On the Unification of Centralized and Decentralized Clearing Mechanisms in Financial Networks

Ketelaars, Martijn; Borm, Peter

Publication date:
2021

Document Version
Early version, also known as pre-print

Citation for published version (APA):
Ketelaars, M., & Borm, P. (2021). *On the Unification of Centralized and Decentralized Clearing Mechanisms in Financial Networks*. (CentER Discussion Paper; Vol. 2021-015). CentER, Center for Economic Research.

CentER

Discussion Paper

No. 2021-015

ON THE UNIFICATION OF CENTRALIZED AND DECENTRALIZED CLEARING MECHANISMS IN FINANCIAL NETWORKS

By

Martijn Ketelaars, Peter Borm

17 June 2021

ISSN 0924-7815
ISSN 2213-9532

Tilburg University

On the unification of centralized and decentralized clearing mechanisms in financial networks

Martijn Ketelaars[1,2] Peter Borm[2]

Abstract

We analyze clearing mechanisms in financial networks in which agents may have both monetary individual assets and mutual liabilities. A clearing mechanism prescribes mutual payments between agents to settle their mutual liabilities. The corresponding payments, summarized in a payment matrix, are made in accordance with agent-specific claims rules that stem from the vast literature on claims problems. We show that large classes of centralized and decentralized clearing mechanisms all prescribe the same payment matrix under the condition that the underlying claims rules satisfy composition; a property satisfied by the proportional rule that is often applied in insolvency proceedings. This payment matrix is the one that contains the minimal amount of payments required to clear the network. In fact, we show that composition guarantees unification of clearing mechanisms in which agents pay simultaneously and clearing mechanisms in which agents pay sequentially in any arbitrary order. Therefore, for a given financial network, each clearing mechanism gives rise to the same transfer allocation. Moreover, we provide an axiomatic characterization of the corresponding mutual claims rule on the basis of five axioms: scale invariance, equal treatment of equals, composition, path independence and consistency. This characterization extends the analogous characterization for claims rules as given by Moulin (2000).

Keywords: Clearing mechanisms, decentralization, financial networks and contagion, mutual claims rules, composition.

JEL Classification Number: C71, G33, G10.

[1]Corresponding author. Email: M.W.Ketelaars@tilburguniversity.edu

[2]Tilburg University, Department of Econometrics and Operations Research, The Netherlands.

1 Introduction

Unlike classical bankruptcy situations in which claimants claim from a single bankrupt entity, bankruptcy situations in a network setting are more intricate because of the interdependence between agents. Exogenous or endogenous shocks can disrupt a financial system and can cause agents to default which may trigger default of other agents or systems. This phenomenon is known as *financial contagion*; see Glasserman and Young (2016) for an overview. Although the network perspective became increasingly important as a result of the global financial and banking crises between mid 2007 and early 2009, the recent COVID-19 pandemic stresses the importance of studying bankruptcy situations in the network setting once more.

Our model of a financial network follows the seminal paper of Eisenberg and Noe (2001). A financial network comprises a finite set of agents and is characterized by a non-negative estates vector containing each agent's monetary estate and a non-negative claims matrix containing mutual rightful claims between pairs of agents. The unilateral setting is characterized by a single agent's estate and a vector of rightful claims on the estate. A bankruptcy problem arises if this agent is insolvent and we need to allocate the available estate among the claimants. After the early work of O'Neill (1982) on bankruptcy problems, many so-called claims rules have been introduced; see Thomson (2003, 2013, 2015) for an overview. A claims rule provides, for each bankruptcy problem, an allocation of the estate among the claimants. In essence, claims rules are clearing mechanisms in the unilateral setting, and thus naturally form the basis for clearing mechanisms in the network setting. We follow this convention, but want to emphasize that clearing mechanisms without underlying claims rules can be analyzed as well; see, e.g., Ketelaars (2020).

In this paper, we study clearing mechanisms and their corresponding allocation mechanisms in financial networks. A clearing mechanism prescribes mutual payments between agents to settle their mutual liabilities. The corresponding payments, summarized in a payment matrix, are made in accordance with claims rules. Most of the literature on financial contagion follows Eisenberg and Noe (2001) and uses the proportional rule as the underlying payment mechanism. This is in line with current insolvency proceedings where payments are in accordance with the *pari passu principle* — Latin for "equally and without preference". However, the proportional rule does not fully encompass insolvency proceedings in, for instance, the United States and the European Union. In practice, claimants in insolvency proceedings are partitioned into priority classes and within each class there is pari passu treatment.[1] Moulin (2000) provides an axiomatic characterization of such priority claims rules in bankruptcy problems. Since we want to take such priority claims rules and the individuality of agents into account, we allow for general agent-specific claims rules like Csóka and Herings (2018). Nevertheless, contrary to Csóka and Herings (2018) but in line with all research on financial contagion, we consider financial networks in the perfectly divisible setup.

