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Publication date:
1995

Link to publication in Tilburg University Research Portal

Citation for published version (APA):
Miyazawa, M., & Nieuwenhuis, G. (1995). *Modified Palm and Modified Time-Stationary Distributions for Random Measures and Applications*. (FEW Research Memorandum; Vol. 670). Department of Econometrics and Operations Research.

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Modified Palm and modified time-stationary distributions for random measures and applications

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July 2, 1994

Abstract

Palm distributions are known to be useful to obtain relations among various characteristics concerning stationary processes, in particular, arising in queueing theory. By assuming ergodicity, one can derive sample-path formulas from these relations. However, if ergodicity does not hold, there has been a certain gap between the expectation and the sample-path formulas. This gap has recently been filled by introducing modified Palm distributions, which also have nice physical interpretations. Under this setting, conditional expectation versions of relations such as Campbell's formula have been obtained, which can be expressed in terms of long-run sample averages. The main purpose of this paper is to generalize those formulas for the modified Palm distribution with respect to a stationary random measure. We also introduce a modified time-stationary distribution, and apply those modified distributions for characterizing the limit distributions of asymptotic stationary processes.

Keywords: Sample average, Conditional Palm distribution, Modified Palm distribution, Asymptotic stationary process, Queues, Extensions of $H = \lambda G$.

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[0]AMS 1991 subject classifications: 60G57, 60K25; secondary 60G55, 60G10

1. Introduction

Palm distributions are often introduced to obtain relations between various characteristics concerning stationary processes, in particular, arising in queueing theory. Campbell's formula, which is equivalent to the sample path formula $H = \lambda G$, is a typical example of them (see, e.g., Whitt (1991)). There is much literature about those relations in queues. For example, see Franken et al. (1982), Baccelli and Brémaud (1987) and Miyazawa (1994a,b). Under ergodicity restrictions, those formulas of Palm distributions, which are referred as expectation formulas in the following, can be interpreted in terms of sample-paths. However, if ergodicity does not hold, there has been a certain gap between the expectation and sample-path formulas. This gap has recently been filled by introducing conditional versions of Palm and modified Palm distributions. Here, the latter present limit distributions for Césaro averages, which can describe distributions of long-run sample averages. Nieuwenhuis (1993, 1994) considered this and derived several interesting relations concerning conditional modified Palm distributions. Sigman (1994) discussed it from view of sample averages. He derived several relations such as Little's and Campbell's formulas in terms of conditional modified Palm distributions. Those results are considered for marked point processes on the half lines $[0, \infty)$. That is, the underlying event processes are point processes. However, we meet more general underlying processes, which can be expressed as random measures, in connection to fluid approximations of queueing models and extensions of $H = \lambda G$ (see, e.g., Glynn and Whitt (1989), Miyazawa (1994a,b,1995)).

The main purpose of this paper is to generalize those formulas of conditional modified Palm distributions for stationary random measures and derive more general relations such as Mecke's (1967) formula in terms of these conditional distributions. To realize this, we define shift operators concerning the random measures, which correspond to the shifts of the sequence of occurrences and marks of marked point processes. These shift operators are shown to be stationary with respect to appropriate Palm and *modified Palm distributions*, under the time-stationary framework in which the original probability measure is unchanged by time-shift. Here, the modified Palm distributions are extended for random measures. Then, we can apply Birkhoff's ergodic theorem to relate sample average formulas to conditional expectations of the modified Palm distributions. It is shown that these formulas also hold under the probability measures from which we started in different ways. This is called the cross ergodic theorem. The similar story goes through under the shift-stationary framework in which the original probability measure is unchanged by the shift operation generated by a random measure. Here, we will introduce a *modified time-stationary distribution* for handling time-stationary processes in a suitable way. Those modified distributions are key issues of the paper.

We then derive various relations between conditional modified Palm distributions and conditional time-stationary probability measures, which enable us to express the relations in terms of sample averages. We also discuss the limit distributions of asymptotic stationary processes, which means that the limits are taken as Césaro averages. We characterize the existence of these distributions with respect to different shift operations by each other, and give relations between them by using the modified Palm and modified time-stationary distributions. We finally discuss applications of those results to the extensions of $H = \lambda G$.

This paper is composed of six sections. Section 2 introduces a framework for our formulation for stationary processes and ordinary Palm distributions. We derive an inversion formula and cross ergodic results. In Section 3, we introduce the modified Palm and modified time-stationary distribution and reformulate the results of Section 2 under these frameworks. Several relations among conditional modified Palm distributions are obtained in Section 4. Section 5 discusses asymptotic stationarity. Applications for relations in terms of long-run sample averages are discussed in Section 6.

2. Stationary random measure and Palm distribution

Let $(\Omega, \mathcal{F})$ be a measurable space, and $\{\theta_t\}_{t \in I\!R}$ be an $\mathcal{F}$-measurable operator group on Ω, i.e., $\theta_t(\omega)$ $((t,\omega) \in I\!R \times \Omega)$ is a measurable mapping from $(I\!R \times \Omega, \mathcal{B}(I\!R) \times \mathcal{F})$ to $(\Omega, \mathcal{F})$, and θ_0 is the identity on Ω. Here, $I\!R = (-\infty, +\infty)$ and $\mathcal{B}(I\!R)$ is the Borel σ-field on $I\!R$. In our applications, θ_t represents a time shift by t, but we do not need this interpretation in general. Let Λ be a random measure on $(I\!R, \mathcal{B}(I\!R))$. That is, Λ is a measure for each fixed $\omega \in \Omega$, and $\Lambda(B)$ is a random variable for each $B \in \mathcal{B}(I\!R)$. We assume that $\Lambda(B) < \infty$ for all $\omega \in \Omega$ and bounded $B \in \mathcal{B}(I\!R)$. Furthermore, assume that

(i) Λ is consistent with $\{\theta_t\}$, i.e.

$$\Lambda(B) \circ \theta_t = \Lambda(B + t) \qquad\qquad (\forall B \in \mathcal{B}(I\!\!R), \forall t \in I\!\!R) ,$$

where $\Lambda(B) \circ \theta_t(\omega) = \Lambda(B)(\theta_t(\omega))$ and $B + t = \{s + t | s \in B\}$. Define

$$\Lambda(t) = \begin{cases} \Lambda((0, t]) & \text{if } t \geq 0 \\ -\Lambda((t, 0]) & \text{if } t < 0 . \end{cases}$$

Thus, we use the same notation $\Lambda(\cdot)$ in two different ways, but they are clearly distinguished by their arguments. Note that

$$\Lambda(t) \circ \theta_u = \Lambda(t + u) - \Lambda(u) .$$

We will frequently use this equality. Set $S = \{\Lambda(t) | t \in I\!\!R\}$. Assume

(ii) S is identical for all $\omega \in \Omega$ and is equal to either $I\!\!R$ or $Z\!\!\!Z$,

where $Z\!\!\!Z = \{0, \pm 1, \pm 2, \cdots\}$. This assumption implies that Λ is either a simple point process, which we denote by N, or it is such that $\Lambda(t)$ is a continuous function of t for each fixed $\omega \in \Omega$. Of course, we can discuss these two cases separately, but a unified argument is possible by representing them by a single Λ.

We now define an inverse function Λ^{-1} for Λ by,

$$\Lambda^{-1}(x) = \inf\{u | \Lambda(u) \geq x\} \qquad\qquad (x \in S) .$$

Then, by (ii), we can see that $\Lambda^{-1}(x)$ is strictly increasing in $x \in S$, which leads to $\Lambda(\Lambda^{-1}(x)) = x$. This is a key observation, which enables us to define a desirable operator η_x on Ω for Λ by

$$\eta_x = \theta_{\Lambda^{-1}(x)} \qquad\qquad (x \in S) . \tag{2.1}$$

Lemma 2.1 Under assumptions (i) and (ii), $\{\eta_x\}$ is a measurable operator group. Especially, $\eta_x \circ \eta_y = \eta_{x+y}$ for $x, y \in S$.

