Minimum cost spanning extension problems

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Publication date:
1994

Minimum cost spanning extension problems : the proportional rule and the decentralized rule

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October 27, 1994

Abstract

Minimum cost spanning extension problems are generalizations of minimum cost spanning tree problems (see Bird 1976) where an existing network has to be extended to connect users to a source. In this paper, we present two cost allocation rules for these problems, viz. the proportional rule and the decentralized rule. We introduce algorithms that generate these rules and prove that both rules are refinements of the irreducible core, as defined in Feltkamp, Tijs and Muto (1994b). We then proceed to axiomatically characterize the proportional rule.
Key words : TU-games, cost allocation rules, minimum cost spanning trees.

1 Introduction

Consider a group of villages, each of which needs to be connected directly or via other villages to a source. Such a connection needs costly links. Each village could connect itself directly to the source, but by cooperating, the linking costs might be reduced. Suppose some of the links are already present and can freely be used by the villages. This situation gives rise to two problems, an Operations Research problem of finding the minimum cost extension that together with the original network connects every village

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[2]This author wishes to acknowledge the Canon Foundation in Europe Visiting Research Fellowship which made his visit to CentER possible.

to the source, and a cost allocation problem of allocating the cost of a minimal cost extension to the villages in a reasonable way.

The special case of the cost minimization problem where no network is initially present is an old problem in Operations Research, and Borůvka (1926) provided algorithms to construct a tree connecting every village to the source with minimal total cost. Later, Kruskal (1956), Prim (1957) and Dijkstra (1959) found similar algorithms. A historic overview of this minimization problem can be found in Graham and Hell (1985).

Claus and Kleitman (1973) introduced the cost allocation problem for the special case of *minimum cost spanning tree problems*, in which no network is initially present, whereupon Bird (1976) treated this problem with game-theoretic methods and for each minimum cost spanning tree proposed a cost allocation associated with it.

In this paper we treat the Operations Research problem and the cost allocation problem simultaneously. One reason is that they are two sides of the same problem, and solving one side gives insight into the other side. For example, examining Bird's tree allocation rule for minimum cost spanning tree problems, one sees that it is intimately linked to the algorithm for finding minimum cost spanning trees that is described in Prim (1957) and Dijkstra (1959) (cf Feltkamp, Tijs and Muto (1994a)). This suggested allocation rules that correspond to the other algorithms for finding minimum cost spanning trees, viz. the algorithm of Kruskal (1956), and the decentralized algorithm that was first described in Borůvka (1926). Furthermore, when axiomatizing cost allocation solutions that associate a cost allocation to each minimal extension, knowing which extension a particular allocation is associated with is useful. Hence, in our approach, a solution to a minimum cost spanning extension (mcse) problem specifies a set of extensions, and for each extension, associated allocations.

Moreover, we extend the class of problems to include problems where a network is initially present. This is motivated by the consideration that an mcse problem that is half solved can now be reconsidered as an mcse problem, after which the solution given for the original problem and the continuation problem can be compared.

In Feltkamp, Tijs and Muto (1994b), we also address the minimum cost spanning extension problem and generalize the irreducible core, that was introduced in Bird (1976) for minimum cost spanning tree problems. In most cases, this irreducible core consists of a continuum of points. We here propose two refinements of the irreducible core, viz. the proportional and the decentralized rule, and axiomatically characterize the proportional rule.

The outline of this paper is as follows.

Section 2 formally presents minimum cost spanning extension problems and introduces the algorithm that generates the proportional rule, while section 3 introduces the algorithm for the decentralized solution. Instead of solving the Operations Research and cost allocation problems consecutively, they are integrated : the cost of a link in a minimum cost spanning extension is allocated at the same moment the link is constructed. Section 4 characterizes the set of allocations generated by the proportional rule axiomatically, using efficiency and converse consistency. Section 5 concludes.

Preliminaries and notations

We refer to any elementary textbook on graph theory for an understanding of graph theory, but recall some definitions to show the notational conventions. A graph $G = < V, E >$ consists of a set V of vertices and a set E of edges. An edge e incident with two vertices i and j is identified with $\{i, j\}$[3]. For a graph $G = < V, E >$ and a set $W \subseteq V$,

$$E(W) := \{e \in E \mid e \subseteq W\}$$

is the set of edges linking two vertices in W. Similarly, for a subset E' of E,

$$V(E') := \{v \in V \mid \text{there exists an edge } e \in E' \text{ with } v \in e\}$$

is the set of vertices incident with E'.

The complete graph on a vertex set V is the graph $K_V = < V, E_V >$, where

$$E_V := \{\{v, w\} \mid v, w \in V \text{ and } v \neq w\}.$$

A *path* from i to j in a graph $< V, E >$ is a sequence $(i = i_0, i_1, \ldots, i_k = j)$ of vertices such that for all $1 \leq l \leq k$, the edge $\{i_{l-1}, i_l\}$ lies in E. A *cycle* is a path of which the begin and end-points coincide. Two vertices $i, j \in V$ are *connected* in a graph $< V, E >$ if there is a path from i to j in $< V, E >$. A subset W of V is *connected* in $< V, E >$ if every two vertices $i, j \in W$ are connected in the subgraph $< W, E(W) >$. A connected set W is a *connected component* of the graph $< V, E >$ if no superset of W is connected. If $W \subseteq V$, the set of connected components of the graph $< W, E(W) >$ is denoted by W/E. A *connected graph* is a graph $< V, E >$ with V connected in $< V, E >$. A *tree* is a connected graph that contains no cycles.