The goal of this paper is to unify centralized and decentralized clearing mechanisms. In centralized clearing mechanisms, an independent authority is in charge of the clearing process. In accordance with the literature on financial contagion, we characterize a payment matrix of a centralized clearing mechanism by a fixed point (equilibrium) of an appropriate mapping. We call payment matrices corresponding to centralized clearing mechanisms ϕ-transfer schemes, where ϕ is a vector of claims rules. A ϕ-transfer scheme satisfies three criteria. First, payments are made in accordance with claims rules. Second, the total payment by an agent cannot exceed the amount he has to his disposal; a *limited liability* requirement. Third, each agent either pays off all his debts, or everything he has to his disposal is paid to his claimants; an *absolute priority of debt over equity* requirement. The set of ϕ-transfer schemes is a complete lattice so that there exists a bottom ϕ-transfer scheme and a top ϕ-transfer scheme; see, e.g., Csóka and Herings (2018). Despite the existence of (infinitely) many ϕ-transfer schemes, Groote Schaarsberg, Reijnierse, and Borm (2018) show that any two ϕ-transfer schemes lead to the same transfer allocation. A transfer allocation is a redistribution of the estates vector according to a payment matrix. The literature on contagion is concerned with computing the top ϕ-transfer scheme. For example, Eisenberg and Noe (2001) and Rogers and Veraart (2013) propose efficient algorithms to compute the top ϕ-transfer scheme, where ϕ comprises proportional rules. We, on the other hand, focus on the bottom ϕ-transfer scheme. We propose a recursive procedure that generates a monotonically increasing sequence of payment matrices that converges to the bottom ϕ-transfer scheme. This procedure need not terminate in a finite number of steps, but, in practice, converges rather quickly and stable.

The reason we consider the bottom ϕ-transfer scheme is because it provides a direct connection with decentralized clearing mechanisms. In decentralized clearing mechanisms, agents as individuals are in charge of the clearing process. One advantage of such mechanisms is that it relies only on

[1] We refer to Kaminski (2000) and Chapter 4 of Wessels and Madaus (2017) for more details.

local information. As far as we know, there exist two decentralized clearing mechanisms in the literature that adhere to the same three aforementioned criteria.

The first one is the ϕ-based individual settlement allocation procedure (ISAP) put forward by Ketelaars, Borm, and Quant (2020). In each step of this recursive procedure, each agent pays the other agents by allocating his current estate on the basis of his own claims rule. As a result of these payments, there is a redistribution of the individual estates and a reduction in the mutual liabilities. Such a procedure need not be finite, but has a corresponding limiting payment matrix nonetheless.

The second one is the class of ϕ-based decentralized clearing processes put forward by Csóka and Herings (2018), but in a discrete setup. We first generalize this process for a perfectly divisible setup. In each step of this recursive procedure, exactly one agent is selected that communicates to the other agents what he would pay according to his own claims rule based on what he currently has to his disposal. The amount each agent has to his disposal in a step is his initial estate plus provisional payments from the other agents. The selection process need not be deterministic. In general, the selection process can be history dependent, stochastic or both. Csóka and Herings (2018) show that in the discrete setup any ϕ-based decentralized clearing process terminates in a finite number of iterations and converges to the bottom ϕ-transfer scheme. In the perfectly divisible setup, a ϕ-based decentralized clearing process need not be a finite process but we show that it still converges to the bottom ϕ-transfer scheme, irrespective of the selection process. The actual payments that take place in a ϕ-based decentralized clearing process are thus those with respect to the bottom ϕ-transfer scheme.

In addition to the two above decentralized clearing mechanisms, we introduce one that is a variation on ϕ-based ISAP, a procedure in which agents pay simultaneously in each step. Now, exactly one agent is selected in each step that pays the other agents by allocating his current estate on the basis of his own claims rule. Notice that such a mechanism differs from a ϕ-based decentralized clearing process since actual transfers between agents take place in each step.

Our main result is one of unification: the centralized clearing mechanism based on ϕ-transfer schemes and all existing decentralized clearing mechanisms prescribe the bottom ϕ-transfer scheme under the condition that the underlying claims rules satisfy composition. Composition (Young, 1988) is a property that pertains to situations in which an agent allocates a provisional estate value but later learns that the true value is larger than expected. If a claims rule satisfies composition, then, for any bankruptcy problem, allocating the surplus value and adding this to the initial allocation is equivalent to simply reallocating the actual (larger) value.

We want to emphasize that our main result has applications beyond finance. One example is the distribution of COVID-19 vaccines. Here, the network consists of governments, pharmaceutical companies and health care institutions such as hospitals, where agents have mutual (rightful) claims on each other in the form of vaccine doses. However, currently, the number of vaccine doses available in the network, which are owned by individual agents, falls short of meeting the demand. There could be several reasons for a shortage in the number of vaccine doses. One can think about disruptions to the production process, like technical or upscaling issues, or disruptions to the underlying supply chain. In other words, vaccine doses become available in stages. Consequently, composition is a desirable property as it guarantees invariance of clearing and allocating in stages.

Finally, our main result enables us to extend the axiomatic characterization for claims rules as given by Moulin (2000). To this end, we build on the literature on mutual claims problems and mutual claims rules introduced by Groote Schaarsberg et al. (2018). Mutual claims problems generalize bankruptcy problems in the sense that insolvency of (some of the) agents calls for an allocation of the total estate among the agents. A mutual claims rule prescribes, for each mutual claims problem, an allocation of the total estate among the agents. For our purpose, we consider two types of such rules: ϕ-based mutual claims rules corresponding to the centralized mechanism based on ϕ-transfer schemes, and recursive ϕ-based mutual claims rules corresponding to ϕ-based ISAP. Both types of rules prescribe a transfer allocation based on payments with respect to ϕ that follow from its respective clearing mechanism. Moreover, we provide adequate extensions of the properties that Moulin (2000) considers: scale invariance, composition, path independence and consistency. While Moulin (2000) leaves out the equal treatment of equals axiom, we include it since then exactly three claims rules remain: the proportional, equal-awards and equal-losses rules. In the literature, these three rules are otherwise known as the three musketeers (Herrero & Villar, 2001). We show that under the extensions of these five properties exactly 3^n (recursive) ϕ-based mutual claims rules remain, where n is the number of agents in the network. There are 3^n of such rules because each coordinate of the vector of claims rules ϕ can be one of the three musketeers.

