline{\mu} - v + k_x + k_y$. By using that $k_1 + k_2 = \mu + v - \overline{\mu} - 1$, the result follows. □

Both Theorem 2.1 and Lemma 2.3 imply the following.

COROLLARY 2.4. *A graph with constant μ and $\overline{\mu}$ is regular if and only if it is strongly regular.* □

Observe that G is regular if and only if $(\mu + v - \overline{\mu} - 1)^2 = 4\mu(v - 1)$ or $n_1 = 0$ or $n_2 = 0$. Since we can express all parameters in terms of the Laplace spectrum, it follows that it can be recognized from the Laplace spectrum whether a graph is strongly regular or not. This is no surprise, since it is in general true that regularity of a graph follows from its Laplace spectrum.

Before proving the next lemma we first look at the disconnected graphs. Since the number of components of a graph equals the multiplicity of its Laplace eigenvalue 0, a graph with constant μ and $\overline{\mu}$ is disconnected if and only if one of its restricted Laplace eigenvalues equals 0. Consequently this is the case if and only if $\mu = 0$. So in a disconnected graph G with constant μ and $\overline{\mu}$ any two vertices that are not adjacent have no common neighbours. This implies that two vertices that are not adjacent are in a different component of G. So G is the disjoint union of cliques. Since the only two vertex degrees that can occur are $v - \overline{\mu} - 1$ and 0, G is the disjoint union of $(v - \overline{\mu})$-cliques and isolated vertices.

LEMMA 2.5. *Let G be a graph with two restricted Laplace eigenvalues* $\theta_1 > \theta_2$ *and vertex degrees* $k_1 \geq k_2$. *Then* $\theta_1 - 1 \geq k_1 \geq k_2 \geq \theta_2$, *with* $k_2 = \theta_2$ *if and only if G or* $\overline{G}$ *is disconnected.*

Proof. Assume that G is not regular, otherwise G is strongly regular and the result easily follows. First, suppose that the induced graph on the vertices of degree k_1 is not a coclique. So there are two vertices of degree k_1 that are adjacent. Then the 2×2 submatrix of the Laplace matrix Q of G induced by these two vertices has eigenvalues $k_1 \pm 1$, and since these interlace (cf. [5]) the eigenvalues of Q, we have that $k_1 + 1 \leq \theta_1$. Since $k_1 + k_2 + 1 = \theta_1 + \theta_2$, then also $k_2 \geq \theta_2$.

Next, suppose that the induced graph on the vertices of degree k_2 is not a clique. So there are two vertices of degree k_2 that are not adjacent. Now the 2×2 submatrix of Q induced by these two vertices has two eigenvalues k_2, and since these also interlace the eigenvalues of Q, we have that $k_2 \geq \theta_2$, and then also $\theta_1 - 1 \geq k_1$.

The remaining case is that the induced graph on the vertices of degree k_1 is a coclique and the induced graph on the vertices of degree k_2 is a clique. Suppose we have such a graph. Since a vertex of degree k_1 only has neighbours of degree k_2, and $\lambda_{12} = \mu - 1$, we find that $k_1 = \mu$. Since any two vertices of degree k_1 have μ common neighbours, it follows that every vertex of degree k_1 is adjacent with every vertex of degree k_2, and we find that $k_2 \geq k_1$, which is a contradiction. So the remaining case cannot occur, and we have proven the inequalities.

Now suppose that G or $\overline{G}$ is disconnected. Then it follows from the observations before the lemma or looking at the complement that $k_2 = \theta_2$.

On the other hand, suppose that $k_2 = \theta_2$. Then it follows that $k_1 = \theta_1 - 1$ and from the equation $\theta_1\theta_2 = k_1k_2 + \mu$ it then follows that $k_2 = \mu$. Now take a vertex x_2 of degree k_2 that is adjacent with a vertex x_1 of degree k_1. If there are no such vertices then G is disconnected and we are done. It follows that every vertex that is not adjacent with x_2, is adjacent with all neighbours of x_2, so also with x_1. Since x_1 and x_2 have $\mu - 1$ common neighbours, x_1 is also adjacent with all neighbours of x_2. So x_1 is adjacent with all other vertices, and so $\overline{G}$ is disconnected. □

We conclude this section with so-called Bruck-Ryser conditions.

LEMMA 2.6. *Let G be a graph with constant* μ *and* $\overline{\mu}$ *on v vertices, with v odd, and with restricted Laplace eigenvalues* θ_1 *and* θ_2. *Then the Diophantine equation*

$$x^2 = (\theta_1 - \theta_2)^2 y^2 + (-1)^{(v-1)/2}\mu z^2$$

has a nontrivial integral solution (x, y, z).