Proof The measurability of η_x is a direct consequence of those of θ_t and Λ. To see the group property, we first note that

$$\Lambda^{-1}(x) \circ \eta_y = \inf\{u | \Lambda(u) \circ \theta_{\Lambda^{-1}(y)} \geq x\}$$

$$
\begin{aligned}
&= \inf\{u|\Lambda(u+\Lambda^{-1}(y)) - \Lambda(\Lambda^{-1}(y)) \geq x\} \\
&= \inf\{u|\Lambda(u+\Lambda^{-1}(y)) \geq x+y\} \\
&= \inf\{u|\Lambda(u) \geq x+y\} - \Lambda^{-1}(y) \\
&= \Lambda^{-1}(x+y) - \Lambda^{-1}(y) \ .
\end{aligned}
$$

Hence, we have, for all $\omega \in \Omega$,

$$
\begin{aligned}
\eta_x \circ \eta_y(\omega) &= \theta_{\Lambda^{-1}(x)\circ\eta_y(\omega)}(\eta_y(\omega)) \\
&= \theta_{\Lambda^{-1}(x+y)(\omega)-\Lambda^{-1}(y)(\omega)}(\theta_{\Lambda^{-1}(y)(\omega)}(\omega)) \\
&= \theta_{\Lambda^{-1}(x+y)(\omega)}(\omega) = \eta_{x+y}(\omega) \ . \qquad \Box
\end{aligned}
$$

Remark 2.1 $\eta_0(\omega)$ is not necessarily equal to ω. A similar operator to η_x was introduced in Miyazawa (1994b). But those two η_x's are different. The latter is defined for a function space on $I\!\!R \times \Omega$ and allows multiple jumps at the same time points, while the present η_x is defined on Ω and has the extended parameter set S but does not allow multiple jumps.

We next introduce shift-invariant sets. Define

$$
\mathcal{I}^{(\theta)} = \{A \in \mathcal{F}|\theta_t^{-1}(A) = A \text{ for all } t \in I\!\!R\} \ , \quad \mathcal{I}^{(\eta)} = \{A \in \mathcal{F}|\eta_x^{-1}(A) = A \text{ for all } x \in S\} \ .
$$

The next lemma is an extension of Lemma 2 of Nieuwenhuis (1994), in which the case of $S = Z\!\!\!Z$ is obtained.

Lemma 2.2 Under assumptions (i) and (ii), $\mathcal{I}^{(\theta)} = \mathcal{I}^{(\eta)}$.

Proof Suppose $A \in \mathcal{I}^{(\theta)}$. Then, for each fixed $\omega' \in \Omega$,

$$
\omega' \in A \quad \text{if and only if} \quad \theta_t(\omega') \in A \text{ for all } t \in I\!\!R.
$$

Fix $\omega \in \Omega$. If $\omega \in A$, then $\eta_x(\omega) = \theta_{\Lambda^{-1}(x)(\omega)}(\omega) \in A$ for all $x \in S$, because of the only-if part of the above iff statement. If $\eta_x(\omega) \in A$, then apply this only-if part again with $\omega' = \eta_x(\omega)$ and $t = -\Lambda^{-1}(x)(\omega)$. Since $\theta_t \circ \eta_x(\omega) = \theta_{t+\Lambda^{-1}(x)(\omega)}(\omega)$, we have $\omega \in A$. So, $\mathcal{I}^{(\theta)} \subset \mathcal{I}^{(\eta)}$. To show the reverse direction, we first note that, for $x \in S$ and $t \in I\!\!R$,

$$
\begin{aligned}
\Lambda^{-1}(x) \circ \theta_t &= \inf\{u|\Lambda(u+t) - \Lambda(t) \geq x\} \\
&= \inf\{u|\Lambda(u) \geq x + \Lambda(t)\} - t \\
&= \Lambda^{-1}(x+\Lambda(t)) - t \ , \qquad (2.2)
\end{aligned}
$$

and therefore,

$$\eta_x \circ \theta_t = \theta_{\Lambda^{-1}(x)} \circ \theta_t = \theta_{\Lambda^{-1}(x+\Lambda(t))} = \eta_{x+\Lambda(t)} \,. \tag{2.3}$$

Now suppose $A \in \mathcal{I}^{(\eta)}$. Then, for each fixed $\omega' \in \Omega$,

$$\omega' \in A \quad \text{if and only if} \quad \eta_y(\omega') \in A \text{ for all } y \in S.$$

Fix $\omega \in \Omega$. If $\omega \in A$ and $t \in I\!R$, then

$$\eta_x(\omega) = \eta_x \circ \theta_t(\theta_{-t}(\omega)) = \eta_{x+\Lambda(t)(\theta_{-t}(\omega))}(\theta_{-t}(\omega))$$

(see (2.3)) is also an element of A by the above iff statement. Choosing $\omega' = \theta_{-t}(\omega)$ and $y = x + \Lambda(t)(\theta_{-t}(\omega))$ in the only-if part yields $\theta_{-t}(\omega) \in A$. Since $\theta_0(\omega) = \omega$, this concludes $\mathcal{I}^{(\eta)} \subset \mathcal{I}^{(\theta)}$. $\Box$

In view of Lemma 2.2, we denote $\mathcal{I}^{(\theta)}$ and $\mathcal{I}^{(\eta)}$ by a single notation $\mathcal{I}$. We now introduce a probability measure P on $(\Omega, \mathcal{F})$. Set $\lambda = E(\Lambda(1))$, the intensity of Λ. We make two assumptions.

(iii) $\{\theta_t\}$ is stationary with respect to P, i.e. $P(\theta_t^{-1}(A)) = P(A)$ for all $A \in \mathcal{F}$ and $t \in I\!R$.

(iv) $0 < \lambda < \infty$.

Assumption (iii) implies that Λ is a stationary random measure.

Definition 2.1 Under assumptions (i)-(iv), define a probability measure P_Λ on $(\Omega, \mathcal{F})$ by

$$P_\Lambda(A) = \lambda^{-1} E\left(\int_0^1 1_A \circ \theta_u \Lambda(du)\right) \qquad (A \in \mathcal{F})\,, \tag{2.4}$$

where 1_A is the indicator function of a set A, and the integration in the expectation is taken over the left open interval $(0,1]$. P_Λ is called Palm probability measure of P with respect to Λ.

This definition is essentially due to Mecke (1967) and a natural extension of the case where Λ is a point process (see Miyazawa (1994a,b)). By changing variables in the expectation of (2.4), we can see that (2.4) is equivalent to

$$P_\Lambda(A) = \lambda^{-1} E\left(\int_0^{\Lambda(1)} 1_A \circ \eta_x m(dx)\right) \qquad (A \in \mathcal{F})\,, \tag{2.5}$$

where m is the measure on $(I\!\!R, \mathcal{B}(I\!\!R))$ that is Lebesgue measure if $S = I\!\!R$, and lattice measure concentrated on $Z\!\!\!Z$ satisfying $m(\{n\}) = 1$ for $n \in Z\!\!\!Z$ if $S = Z\!\!\!Z$. Then, we have the following lemma.

Lemma 2.3 $\{\eta_x\}$ is stationary with respect to P_Λ.

Proof We use the same argument as in the proof of Lemma 3.3 of Miyazawa (1994b).Let $A \in \mathcal{F}$. We first note that the stationarity of θ_t with respect to P and (2.3) imply that, for $x \in S$,

$$\begin{aligned} E\left(\int_{\Lambda(1)}^{\Lambda(1)+x} 1_A \circ \eta_y m(dy)\right) &= E\left((\int_{\Lambda(1)}^{\Lambda(1)+x} 1_A \circ \eta_y m(dy)) \circ \theta_{-1}\right) \\ &= E\left(\int_{-\Lambda(-1)}^{-\Lambda(-1)+x} 1_A \circ \eta_{y+\Lambda(-1)} m(dy)\right) \\ &= E\left(\int_0^x 1_A \circ \eta_y m(dy)\right) . \end{aligned}$$

Hence, from (2.5), we have

$$\begin{aligned} P_\Lambda(\eta_x^{-1}(A)) &= \lambda^{-1} E\left(\int_0^{\Lambda(1)} 1_A \circ \eta_{x+y} m(dy)\right) \\ &= \lambda^{-1} E\left(\int_x^{\Lambda(1)+x} 1_A \circ \eta_y m(dy)\right) = \lambda^{-1} E\left(\int_0^{\Lambda(1)} 1_A \circ \eta_y m(dy)\right) . \quad \Box \end{aligned}$$

We next consider the inversion formula for (2.4). This is well known for the case of a point process, but it seems not to have been reported for a stationary random measure, as far as the authors know. We start with Mecke's (1967) formula:

$$E\left(\int_{-\infty}^{+\infty} f(u, \theta_u)\Lambda(du)\right) = \lambda E_\Lambda\left(\int_{-\infty}^{+\infty} f(-u, \theta_0) du\right) , \tag{2.6}$$

where $f(u, \theta_v(\omega))$ is a nonnegative-valued measurable function of (u, v, ω) from $(I\!\!R^2 \times \Omega, \mathcal{B}(I\!\!R^2) \times \mathcal{F})$ to $(I\!\!R, \mathcal{B}(I\!\!R))$ (see also Miyazawa (1995)). If Λ is a simple point process N and if Ψ is a marked point process , then (2.6) reduces to Campbell's formula:

$$E\left(\sum_{n=-\infty}^{+\infty} f(T(n), \eta_n)\right) = \lambda E_N\left(\int_{-\infty}^{+\infty} f(u, \theta_0) du\right) , \tag{2.7}$$

where $T(n) = \Lambda^{-1}(n)$ is the n-th occurrence time of N such that $T(0) \le 0 < T(1)$.