As far as we know, there exist two other axiomatic characterizations of mutual claims rules in the literature on clearing in financial networks. First, Groote Schaarsberg et al. (2018) provide an axiomatic characterization of a

ϕ-based mutual claims rule on the basis of the *concede-and-divide principle* and consistency. In this characterization, ϕ is the Talmud rule (Aumann & Maschler, 1985). Second, Csóka and Herings (2021) provide an axiomatic characterization of the proportional rule in financial networks.

The paper is organized as follows. In Section 2, we review existing claims rules such as the three musketeers. Section 3 contains our main result on the unification of the centralized clearing mechanism based on ϕ-transfer schemes and the decentralized clearing mechanism based on ϕ-based ISAP. In Section 4, we introduce extensions of five properties of claims rules and correspondingly provide the extension of the axiomatic characterization of Moulin (2000). In Section 5, we extend our main result by considering decentralized sequential clearing mechanisms. Section 6 concludes.

2 Claims rules

A *claims problem* is modeled by a pair $(e, c) \in \mathbb{R}_+ \times \mathbb{R}^N_+$ where N is a finite set of *claimants*, e is an *estate* and $c = (c_i)_{i\in N}$ is a vector of rightful claims on the estate. The class of all claims problems on N is denoted by $\mathcal{C}^N$ and the class of all claims problems with arbitrary but finite N is denoted by $\mathcal{C}$. A subclass of claims problems on N where the sum of claims exceeds the value of estate, i.e., $\sum_{i\in N} c_i > e$, is the class of *bankruptcy problems* and is denoted by $\mathcal{B}^N$.

A *claims rule* $\varphi\colon \mathcal{C}^N \to \mathbb{R}^N$ prescribes how the estate in each claims problem will be allocated among the claimants. For all $(e, c) \in \mathcal{C}^N$, the allocation vector $\varphi(e, c)$ satisfies

(i) $0 \leq \varphi_i(e, c) \leq c_i$ for all $i \in N$,

(ii) $\displaystyle\sum_{i\in N} \varphi_i(e, c) = \min\{e, \sum_{i\in N} c_i\}$.

The second condition (ii) boils down to $\sum_{i\in N} \varphi_i(e, c) = e$ in bankruptcy problems. On the other hand, if the estate can cover all claims, then conditions (i) and (ii) imply $\varphi(e, c) = c$.

From the outset we assume that a claims rule satisfies estate monotonicity, which requires that no claimant should receive less than what he did receive initially when it turns out there is more to be allocated.

Definition 2.1. A claims rule φ satisfies *estate monotonicity* if for all $(e, c) \in \mathcal{C}^N$ and $(\tilde{e}, c) \in \mathcal{C}^N$ with $e \leq \tilde{e}$ it holds that $\varphi(e, c) \leq \varphi(\tilde{e}, c)$.[2]

[2]Note that the inequality here is a vector inequality, i.e., for two vectors $x, y \in \mathbb{R}^N$

Estate monotonicity implies another desirable property, namely estate continuity.

Definition 2.2. A claims rule φ satisfies *estate continuity* if for all $(e, c) \in \mathcal{C}^N$ and for any sequence of non-negative estates $\{e^k\}_{k=1}^{\infty}$ that converges to e, the sequence $\{\varphi(e^k, c)\}_{k=1}^{\infty}$ converges to $\varphi(e, c)$.

The following three claims rules are well known in the literature on claims problems. For an extensive survey on claims rules, see Thomson (2003, 2013, 2015). The *proportional rule* PROP prescribes a proportional allocation of the estate on the basis of the proportion of a claimant's claim to the total amount of claims. For all $(e, c) \in \mathcal{B}^N$, it is defined by

$$\mathrm{PROP}_i(e, c) = \frac{c_i}{\sum_{j \in N} c_j}\, e$$

for all $i \in N$.

The *constrained equal-awards rule* CEA interprets equality in terms of gains. It allocates the estate among the claimants as equal as possible provided that no claimant receives more than he claims. For all $(e, c) \in \mathcal{B}^N$, it is defined by

$$\mathrm{CEA}_i(e, c) = \min\{c_i, \alpha\}$$

for all $i \in N$, with $\alpha \geq 0$ such that $\sum_{j \in N} \min\{c_j, \alpha\} = e$.

The natural dual of the constrained equal-awards rule is the *constrained equal-losses rule* CEL in the sense that it interprets equality in terms of losses with respect to the claims. The losses are incurred as equal as possible by the claimants provided that no claimant's loss exceeds his claim. For all $(e, c) \in \mathcal{B}^N$, it is defined by

$$\mathrm{CEL}_i(e, c) = \max\{0, c_i - \beta\}$$

for all $i \in N$, with $\beta \geq 0$ such that $\sum_{j \in N} \max\{0, c_j - \beta\} = e$. We adopt the terminology of Herrero and Villar (2001) and refer to the three above rules as *the three musketeers*.

Likewise, we complement these three rules with a fourth rule which plays the role of *d'Artagnan*: the *Talmud rule* TAL (Aumann & Maschler, 1985).

with N being a finite set, we have $x \leq y$ if and only if $x_i \leq y_i$ for all $i \in N$. We have $x < y$ if and only if $x \leq y$ and $x_i < y_i$ for at least one $i \in N$.