Proof. Let Q be the Laplace matrix of G, then

$$(Q - \tfrac{1}{2}(\theta_1 + \theta_2)I)(Q - \tfrac{1}{2}(\theta_1 + \theta_2)I)^T = Q^2 - (\theta_1 + \theta_2)Q + \tfrac{1}{4}(\theta_1 + \theta_2)^2 I =$$

$$\mu J + (\tfrac{1}{4}((\theta_1 + \theta_2)^2 - \theta_1\theta_2)I = \tfrac{1}{4}(\theta_1 - \theta_2)^2 I + \mu J.$$

Since $Q - \tfrac{1}{2}(\theta_1 + \theta_2)I$ is a rational matrix, it follows from a lemma by Bruck and Ryser (cf. [1]) that the Diophantine equation

$$x^2 = \tfrac{1}{4}(\theta_1 - \theta_2)^2 y^2 + (-1)^{(v-1)/2}\mu z^2$$

has a nontrivial integral solution, which is equivalent to stating that the Diophantine equation above has a nontrivial integral solution. □

3. Cocliques

If $k_1 - k_2 > \mu - 1$, then the induced graph on the set of vertices of degree k_2 is a coclique, since two adjacent vertices of degree k_2 would have a negative number of common neighbours. It turns out (see the table in Section 8) that this is the case in many examples. Therefore we shall have a closer look at cocliques. If G is a graph, then we denote by $\alpha(G)$ the maximal size of a coclique in G.

LEMMA 3.1. *Let G be a graph on v vertices with largest Laplace eigenvalue θ_1 and smallest vertex degree k_2. Then* $\alpha(G) \le v(\theta_1 - k_2)/\theta_1$.

Proof. Let C be a coclique of size $\alpha(G)$. Partition the vertices of G into C and the set of vertices not in C, and partition the Laplace matrix Q of G according to this partition of the vertices. Let B be the matrix of average row sums of the blocks of Q, then

$$B = \begin{pmatrix} k & -k \\ -k\dfrac{\alpha(G)}{v-\alpha(G)} & k\dfrac{\alpha(G)}{v-\alpha(G)} \end{pmatrix},$$

where k is the average degree of the vertices in C. Since B has eigenvalues 0 and $kv/(v - \alpha(G))$, and since these interlace the eigenvalues of Q (cf. [5]), we have that $kv/(v - \alpha(G)) \le \theta_1$. The result now follows from the fact that $k_2 \le k$. □

As remarked before, if G is a graph with constant μ and $\bar{\mu}$ with $\lambda_{22} < 0$, then the vertices of degree k_2 form a coclique, and so

$$n_2 \le v(\theta_1 - k_2)/\theta_1.$$

If the bound is tight, then it follows from tight interlacing that the partition of the vertices is regular, that is, every block in the partitioned matrix has constant row sums. If

$$Q = \begin{pmatrix} k_2 I & -N \\ -N^T & Q_1 \end{pmatrix},$$

then it follows that N is the incidence matrix of a 2-(n_2, κ, μ) design, where $\kappa = n_2k_2/n_1$. Furthermore, it follows from a lemma by Haemers and Higman [6] that if G has Laplace spectrum $\{[\theta_1]^{m_1}, [\theta_2]^{m_2}, [0]^1\}$, then the adjacency matrix of the induced graph G_1 on the vertices of degree k_1 has spectrum

$$\{[k_1 - \kappa]^1,\ [k_1 - \theta_2]^{m_2+1-n_2},\ [-1]^{n_2-1},\ [k_1 - \theta_1]^{m_1-n_2}\},$$

so G_1 is a regular graph with at most four eigenvalues. It follows from the multiplicities that θ_1 and θ_2 must be integral.

In this way it can be proven that there is no graph on 25 vertices with constant $\mu = 2$ and $\bar{\mu} = 12$, with 10 vertices of degree 6. These 10 vertices induce a coclique for which the bound is tight. The induced graph on the remaining 15 vertices has spectrum $\{[4]^1, [3]^3, [-1]^9, [-2]^2\}$, but such a graph cannot exist (cf. [3]).

Examples for which the bound is tight are obtained by taking an affine plane for the design and a disjoint union of cliques for G_1. This is family *b* of Section 4. Another example is constructed from a polarity with $q\sqrt{q} + 1$ absolute points in $PG(2, q)$ where q is a square prime power (cf. Section 5).

Another bound is given by the multiplicities of the eigenvalues.

LEMMA 3.2. *Let G be a connected graph with Laplace spectrum* $\{[\theta_1]^{m_1}, [\theta_2]^{m_2}, [0]^1\}$, *where* $\theta_1 > \theta_2 > 0$, *such that* $\bar{G}$ *is also connected. Then* $\alpha(G) \leq \min\{m_1, m_2 + 1\}$.

Proof. Suppose C is a coclique with size greater than m_1. Consider the submatrix of the Laplace matrix Q induced by the vertices of C. This matrix only has eigenvalues k_1 and k_2, and since these interlace the eigenvalues of Q, we find that $k_2 \leq \theta_2$. This is in contradiction with Lemma 2.5, since G and $\bar{G}$ are connected. If C is a coclique of size greater than $m_2 + 1$, we find by interlacing that $k_1 \geq \theta_1$, which is again a contradiction. □

In Section 6 we find a large family of graphs for which this bound is tight.