For a nonnegative random variable X on $(\Omega, \mathcal{F})$, define f by

$$f(u, \theta_0) = X \circ \theta_{-u} 1_{\{0 \le -u < \Lambda^{-1}(1)\}} . \tag{2.8}$$

Since

$$\int_{-\infty}^{+\infty} f(-u,\theta_0)du = \int_0^{\Lambda^{-1}(1)} X \circ \theta_u du \ ,$$

and

$$\begin{aligned}
\int_{-\infty}^{+\infty} f(u,\theta_u)\Lambda(du) &= X \int_{-\infty}^{+\infty} 1_{\{u\le 0|\Lambda^{-1}(1+\Lambda(u))>0\}}\Lambda(du) \\
&= X \int_{-\infty}^{+\infty} 1_{\{u\le 0|1+\Lambda(u)>0\}}\Lambda(du) \\
&= X\Lambda((\Lambda^{-1}(-1),0]) = X \ .
\end{aligned}$$

So, (2.6) leads to the following inversion formula.

Lemma 2.4 Assume (i)-(iv). Then

$$E(X) = \lambda E_\Lambda \left(\int_0^{\Lambda^{-1}(1)} X \circ \theta_u du \right) \ . \tag{2.9}$$

Remark 2.2 When Λ is a point process, then $\Lambda^{-1}(1)$ is the first occurrence after time 0. Then, (2.9) becomes the well-known inversion formula for point processes (see, e.g., Franken et al. (1982)).

For $\mathcal{I}$-measurable functions $g : \Omega \to I\!R$, we have:

$$g \circ \theta_t = g \qquad \text{and} \qquad g \circ \eta_x = g, \tag{2.10}$$

for all $t \in I\!R$ and $x \in S$. We will frequently use these properties. Set $\lambda_{\mathcal{I}} = E(\Lambda(1)|\mathcal{I})$. As a consequence of (2.10), (2.4) and (2.9), we obtain for all $A \in \mathcal{I}$,

$$P_\Lambda(A) = \lambda^{-1} E(1_A \lambda_{\mathcal{I}}) \qquad \text{and} \qquad P(A) = \lambda E_\Lambda \left(1_A E_\Lambda(\Lambda^{-1}(1)|\mathcal{I}) \right) \ . \tag{2.11}$$

Write $P|_{\mathcal{I}}$ and $P_\Lambda|_{\mathcal{I}}$ for the restrictions of P and P_Λ to $\mathcal{I}$.

Lemma 2.5 The probability measures $P|_{\mathcal{I}}$ and $P_\Lambda|_{\mathcal{I}}$ have the same null sets. Especially, $P(0 < \lambda_{\mathcal{I}} < \infty) = P_\Lambda(0 < \lambda_{\mathcal{I}} < \infty) = 1$. The Radon-Nikodym derivatives can be expressed as follows:

$$\begin{aligned}
\frac{dP_\Lambda|_{\mathcal{I}}}{dP|_{\mathcal{I}}} &= \frac{\lambda_{\mathcal{I}}}{\lambda} \qquad P - a.s. \ , \\
\frac{dP|_{\mathcal{I}}}{dP_\Lambda|_{\mathcal{I}}} &= \frac{\lambda}{\lambda_{\mathcal{I}}} = \lambda E_\Lambda(\Lambda^{-1}(1)|\mathcal{I}) \qquad P_\Lambda - a.s. \ .
\end{aligned}$$

Proof Because of (2.11), only the last expression needs an argument. By the left-hand side of (2.11) and the first part of the lemma, we obtain that $\lambda_{\mathcal{I}} > 0$ $a.s.$ under both $P|_{\mathcal{I}}$ and $P_\Lambda|_{\mathcal{I}}$. By the right-hand side of (2.11), the last equality of the lemma follows. □

By applying Birkhoff's ergodic theorem, we obtain the following direct consequences of the stationarity of $\{\theta_t\}$ and $\{\eta_x\}$ with respect to P and P_Λ, respectively.

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t 1_A \circ \theta_u du \quad = \quad P(A|\mathcal{I}) \qquad\qquad P - a.s, \tag{2.12}$$

$$\lim_{x\to\infty} \frac{1}{x}\int_0^x 1_A \circ \eta_y m(dy) \quad = \quad P_\Lambda(A|\mathcal{I}) \qquad\qquad P_\Lambda - a.s, \tag{2.13}$$

for all $A \in \mathcal{F}$. Note that the ω-sets described in (2.12) and (2.13), are elements of $\mathcal{I}$. By the first part of Lemma 2.5, we obtain the following cross-ergodic results.

Theorem 2.1 Under assumptions (i)-(iv), (2.12) and (2.13) hold as well under P_Λ and P, respectively.

Remark 2.3 (2.13) gives the nice interpretation that the conditional Palm probability measure $P_\Lambda(\cdot|\mathcal{I}^{(\eta)})$ is obtained by η_x-sample averaging. For the case of point processes, similar results as in Lemma 2.5 and Theorem 2.1 can be found in the literature. See, e.g., Franken et al. (1982), Miyazawa (1977) and Nieuwenhuis (1994).

3. Modified Palm and modified time-stationary distributions

In this section we modify the Palm distribution P_Λ in such a way that the resulting probability measure is just the limit of a Cesaro averaging procedure. This distribution has a strong physical meaning, which will be demonstrated in Section 5. Furthermore we can consider the results of Section 2 under a weaker assumption.

Assume that (i)-(iii) are satisfied. We replace (iv) by the following condition.

(iv-a) $P(0 < E(\Lambda(1)|\mathcal{I}) < \infty) = 1$.

By Lemma 2.5, it is obvious that (iv) implies (iv-a). Again, set $\lambda_{\mathcal{I}} = E(\Lambda(1)|\mathcal{I})$.

Definition 3.2 Under assumptions (i)-(iii) and (iv-a), define

$$\overline{P}_\Lambda(A) = E\left(\frac{1}{\lambda_{\mathcal{I}}}\int_0^1 1_A \circ \theta_u \Lambda(du)\right) \qquad\qquad (A \in \mathcal{F})\ . \tag{3.1}$$

Note that, indeed, $\overline{P}_\Lambda$ is a probability measure on $(\Omega, \mathcal{F})$. $\overline{P}_\Lambda$ is called the *modified Palm distribution* of P with respect to Λ.

Remark 3.1 $\lambda < \infty$ is not necessary for (3.1). If $0 < \lambda < \infty$, the modified Palm distribution arises from the Palm distribution P_Λ by shifting λ^{-1} in (2.4) behind E and replacing the intensity λ by the conditional intensity $\lambda_{\mathcal{I}}$. For the case of point processes $\Lambda = N$ on the whole real line, $\overline{P}_\Lambda$ was first introduced in Nawrotzki (1978; p.248), and further characterized in Nieuwenhuis (1994). See also Sigman (1994).

The results for P_Λ, derived so far, can be reformulated for $\overline{P}_\Lambda$ by replacing λ by $\lambda_{\mathcal{I}}$ and P_Λ by $\overline{P}_\Lambda$. For instance, (2.5) and Lemma 2.3 turn into

$$\overline{P}_\Lambda(A) = E\left(\frac{1}{\lambda_{\mathcal{I}}}\int_0^{\Lambda(1)} 1_A \circ \eta_x m(dx)\right) \qquad (A \in \mathcal{F}), \tag{3.2}$$

$$\left(= \lambda E_\Lambda\left(1_A \frac{1}{\lambda_{\mathcal{I}}}\right) \text{ if, additionally, (iv) holds}\right),$$

$$\{\eta_x\} \quad \text{is} \quad \text{stationary with respect to } \overline{P}_\Lambda . \tag{3.3}$$

To get a corresponding formula to (2.9), we note that originally Mecke's formula was derived without (iv) (see Mecke (1967) and the proof of Theorem 1.2.8 of Franken et al. (1982)). That is, (2.6) can be rewritten as:

$$E\left(\int_{-\infty}^{+\infty} f(u, \theta_u)\Lambda(du)\right) = E\left(\int_0^1 \left(\int_{-\infty}^{+\infty} f(-u, \theta_0)du\right) \circ \theta_v \Lambda(dv)\right) . \tag{3.4}$$

For this formula to hold, assumption (iv) is not needed. Then, by applying the same arguments as in the proof of Lemma 2.4, we obtain the inversion formula for the modified Palm distribution.