For all $(e,c) \in \mathcal{B}^N$, it is defined in terms of CEA as follows:

$$\mathrm{TAL}(e,c) = \begin{cases} c - \mathrm{CEA}(\sum_{j \in N} c_j - e, \frac{1}{2}c) & \text{if } \sum_{j \in N} c_j \leq 2e, \\ \mathrm{CEA}(e, \frac{1}{2}c) & \text{if } \sum_{j \in N} c_j \geq 2e. \end{cases}$$

Finally, Young (1988) introduces a large class of so-called *equal sacrifice methods*. We consider one such method which will be used in an example later on. The *equal sacrifice methods rule* ESM is, for all $(e,c) \in \mathcal{B}^N$ defined by

$$\mathrm{ESM}_i(e,c) = \frac{c_i}{1 + \lambda c_i}$$

for all $i \in N$, with $\lambda \geq 0$ such that $\sum_{j \in N} \frac{c_j}{1+\lambda c_j} = e$.

3 Clearing and allocation mechanisms

Mutual claims problems generalize claims problems by allowing for multiple estates and mutual claims. A mutual claims problem is modeled by a pair $(E,C) \in \mathbb{R}^N_+ \times \mathbb{R}^{N \times N}_+$ where N is a finite set of agents, $E = (e_i)_{i \in N}$ is an *estates vector* and $C = (c_{ij})_{i,j \in N}$ is a *claims matrix*. Each coordinate e_i of E represents the estate belonging to agent $i \in N$. The claims matrix C represents mutual liabilities between agents. Each cell c_{ij} of C represents the rightful claim of agent $j \in N$ on agent $i \in N$. Row i in C thus captures creditors of agent i, while column i of C captures debtors of agent i. By assumption, agents have no claim on themselves, so, for all $i \in N$, $c_{ii} = 0$. No additional conditions are imposed on the claims matrix, in particular, there is no condition on the relation between claims c_{ij} and c_{ji} for $i \neq j$. We denote the i-th row of C by $\bar{c}_i = (c_{ij})_{j \in N}$.

The class of all mutual claims problems on N is denoted by $\mathcal{L}^N$ and the class of all mutual claims problems with arbitrary but finite N is denoted by $\mathcal{L}$. A *mutual claims rule* $\mu\colon \mathcal{L}^N \to \mathbb{R}^N$ prescribes how the total estate value in the network will be allocated among its agents. For all $(E,C) \in \mathcal{L}^N$, the allocation vector $\mu(E,C)$ satisfies

(i) $\mu_i(E,C) \geq 0$ for all $i \in N$,

(ii) $\sum_{i \in N} \mu_i(E,C) = \sum_{i \in N} e_i$.

Condition (i) is a non-negativity condition and condition (ii) implies that a reallocation of the estates neither generates nor destroys value.

In the definition of a mutual claims rule, we do not specify how the reallocation of the total estate value, in the form of transfers (e.g. payments) between agents, should take place. One way to pin down the clearing mechanism is to impose that transfers between agents constitute a ϕ-transfer scheme which then gives rise to a transfer allocation. We will elaborate on this in the next section on transfer schemes.

3.1 Transfer schemes

All mutual claims rules analyzed in this paper rely on an underlying *payment matrix*. A payment matrix is a non-negative matrix $P = (p_{ij})_{i,j \in N}$ where cell p_{ij} indicates the payment of agent i to agent j. A (bilateral) transfer scheme is a specific type of payment matrix that contains feasible and reasonable (bilateral) payments.

Definition 3.1. Let $(E, C) \in \mathcal{L}^N$. The payment matrix $P = (p_{ij}) \in \mathbb{R}_+^{N \times N}$ is called a *transfer scheme* for (E, C) if,

(i) for all $i \in N$, $p_{ii} = 0$,

(ii) for all $i, j \in N$, $0 \leq p_{ij} \leq c_{ij}$,

(iii) for all $i \in N$, $\sum_{m \in N} p_{im} \leq e_i + \sum_{m \in N} p_{mi}$.

The set of all possible transfer schemes for (E, C) is denoted by $\mathcal{P}(E, C)$.

Condition (i) is a normalization assumption; condition (ii) bounds the payment of agent $i \in N$ to agent $j \in N$ by zero and the claim of agent j on agent i; condition (iii) is a limited liability assumption in the sense that the total payment by agent $i \in N$ cannot exceed the amount agent i has to his disposal.

A transfer scheme can directly be translated into a transfer allocation, where the allocation to each agent equals his initial estate plus his net payments.

Definition 3.2. Let $(E, C) \in \mathcal{L}^N$ and let $P \in \mathcal{P}(E, C)$. The vector $\alpha^P \in \mathbb{R}^N$ is a *transfer allocation* if, for all $i \in N$,

$$\alpha_i^P = e_i + \sum_{j \in N} p_{ji} - \sum_{j \in N} p_{ij}. \tag{3.1}$$

Note that condition (iii) of a transfer scheme implies that the transfer allocation is non-negative.

In contrast to Groote Schaarsberg et al. (2018) and Ketelaars et al. (2020), we allow for agent-specific claims rules. Formally, we let $\phi = (\varphi^i)_{i \in N}$ be a vector of claims rules, where φ^i is the claims rule associated with agent $i \in N$. Correspondingly, a payment matrix is called a ϕ-transfer scheme if ϕ forms the basis for a payment matrix with the additional requirement that payments satisfy an equilibrium condition.