Also if $\lambda_{22} = 0$, we find a bound on the number of vertices n_2 of degree k_2.

LEMMA 3.3. *If* $k_1 - k_2 \geq \mu - 1$, *then* $n_2 \leq v - \mu$.

Proof. Fix a vertex x_1 of degree k_1. If x_1 has no neighbours of degree k_2 then $n_1 \geq k_1 + 1 \geq \mu + k_2 \geq \mu$, and so $n_2 \leq v - \mu$. If x_1 has a neighbour x_2 of degree k_2, then x_1 and x_2 cannot have a common neighbour y_2 of degree k_2, since otherwise x_2 and y_2 have a common neighbour x_1, so that $0 \geq \mu - 1 + k_2 - k_1 = \lambda_{22} > 0$, which is a contradiction. So all common neighbours of x_1 and x_2 have degree k_1, so $n_1 \geq \lambda_{12} + 1 = \mu$, and so $n_2 \leq v - \mu$. □

4. Geodetic graphs of diameter two

A geodetic graph is a graph in which any two vertices are connected by a unique shortest path. Thus a geodetic graph of diameter two is a graph with constant $\mu = 1$. It is proven (see [2, Thm. 1.17.1]) that if G is a geodetic graph of diameter two, then either

(i) G contains a vertex adjacent to all other vertices, or
(ii) G is strongly regular, or
(iii) precisely two vertex degrees $k_1 > k_2$ occur. If X_1 and X_2 denote the sets of vertices with degrees k_1 and k_2, respectively, then X_2 induces a coclique, maximal cliques meeting both X_1 and X_2 have size two, and maximal cliques contained in X_1 have size $k_1 - k_2 + 2$. Moreover, $v = k_1k_2 + 1$.

If G is of type (i), then G need not have constant $\bar{\mu}$. Note that its complement is disconnected, so see Section 4. If G is of type (ii) or (iii), then it has constant $\bar{\mu}$. If G is of type (ii) then it is clear. Now suppose that G is of type (iii). Since $\mu = 1$, every edge in in a unique maximal clique. Let x and y be two adjacent vertices, then x and y cannot both be in X_2. If one is in X_1, and the other in X_2, then they have no common neighbour, since maximal cliques meeting both X_1 and X_2 have size 2. So $\lambda_{12} = 0$ and then $\bar{\mu}_{12} = v - k_1 - k_2$. If both x and y are in X_1, then by the previous argument they have no common neighbours in X_2, and since every maximal clique contained in X_1 has size $k_1 - k_2 + 2$, they have $k_1 - k_2$ common neighbours in X_1. So $\lambda_{11} = k_1 - k_2$, and then also $\bar{\mu}_{11} = v - k_1 - k_2$. So G has constant $\bar{\mu}$.

The following four families of graphs are all known examples of type (iii).

a. Take a clique and a coclique of size k_1, and an extra vertex. Join the vertices of the clique and the coclique by a matching, and join the extra vertex to every vertex of the coclique (see also Section 6).

b. Take an affine plane. Take as vertices the points and lines of the plane. A point is adjacent to a line if it is on the line, and two lines are adjacent if they are parallel.

c. Take the previous example and add the parallel classes to the vertices. Join each line to the parallel class it is in, and join all parallel classes mutually.

d. Take a projective plane with a polarity σ. Take as vertices the points of the plane, and join two points x and y if x is on the line y^σ (cf. Section 5).

5. Symmetric designs with a polarity

Let D be a symmetric design. A polarity of D is a one-one correspondence σ between its points and blocks such that for any point p and any block b we have that $p \in b$ if and only if $b^\sigma \in p^\sigma$. A point is called absolute (with respect to σ) if $p \in p^\sigma$. Now D has a polarity if and only if it has a symmetric incidence matrix A. The number of absolute points is the number of ones on the diagonal of A.

Suppose that D is a symmetric 2-(v, k, λ) design with a polarity σ. Let $G = P(D)$ be the graph on the points of D, where two distinct points x and y are adjacent if $x \in y^\sigma$. Then the only vertex degrees that can occur are k and $k - 1$. The number of vertices with degree $k - 1$ is the number of absolute points of σ. Let A be the corresponding symmetric incidence matrix, then $Q = kI - A$ is the Laplace matrix of G. Since A is a symmetric incidence matrix of D, we find that $(kI - Q)^2 = A^2 = AA^T = (k - \lambda)I + \lambda J$, so $Q^2 - 2kQ + (k^2 - k + \lambda)I = \lambda J$. Thus Q has two distinct restricted eigenvalues $k \pm \sqrt{(k - \lambda)}$. The converse is also true.