With many economic situations in which costs have to be divided one can associate a *cost game* (N, c) consisting of a finite set N of players and a *characteristic function* $c : 2^N \to \mathbb{R}$, with $c(\emptyset) = 0$. Here $c(S)$ represents the minimal cost for coalition $S \subseteq N$ if it secedes, i.e. if people of S cooperate and can not count upon help from people outside S.

The economic situations in the sequel involve a set N of users of a source $*$. For a coalition $S \subseteq N$, we denote $S \cup \{*\}$ by S^*.

The *core* of a cost game (N, c), is defined by

$$\text{Core}(c) := \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = c(N) \text{ and } \sum_{i \in S} x_i \leq c(S) \text{ for all } S \subseteq N\}.$$

A game with a non-empty core is called *balanced.*

The cardinality of a set S will be denoted by $|S|$. For two vectors $x \in \mathbb{R}^S$ and $y \in \mathbb{R}^T$, where S and T are two disjoint coalitions, we denote $(x, y) \in \mathbb{R}^{S \cup T}$ the vector with components

$$(x, y)_k := \begin{cases} x_k & \text{if } k \in S, \\ y_k & \text{if } k \in T. \end{cases}$$

Furthermore, for two coalitions $S \subseteq T$ and a vector $x \in \mathbb{R}^T$, we denote x^S the restriction of x to S. For a coalition $S \subseteq N$, the symbol 1_S is used to denote the vector in $\mathbb{R}^N$ with coordinates

$$(1_S)_k := \begin{cases} 1 & \text{if } k \in S, \\ 0 & \text{if } k \in N \setminus S. \end{cases}$$

[3]Because we do not consider multigraphs : two vertices are connected by at most one edge.

For any coalition S, the simplex Δ^S is defined by

$$\Delta^S := \{x \in \mathbb{R}^S_+ \mid \sum_{i \in S} x_i = 1\}.$$

2 Mcse problems and the proportional solution

Formally, a *minimum cost spanning extension (mcse) problem* $\mathcal{M} = < N, *, w, E >$ consists of a finite set N of agents, each of whom wants to be connected to a common source, denoted by $*$. The non-negative cost of constructing a link $\{i, j\}$ between the vertices i and j in $N^* \equiv N \cup \{*\}$ is denoted by $w(i, j)$. There is a set E of already constructed edges, which can be used when connecting agents to the source.

The problem we address is how to construct a network connecting all agents to the source, in the cheapest possible way, and how to allocate the costs of such a network among the agents. We address the two questions simultaneously by allocating the cost of an edge at the moment it is constructed. Inspired by Bird (1976), we analyze the cost allocations that we provide with game theoretic methods.

To an mcse problem $\mathcal{M} = < N, *, w, E >$, we associate a cooperative cost game $< N, c^{\mathcal{M}} >$, where the worth $c^{\mathcal{M}}(S)$ of a coalition S is the minimal cost of an extension of the present network by building edges between components containing members of S^*, such that in the extended network, S is connected to the source. This means players in S are connected to the source via a path which can use the edges that are present, but which does not use components disjoint with S^*.

Example 2.1 In the mcse game associated with the graph depicted in figure 1, coalition $\{1\}$ can link itself to the root using player 2, but not player 3. Hence, $c(\{1\}) = 3$. Similarly, the costs for other coalitions are : $c(\{2\}) = 3$, $c(\{3\}) = 1$, $c(\{1, 2\}) = 3$, $c(\{1, 3\}) = 1 + 1 = c(\{2, 3\}) = c(\{1, 2, 3\})$.

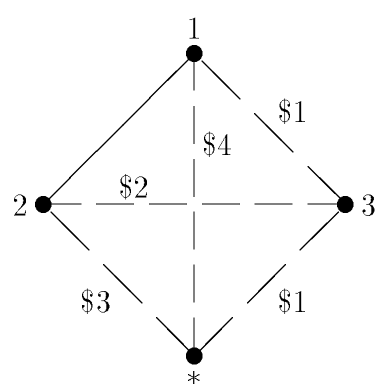

Figure 1: The edge $\{1, 2\}$ is initially present.

In general,

$$c^{\mathcal{M}}(S) := \min\left\{\sum_{e\in E'} w(e) \mid \begin{array}{l} S^* \subset C_*^{E'} \text{ and } E' \text{ contains only edges} \\ \text{between components containing members of } S^* \end{array}\right\}$$

for all $S \subseteq N$, where $C_*^{E'}$ is the component of the source $*$ in the graph $< N^*, E \cup E' >$.

Other definitions of cost games are possible, notably by allowing a coalition S to use only the edges between members of S^*, but this variant has a larger core than the game we define. Hence, if we prove that our game is balanced, we also prove that the variant is balanced. Furthermore, in our game, the players in the component of the source are dummy players, and that is not true for the variant. Note also that $c^{\mathcal{M}}(S)$ can be attained by an extension with precisely $|S^*/E| - 1$ edges.