Lemma 3.1 Assume (i)-(iii) and (iv-a). Then

$$P(A) = \overline{E}_\Lambda\left(\lambda_{\mathcal{I}} \int_0^{\Lambda^{-1}(1)} 1_A \circ \theta_u du\right) \qquad (A \in \mathcal{F}), \tag{3.5}$$

where $\overline{E}_\Lambda$ denotes expectation with respect to $\overline{P}_\Lambda$.

By (3.1) and (2.10), it is obvious that

$$P = \overline{P}_\Lambda \quad \text{on } \mathcal{I} . \tag{3.6}$$

If, additionally, (iv) is satisfied, then we have (by (2.10) and (3.6)):

$$\overline{E}_\Lambda\left(X|\mathcal{I}\right) = E_\Lambda\left(X|\mathcal{I}\right) \qquad \overline{P}_\Lambda -, P-, \text{ and } P_\Lambda - a.s.\ , \tag{3.7}$$

for all nonnegative random variables X.

Lemma 3.2 Assume (i)-(iii) and (iv-a). Then

$$\overline{E}_\Lambda\left(\Lambda^{-1}(1)\middle|\mathcal{I}\right) = \frac{1}{\lambda_\mathcal{I}} \qquad \overline{P}_\Lambda -, \text{ and } P - a.s.\ . \tag{3.8}$$

If, additionally, (iv) is satisfied, then $\overline{P}_\Lambda$ and P_Λ have the same null sets. The corresponding Radon-Nikodym densities $d\overline{P}_\Lambda/dP_\Lambda$ and $dP_\Lambda/d\overline{P}_\Lambda$ can be expressed as $\lambda E_\Lambda(\Lambda^{-1}(1)|\mathcal{I})$ and $(\lambda E_\Lambda(\Lambda^{-1}(1)|\mathcal{I}))^{-1}$, respectively. $\overline{P}_\Lambda$ and P_Λ coincide iff $E_\Lambda(\Lambda^{-1}(1)|\mathcal{I}) = 1/\lambda\ \overline{P}_\Lambda -$ $a.s.$.

Proof By (3.6) and (3.5), we obtain, for all $A \in \mathcal{I}$,

$$\overline{P}_\Lambda(A) = P(A) = \overline{E}_\Lambda\left(1_A \lambda_\mathcal{I} \overline{E}_\Lambda(\Lambda^{-1}(1)|\mathcal{I})\right)\ .$$

Hence, $\lambda_\mathcal{I}\overline{E}_\Lambda(\Lambda^{-1}(1)|\mathcal{I}) = 1\ \overline{P}_\Lambda -$ and $P - a.s.$ The other parts of the lemma are consequences of the expression following (3.2). □

Since $\{\eta_x\}$ is stationary with respect to $\overline{P}_\Lambda$, it is obvious that (2.13) remains valid if P_Λ is replaced by $\overline{P}_\Lambda$. By (3.6), we obtain new cross-ergodic results:

Theorem 3.1 Under assumptions (i)-(iii) and (iv-a),

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t 1_A \circ \theta_u du = P(A|\mathcal{I}) \qquad \overline{P}_\Lambda - a.s, \tag{3.9}$$

$$\lim_{x\to\infty} \frac{1}{x}\int_0^x 1_A \circ \eta_y m(dy) = \overline{P}_\Lambda(A|\mathcal{I}) \qquad P - a.s, \tag{3.10}$$

for all $A \in \mathcal{F}$.

Take expectations of (3.9) and (3.10) with respect to $\overline{P}_\Lambda$ and P, respectively. Then, by (3.6) and by the dominated convergence theorem, we obtain results which describe $\overline{P}_\Lambda$ and P in terms of Césaro averages:

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \overline{P}_\Lambda(\theta_u^{-1}(A))du = P(A)\ , \tag{3.11}$$

$$\lim_{x\to\infty} \frac{1}{x}\int_0^x P(\eta_y^{-1}(A))m(dy) = \overline{P}_\Lambda(A)\ , \tag{3.12}$$

for all $A \in \mathcal{F}$. Especially, note that the limit in (3.12) is not necessarily equal to $P_\Lambda(A)$. See also Nieuwenhuis (1993, 1994) for the case that $\Lambda = N$.

We end this section with a theorem characterizing the stationarity of $\{\eta_x\}$, not assuming (iii), and introduce another modified distribution. These results will be applied in Section 5. The theorem is similar to Satz 3.9 in Nawrotzki (1978) for the case $\Lambda = N$. His proof may be extended for general random measures, but we here give another proof along the proof of Lemma 2.3. In the following, we denote the expectation concerning a probability measure Q by E_Q.

Theorem 3.2 Assume (i) and (ii). Let Q^0 be a probability measure on $(\Omega, \mathcal{F})$. Then, there exists a probability measure $\overline{Q^0}$ satisfying (iii) and (iv-a), with the property that its modified Palm distribution with respect to Λ is equal to Q^0 if and only if the following two conditions hold.

(iii-a) $\{\eta_x\}$ is stationary with respect to Q^0.

(iv-b) $Q^0(0 < E_{Q^0}(\Lambda^{-1}(1)|\mathcal{I}) < \infty) = 1$

In this case

$$E_{\overline{Q^0}}(\Lambda(1)|\mathcal{I}) = 1/E_{Q^0}(\Lambda^{-1}(1)|\mathcal{I}) \qquad Q^0-, \overline{Q^0} - a.s.\,, \tag{3.13}$$

$$\overline{Q^0}(A) = E_{Q^0}\left(\frac{1}{E_{Q^0}(\Lambda^{-1}(1)|\mathcal{I})}\int_0^{\Lambda^{-1}(1)} 1_A \circ \theta_u du\right) \qquad (A \in \mathcal{F})\,. \tag{3.14}$$

Proof For convenience, we write $\mu_{\mathcal{I}}$ for $E_{Q^0}(\Lambda^{-1}(1)|\mathcal{I})$. The only-if part and (3.14) are direct consequences of (3.3), (3.5), and Lemma 3.2. Next, suppose that $\{\eta_x\}$ is stationary with respect to Q^0 and $Q^0(0 < \mu_{\mathcal{I}} < \infty) = 1$. Define the probability measure $\overline{Q^0}$ by (3.14). We first prove that $\{\theta_t\}$ is stationary with respect to $\overline{Q^0}$. Note that the stationarity of $\{\eta_x\}$ implies

$$\begin{aligned}
E_{Q^0}\left(\frac{1}{\mu_{\mathcal{I}}}\int_{\Lambda^{-1}(1)}^{\Lambda^{-1}(1)+t} 1_A \circ \theta_u du\right) &= E_{Q^0}\left(\frac{1}{\mu_{\mathcal{I}}}\int_0^t 1_A \circ \theta_{u+\Lambda^{-1}(1)} du\right) \\
&= E_{Q^0}\left(\left(\frac{1}{\mu_{\mathcal{I}}}\int_0^t 1_A \circ \theta_u du\right) \circ \eta_1\right) \\
&= E_{Q^0}\left(\frac{1}{\mu_{\mathcal{I}}}\int_0^t 1_A \circ \theta_u du\right)
\end{aligned}$$

for all $t \in I\!R$ and $A \in \mathcal{F}$. Hence, we have

$$\begin{aligned}
\overline{Q^0}(\theta_t^{-1}(A)) &= E_{Q^0}\left(\frac{1}{\mu_{\mathcal{I}}}\int_0^{\Lambda^{-1}(1)} 1_A \circ \theta_{u+t} du\right) \\
&= E_{Q^0}\left(\frac{1}{\mu_{\mathcal{I}}}\int_t^{\Lambda^{-1}(1)+t} 1_A \circ \theta_u du\right) \\
&= E_{Q^0}\left(\frac{1}{\mu_{\mathcal{I}}}\int_0^{\Lambda^{-1}(1)} 1_A \circ \theta_u du\right) = \overline{Q^0}(A) \ .
\end{aligned}$$

Note that (3.14) implies

$$Q^0 = \overline{Q^0} \quad \text{on} \quad \mathcal{I} \ . \tag{3.15}$$

We next prove that

$$\overline{\left(\overline{Q^0}\right)_\Lambda}(A) = Q^0(A) \qquad\qquad (A \in \mathcal{F}) \ . \tag{3.16}$$

Since $\{\theta_t\}$ is stationary with respect to $\overline{Q^0}$, (3.6) implies $\overline{Q^0} = \overline{\left(\overline{Q^0}\right)_\Lambda}$ on $\mathcal{I}$. This and (3.15) concludes

$$Q^0 = \overline{\left(\overline{Q^0}\right)_\Lambda} \quad \text{on} \quad \mathcal{I} \ . \tag{3.17}$$