THEOREM 5.1. *Let G be a graph with constant μ and $\bar{\mu}$ on v vertices, with vertex degrees k and $k - 1$. Then G comes from a symmetric 2-(v, k, λ) design with a polarity.*

Proof. Let G have restricted Laplace eigenvalues θ_1 and θ_2, then $\theta_1 + \theta_2 = 2k$ and $\theta_1\theta_2 = vk(k - 1)/(v - 1)$. Define $\lambda = k(k - 1)/(v - 1)$, then $Q^2 - 2kQ + v\lambda I = \lambda J$. Now let $A = kI - Q$, then A is a symmetric (0, 1)-matrix with row sums k, and $AA^T = A^2 = k^2I - 2kQ + Q^2 = (k^2 - v\lambda)I + \lambda J = (k - \lambda)I + \lambda J$, so A is the incidence matrix of a symmetric 2-(v, k, λ) design with a polarity. □

Since the polarities in the unique 2-(7, 3, 1), 2-(11, 5, 2) and 2-(13, 4, 1) designs are unique, the graphs we obtain from these designs are also uniquely determined by their parameters.

In a projective plane of order n, where n is not a square, any polarity has $n + 1$ absolute points. If n is a square, then the number of absolute points in a polarity lies between $n + 1$ and $n\sqrt{n} + 1$. $PG(2, q)$ admits a polarity with $q + 1$ absolute points for every prime power q and a polarity with $q\sqrt{q} + 1$ absolute points for every square prime power (cf. [1, § VIII.9]).

By Paley's construction of Hadamard matrices (cf. [1, Thm. I.9.11]) we obtain symmetric 2-$(2^e(q + 1) - 1, 2^{e-1}(q + 1) - 1, 2^{e-2}(q + 1) - 1)$ designs with a polarity with $2^{e-1}(q + 1) - 1$ absolute points, for every odd prime power q and every $e > 0$.

Furthermore, we found polarities with 0, 4, 8, 12 and 16 absolute points in a 2-(16, 6, 2) design, a polarity in the 2-(37, 9, 2) design from the difference set (cf. [1, Ex. VI.4.3]) and a polarity with 16 absolute points in the 2-(40, 13, 4) design $PG_2(3, 3)$. Spence [personal communication] found polarities with 3, 7, 11 and 15 absolute points in 2-(15, 7, 3) designs, polarities in 2-(25, 9, 3) and 2-(30, 13, 3) designs, polarities with 5, 11, 17, 23 and 29 absolute points in 2-(35, 17, 8) designs, polarities with 0, 6, 12, 18, 24, 30 and 36 absolute points in 2-(36, 15, 6) designs and polarities with 10, 16, 22, 28 and 34 absolute points in 2-(40, 13, 4) designs.

6. Other graphs from symmetric designs.

Let D be a symmetric 2-(w, k, λ) design. Fix a point x. We shall construct a graph $G = G(D)$ that has constant μ and $\bar{\mu}$. The vertices of G are the points and the blocks of D, except for the point x. Between the points there are no edges. A point y and a block b will be adjacent if and only if precisely one of x and y is incident with b. Two blocks will be adjacent if and only if both blocks are incident with x or both blocks are not incident with x. It is not hard to show that the resulting graph G has constant $\mu = k - \lambda$ and constant $\bar{\mu} = w - k - 1 + \lambda$. In G the $n_1 = w$ blocks have degrees $k_1 = w - 1$, and the $n_2 = w - 1$ points have degrees $k_2 = 2(k - \lambda)$. Note that D and the complement of D give rise to the same graph G. We have the following characterization of $G(D)$.

THEOREM 6.1. *Let G be a graph with constant μ and $\bar{\mu}$ on $2w - 1$ vertices, such that both G and $\bar{G}$ are connected. Suppose G has w vertices of degree k_1, and $w - 1$ vertices of degree k_2, and suppose that the vertices of degree k_2 induce a coclique. Then $k_1 = w - 1$, $k_2 = 2\mu$, and $G = G(D)$, where D is a symmetric 2-$(w, k, k - \mu)$ design.*

Proof. Let

$$A = \begin{pmatrix} A_1 & N^T \\ N & O \end{pmatrix}$$

be the adjacency matrix of G, where the partition is induced by the degrees of the vertices.

Now N cannot have constant column sums. Since N has constant row sums k_2, it follows that N has average column sum $k_2(w - 1)/w$, so if N would have constant column sums, then it would follow that $k_2 = 0$ or w, but then G or $\bar{G}$ is disconnected, which is a contradiction.

Two vertices of degrees k_2 have μ common neighbours, so $NN^T = k_2 I + \mu(J - I)$. A vertex of degree k_2 and a vertex of degree k_1 have $\mu - 1$ or μ common neighbours, depending on whether they are adjacent or not, so $NA_1 = \mu J - N$.