In this section and the next one, we propose cost allocation rules for mcse problems that will prove to yield core elements of the associated mcse games.

First, define for an mcse problem $\mathcal{M}$ the *initial obligation* o_i of a player i by

$$o_i := \begin{cases} \frac{1}{|C_i|} & \text{if } * \notin C_i \\ 0 & \text{if } * \in C_i \end{cases} \tag{2.1}$$

where C_i is the component of $< N^*, E >$ containing player i. The interpretation is that if player i is in the component of the source, he has no obligation to contribute to the edge costs, but if he is not in the component of the source, then the component C_i of i has to pay one edge, or more precisely, portions of edges summing up to 1, and this obligation is divided equally among the players in C_i.

The *proportional* solution is constructed by the following algorithm : construct the edges of a minimum cost spanning extension as in Kruskal's algorithm. Each time an edge is constructed, its cost is divided proportionally to the remaining obligations, among the players in the components being linked. More precisely :

Algorithm 2.2 (the proportional rule)

input : an mcse problem $< N, *, w, E >$
output : a sequence of edges leading to an mcse and a cost allocation

1. Given $\mathcal{M} \equiv < N, *, w, E >$, define

$$\begin{array}{rcll} t & = & 0 & \text{the initial stage,} \\ \tau & = & |N^*/E| - 1 & \text{the number of stages,} \\ E^0 & = & E & \text{the initial edge set,} \\ o_i^0 & = & o_i & \text{for all } i \in N\text{, the initial obligation is defined} \\ & & & \text{in equation 2.1.} \end{array}$$

2. $t := t + 1$.

3. At stage t, given E^{t-1}, choose a cheapest edge e^t such that the graph $< N^*, E^{t-1} \cup \{e^t\} >$ does not contain more cycles than $< N^*, E^{t-1} >$.

4. If C^t is the connected component just formed by adding the edge e^t to the graph $< N^*, E^{t-1} >$, define the vector $f^t = (f_i^t)_{i \in N}$ of fractions the players contribute by

$$f_k^t = \begin{cases} \dfrac{o_i^{t-1}}{\sum\limits_{l \in C^t} o_l^{t-1}} & \text{if } i \in C^t, \\ 0 & \text{if } i \notin C^t. \end{cases}$$

5. Define the *remaining obligation* after stage t by $o_k^t := o_k^{t-1} - f_k^t$ for all $k \in N$.

6. Define $E^t := E^{t-1} \cup \{e^t\}$.

7. If $t < \tau$, go to step 2.

8. Define $\mathcal{E} = (e^1, \ldots, e^\tau)$.

9. Define $PRO^{\mathcal{E}}(\mathcal{M}) := \sum\limits_{t=1}^{\tau} f^t w(e^t)$.

Note that at every stage, the total obligation of a component that does not contain the source equals 1, and the obligation of any player in the component of the source equals 0. Hence, in step 4, the denominator equals 1 or 2, depending on whether C^t contains the source or not. In particular, it is never zero.

As the allocation generated by this algorithm depends on the choices of edges made, we define the proportional rule (or solution) by

$$PRO(\mathcal{M}) := \bigcup \{(\mathcal{E}, PRO^{\mathcal{E}}(\mathcal{M})) \mid \mathcal{E} \text{ is obtained by the algorithm 2.2}\}.$$

Note however, this algorithm constructs exactly one sequence of fraction vectors per sequence of edges chosen. As there are only finitely many minimal cost spanning extensions, $PRO(\mathcal{M})$ is finite for any mcse problem $\mathcal{M}$.

Applying the algorithm the the problem of example 2.1, we see that $o^0 = (.5, .5, 1)$. A possible first edge is edge $\{*, 3\}$. Then $f^1 = (0, 0, 1)/1$ and the remaining obligation of player three is zero, while that of the others is unchanged. The next edge has to be $\{1, 3\}$, which implies $f^2 = (.5, .5, 0)/1$. Hence the allocation is $1(.5, .5, 0) + 1(0, 0, 1) = (.5, .5, 1)$.

The only other possible first edge is $\{1, 3\}$, yielding $f^1 = (.5, .5, 1)/2$ and $o^2 = (.5, .5, 1) - (.5, .5, 1)/2 = (.25, .25, .5)$. Then $\{*, 3\}$ is the second edge, yielding $f^2 = (.25, .25, .5)/1$. Hence, the allocation is $1(.25, .25, .5) + 1(.25, .25, .5) = (.5, .5, 1)$. So the two sequences yield the same allocation. This is due to the small size of the problem.

Feltkamp, Tijs and Muto (1994b) defined the irreducible core $IC(\mathcal{M})$ of an mcse problem $\mathcal{M}$ and proved that it is generated by an algorithm that constructs a network in the same way as is done in algorithm 2.2, but that associates with every sequence of edges constructed a set of *valid* sequences of fraction vectors. For a sequence $(e^1, \ldots, e^\tau)$ of edges, valid sequences of fraction vectors are those sequences $(f^1, \ldots, f^\tau)$ that satisfy :

- Each component of the original graph that does not contain the source has to pay fractions of edges that total 1.