On the other hand, by Birkhoff's ergodic theorem, (3.15) and (3.17), we have

$$\lim_{x\to\infty} \frac{1}{x}\int_0^x 1_A \circ \eta_y m(dy) = Q^0(A|\mathcal{I}) \qquad Q^0-, \overline{Q^0}-, \overline{\left(\overline{Q^0}\right)_\Lambda} - a.s, \tag{3.18}$$

$$\lim_{x\to\infty} \frac{1}{x}\int_0^x 1_A \circ \eta_y m(dy) = \overline{\left(\overline{Q^0}\right)_\Lambda}(A|\mathcal{I}) \qquad Q^0-, \overline{Q^0}-, \overline{\left(\overline{Q^0}\right)_\Lambda} - a.s \ , \tag{3.19}$$

for all $A \in \mathcal{F}$. Thus, we obtain

$$Q^0(A|\mathcal{I}) = \overline{\left(\overline{Q^0}\right)_\Lambda}(A|\mathcal{I}) \qquad\qquad Q^0-, \overline{Q^0}-, \overline{\left(\overline{Q^0}\right)_\Lambda} - a.s \ .$$

This and (3.17) yield (3.16). (3.13) follows immediately from (3.16) and Lemma 3.2. □

In the context of Theorem 3.2, we will call $\overline{Q^0}$ the *modified time-stationary distribution* of Q^0 with respect to Λ. We finally mention the following cross-ergodic result under assumptions (i), (ii), (iii-a) and (iv-b), which is just a counterpart of (3.18) and derived similarly to (3.18).

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t 1_A \circ \theta_u du = \overline{Q^0}(A|\mathcal{I}) \qquad\qquad Q^0-, \overline{Q^0} - a.s, \tag{3.20}$$

for all $A \in \mathcal{F}$.

4. Conditional modified Palm calculus

It is well known that the relationship between the stationary probability measure P and its Palm distribution P_Λ is useful, particularly in queueing theory (see, e.g. Miyazawa (1994a) and Whitt (1991)). Mecke's (1967) formula expresses one of the most general forms of this relationship. Miyazawa (1995) further generalized the formula so that it can handle extensions of $H = \lambda G$, due to Glynn and Whitt (1989). In this section, we consider a similar relationship for the conditional distributions $\overline{P}_\Lambda(\cdot|\mathcal{I})$ and $P(\cdot|\mathcal{I})$. In view of (3.7), if $0 < \lambda < \infty$, these results simultaneously give relations for $P_\Lambda(\cdot|\mathcal{I})$ and $P(\cdot|\mathcal{I})$. For simple point processes on the half line, they have been obtained by Sigman (1994). We generalize them for random measures on $I\!R$ satisfying (ii). Furthermore, our derivation is straightforward, while Sigman used sample path arguments. Throughout this section, we assume (i)-(iii) and (iv-a), but, by Theorem 3.2, all results of this section are also valid under assumptions (i), (ii), (iii-a) and (iv-b), if we replace P by the corresponding modified time-stationary distribution.

We will consider conditional versions of various relations between P and $\overline{P}_\Lambda$. Recall that $E(\Lambda(1)|\mathcal{I})$ is denoted by $\lambda_\mathcal{I}$, and that $\lambda_\mathcal{I} > 0$ *a.s.* under both P and $\overline{P}_\Lambda$. Let X be a nonnegative valued random variable on $(\Omega, \mathcal{F})$.

Lemma 4.1 Under P and $\overline{P}_\Lambda$,

$$\overline{E}_\Lambda(X|\mathcal{I}) = \lambda_\mathcal{I}^{-1} E\Big(\int_0^1 X \circ \theta_u \Lambda(du)|\mathcal{I}\Big) \qquad a.s.\,. \tag{4.1}$$

Proof By (3.6) and the definition of $\overline{P}_\Lambda$, we have for $A \in \mathcal{I}$,

$$\begin{aligned} E(\lambda_\mathcal{I}\overline{E}_\Lambda(X|\mathcal{I})1_A) &= \overline{E}_\Lambda(\lambda_\mathcal{I}\overline{E}_\Lambda(X1_A|\mathcal{I})) \\ &= \overline{E}_\Lambda(\lambda_\mathcal{I} X 1_A) \\ &= E\Big(\int_0^1 X \circ \theta_u \Lambda(du) 1_A\Big)\,. \end{aligned}$$

Hence, we get (4.1) under P. By (3.6) it is also valid under $\overline{P}_\Lambda$. □

Lemma 4.2 Under P and $\overline{P}_\Lambda$,

$$E(X|\mathcal{I}) = \lambda_\mathcal{I}\overline{E}_\Lambda\left(\int_0^{\Lambda^{-1}(1)} X \circ \theta_u du \,\middle|\, \mathcal{I}\right) \qquad a.s.\,. \tag{4.2}$$

Proof Let $A \in \mathcal{I}$. The proof is similar to Lemma 4.1. That is, (3.6) and Lemma 3.1 yield:

$$\begin{aligned}
\overline{E}_\Lambda(E(X|\mathcal{I})1_A) &= E\left(E(X|\mathcal{I})1_A\right) \\
&= E\left(X1_A\right) \\
&= \overline{E}_\Lambda\left(\lambda_{\mathcal{I}} 1_A \int_0^{\Lambda^{-1}(1)} X \circ \theta_u du\right) . \ \square
\end{aligned}$$

Remark 4.1 For the case of point processes, Lemmas 4.1 and 4.2 are obtained in Nieuwenhuis (1994; p.51). They can also be deduced from Corollaries 2.9 and 2.10 of Sigman (1994). In the latter literature, the derivations are more elaborated because sample path arguments are involved.

We next derive a conditional version of the generalized Mecke's formula of Miyazawa (1995):

$$\lambda_1 E_{\Lambda_1}\left(\int_{-\infty}^{+\infty} f(u,\theta_u)\Lambda_2(du)\right) = \lambda_2 E_{\Lambda_2}\left(\int_{-\infty}^{+\infty} f(-u,\theta_0)\Lambda_1(du)\right) , \tag{4.3}$$

for all measurable functions $f : (I\!R \times \Omega, \mathcal{B}(I\!R) \times \mathcal{F}) \to [0,\infty)$ (see after (2.6) for the details of f). Here Λ_1 and Λ_2 are random measures; conditions (i)-(iv) are assumed for both Λ_1 and Λ_2. Similarly to (3.4), we rewrite (4.3) in the following form.

$$\begin{aligned}
&E\left(\int_0^1 \left(\int_{-\infty}^{+\infty} f(u,\theta_u)\Lambda_2(du)\right) \circ \theta_v \Lambda_1(dv)\right) \\
&\qquad = E\left(\int_0^1 \left(\int_{-\infty}^{+\infty} f(-u,\theta_0)\Lambda_1(du)\right) \circ \theta_v \Lambda_2(dv)\right) .
\end{aligned} \tag{4.4}$$

Again, we do not need condition (iv) for the validity of (4.4). (4.3) and (4.4) clearly reduce to Mecke's formulas (2.6) and (3.4), respectively, if Λ_1 is Lebesgue measure.

Theorem 4.1 Under assumptions (i)-(iii) and (iv-a) for Λ_1 and Λ_2, we have, under P, $\overline{P}_{\Lambda_1}$, and $\overline{P}_{\Lambda_2}$,

$$\lambda_{1,\mathcal{I}}\overline{E}_{\Lambda_1}\left(\int_{-\infty}^{+\infty} f(u,\theta_u)\Lambda_2(du)\middle|\mathcal{I}\right) = \lambda_{2,\mathcal{I}}\overline{E}_{\Lambda_2}\left(\int_{-\infty}^{+\infty} f(-u,\theta_0)\Lambda_1(du)\middle|\mathcal{I}\right) \quad a.s. , \tag{4.5}$$

where $\lambda_{i,\mathcal{I}} = E(\Lambda_i(1)|\mathcal{I})$ for $i = 1, 2$.