Let $\{v_i : i = 1,..., w - 1\}$ be an orthonormal set of eigenvectors of NN^T, with v_1 the constant vector, then $NN^T v_i = (k_2 - \mu)v_i$, $i = 2,..., w - 1$. Now

$$A_1(N^T v_i) = (NA_1)^T v_i = (\mu J - N)^T v_i = -N^T v_i,\ i = 2,..., w - 1.$$

Since $k_2 > \mu$ (otherwise G or $\bar{G}$ is disconnected), it follows that A_1 has -1 as an eigenvalue with multiplicity at least $w - 2$. Let $\lambda_1 \geq \lambda_2$ be the other eigenvalues of A_1. Suppose that $\lambda_2 \leq -1$, then $\lambda_1 = w - 2 - \lambda_2 \geq w - 1$, so $\lambda_1 = w - 1$, and $A_1 = J - I$. But then $\bar{G}$ has a coclique of size w, contradicting Lemma 3.2. Now $A_1 + I$ is positive semidefinite of rank two with diagonal 1, and so it is the Gram matrix of a set of vectors of length 1 in $\mathbb{R}^2$, with mutual inner products 0 or 1. It follows that there can only be two distinct vectors, and A_1 is the adjacency matrix of a disjoint union of two cliques, say of sizes k and $w - k$ ($k \neq w - k$).

Let $N = (N_1\ N_2)$ be partitioned according to the partition of A_1 into two cliques, where N_1 has k columns and N_2 has $w - k$ columns. From the equation $NA_1 = \mu J - N$ we derive that $N_1 J = N_2 J = \mu J$, so both N_1 as N_2 have row sums μ, and so N has row sums $k_2 = 2\mu$. Since $k_1 k_2 = \mu(v - 1)$, it then follows that $k_1 = w - 1$.

Now let

$$M = \begin{pmatrix} j^T & 0^T \\ J - N_1 & N_2 \end{pmatrix},$$

then M is square of size w, with row sums k. Furthermore, we find that $(J - N_1)(J - N_1)^T + N_2 N_2^T = (k - 2\mu)J + NN^T = (k - 2\mu)J + (k_2 - \mu)I + \mu J = \mu I + (k - \mu)J$, and so we have that $MM^T = \mu I + (k - \mu)J$, so M is the incidence matrix of a symmetric 2-(w, k, $k - \mu$) design D, and $G = G(D)$. □

The matrix N that appears in the proof above is the incidence matrix of a structure, that is called a pseudo design by Marrero and Butson [8] and a 'near-square' λ-linked design by Woodall [9]. An alternative proof of Theorem 6.1 uses Theorem 3.4 of [8] that states that a pseudo ($w \neq 4\mu$, $k_2 = 2\mu$, μ)-design comes from a symmetric design in the way described above. The problem then is to prove the case $w = 4\mu$.

For every orbit of the action of the automorphism group of the design D on its points, we get a different graph $G(D)$ by taking the fixed point x from that orbit. Since the trivial 2-($k_1 + 1$, 1, 0) (here we get family *a* of geodetic graphs given in Section 5), the 2-(7, 3, 1), the 2-(11, 5, 2) and the 2-(13, 4, 1) designs are unique and have an automorphism group that acts transitively on the points, the graphs we obtain are uniquely determined by their parameters. According to Spence [personal communication], the five 2-(15, 7, 3) designs have respectively 1, 2, 3, 2 and 2 orbits, the three 2-(16, 6, 2) designs all have a transitive automorphism group, and the six 2-(19, 9, 4) designs have respectively 7, 5, 3, 3, 3 and 1 orbits. Thus we get precisely ten graphs on 29 vertices with constant $\mu = 4$ and $\bar{\mu} = 10$, three graphs on 31 vertices with constant $\mu = 4$ and $\bar{\mu} = 11$, and 22 graphs on 37 vertices with constant $\mu = 5$ and $\bar{\mu} = 13$.

7. Switching in strongly regular graphs

Let G be a strongly regular graph with parameters ($v = 2k + 1$, k, λ, μ^*). Fix a vertex x and "switch" between the set of neighbours of x and the set of vertices (distinct from x) that are not neighbours of x, that is, a vertex that is adjacent with x and a vertex that is not adjacent with x are adjacent if and only if they are not adjacent in G. All other adjacencies remain the same. If the (ordinary) eigenvalues of G are k, r and s, then we obtain a graph with restricted Laplace eigenvalues $2(\lambda + 1) - s$ and $2(\lambda + 1) - r$. The graph has constant $\mu = k - \mu^* = \lambda + 1$ and $\overline{\mu} = \mu^*$, and there is one vertex of degree k and $2k$ vertices of degree $2(\lambda + 1)$. Almost all examples have $k = 2(\lambda + 1) = 2\mu^*$, so that we get a (strongly) regular graph.

The only known (to us) examples for which $k \neq 2(\lambda + 1)$ are the triangular graph $T(7)$ and its complement. (Note that from one pair of complementary graphs we get another pair of complementary graphs.) $T(7)$ is the strongly regular graph on the unordered pairs $\{i, j\}$, $i, j = 1,..., 7$, $i \neq j$, where two distinct pairs are adjacent if they intersect.