- At each stage, the players in the component that contains the source do not contribute to the cost of the edge constructed.
- At each stage, the cost of the edge that is constructed is shared by the players in the two components that it joins.

Moreover, it is proved that if $\mathcal{E} = (e^1, \ldots, e^\tau)$ is a sequence of edges of an mcse of the mcse problem $\mathcal{M}$ ordered by non-decreasing weight and $\mathcal{F} = (f^1, \ldots, f^\tau)$ is valid for $\mathcal{E}$, then

$$x^{\mathcal{E},\mathcal{F}} := \sum_{t=1}^{\tau} f^t w(e^t) \in IC(\mathcal{M}). \tag{2.2}$$

It is straightforward to see that for an mcse problem $\mathcal{M}$, the proportional algorithm generates a list of edges $\mathcal{E}$ ordered by non-decreasing weights and a valid sequence $\mathcal{F}$ of fraction vectors. Hence, $PRO^{\mathcal{E}}(\mathcal{M}) = x^{\mathcal{E},\mathcal{F}} \in IC(\mathcal{M})$. So, the set of allocations generated by the proportional solution is a refinement of the irreducible core, which is a subset of the core. In particular, the allocations generated by the proportional solution are all core elements of the mcse game (see Feltkamp, Tijs and Muto (1994b)). This proves that mcse games are balanced.

3 The decentralized solution

The proportional algorithm, the algorithm for the irreducible core in Feltkamp, Tijs and Muto (1994b) and the algorithm generating Bird's tree allocations rule in Feltkamp, Tijs and Muto (1994a) are centralized algorithms, in the sense that one edge is constructed per stage. However, one might imagine a situation in which at a certain stage in the construction, each component greedily starts to build the cheapest edge that links it to another component. If two components want to build the same edge, they meet in the middle, and each pays half, after which each has a remaining obligation to build half an edge in the following stages. Of course, the component that contains the source never contributes to any edge, so whenever a component is joined to the component of the source, it has to pay the whole edge, and thereafter, it does not contribute any more. The idea of building a minimal cost spanning tree in this way dates back in its first documented full formulation to Borůvka (1926a), (1926b). He considered minimum cost spanning tree problems, but the distinction is minimal if the only goal is to construct a network. It is only if one wants to allocate costs that the difference is essential. This decentralized algorithm will build a network in fewer stages than all previously described (centralized) algorithms, though the stages themselves are larger. A more precise formulation of the algorithm is :

Algorithm 3.1 (the decentralized rule)

input : an mcse problem $< N, *, w, E >$
output : a network and a cost allocation

1. Given $\mathcal{M} \equiv < N, *, w, E >$, define

$$\begin{array}{rcll} t & = & 0 & \text{the initial stage,} \\ E^0 & = & E & \text{the initial edge set.} \\ o_i^0 & = & o_i & \text{for all } i \in N \text{, the initial obligation is defined in equation 2.1 .} \end{array}$$

2. $t := t + 1$.

3. At stage t, each component C of $< N^*, E^{t-1} >$ that does not contain the source chooses a cheapest edge e_C^t linking C to another component of $< N^*, E^{t-1} >$.

4. Define the vector $f^t = (f_k^t)_{k \in N}$ of fractions by

$$f_k^t = \begin{cases} o_k^{t-1} & \text{if no other component chooses } e_C^t \\ o_k^{t-1}/2 & \text{if another component also chooses } e_C^t \\ 0 & \text{if } k \in C_*, \end{cases}$$

   for all $k \in N$. As usual, C_k^{t-1} denotes the component containing k in the graph $< N^*, E^{t-1} >$ constructed at stage $t-1$.

5. Define the remaining obligation after stage t by $o_k^t := o_k^{t-1} - f_k^t$ for all $k \in N$.

6. Define $E^t := E^{t-1} \cup \{e_C^t \mid C \in N^*/E^{t-1} \text{ and } * \notin C\}$.

7. If the graph $< N^*, E^t >$ is not yet connected, go to step 2.

8. Define τ to be the number of stages.

9. Define the decentralized value $DEC(\mathcal{M})$ by

$$DEC_k(\mathcal{M}) := \sum_{s=1}^{\tau} f_k^s w(e^s_{C_k^{s-1}})$$

   for all $k \in N$.

This algorithm can generate a network with cycles when applied to an arbitrary mcse problem, but on generic mcse problems, where all weights are different, it does not.

Definition 3.2 An mcse problem $< N, *, w, E >$ is called *generic* if for every pair $e \neq \tilde{e}$ of edges,

$$w(e) \neq w(\tilde{e}).$$

Note that on the class of generic mcse problems, for each component $C \in N^*/E^t$, there is only one edge e_c^t that can be chosen in step 3, so the decentralized solution constructs a unique mcse and allocation on this class of problems.

Theorem 3.3 If the mcse problem $< N, *, w, E >$ is generic, the decentralized algorithm generates an mcse.