Proof For $A \in \mathcal{I}$, we have

$$E\left(\overline{E}_{\Lambda_1}\left(\int_{-\infty}^{+\infty} f(u,\theta_u)\Lambda_2(du)\middle|\mathcal{I}\right)1_A\right)$$

$$
\begin{aligned}
&= \overline{E}_{\Lambda_1}\left(\overline{E}_{\Lambda_1}\left(\int_{-\infty}^{+\infty} f(u,\theta_u)\Lambda_2(du)\middle|\mathcal{I}\right)1_A\right) && \text{(by (3.6))}\\
&= \overline{E}_{\Lambda_1}\left(\int_{-\infty}^{+\infty} f(u,\theta_u)\Lambda_2(du)1_A\right)\\
&= E\left(\frac{1}{\lambda_{1,\mathcal{I}}}\int_0^1\left(\int_{-\infty}^{+\infty} f(u,\theta_u)\Lambda_2(du)1_A\right)\circ\theta_v\Lambda_1(dv)\right) && \text{(by the definition)}\\
&= E\left(\frac{1}{\lambda_{1,\mathcal{I}}}\int_0^1\left(\int_{-\infty}^{+\infty} f(-u,\theta_0)1_A\Lambda_1(du)\right)\circ\theta_v\Lambda_2(dv)\right) && \text{(by (4.4))}\\
&= \overline{E}_{\Lambda_2}\left(\frac{\lambda_{2,\mathcal{I}}}{\lambda_{1,\mathcal{I}}}1_A\int_{-\infty}^{+\infty} f(-u,\theta_0)\Lambda_1(du)\right) && \text{(by the definition)}\\
&= \overline{E}_{\Lambda_2}\left(\frac{\lambda_{2,\mathcal{I}}}{\lambda_{1,\mathcal{I}}}1_A\,\overline{E}_{\Lambda_2}\left(\int_{-\infty}^{+\infty} f(-u,\theta_0)\Lambda_1(du)\middle|\mathcal{I}\right)\right)\\
&= E\left(\frac{\lambda_{2,\mathcal{I}}}{\lambda_{1,\mathcal{I}}}\,\overline{E}_{\Lambda_2}\left(\int_{-\infty}^{+\infty} f(-u,\theta_0)\Lambda_1(du)\middle|\mathcal{I}\right)1_A\right) && \text{(by (3.6))}\ . \ \square
\end{aligned}
$$

Remark 4.2 The theorem can be used to obtain for $\overline{P}_{\Lambda_1}$ and $\overline{P}_{\Lambda_2}$ the obvious unconditional relation which is similar to (4.5). Just take the expectation $\overline{E}_{\Lambda_1}$ in (4.5) and use the fact that $\overline{P}_{\Lambda_1} = \overline{P}_{\Lambda_2}$ on $\mathcal{I}$. Sigman (1994) obtained (4.5) for Campbell's formula (2.7), i.e. for the case where Λ_1 is Lebesgue measure and Λ_2 is a point process on $[0,\infty)$. His derivation uses sample path relations, while ours is purely analytical.

Corollary 4.1 Under the assumptions of Theorem 4.1, we have, under P, $\overline{P}_{\Lambda_1}$ and $\overline{P}_{\Lambda_2}$,

$$
\lambda_{1,\mathcal{I}}\overline{E}_{\Lambda_1}\left(X\middle|\mathcal{I}\right) = \lambda_{2,\mathcal{I}}\overline{E}_{\Lambda_2}\left(\int_0^{\Lambda_2^{-1}(1)} X\circ\theta_u\Lambda_1(du)\middle|\mathcal{I}\right) \quad a.s.\ , \tag{4.6}
$$

where X is a nonnegative random variable on $(\Omega,\mathcal{F})$, and, in particular,

$$
\frac{\lambda_{1,\mathcal{I}}}{\lambda_{2,\mathcal{I}}} = \overline{E}_{\Lambda_2}\left(\Lambda_1(\Lambda_2^{-1}(1))\middle|\mathcal{I}\right) \quad a.s.\ . \tag{4.7}
$$

Proof To Theorem 4.1, apply the function f defined by

$$
f(u,\omega) = X\circ\theta_{-u}(\omega)1_{\{0<-u\le\Lambda_2^{-1}(1)\}}(\omega)\ .
$$

Then we immediately obtain (4.6). $\square$

Remark 4.3 Corollary 4.1 implies Lemma 4.2 as the special case where Λ_1 is Lebesgue measure. For the case where Λ_1 and Λ_2 are simple point processes, the expected Palm version of (4.6) is known as Neveu's formula (1976). (4.7) is a generalization of Lemma 2.4 in Nieuwenhuis (1993).

Thus, we have seen that the conditional versions (4.1), (4.2) and (4.5) of the modified Palm formulas are just obtained by replacing λ, λ_i, $\overline{P}_\Lambda$ and $\overline{P}_{\Lambda_i}$ by their conditional substitutes. This is very nice to interpret Palm formulas in terms of sample averages. We will discuss this in Sections 5 and 6.

5. Application to asymptotic stationarity

We now relax assumption (iii) in the following way. Let Λ satisfy (i) and (ii), η_x be defined by (2.1), and P be a probability measure on $(\Omega, \mathcal{F})$. We do not assume (iii) for P, but consider the following situation.

(iii-b) There exists a probability measure Q^0 on $(\Omega, \mathcal{F})$ satisfying

$$\lim_{x\to\infty} \frac{1}{x}\int_0^x P\left(\eta_y^{-1}(A)\right) m(dy) = Q^0(A) \qquad (A \in \mathcal{F}) . \tag{5.1}$$

If this condition holds, $\{\eta_x\}_{x\in S}$ is said to be Λ-asymptotically stationary (with respect to P) with limit distribution Q^0. This is abbreviated as $\{\eta_x\}$ is Λ-AS(Q^0). In particular, if Λ is Lebesgue measure, then $\eta_x = \theta_x$. Hence, by replacing Q^0 in (5.1) by Q, we have

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t P\left(\theta_u^{-1}(A)\right) du = Q(A) \qquad (A \in \mathcal{F}) . \tag{5.2}$$

In this case, $\{\theta_t\}$ is said to be asymptotically stationary (with respect to P) with limit distribution Q, which is abbreviated as $\{\theta_t\}$ is AS(Q).

Note that, in these situations, $\{\eta_x\}$ and $\{\theta_t\}$ are stationary with respect to Q^0 and Q, respectively. If conditions (i), (ii), and (iv-a) are satisfied, then (iii) implies (iii-b) for $Q^0 = \overline{P}_\Lambda$ by Theorem 3.1.

Theorem 5.1 Assume (i) and (ii). Then, there exists a probability measure Q on $(\Omega, \mathcal{F})$ such that $\{\theta_t\}$ is AS(Q) and $Q\left(0 < E_Q(\Lambda(1)|\mathcal{I}) < \infty\right) = 1$ if and only if there exists a probability measure Q^0 on $(\Omega, \mathcal{F})$ such that $\{\eta_x\}$ is Λ-AS(Q^0) and $Q^0\left(0 < E_{Q^0}(\Lambda^{-1}(1)|\mathcal{I}) < \infty\right)$ $= 1$. In this case, Q and Q^0 are uniquely determined by each other, and the following formulas hold.

$$E_Q(\Lambda(1)|\mathcal{I}) = \left(E_{Q^0}(\Lambda^{-1}(1)|\mathcal{I})\right)^{-1} \qquad P-,\ Q-,\ Q^0 - a.s. , \tag{5.3}$$

$$Q(A) = \lim_{t\to\infty} \frac{1}{t}\int_0^t Q^0\left(\theta_u^{-1}(A)\right) du , \tag{5.4}$$

$$Q^0(A) = \lim_{x\to\infty} \frac{1}{x}\int_0^x Q\left(\eta_y^{-1}(A)\right) m(dy) , \tag{5.5}$$

for all $A \in \mathcal{F}$. Furthermore, $Q^0 = \overline{Q}_\Lambda$, the modified Palm distribution of Q with respect to Λ, and $Q = \overline{Q^0}$, the modified time-stationary distribution of Q^0 with respect to Λ.

Proof Let $\{\theta_t\}$ be AS(Q) and assume that $Q\,(0 < E_Q(\Lambda(1)|\mathcal{I}) < \infty) = 1$. Note that $P = Q$ on $\mathcal{I}$. Hence, by (3.6), we have

$$\overline{Q}_\Lambda = Q = P \quad \text{on } \mathcal{I}. \tag{5.6}$$

Since $\{\eta_x\}$ is stationary with respect to the modified Palm distribution $\overline{Q}_\Lambda$, we have

$$\lim_{x\to\infty} \frac{1}{x}\int_0^x 1_A \circ \eta_y m(dy) = \overline{Q}_\Lambda(A|\mathcal{I}) \qquad \overline{Q}_\Lambda -, Q-, \text{ and } P - a.s. \tag{5.7}$$

Define $Q^0 = \overline{Q}_\Lambda$, and take the expectation of (5.7) under P. Then we have (5.1) by the dominated convergence theorem and (5.6), which simultaneously shows that Q^0 is uniquely determined by Q. Also take the expectation of (5.7) under Q and apply (5.6). Then we have (5.5). (5.4) is obtained similarly by using (5.6) and Birkhoff's ergodic theorem. The reverse implication in the iff statement can be proved in a similar way by defining $Q = \overline{Q^0}$, the modified time-stationary distribution of Q^0 with respect to Λ, and using Theorem 3.2. (5.3) is nothing but (3.13) of Theorem 3.2. □

The proof of Theorem 5.1 immediately leads to the following results.