Definition 3.3. Let $(E, C) \in \mathcal{L}^N$ and let $\phi = (\varphi^i)_{i \in N}$ be a vector of claims rules. The payment matrix $P = (p_{ij}) \in \mathbb{R}_+^{N \times N}$ is called a *ϕ-transfer scheme* for (E, C) if, for all $i, j \in N$,

$$p_{ij} = \varphi^i_j(e_i + \sum_{m \in N} p_{mi}, \bar{c}_i). \tag{3.2}$$

The set of all possible ϕ-transfer schemes for (E, C) is denoted by $\mathcal{P}^\phi(E, C)$.

One of the main results of Groote Schaarsberg et al. (2018) is that any two ϕ-transfer schemes lead to the same transfer allocation. Note that in Groote Schaarsberg et al. (2018) agents share a common claims rule, i.e., $\phi = (\varphi^i)_{i \in N}$ with $\varphi^i = \varphi$ for all $i \in N$. It is readily verified that the result carries over to the situation with agent-specific claims rules.

Theorem 3.1 (cf. Groote Schaarsberg et al. (2018))**.** *Let $(E, C) \in \mathcal{L}^N$, let $\phi = (\varphi^i)_{i \in N}$ be a vector of claims rules and let $P, P' \in \mathcal{P}^\phi(E, C)$. Then, $\alpha^P = \alpha^{P'}$.*

The above theorem implies that the resulting allocation vector depends only on ϕ and not on the underlying ϕ-transfer scheme.

Definition 3.4. A *ϕ-based mutual claims rule* ρ^ϕ on $\mathcal{L}$ is, for all finite N and all $(E, C) \in \mathcal{L}^N$, defined by

$$\rho^\phi(E, C) = \alpha^P,$$

where P is a ϕ-transfer scheme for (E, C).

So far we have not discussed the existence of ϕ-transfer schemes. For our specific purposes, we show existence of a ϕ-transfer scheme by means of the following recursive procedure, provided that all claims rules in ϕ satisfy estate monotonicity.[3]

[3]Alternatively, one can define a monotone mapping on a complete lattice and use Tarski's fixed-point theorem (Tarski, 1955), see, e.g., Csóka and Herings (2018) or Ketelaars (2020).

Let $(E, C) \in \mathcal{L}^N$. A ϕ-transfer scheme $P = (p_{ij})_{i,j \in N}$ with respect to (E, C) can be constructed recursively as follows. Define for all $i \in N$ and $k \in \mathbb{N}$,

$$\gamma_i(k+1) = e_i + \sum_{j \in N} \varphi_i^j(\gamma_j(k), \bar{c}_j), \tag{3.3}$$

with $\gamma_i(1) = e_i$. Then, for $i, j \in N$, set

$$p_{ij} = \lim_{k \to \infty} \varphi_j^i(\gamma_i(k), \bar{c}_i). \tag{3.4}$$

First, let us verify that the limit in (3.4) exists. Let $i \in N$ and note that

$$\gamma_i(1) = e_i \leq e_i + \sum_{j \in N} \varphi_i^j(\gamma_j(1), \bar{c}_j) = \gamma_i(2).$$

Let $k \in \mathbb{N}$ and assume that $\gamma(k) \leq \gamma(k+1)$. Estate monotonicity of the claims rules in ϕ then implies that

$$\gamma_i(k+1) = e_i + \sum_{j \in N} \varphi_i^j(\gamma_j(k), \bar{c}_j) \leq e_i + \sum_{j \in N} \varphi_i^j(\gamma_j(k+1), \bar{c}_j) = \gamma_i(k+2).$$

By induction it follows that (3.3) constitutes a monotonically increasing sequence

$$\gamma_i(1) \leq \gamma_i(2) \leq \gamma_i(3) \leq \dots,$$

that is bounded from above by $e_i + \sum_{j \in N} c_{ji}$. By the monotone convergence theorem, the sequence $\{\gamma_i(k)\}_{k \in \mathbb{N}}$ has a limit.

Now, to verify that P is in fact a ϕ-transfer scheme, one needs to check condition (3.2). Let $i, j \in N$, then

$$\begin{aligned} p_{ij} &= \lim_{k \to \infty} \varphi_j^i(\gamma_i(k), \bar{c}_i) \\ &= \varphi_j^i(\lim_{k \to \infty} \gamma_i(k), \bar{c}_i) \\ &= \varphi_j^i(e_i + \lim_{k \to \infty} \sum_{m \in N} \varphi_i^m(\gamma_m(k), \bar{c}_m), \bar{c}_i) \\ &= \varphi_j^i(e_i + \sum_{m \in N} p_{mi}, \bar{c}_i). \end{aligned}$$

The first equality follows from (3.4), the second equality follows from estate continuity of φ^i, the third equality follows from (3.3) and the last equality follows from (3.4).

Remark 3.1. *If one replaces the starting point* $\gamma_i(1) = e_i$ *with* $\gamma_i(1) = e_i + \sum_{j\in N} c_{ji}$ *for each* $i \in N$*, then the recursive procedure generates a monotonically decreasing sequence of payment matrices that converges to a* ϕ*-transfer scheme as well. However, this* ϕ*-transfer scheme need not be the same as the one defined by* (3.4).

A transfer scheme that is constructed as before is henceforth called the bottom ϕ-transfer scheme.