Proof :

Let $< N, *, w, E >$ be a generic mcse problem. Clearly, algorithm 3.1 leads to a connected graph. The only way that a cycle can be introduced in this graph is that after a stage $t-1$, there are a number (say p) of components $C_1, \ldots, C_p$, such that at stage t, for each $1 \leq q < p$, component C_q constructs an edge e_q connecting it to component C_{q+1}, while component C_p constructs an edge e_p connecting it to component C_1. Now because C_q prefers e_q to e_{q-1} for all $1 < q \leq p$ and C_1 prefers e_1 to e_p, it follows that

$$w(e_1) \geq w(e_2) \geq \cdots \geq w(e_p) \geq w(e_1). \tag{3.1}$$

But this can only hold if all inequalities in 3.1 are equalities, which is impossible in a generic mcse problem.

Suppose the network $< N^*, E^\tau >$ constructed by algorithm 3.1 is not an mcse of $< N^*, E >$. Then there exists a minimum cost spanning extension $< N^*, \tilde{E} >$ that satisfies

$$\sum_{e \in \tilde{E}} w(e) < \sum_{e \in E^\tau} w(e).$$

Now consider the earliest stage t in which there is a component C of $< N^*, E^{t-1} >$ that constructs an edge e_C^t that is not present in $\tilde{E}$. This means that all edges in $< N^*, E^{t-1} >$ are present in $\tilde{E}$. Adding e_C^t to $\tilde{E}$ introduces a cycle, which has to include another edge $\tilde{e}$ linking C to another component of $< N^*, E^{t-1} >$ because e_C^t has only one end-point in C. Now e_C^t was the cheapest edge linking C to another component of $< N^*, E^{t-1} >$, so $w(\tilde{e}) > w(e_C^t)$, and deleting $\tilde{e}$ from $\tilde{E} \cup \{e_C^t\}$ produces a spanning extension which has smaller cost than $< N^*, \tilde{E} >$. This is a contradiction, hence the algorithm produces an mcse. □

The next theorem states that applied to a generic mcse problem, the decentralized algorithm constructs core elements of the associated mcse game.

Theorem 3.4 On the class of generic mcse problems, the allocations generated by the decentralized algorithm are elements of the irreducible core.

Proof : To prove this, we only need to prove that the allocations generated by the decentralized algorithm can also be generated by a sequence of edges that can be generated by algorithm 2.2, together with fraction vectors that are valid for this sequence, as defined in section 2.

Let $\mathcal{M} \equiv < N, *, w, E >$ be a generic mcse problem. Construct the sequence $\widetilde{\mathcal{E}}$ of edges as follows : at each stage t, order the edges constructed at stage t by the decentralized algorithm by increasing cost into a sequence $\mathcal{E}^t$. Then chain these sequences together and define

$$\widetilde{\mathcal{E}} \equiv (\tilde{e}^1, \ldots, \tilde{e}^n) := \mathcal{E}^1 \cdots \mathcal{E}^\tau,$$

where τ is the number of stages of the algorithm. Construct the sequence $\widetilde{\mathcal{F}} = (\tilde{f}^1, \ldots, \tilde{f}^n)$ of fraction vectors by

$$\tilde{f}_k^s := \begin{cases} f_k^t & \text{if } \tilde{e}^s = e^t_{C_k^{t-1}}, \\ 0 & \text{otherwise.} \end{cases}$$

One easily sees that $\tilde{\mathcal{F}}$ is valid for $\tilde{\mathcal{E}}$ and that

$$DEC(\mathcal{M}) = \sum_{s=1}^{n} \tilde{f}^s w(\tilde{e}^s).$$

Moreover, in any component C^t of $< N^*, E^t >$ that does not contain the source and that is constructed at stage t by the decentralized algorithm, there is exactly one edge e^t that is constructed by *two components* (call these C_0 and C_1) of $< N^*, E^{t-1} >$, all other edges are constructed by only *one of the two components* which they connect. Now this edge e^t is cheaper than any other edge $\tilde{e}^t_C$ that is constructed at stage t by any component C of $< N^*, E^{t-1} >$ that is a subset of C^t. To see this, consider the 'path' $C = C_p, e_p = \tilde{e}^t_C, \ldots, C_2, e_2, C_1, e_1 = e^t, C_0$ constructed at stage t (see figure 2).

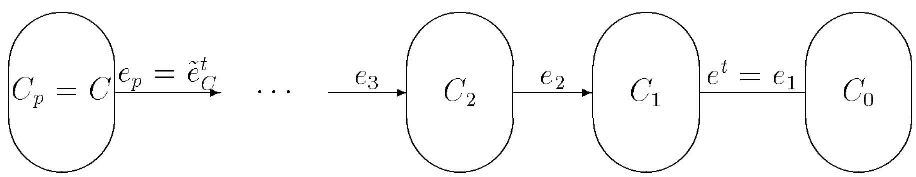

Figure 2: The 'path' between C_p and C_0.

For all $0 < q < p$, the component C_q prefers edge e_q to edge e_{q+1}. Hence,

$$w(e^t) = w(e_1) < w(e_2) < \ldots < w(e_p) = w(\tilde{e}^t). \tag{3.2}$$

Moreover, any edge e that has exactly one end-point in C^t, has exactly one end-point in a component C^{t-1} that is a 'building stone' of C^t. Hence, because e was not chosen by C^{t-1}, it has higher cost than $e^t_{C^{t-1}}$, the edge that C^{t-1} chose. Using equation 3.2, we obtain

$$w(e) > w(e^t_{C^{t-1}}) > w(e^t),$$

where e^t is the unique edge in C^t that is constructed at stage t by two components of $< N^*, E^{t-1} >$. This holds for all edges e that have exactly one end-point in C^t, hence also for the edge $e^{t+1}_{C^t}$ chosen by C^t at stage $t+1$.