Corollary 5.1 Assume (i) and (ii). Let Q and Q^0 be probability measures on $(\Omega, \mathcal{F})$ such that $\{\theta_t\}$ is stationary with respect to Q and $\{\eta_x\}$ with respect to Q^0. Then:

(a) $\{\theta_t\}$ is AS(Q) iff $P = Q$ on $\mathcal{I}$.

(b) $\{\eta_x\}$ is Λ-AS(Q^0) iff $P = Q^0$ on $\mathcal{I}$.

Theorem 5.1 says that, for a given random measure Λ, we can construct both asymptotically stationary distributions by starting with either one of them. This is analogous to the relation between a stationary distribution and its Palm distribution. In Sigman (1994), similar results are derived for point processes on the half line $[0, \infty)$, using coupling methods. We can further generalize Theorem 5.1 for Λ_1-AS(Q^1) and Λ_2-AS(Q^2) with random measures Λ_1 and Λ_2.

Theorem 5.2 Assume (i) and (ii) for both Λ_1 and Λ_2. For Λ_i ($i = 1, 2$) denote the corresponding η_x and m by $\eta_{i,x}$ and m_i, respectively. Then, there exists a probability measure Q^1 on $(\Omega, \mathcal{F})$ such that $\{\eta_{1,x}\}$ is Λ_1-AS(Q^1), $Q^1\left(0 < E_{Q^1}(\Lambda_1^{-1}(1)|\mathcal{I}) < \infty\right) = 1$ and $Q^1\left(0 < E_{\overline{Q^1}}(\Lambda_2(1)|\mathcal{I}) < \infty\right) = 1$ if and only if there exists a probability measure Q^2 on $(\Omega, \mathcal{F})$ such that $\{\eta_{2,x}\}$ is Λ_2-AS(Q^2), $Q^2\left(0 < E_{Q^2}(\Lambda_2^{-1}(1)|\mathcal{I}) < \infty\right) = 1$ and $Q^2\left(0 < E_{\overline{Q^2}}(\Lambda_1(1)|\mathcal{I}) < \infty\right) = 1$. In this case, $\overline{Q^1} = \overline{Q^2}$, and, under P, Q^1 and Q^2,

$$E_{Q^2}\left(\Lambda_1(\Lambda_2^{-1}(1))\Big|\mathcal{I}\right) = \left(E_{Q^1}\left(\Lambda_2(\Lambda_1^{-1}(1))\Big|\mathcal{I}\right)\right)^{-1} = \frac{E_{\overline{Q^1}}(\Lambda_1(1)|\mathcal{I})}{E_{\overline{Q^1}}(\Lambda_2(1)|\mathcal{I})} \quad a.s.\,. \tag{5.8}$$

$$Q^1(A) = E_{Q^2}\left(\frac{1}{E_{Q^2}\left(\Lambda_1(\Lambda_2^{-1}(1))\Big|\mathcal{I}\right)}\int_0^{\Lambda_2^{-1}(1)} 1_A \circ \theta_u \Lambda_1(du)\right)\,, \tag{5.9}$$

$$Q^2(A) = E_{Q^1}\left(\frac{1}{E_{Q^1}\left(\Lambda_2(\Lambda_1^{-1}(1))\Big|\mathcal{I}\right)}\int_0^{\Lambda_1^{-1}(1)} 1_A \circ \theta_u \Lambda_2(du)\right)\,, \tag{5.10}$$

for all $A \in \mathcal{F}$. Also,

$$Q^1(A) = \lim_{x\to\infty}\frac{1}{x}\int_0^x Q^2\left(\eta_{1,y}^{-1}(A)\right) m_1(dy)\,, \tag{5.11}$$

$$Q^2(A) = \lim_{x\to\infty}\frac{1}{x}\int_0^x Q^1\left(\eta_{2,y}^{-1}(A)\right) m_2(dy)\,. \tag{5.12}$$

Proof Assume that $\{\eta_{2,x}\}$ is Λ_2-AS(Q^2), $Q^2\left(0 < E_{Q^2}(\Lambda_2^{-1}(1)|\mathcal{I}) < \infty\right) = 1$ and Q^2 $\left(0 < E_{\overline{Q^2}}(\Lambda_1(1)|\mathcal{I}) < \infty\right) = 1$. So, (iii-a) and (iv-b) hold for Q^2 and Λ_2, and $\overline{Q^2}$ is well-defined. For $\overline{Q^2}$ and Λ_1, we have (iii) and (iv-a) (since $Q^2 = \overline{Q^2}$ on $\mathcal{I}$). Hence, the probability measure Q^1 defined as $\overline{(\overline{Q^2})_{\Lambda_1}}$ is also well-defined. Obviously, $\{\eta_{1,x}\}$ is stationary with respect to Q^1, and $E_{Q^1}(\Lambda_1^{-1}(1)|\mathcal{I}) = 1/E_{\overline{Q^2}}(\Lambda_1(1)|\mathcal{I})$ $Q^1-a.s.$ by Lemma 3.2. So, $Q^1(0 < E_{Q^1}(\Lambda_1^{-1}(1)|\mathcal{I}) < \infty) = 1$. Since Q^1 and Q^2 are both modified Palm distributions of $\overline{Q^2}$, we can apply Corollary 4.1, yielding, under $\overline{Q^2}$, Q^1, and Q^2,

$$E_{Q^1}(1_A|\mathcal{I}) = \frac{E_{\overline{Q^2}}(\Lambda_2(1)|\mathcal{I})}{E_{\overline{Q^2}}(\Lambda_1(1)|\mathcal{I})} E_{Q^2}\left(\int_0^{\Lambda_2^{-1}(1)} 1_A \circ \theta_u \Lambda_1(du)\Big|\mathcal{I}\right) \qquad a.s.\,. \tag{5.13}$$

Let $A = \Omega$ in (5.13), then the equality of the first and third terms of (5.8) follows. By applying this equality to (5.13), we obtain (5.9), since $\overline{Q^2} = Q^1 = Q^2$ on $\mathcal{I}$. Note that $\overline{Q^2}\left(0 < E_{\overline{Q^2}}(\Lambda_1(1)|\mathcal{I}) < \infty\right) = 1$. Hence, the fact that $\{\eta_{1,x}\}$ is Λ_1-AS(Q^1), follows from Theorem 5.1. Since

$$\lim_{x\to\infty}\frac{1}{x}\int_0^x 1_A \circ \eta_{1,y} m_1(dy) = Q^1(A|\mathcal{I}) \qquad Q^1-, \overline{Q^2}-, \text{ and } Q^2 - a.s.\,, \tag{5.14}$$

(5.11) follows. Since $E_{Q^2}(\Lambda_2^{-1}(1)|\mathcal{I}) = (E_{\overline{Q^2}}(\Lambda_2(1)|\mathcal{I}))^{-1}$, we also have $Q^1(0 < E_{\overline{Q^1}}(\Lambda_2(1)|\mathcal{I}) < \infty) = 1$ (note that $\overline{Q^1} = \overline{Q^2}$). The second equation of (5.8), (5.10) and (5.12) can be proved similarly by using the only-if part, which we have just proved. □

Remark 5.1 In the context of stationary, simple, marked point processes on $I\!R$, similar results are derived in Nieuwenhuis (1993; Sections 5 and 6).

6. Application to the extensions of $H = \lambda G$

Under the ergodicity assumption, Mecke's formula (2.6) and the generalized Mecke's formula (4.3) can be expressed in terms of long-run sample averages. So we can get sample-path versions of relations such as Little's formula $L = \lambda W$, $H = \lambda G$ and its extensions under the stationary framework. See, e.g. Franken et al. (1982) and Miyazawa (1994a, 1995). The important conclusion of Theorems 3.1 and 4.1 is that we can remove the ergodicity assumption for deriving those sample-path formulas. In this section, we exemplify this.

Sigman (1994) discussed this for $L = \lambda W$ and $H = \lambda G$ from the opposite side. That is, he derived conditional Palm formulas for $H = \lambda G$, i.e. the conditional version of Campbell's formula from the corresponding sample-path relations. We here derive sample average formulas by using Theorems 3.1 and 4.1.