From the complement of $T(7)$ we get a graph with constant $\mu = 4$ and $\overline{\mu} = 6$ on 21 vertices with one vertex of degree 10 and 20 vertices of degree 8. The subgraph induced by the neighbours of the vertex x of degree 10 is the Petersen graph (the complement of $T(5)$).

This construction can be reversed, that is, if G is a graph on v vertices with constant μ and $\overline{\mu}$, such that there is one vertex of degree $k = (v - 1)/2$ and $2k$ vertices of degree 2μ, then it must be constructed from a strongly regular graph in the above way. Since $T(7)$ is uniquely determined by its parameters, and it has a transitive automorphism group it follows that there is precisely one graph with constant $\mu = 4$ and $\overline{\mu} = 6$ on 21 vertices with one vertex of degree 10 and 20 vertices of degree 8.

Next, let G be a strongly regular graph with parameters ($v^* = 2k + 1$, k, λ, μ^*) with a regular partition into two parts, where one part has k_2 vertices and the induced graph is regular of degree $k_2 - \mu^* - 1$, and the other part has $v^* - k_2$ vertices and the induced graph is regular of degree $k - \mu^*$. (Then $k_2(k - k_2 + \mu^* + 1) = (v^* - k_2)\mu^*$.) Add an isolated vertex to the second part and then switch with respect to this partition, that is, two vertices from different parts will be adjacent if and only if they are not adjacent in G, and two vertices from the same part will be adjacent if and only if they also are adjacent in G.

The obtained graph has one vertex of degree k_2 and v^* vertices of degree $k_1 = k_2 + k - 2\mu^*$. If the (ordinary) eigenvalues of G are k, r and s, then we obtain a graph with restricted Laplace eigenvalues $k_1 - s$ and $k_1 - r$, and it has constant $\mu = k_2 - \mu^*$ and $\overline{\mu} = k + 1 - k_2 + \mu$. Again, we obtain a (strongly) regular graph if $k = 2\mu^*$.

Also here the construction can be reversed. A graph on v vertices with constant μ and $\overline{\mu}$, such that $\mu + \overline{\mu} = v/2$ and there is one vertex of degree k_2 must be constructed from a strongly regular graph in the above way.

If we take $T(7)$ and take for one part of the partition a 7-cycle or the disjoint union of a 3-cycle and a 4-cycle, then we find that there are precisely two nonisomorphic graphs on 22 vertices with constant $\mu = 3$ and $\overline{\mu} = 8$, with 21 vertices of degree 9 and one vertex of degree 7.

In $T(7)$ there cannot be a regular partition with $k_2 = 12$ (which is the other value satisfying the quadratic equation) since this would give a graph which is the complement of a graph with $\lambda_{22} = 0$ and $n_1 < \mu$, contradicting Lemma 3.3.

8. Feasible parameter sets

By computer we generated all feasible parameter sets for graphs on v vertices with constant μ and $\bar{\mu}$, having restricted Laplace eigenvalues $\theta_1 > \theta_2$ and vertex degrees $k_1 > k_2$, for $v \le 40$, satisfying $0 < \mu \le \bar{\mu}$. If $\lambda_{22} < 0$, then also the condition $n_2 \le v(\theta_1 - k_2)/\theta_1$ is satisfied. By # we denote the number of (nonregular) graphs. By `Bruck-Ryser(`*p*`)` we denote that the Bruck-Ryser condition is not satisfied modulo p.