Hence an edge that is chosen by a component C at a stage t is more expensive than the edge chosen in the previous stage by two components of $< N^*, E^{t-1} >$ that are subsets of the component C. Now for a player k, this means that the edges for which player k pays according to the algorithm 3.1, are ordered by increasing cost in the sequence $\mathcal{E}$. Hence if the sequence $\mathcal{E}'$ is constructed by sorting the edges in $\tilde{\mathcal{E}}$ by increasing cost and $\mathcal{F}'$ is defined accordingly :

$$f'^s := \tilde{f}^t \qquad \text{for the unique } t \text{ such that } e'^s = \tilde{e}^t,$$

then $\mathcal{E}'$ is a sequence of edges of an mcse ordered by non-decreasing weight and because for any player $k \in N$ the relative order of the subsequence of edges to which k contributes is unperturbed in the transition from $\mathcal{E}$ to $\mathcal{E}'$, $\mathcal{F}'$ is valid for $\mathcal{E}'$. Moreover,

$$DEC(\mathcal{M}) = x^{\widetilde{\mathcal{E}},\widetilde{\mathcal{F}}} = x^{\mathcal{E}',\mathcal{F}'}$$

which lies in the irreducible core of $\mathcal{M}$. □

4 Axiomatic characterization of the proportional rule

In sections 2 and 3, we introduced two solution rules of mcse problems. We axiomatically characterize the proportional rule in this section.

We define a *solution* of mcse problems as a function ψ assigning to every mcse problem $< N, *, w, E >$, a set

$$\psi(< N, *, w, E >) \subseteq \left\{ ((e^1, \ldots, e^\tau), x) \mid \begin{array}{l} < N^*, E \cup \{e^1, \ldots, e^\tau\} > \text{ is a} \\ \text{connected graph and } x \in \mathbb{R}^N \text{ satisfies} \\ \sum_{i \in N} x_i \geq \sum_{t=1}^{\tau} w(e^t). \end{array} \right\}.$$

We enumerate a few possible properties of a solution ψ of mcse problems.

Definition 4.1

Eff ψ is *efficient* if for all $((e^1, \ldots, e^\tau), x) \in \psi(\mathcal{M})$, for all $\mathcal{M}$, $E \cup \{e^1, \ldots, e^\tau\}$ is a minimal cost spanning extension and

$$\sum_{i \in N} x_i = \sum_{t=1}^{\tau} w(e^t).$$

MC ψ has the *minimal contribution property* if in every mcse problem, every component that does not contain the source contributes at least the cost of a minimum cost edge that connects two components. In formula : for all $\mathcal{M} \equiv < N, *, w, E >$, for all $(\mathcal{E}, x) \in \psi(\mathcal{M})$, for each component $C \in N^*/E$ that does not contain the source,

$$\sum_{i \in C} x_i \geq \min\{w(e) \mid e \text{ connects two components of } < N^*, E >\}.$$

FSC ψ has the *free for source component* property if for all $\mathcal{M}$, for all $(\mathcal{E}, x) \in \psi(\mathcal{M})$, we have

$$x_i = 0$$

for all i in the component of the source in the graph $< N^*, E >$.

ET ψ satisfies *equal treatment* if for all $\mathcal{M}$, for all $(\mathcal{E}, x) \in \psi(\mathcal{M})$, for all components $C \in N^*/E$, and for all players i and $j \in C$,

$$x_i = x_j.$$

We now introduce edge-reduced mcse problems.

Definition 4.2 Given an mcse problem $\mathcal{M} \equiv < N, *, w, E >$ and an edge $e = \{i, j\}$ that connects two components of $< N^*, E >$, define the *edge-reduced mcse problem*

$$\mathcal{M}^e =< N, *, w, E \cup \{e\} > .$$

Note that the edge-reduced problem is a smaller problem than the original problem : less edges have to be constructed, but it has the same number of players as the original problem.

The next three properties of a solution ψ relate the solution of a mcse problem and the solutions of its edge-reduced mcse problems.

Definition 4.3

ES ψ satisfies *equal share* if for any $\mathcal{M}$, for all all $((e^1, \ldots, e^\tau), x) \in \psi(\mathcal{M})$ with e^1 connecting two components C_1 and C_2, there exists a $(\widetilde{\mathcal{E}}, \tilde{x}) \in \psi(\mathcal{M}^{e^1})$ such that

$$\sum_{i \in C_1} (x_i - \tilde{x}_i) = \sum_{i \in C_2} (x_i - \tilde{x}_i).$$

In effect, this axiom requires that the two components connected in the first step of a solution participate in equal amounts in the cost of the edge which connects them.

Loc ψ is *local* if for all $\mathcal{M}$, for all $((e^1, \ldots, e^\tau), x) \in \psi(\mathcal{M})$, where e^1 connects the components C_1 and C_2 into a component C, there exists an $\tilde{x} \in \mathbb{R}^C$ such that

$$((e^2, \ldots, e^\tau), (\tilde{x}, x^{N \setminus C})) \in \psi(\mathcal{M}^{e^1}).$$

This axiom requires that adding an extra (minimum cost) edge in an mcse problem should not affect the players outside the component formed by adding this edge.