Let $T_i(x) = \Lambda_i^{-1}(x)$, $\eta_{i,x} = \theta_{\Lambda_i^{-1}(x)}$ for $i = 1, 2$. Let m_i be the corresponding measure m of Λ_i for $i = 1, 2$. Note that $T_i(x)$ is the time t at which $\Lambda_i(t)$ attains the level x. For the case where Λ_i is a simple point process, the notation $T(n)$ has been introduced in Section 2 (see (2.7)). Then, we get the following result.

Theorem 6.1 Assume (i) and (ii) for both Λ_1 and Λ_2. Also assume that $\{\theta_t\}$ is AS(Q) with respect to P, and that $Q(0 < E_Q(\Lambda_i(1)|\mathcal{I}) < \infty) = 1$ for $i = 1, 2$, Then we have, under P, Q, $\overline{Q}_{\Lambda_1}$ and $\overline{Q}_{\Lambda_2}$,

$$\begin{aligned} &\lambda_{1,\mathcal{I}} \lim_{x\to\infty} \frac{1}{x} \int_0^x \left(\int_{-\infty}^{+\infty} f(T_1(y) - u, \theta_u)\Lambda_2(du) \right) m_1(dy) \\ &\quad = \lambda_{2,\mathcal{I}} \lim_{x\to\infty} \frac{1}{x} \int_0^x \left(\int_{-\infty}^{+\infty} f(u - T_2(y), \eta_{2,y})\Lambda_1(du) \right) m_2(dy) \ \ a.s.. \end{aligned} \quad (6.1)$$

Proof Note that, for all $\omega \in \Omega$, and $v \in I\!R$,

$$\left(\int_{-\infty}^{+\infty} f(u,\theta_u)\Lambda_2(du)\right)\circ\theta_v = \int_{-\infty}^{+\infty} f(u,\theta_{u+v})\Lambda_2(du+v) = \int_{-\infty}^{+\infty} f(u-v,\theta_u)\Lambda_2(du)\ , \tag{6.2}$$

and

$$\left(\int_{-\infty}^{+\infty} f(-u,\theta_0)\Lambda_1(du)\right)\circ\theta_v = \int_{-\infty}^{+\infty} f(-u,\theta_v)\Lambda_1(du+v) = \int_{-\infty}^{+\infty} f(v-u,\theta_v)\Lambda_1(du)\ . \tag{6.3}$$

Since Q satisfies conditions (iii) and (iv-a) for $\Lambda = \Lambda_i$ $(i = 1,2)$, we may apply Theorem 3.1, yielding

$$\lim_{x\to\infty}\frac{1}{x}\int_0^x 1_A\circ\eta_{i,y}m_i(dy) = \overline{Q}_{\Lambda_i}(A|\mathcal{I}) \quad a.s.$$

under Q (and hence also under P, $\overline{Q}_{\Lambda_1}$ and $\overline{Q}_{\Lambda_2}$). Apply (6.2) and (6.3) to this limit result, and rewrite $f(t,\cdot)$ by $f(-t,\cdot)$. Then (6.1) is an immediate consequence of Theorem 3.1. $\square$

The following corollary is a direct consequence of Theorems 3.2, 6.1 and Corollary 4.1.

Corollary 6.1 Assume (i) and (ii) for both Λ_1 and Λ_2. Also assume either that $i = 1$ and $j = 2$ or that $i = 2$ and $j = 1$. Then, if $\{\eta_{i,x}\}$ is Λ_i-AS(Q^i) with respect to P and if $Q^i(0 < E_{Q^i}(\Lambda_i^{-1}(1)|\mathcal{I}) < \infty, 0 < E_{\overline{Q^i}}(\Lambda_j(1)|\mathcal{I}) < \infty) = 1$, then (6.1) holds under P, Q^i, $\overline{Q^i}$ and Q^j, where Q^j is defined by (5.8) or (5.9), respectively, for $i = 2$ or $i = 1$.

A variant of Theorem 6.1 was obtained under the ergodicity assumption and conditions (iii) and (iv) for Λ_1 and Λ_2 in Miyazawa (1995). In his result, the expression corresponding to (6.1) is somehow different, because condition (ii) is not assumed. It is not hard to see that we can also get his result without the ergodicity assumption. However, condition (ii) enables us to have (6.1), which has a nice interpretation. Indeed, suppose that Λ_1 and Λ_2 are integrators (or weight functions) over space and time, respectively, and that $g(t,\Psi)$ is a reward at time t for an investment Ψ, where Ψ itself is a stochastic process. We assume that Ψ is consistent with θ_t, i.e. if $\Psi = \{X(t)\}$, then

$\Psi \circ \theta_s = \{X(t+s)\}$, and that $f(u, \theta_v) \equiv g(u, \Psi \circ \theta_v)$ satisfies the condition for f of (2.6). Define

$$\begin{aligned} h_1(y) &= \int_{-\infty}^{+\infty} g(u - T_2(y), \Psi \circ \eta_{2,y}) \Lambda_1(du) \\ h_2(y) &= \int_{-\infty}^{+\infty} g(T_1(y) - u, \Psi \circ \theta_u) \Lambda_2(du) \ . \end{aligned}$$

Then, $h_2(y)$ represents the total time average reward measured at time $T_1(y)$ as the origin, while $h_1(y)$ represents the total space average reward for the investment arriving at time $T_2(y)$. Then, (6.1) is equivalent to

$$\lambda_{1,\mathcal{I}} \lim_{x\to\infty} \frac{1}{x} \int_0^x h_2(y) m_1(dy) = \lambda_{2,\mathcal{I}} \lim_{x\to\infty} \frac{1}{x} \int_0^x h_1(y) m_2(dy) \ \ a.s. \ . \tag{6.4}$$

This formula can be considered as an extension of $H = \lambda G$. For example, let Λ_1 be Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, and let a stationary, simple, marked point process Φ be represented by $\{(T_i, X_i)\}_{i=-\infty}^{+\infty}$ with marks X_i in $\mathbb{R}$. We assume that Φ is consistent with $\{\theta_t\}$, i.e., $\Phi \circ \theta_t$ can be represented by $\{(T_i - t, X_i)\}_{i=-\infty}^{+\infty}$. Let Λ_2 be the point process N generated by $\{T_i\}$, and $\Psi = X_0$. We assume that $\lambda_{\mathcal{I}} \equiv E(N(1)|\mathcal{I}) < \infty$ $a.s.$. We rewrite $\eta_{2,y}$ as η_y. Hence, we have

$$\begin{aligned} h_1(n) &= \int_{-\infty}^{+\infty} g(u - T_n, \Psi \circ \eta_n) du = \int_{-\infty}^{+\infty} g(u - T_n, X_n) du \\ h_2(y) &= \int_{-\infty}^{+\infty} g(y - u, \Psi \circ \theta_u) N(du) = \sum_{k=-\infty}^{+\infty} g(y - T_k, X_k) \ . \end{aligned}$$

Then, (6.4) is nothing but Little's formula, if T_n and X_n are interpreted as the n-th arriving customer and his sojourn time, respectively, and if $g(t, x) = 1_{[0,x)}(t)$. $H = \lambda G$ is also obtained under the same setting, but replacing $\Psi = X_0$ by $\Psi = \Phi$. In this case, we rewrite $g(u - T_n, \Phi \circ \eta_n)$ as $g_n(u)$.

In fluid models for queues, we really need that Λ_2 is a random measure. Suppose that the fluid flows into a buffer and it is released with a stationary rate. Let $\Lambda = \Lambda_2(t)$ be the accumulated input up to time t. We assume that $\Lambda_2(t)$ is continuous in t, and that Λ_2 satisfies (i) and (ii). Let Λ_1 be still Lebesgue measure, and Ψ be the sojourn time of a fluid arriving at time 0, which is denoted by $X(0)$. We assume that $X(0) \circ \theta_t = X(t)$. Define $g(t, x) = 1_{[0,x)}(t)$. Then, h_1 and h_2 become

$$\begin{aligned} h_1(y) &= \int_{-\infty}^{+\infty} g(u - T(y), X(T(y))) du = X(T(y)) \ , \\ h_2(y) &= \int_{-\infty}^{+\infty} g(y - u, X(u)) \Lambda(du) = \int_{u \le y < u + X(u)} \Lambda(du) \ . \end{aligned}$$

$h_2(y)$ can be interpreted as a buffer content at time y. Thus we get the continuous version of Little's formula, which was originally derived by Rolski and Stidham (1983) (see also Miyazawa (1994a)).

Acknowledgement

A part of this paper was written while the first author visited Dept. of IEOR of University of California at Berkeley, in June, 1994. The authors are grateful to Ronald W. Wolff for helping them to do this work. The first author is partly supported by NEC C&C Laboratories.

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