v	μ	$\bar{\mu}$	θ_1	θ_2	k_1	k_2	n_1	n_2	λ_{22}	#	Notes	Section
7	1	2	4.4142	1.5858	3	2	4	3	-1	1	*G*(4,1,0), *P*(7,3,1)	4.*a*,*d* , 5, 6
9	1	3	5.3028	1.6972	4	2	5	4	-2	1	*G*(5,1,0)	4.*a*, 6
11	1	4	6.2361	1.7639	5	2	6	5	-3	1	*G*(6,1,0)	4.*a*, 6
11	2	3	6.7321	3.2679	5	4	6	5	0	1	*P*(11,5,2)	5
13	1	5	7.1926	1.8074	6	2	7	6	-4	1	*G*(7,1,0)	4.*a*, 6
13	1	6	5.7321	2.2679	4	3	9	4	-1	1	*P*(13,4,1)	4.*c*,*d*, 5
13	2	4	7.5616	3.4384	6	4	7	6	-1	1	*G*(7,3,1)	6
15	1	6	8.1623	1.8377	7	2	8	7	-5	1	*G*(8,1,0)	4.*a*, 6
15	2	5	8.4495	3.5505	7	4	8	7	-2	0	*G*(*D*)	6
15	3	4	9	5	7	6			1	≥ 3	*P*(15,7,3)	5
16	2	6	8	4	6	5			0	≥ 3	*P*(16,6,2)	5
17	1	7	9.1401	1.8599	8	2	9	8	-6	1	*G*(9,1,0)	4.*a*, 6
17	2	6	9.3723	3.6277	8	4	9	8	-3	0	Bruck-Ryser(3), *G*(*D*)	2, 6
17	3	5	9.7913	5.2087	8	6	9	8	0	0	Bruck-Ryser(7)	2
19	1	8	10.1231	1.8769	9	2	10	9	-7	1	*G*(10,1,0)	4.*a*, 6
19	1	10	7.4495	2.5505	6	3	11	8	-3	0	Bruck-Ryser(3)	2, 4
19	2	7	10.3166	3.6834	9	4	10	9	-4	0	*G*(*D*)	6
19	4	5	11.2361	6.7639	9	8	10	9	2	≥ 1	*P*(19,9,4)	5
21	1	9	11.1098	1.8902	10	2	11	10	-8	1	*G*(11,1,0)	4.*a*, 6
21	1	12	7	3	5	4			-1	≥ 2	*P*(21,5,1)	4.*b*,*d*, 5
21	2	8	11.2749	3.7251	10	4	11	10	-5	0	Bruck-Ryser(3), *G*(*D*)	2, 6
21	3	7	11.5414	5.4586	10	6	11	10	-2	1	*G*(11,5,2)	6
21	4	6	12	7	10	8			1	≥ 1	switched *T*(7)	7
22	3	8	11	6	9	7			0	≥ 2	switched *T*(7)	7
23	1	10	12.0990	1.9010	11	2	12	11	-9	1	*G*(12,1,0)	4.*a*, 6
23	2	9	12.2426	3.7574	11	4	12	11	-6	0	*G*(*D*)	6
23	3	8	12.4641	5.5359	11	6	12	11	-3	0	*G*(*D*)	6
23	4	7	12.8284	7.1716	11	8	12	11	0			
23	5	6	13.4495	8.5505	11	10	12	11	3	≥ 1	*P*(23,11,5)	5
25	1	11	13.0902	1.9098	12	2	13	12	-10	1	*G*(13,1,0)	4.*a*, 6
25	1	15	7.7913	3.2087	6	4	16	9	-2	1		4.*c*
25	2	10	13.2170	3.7830	12	4	13	12	-7	0	*G*(*D*)	6
25	2	12	10	5	8	6			-1			
25	3	9	13.4051	5.5949	12	6	13	12	-4	1	*G*(13,4,1)	6
25	3	10	11.4495	6.5505	9	8	16	9	1	≥ 1	*P*(25,9,3)	5
25	5	7	14.1926	8.8074	12	10	13	12	2			
27	1	12	14.0828	1.9172	13	2	14	13	-11	1	*G*(14,1,0)	4.*a*, 6
27	2	11	14.1962	3.8038	13	4	14	13	-8	0	*G*(*D*)	6
27	3	10	14.3589	5.6411	13	6	14	13	-5	0	*G*(*D*)	6
27	5	8	15	9	13	10			1			
27	6	7	15.6458	10.3542	13	12	14	13	4	≥ 1	*P*(27,13,6)	5
28	4	10	14	8	12	9			0			
29	1	13	15.0765	1.9235	14	2	15	14	-12	1	*G*(15,1,0)	4.*a*, 6
29	2	12	15.1789	3.8211	14	4	15	14	-9	0	Bruck-Ryser(3), *G*(*D*)	2, 6
29	2	15	10.4495	5.5505	8	7	21	8	0	0	Bruck-Ryser(3), *P*(*D*)	2, 5
29	3	11	15.3218	5.6782	14	6	15	14	-6	0	Bruck-Ryser(31), *G*(*D*)	3, 6
29	4	10	15.5311	7.4689	14	8	15	14	-3	10	*G*(15,7,3)	6
29	5	9	15.8541	9.1459	14	10	15	14	0			
29	6	8	16.3723	10.6277	14	12	15	14	3	0	Bruck-Ryser(11)	2
31	1	14	16.0711	1.9289	15	2	16	15	-13	1	*G*(16,1,0)	4.*a*, 6
31	1	20	8.2361	3.7639	6	5	25	6	-1	≥ 1	*P*(31,6,1)	4.*d*, 5
31	2	13	16.1644	3.8356	15	4	16	15	-10	0	*G*(*D*)	6
31	3	12	16.2915	5.7085	15	6	16	15	-7	0	*G*(*D*)	6
31	3	14	12.6458	7.3542	10	9	21	10	1	≥ 1	*P*(31,10,3)	5
31	4	11	16.4721	7.5279	15	8	16	15	-4	3	*G*(16,6,2)	6
31	6	9	17.1623	10.8377	15	12	16	15	2			
31	7	8	17.8284	12.1716	15	14	16	15	5	≥ 1	*P*(31,15,7)	5