Definition 3.5. Let $(E, C) \in \mathcal{L}^N$ and let $\phi = (\varphi^i)_{i\in N}$ be a vector of claims rules. The payment matrix $\underline{P}^\phi = (p_{ij}) \in \mathbb{R}_+^{N\times N}$ is the *bottom* ϕ*-transfer scheme* if, for all $i, j \in N$,

$$p_{ij} = \lim_{k\to\infty} \varphi^i_j(\gamma_i(k), \bar{c}_i),$$

where, for all $i \in N$ and $k \in \mathbb{N}$,

$$\gamma_i(k+1) = e_i + \sum_{j\in N} \varphi^j_i(\gamma_j(k), \bar{c}_j),$$

with $\gamma_i(1) = e_i$.

The recursive procedure is illustrated in the following example.

Example 3.1. Consider the mutual claims problem $(E, C) \in \mathcal{L}^N$ given by $N = \{1, 2, 3\}$,

$$E = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \text{ and } C = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 5 & 2 & 0 \end{bmatrix}.$$

Let $\phi = (\text{CEA}, \text{CEL}, \text{TAL})$. Initially, we have $\gamma(1) = E = (2, 1, 1)$. To determine $\gamma(2)$, we compute the agents' payments on the basis of $\gamma(1)$, which are given by

$$\text{CEA}(2, (0, 1, 2)) = (0, 1, 1), \quad \text{(Agent 1)}$$
$$\text{CEL}(1, (1, 0, 1)), = (\frac{1}{2}, 0, \frac{1}{2}), \quad \text{(Agent 2)}$$
$$\text{TAL}(1, (5, 2, 0)) = (\frac{1}{2}, \frac{1}{2}, 0). \quad \text{(Agent 3)}$$

Hence, $\gamma(2) = (2, 1, 1) + (1, 1\frac{1}{2}, 1\frac{1}{2}) = (3, 2\frac{1}{2}, 2\frac{1}{2})$. Next, the payments under $\gamma(2)$ are given by

$$\text{CEA}(3, (0, 1, 2)) = (0, 1, 2), \quad \text{(Agent 1)}$$

$$\mathrm{CEL}(2\frac{1}{2}, (1,0,1)) = (1,0,1), \qquad \text{(Agent 2)}$$

$$\mathrm{TAL}(2\frac{1}{2}, (5,2,0)) = (1\frac{1}{2}, 1, 0). \qquad \text{(Agent 3)}$$

Hence, $\gamma(3) = (2,1,1) + (2\frac{1}{2}, 2, 3) = (4\frac{1}{2}, 3, 4)$. We see that under $\gamma(2)$ agents 1 and 2 have paid off all their debts, which means that their payments do not change in subsequent steps. Since agent 3 has not yet paid off all his debts, his payment will change under $\gamma(3)$:

$$\mathrm{CEA}(4\frac{1}{2}, (0,1,2)) = (0,1,2), \qquad \text{(Agent 1)}$$

$$\mathrm{CEL}(3, (1,0,1)) = (1,0,1), \qquad \text{(Agent 2)}$$

$$\mathrm{TAL}(4, (5,2,0)) = (3,1,0). \qquad \text{(Agent 3)}$$

Hence, $\gamma(4) = (2,1,1) + (4,2,3) = (6,3,4)$. Note that agent 3 can only allocate a maximum of 4 since agents 1 and 2 have paid off all their debts. Therefore, no more updates will take place in subsequent steps. We have $\gamma(4) = \gamma(5) = \gamma(6) = \dots$, so that the limit of $\{\gamma(k)\}_{k \in \mathbb{N}}$ is obtained in a finite number of steps. The corresponding ϕ-transfer scheme is given by[4]

$$P = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix};$$

the transfer allocation is given by

$$\rho^{\phi}(E, C) = (2,1,1) + (4,2,3) - (3,2,4) = (3,1,0).$$

△

The following theorem justifies the term bottom ϕ-transfer scheme.

Theorem 3.2. *Let $(E, C) \in \mathcal{L}^N$ and let $\phi = (\varphi^i)_{i \in N}$ be a vector of claims rules. Then, $\underline{P}^{\phi} \leq P$ for all $P \in \mathcal{P}^{\phi}(E, C)$.*

Proof. Let $\{P^k\}_{k \in \mathbb{N}}$ be the sequence of matrices where, for each $k \in \mathbb{N}$, $P^k = (p^k_{ij})_{i,j \in N}$ is given by

$$p^k_{ij} = \varphi^i_j(\gamma_i(k), \bar{c}_i),$$

[4] If we replace the starting point $\gamma_i(1) = e_i$ by $\gamma_i(1) = e_i + \sum_{j \in N} c_{ji}$ for all $i \in N$, then we obtain $\gamma(1) = (2,1,1) + (6,3,3) = (8,4,4)$. Consequently, it is immediate that $\gamma(k) = (2,1,1) + (4,2,3) = (6,3,4)$ for all $k \in \{2,3,\dots\}$ so that the resulting payment matrix is P as well.

for $i, j \in N$. Let $i \in N$. Then, as we have seen before,

$$\gamma_i(1) \leq \gamma_i(2) \leq \gamma_i(3) \leq \ldots,$$

and, by estate monotonicity of φ^i,

$$P_i^1 \leq P_i^2 \leq P_i^3 \leq \ldots, \tag{3.5}$$

where, for all $k \in \mathbb{N}$, P_i^k is the i-th row of matrix P^k. Then, for the bottom ϕ-transfer scheme it follows that $\underline{P}^\phi = (\lim_{k\to\infty} p_{ij}^k)_{i,j\in N} = \lim_{k\to\infty}\{P^k\}$.