Cocons ψ satisfies *converse consistency* if for all $\mathcal{M}$, for all $(\mathcal{E}, x) \in E^\tau_{N^*} \times \mathbb{R}^N$ such that the solution ψ' defined by

$$\psi'(\mathcal{M}') = \begin{cases} \psi(\mathcal{M}) \cup \{(\mathcal{E}, x)\} & \text{if } \mathcal{M}' = \mathcal{M} \\ \psi(\mathcal{M}') & \text{if } \mathcal{M}' \neq \mathcal{M} \end{cases} \tag{4.1}$$

satisfies Eff, MC, FSC, ET, ES and Loc, it holds that

$$(\mathcal{E}, x) \in \psi(\mathcal{M}).$$

The upshot of this last axiom is that one should not be able to enlarge a solution without losing at least one of the previous axioms.

Proposition 4.4 The proportional solution satisfies Eff, MC, FSC, ET, ES, Loc and CoCons.

Proof : The proportional solution has Eff, FSC and MC because the set of allocations generated by the proportional algorithm is a refinement of the irreducible core and all allocations in the irreducible core satisfy these properties (cf. Feltkamp, Tijs and Muto (1994b)). ET is a direct consequence of the definition of the proportional solution.

The proportional solution is local : take $((e^1, \ldots, e^\tau), x) \in PRO(\mathcal{M})$. Let e^1 connect two components C_1 and C_2 into a component C. Then $\widetilde{\mathcal{E}} = (e^2, \ldots, e^\tau)$ leads to a minimum cost spanning extension of $\mathcal{M}^{e^1}$. Let $\widetilde{\mathcal{F}}$ be the unique sequence of fractions that corresponds to $\widetilde{\mathcal{E}}$ in the algorithm 2.2 and define $\tilde{x} = x^{\widetilde{\mathcal{E}},\widetilde{\mathcal{F}}}$. Then $\tilde{x}_k = x_k$ for $k \notin C$. Hence the proportional solution is local.

To prove that the proportional solution satisfies the equal share property, take an mcse problem $\mathcal{M}$ and take a component C of $< N^*, E >$. Any two players i and $j \in C$ have the same initial obligations. For any sequence $\mathcal{E}$ constructed by the algorithm, the remaining obligations at a stage t are only dependent on the remaining obligations in the previous stage, so by an induction argument, i and j have the same remaining obligations throughout all stages. Since in the unique sequence of fractions $\mathcal{F}$ corresponding to $\mathcal{E}$ in the proportional algorithm, the fractions of edges that i and j pay are proportional to the remaining obligations, it follows that $f_i^t = f_j^t$ for all t and hence $x_i^{\mathcal{E},\mathcal{F}} = x_j^{\mathcal{E},\mathcal{F}}$. So the proportional solution has the equal share property.

To prove it satisfies CoCons, take an mcse problem $\mathcal{M}$ and take

$$((e^1, \ldots, e^\tau), x) \in E_{N^*}^\tau \times \mathbb{R}^N$$

such that the solution PRO' as defined in equation 4.1 satisfies Eff, MC, FSC, ET, ES and Loc. Suppose e^1 connects the two components C_1 and C_2 into C. By locality, there exists an $\tilde{x} \in \mathbb{R}^C$ such that

$$((e^2, \ldots, e^\tau), (\tilde{x}, x^{N \setminus C})) \in PRO(\mathcal{M}^{e^1}).$$

Hence, there exist fractions vectors $(f^2, \ldots, f^\tau)$ that are constructed by the proportional algorithm together with the sequence $(e^2, \ldots, e^\tau)$, such that

$$(\tilde{x}, x^{N \setminus C}) = x^{(e^2, \ldots, e^\tau),(f^2, \ldots, f^\tau)}. \tag{4.2}$$

By efficiency of the proportional solution on $\mathcal{M}$ and $\mathcal{M}^{e^1}$,

$$\sum_{k \in C} (x_k - \tilde{x}_k) = w(e^1). \tag{4.3}$$

We now distinguish two cases :

- If either of C_1 or C_2 (say C_1) contains the source, by FSC and equal treatment, we obtain
$$x_k - \tilde{x}_k = \begin{cases} 0 & \text{if } k \in C_1, \\ w(e^1)/|C_2| & \text{if } k \in C_2. \end{cases}$$

In this case, define f^1 by

$$f^1_k = \begin{cases} 0 & \text{if } k \notin C_2, \\ 1/|C_2| & \text{if } k \in C_2. \end{cases}$$

Then $x = x^{(e^1,\ldots,e^\tau),(f^1,\ldots,f^\tau)}$ and $((e^1,\ldots,e^\tau),x) \in PRO(\mathcal{M})$.