v	μ	$\bar{\mu}$	θ_1	θ_2	k_1	k_2	n_1	n_2	λ_{22}	#	Notes	Section
33	1	15	17.0664	1.9336	16	2	17	16	-14	1	*G*(17,1,0)	4.*a*, 6
33	1	21	9.5414	3.4586	8	4	19	14	-4	0		4
33	2	14	17.1521	3.8479	16	4	17	16	-11	0	Bruck-Ryser(3), *G*(*D*)	2, 6
33	3	13	17.2663	5.7337	16	6	17	16	-8	0	Bruck-Ryser(7), *G*(*D*)	2, 6
33	4	12	17.4244	7.5756	16	8	17	16	-5	0	*G*(*D*)	6
33	6	10	18	11	16	12			1			
33	7	9	18.5414	12.4586	16	14	17	16	4			
34	5	12	17	10	15	11			0			
35	1	16	18.0623	1.9377	17	2	18	17	-15	1	*G*(18,1,0)	4.*a*, 6
35	2	15	18.1414	3.8586	17	4	18	17	-12	0	*G*(*D*)	6
35	3	14	18.2450	5.7550	17	6	18	17	-9	0	*G*(*D*)	6
35	4	13	18.3852	7.6148	17	8	18	17	-6	0	*G*(*D*)	6
35	6	11	18.8730	11.1270	17	12	18	17	0			
35	7	10	19.3166	12.6834	17	14	18	17	3			
35	8	9	20	14	17	16			6	≥ 5	*P*(35,17,8)	5
36	1	24	9	4	7	5			-2	1		3, 4.*b*
36	2	20	12	6	10	7			-2			
36	4	15	16	9	14	10			-1			
36	6	12	18	12	15	14			4	≥ 5	*P*(36,15,6)	5
37	1	17	19.0586	1.9414	18	2	19	18	-16	1	*G*(19,1,0)	4.*a*, 6
37	2	16	19.1322	3.8678	18	4	19	18	-13	0	*G*(*D*)	6
37	2	20	13.5311	5.4689	12	6	20	17	-5	0	Bruck-Ryser(5)	2
37	2	21	11.6458	6.3542	9	8	28	9	0	≥ 1	*P*(37,9,2)	5
37	3	15	19.2268	5.7732	18	6	19	18	-10	0	*G*(*D*)	6
37	4	14	19.3523	7.6477	18	8	19	18	-7	0	*G*(*D*)	6
37	5	13	19.5249	9.4751	18	10	19	18	-4	22	*G*(19,9,4)	6
37	5	14	17.3166	10.6834	15	12	20	17	1			
37	7	11	20.1401	12.8599	18	14	19	18	2			
37	8	10	20.7016	14.2984	18	16	19	18	5			
39	1	18	20.0554	1.9446	19	2	20	19	-17	1	*G*(20,1,0)	4.*a*, 6
39	2	17	20.1240	3.8760	19	4	20	19	-14	0	*G*(*D*)	6
39	3	16	20.2111	5.7889	19	6	20	19	-11	0	*G*(*D*)	6
39	4	15	20.3246	7.6754	19	8	20	19	-8	0	*G*(*D*)	6
39	5	14	20.4772	9.5228	19	10	20	19	-5	0	*G*(*D*)	6
39	7	12	21	13	19	14			1			
39	8	11	21.4641	14.5359	19	16	20	19	4			
39	9	10	22.1623	15.8377	19	18	20	19	7	≥ 1	*P*(39,19,9)	5
40	3	20	15	8	13	9			-2			
40	4	18	16	10	13	12			2	≥ 5	*P*(40,13,4)	5
40	6	14	20	12	18	13			0			

References

1. Th. Beth, D. Jungnickel and H. Lenz, *Design theory*, Wissenschaftsverlag, Mannheim, 1985.
2. A.E. Brouwer, A.M. Cohen and A. Neumaier, *Distance-regular graphs*, Ergebnisse der Mathematik 3.18, Springer, Heidelberg, 1989.
3. E.R. van Dam, Regular graphs with four eigenvalues, *Linear Algebra Appl.* (to appear).
4. C.D. Godsil and B.D. McKay, Graphs with regular neighbourhoods, pp. 127-140 in: *Combinatorial Mathematics VII* (Proc. Australian Comb. Conference, Newcastle, 1979), Springer, Berlin, 1980.
5. W.H. Haemers, Interlacing eigenvalues and graphs, *Linear Algebra Appl.* (to appear).
6. W.H. Haemers and D.G. Higman, Strongly regular graphs with strongly regular decomposition, *Linear Algebra Appl.* **114/115** (1989), 379-398.
7. T. Kloks, An infinite sequence of ΓΔ-regular graphs, Proceedings of the Oberwolfach meeting "Kombinatorik" (1986), *Discrete Math.* **73** (1989), no. 1-2, 127-132.
8. O. Marrero and A.T. Butson, Modular Hadamard matrices and related designs, *J. Comb. Th.* (A) **15** (1973), 257-269.

9. D.R. Woodall, Square λ-linked designs, *Proc. London Math. Soc.* (3) **20** (1970), 669-687.