Now, let $P \in \mathcal{P}^\phi(E, C)$ and let, for all $i \in N$, P_i and P_i^ϕ denote the i-th rows of $P = (p_{ij})_{i,j\in N}$ and $\underline{P}^\phi = (p_{ij}^\phi)_{i,j\in N}$, respectively. We will prove that $\underline{P}^\phi \leq P$.

Assume that $\underline{P}^\phi \leq P$ does not hold. Then, there exists at least one agent $i \in N$ such that $p_{ij}^\phi > p_{ij}$ for at least one $j \in N$. Without loss of generality, let $i = 1$. If $p_{1j}^\phi > p_{1j}$ for at least one $j \in N$, then by condition (3.2) of a ϕ-transfer scheme

$$p_{1j}^\phi = \varphi_j^1(e_1 + \sum_{m\in N} p_{m1}^\phi, \bar{c}_1) > \varphi_j^1(e_1 + \sum_{m\in N} p_{m1}, \bar{c}_1) = p_{1j}.$$

Consequently, estate monotonicity of claims rule φ^1 implies that we must have

$$e_1 + \sum_{m\in N} p_{m1}^\phi > e_1 + \sum_{m\in N} p_{m1},$$

and thus $p_{1j}^\phi \geq p_{1j}$ for all $j \in N$. Equivalently, in vector notation, we have $P_1^\phi > P_1$. We know from (3.5) that the sequence $\{P_1^k\}_{k\in\mathbb{N}}$ is monotonically increasing and also converges to P_1^ϕ, so there must exist a $K_1 \in \mathbb{N}$ such that $P_1^{K_1} \leq P_1 < P_1^{K_1+1}$, where we take K_1 as small as possible. Condition (3.2) of a ϕ-transfer scheme states that

$$P_1 = \varphi^1(e_1 + \sum_{m\in N} p_{m1}, \bar{c}_1) < \varphi^1(e_1 + \sum_{m\in N} p_{m1}^{K_1}, \bar{c}_1) = P_1^{K_1+1}.$$

Therefore, estate monotonicity of φ^1 implies that agent 1 has received less under P than under P^{K_1+1}, that is, we have

$$\sum_{m\in N} p_{m1} < \sum_{m\in N} p_{m1}^{K_1}.$$

Hence, $p_{m1} < p_{m1}^{K_1}$ for some $m \in N \setminus \{1\}$. Without loss of generality, let $m = 2$ so that $p_{21} < p_{21}^{K_1}$. This means that $p_{21}^{\phi} > p_{21}$ so that we can apply the previous arguments to the case of agent 2. Likewise agent 1, we find that $P_2^{\phi} > P_2$. Therefore, by (3.5), there must exist a $K_2 \in \mathbb{N}$ such that $P_2^{K_2} \leq P_2 < P_2^{K_2+1}$, where we take K_2 as small as possible. In particular, since $p_{21} < p_{21}^{K_1}$, $K_2 + 1$ can be at most K_1, i.e., $K_2 < K_1$. Condition (3.2) of a ϕ-transfer scheme states that

$$P_2 = \varphi^2(e_2 + \sum_{m \in N} p_{m2}, \bar{c}_2) < \varphi^2(e_2 + \sum_{m \in N} p_{m2}^{K_2}, \bar{c}_2) = P_2^{K_2+1}.$$

Therefore, estate monotonicity of φ^2 implies that agent 2 has received less under P than under P^{K_2+1}, that is, we have

$$\sum_{m \in N} p_{m2} < \sum_{m \in N} p_{m2}^{K_2}.$$

Hence, $p_{m2} < p_{m2}^{K_2}$ for some $m \in N \setminus \{1, 2\}$. Agent 1 is excluded since $p_{12} \geq p_{12}^{K_1} \geq p_{12}^{K_2}$, which follows from $P_1^{K_1} \leq P_1$, (3.5) and $K_1 > K_2$. The premise is that we must be able to find an agent that is different from agents 1 and 2. Without loss of generality, let $m = 3$. By repeatedly applying the same arguments, we arrive at a contradiction since N is finite, i.e., we eventually run out of agents to select. □

3.2 Individual settlement allocation procedure

In the ϕ-based individual settlement allocation procedure (ISAP), agents settle their claims individually on the basis of ϕ which eventually leads to a unique redistribution of the estates. Here, the definition of ISAP differs slightly from Ketelaars et al. (2020) since we allow for agent-specific claims rules.

Definition 3.6. Let $(E, C) \in \mathcal{L}^N$ and let $\phi = (\varphi^i)_{i \in N}$ be a vector of claims rules. The *individual settlement allocation procedure* generates a sequence of estates vectors $\{E^k\}_{k \in \mathbb{N}}$, a sequence of claims matrices $\{C^k\}_{k \in \mathbb{N}}$ and a sequence of payment matrices $\{\Phi^k\}_{k \in \mathbb{N}}$ with $\Phi^k = (\Phi_{ij}^k)_{i,j \in N}$ in the following way.

1. Initially, set $E^1 = (e_i^1)_{i \in N} = E$ and $C^1 = (c_{ij}^1)_{i,j \in N} = C$.

Then, recursively for $k = 2, 3, \ldots$

2. For each agent $i \in N$ the payment to agent $j \in N$ in step $k-1$ is equal to

$$\Phi_{ij}^{k-1} = \varphi_j^i(e_i^{k-1}, \bar{c}_i^{k-1}),$$

where $\bar{c}_i^{k-1} \in \mathbb{R}_+^N$ is the i-th row of claims matrix C^{k-1}.