- If neither of C_1 or C_2 contain the source, by ET we obtain

$$x_k = \begin{cases} z_1 & \text{if } k \in C_1 \\ z_2 & \text{if } k \in C_2 \end{cases}$$

for some z_1 and z_2 satisfying

$$z_1|C_1| + z_2|C_2| = \sum_{k\in C} x_k = w(e^1) + \sum_{k\in C} \tilde{x}_k = w(e) + \sum_{k\in C}\sum_{t=2}^{\tau} f^t_k w(e^t),$$

by equations 4.3 and 4.2. Furthermore, by ES,

$$z_1|C_1| = z_2|C_2|.$$

Hence, $z_1 = \frac{w(e)+\sum_{k\in C}\sum_{t=2}^{\tau} f^t_k w(e^t)}{2|C_1|}$ and $z_2 = \frac{w(e)+\sum_{k\in C}\sum_{t=2}^{\tau} f^t_k w(e^t)}{2|C_2|}$, so defining $\tilde{\mathcal{F}}$ by

$$\tilde{f}^t_k = \begin{cases} f^t_k & \text{if } t > 1 \text{ and } k \notin C, \\ 0 & \text{if } t = 1 \text{ and } k \notin C, \\ \frac{\sum_{k\in C} f^t_k}{2|C_1|} & \text{if } t > 1 \text{ and } k \in C_1, \\ \frac{\sum_{k\in C} f^t_k}{2|C_2|} & \text{if } t > 1 \text{ and } k \in C_2, \\ \frac{1}{2|C_1|} & \text{if } t = 1 \text{ and } k \in C_1, \\ \frac{1}{2|C_2|} & \text{if } t = 1 \text{ and } k \in C_2, \end{cases}$$

we obtain $x = x^{(e^1,\ldots,e^\tau),\tilde{\mathcal{F}}}$. As $\tilde{\mathcal{F}}$ is the sequence of share vectors corresponding to $(e^1,\ldots,e^\tau)$ in the proportional algorithm applied to $\mathcal{M}$,

$$((e^1,\ldots,e^\tau),x) \in PRO(\mathcal{M}).$$

$\square$

Lemma 4.5 If a solution ϕ satisfies Eff, MC, FSC, ET, ES and Loc, and a solution ψ satisfies all these properties as well as CoCons, then $\phi(\mathcal{M}) \subseteq \psi(\mathcal{M})$ for all mcse problems $\mathcal{M}$.

Proof : Suppose not. Then there exists an mcse problem $\mathcal{M} = < N, *, w, E >$ and $(\mathcal{E}, x) \in \phi(\mathcal{M}) \setminus \psi(\mathcal{M})$, such that $< N^*, E >$ has the least number of components of all problems with the property that $\phi(\mathcal{M}) \setminus \psi(\mathcal{M}) \neq \emptyset$. Then including $(\mathcal{E}, x)$ in $\psi(\mathcal{M})$

yields a solution that still has the properties Eff, MC, FSC, ET, ES and Loc, so by CoCons, $(\mathcal{E}, x) \in \psi(\mathcal{M})$. □

This implies that if a solution concept satisfies the other axioms, it satisfies CoCons if and only if it is the largest (for the inclusion relation) solution satisfying the other axioms. As a result, the proportional solution is the largest solution satisfying the axioms Eff, MC, FSC, ET, ES, and Loc. The next theorem implies it is also the unique solution that satisfies all mentioned axioms.

Theorem 4.6 The unique solution of mcse problems that satisfies Eff, MC, FSC, ET, ES, Loc and CoCons is the proportional solution.

Proof : We know that the proportional solution has the properties, and by lemma 4.5, if there are two solutions having them, they coincide. □

The decentralized solution has up to now not been characterized axiomatically, but it can be shown that it satisfies the axioms Eff, MC, FSC, ET and Loc. It does not satisfy ES, and hence also CoCons is not satisfied.

5 Concluding remarks

In this paper, we presented mcse problems and two algorithms that compute solutions to these problems. These solutions generate minimum cost spanning extensions as well as cost allocations and so, solve the operations research problem and cost allocation problem simultaneously. This was suggested by our analysis of mcst problems, where Bird's tree allocations appeared to be associated with Prim and Dijkstra's algorithm. Here, we associate cost allocations with generalizations of the other two well-known algorithms of Kruskal (1956) and Borůvka (1926) for computing mcsts.

Second, instead of looking only at the extreme case where no edges are present at the beginning, and a spanning tree has to be constructed, we also consider problems where some network is present already, construct minimum cost spanning extensions and associated cost allocations that lie in the core of the mcse game. This has two advantages. The mathematical advantage is that a half-solved problem is again in the same class of problems, the advantage from an applied viewpoint is that not only problems in which all edges have yet to be constructed are treated, but also problems in which a network has to be extended can be solved. If the original setup was suggested by (among other problems) electrification of Moravia at the beginning of this century, by now the problem is more how to extend an already present network.

We characterized the proportional rule axiomatically. This allows one to evaluate the rule by its properties instead of its definition.

The decentralized algorithm is particularly appealing when considering the problems from a game-theoretic point of view : every connected component constructs links in a greedy way, and this yields mcses for generic mcse problems. It is still an open problem to characterize the decentralized value axiomatically.

Acknowledgements : Inspiring discussions with Harry Aarts, Ruud Jeurissen, Michael Maschler, Gert-Jan Otten, Jos Potters and Hans Reynierse about the subject of the paper are gratefully acknowledged.

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