3. Subsequently, update the estates vector to $E^k = (e_i^k)_{i \in N}$ with

$$e_i^k = e_i^{k-1} + \sum_{m \in N} \Phi_{mi}^{k-1} - \sum_{m \in N} \Phi_{im}^{k-1}.$$

4. Correspondingly, the claims matrix is updated to $C^k = (c_{ij}^k)_{i,j \in N}$ with

$$c_{ij}^k = c_{ij}^{k-1} - \Phi_{ij}^{k-1}.$$

ISAP is a finite procedure if and only if there exists a step $k \in \mathbb{N}$ such that either $e_i^k = 0$ or $\bar{c}_i^k = 0$ for all $i \in N$. Even if ISAP takes an infinite number of steps, the limit of the sequence of estates vectors it generates exists.

Theorem 3.3 (cf. Ketelaars et al. (2020)). *Let $(E, C) \in \mathcal{L}^N$ be a mutual claims problem and let $\phi = (\varphi^i)_{i \in N}$ be a vector of claims rules. Then, the limit of the sequence $\{E^k\}_{k \in \mathbb{N}}$ generated by the individual settlement allocation procedure exists.*

A recursive ϕ-based mutual claims rule is then defined to be equal to the limit of the sequence of estates vectors.

Definition 3.7. A *recursive ϕ-based mutual claims rule* r^ϕ on $\mathcal{L}$ is, for all finite N and all $(E, C) \in \mathcal{L}^N$, defined by

$$r^\phi(E, C) = \lim_{k \to \infty} \{E^k\},$$

where $\{E^k\}_{k \in \mathbb{N}}$ is the sequence generated by ISAP for $(E, C) \in \mathcal{L}^N$ with respect to a vector of claims rules $\phi = (\varphi^i)_{i \in N}$.

Alternatively, the allocation vector of a recursive ϕ-based mutual claims rule can be characterized as a specific transfer allocation.

Proposition 3.1 (cf. Ketelaars et al. (2020)). *Let* $(E,C) \in \mathcal{L}^N$*, let* $\phi = (\varphi^i)_{i\in N}$ *be a vector of claims rules, and let the payment matrix* $P = (p_{ij}) \in \mathbb{R}_+^{N\times N}$ *be given by*

$$p_{ij} = \sum_{k=1}^{\infty} \Phi_{ij}^k$$

for all $i,j \in N$*, where* $\{\Phi^k\}_{k\in\mathbb{N}}$ *with* $\Phi^k = (\Phi_{ij}^k)_{i,j\in N}$ *is the sequence of payment matrices generated by ISAP for* (E,C) *with respect to* ϕ*. Then,* $r^\phi(E,C) = \alpha^P$*.*

Ketelaars et al. (2020) stress the importance of the composition principle in establishing the equivalence between a ϕ-based mutual claims rule ρ^ϕ and a recursive ϕ-based mutual claims rule r^ϕ. In fact, we will show that this equivalence relationship holds for their underlying clearing mechanisms as well.

Composition (Young, 1988) is a property of a claims rule that pertains to situations in which an agent allocates a provisional estate value but later learns that the true value is larger than expected. If a claims rule satisfies composition, then, for any claims problem, allocating the surplus value according to the claims rule and modified claims and adding this to the initial allocation is equivalent to simply reallocating the actual (larger) value. The three musketeers satisfy composition, but the Talmud rule and the equal sacrifice methods rule do not.

Definition 3.8. A claims rule φ on $\mathcal{C}$ satisfies *composition* if, for all finite N, for all $(e,c) \in \mathcal{C}^N$ and $(\tilde{e},c) \in \mathcal{C}^N$ with $e \leq \tilde{e}$ it holds that

$$\varphi(\tilde{e},c) = \varphi(e,c) + \varphi(\tilde{e} - e, c - \varphi(e,c)).$$

Using the composition principle we are able to provide an explicit connection between ϕ-transfer schemes and the payments made in ISAP. Composition of the underlying claims rules guarantees that, for any mutual claims problem, the sequence of cumulative payment matrices generated by ISAP with respect to ϕ converges to the bottom ϕ-transfer scheme.

Theorem 3.4. *Let* $(E,C) \in \mathcal{L}^N$*, let* $\phi = (\varphi^i)_{i\in N}$ *be a vector of claims rules where each claims rule* φ^i *satisfies composition and let the payment matrix* P *be given by* $P = \sum_{k=1}^{\infty} \Phi^k$*, where* $\{\Phi^k\}_{k\in\mathbb{N}}$ *is the sequence of payment matrices generated by ISAP for* (E,C) *with respect to* ϕ*. Then,* $P = \underline{P}^\phi$*.*

Proof. We first show that $P \leq \underline{P}^{\phi}$. Next, we show that $P \in \mathcal{P}^{\phi}(E, C)$ so that also $\underline{P}^{\phi} \leq P$ (Theorem 3.2) which then implies that $P = \underline{P}^{\phi}$.

Let $\{E^k\}_{k\in\mathbb{N}}$, $\{C^k\}_{k\in\mathbb{N}}$ and $\{\Phi^k\}_{k\in\mathbb{N}}$ be the sequences generated by ISAP for (E, C) with respect to ϕ. Set $E^k = (e_i^k)_{i\in N}$, $C^k = (c_{ij}^k)_{i,j\in N}$ and $\Phi^k = (\Phi_{ij}^k)_{i,j\in N}$. For all $k \in \mathbb{N}$, let $P^k = (p_{ij}^k)_{i,j\in N}$ with $p_{ij}^k = \sum_{\ell=1}^{k} \Phi_{ij}^{\ell}$ for all $i, j \in N$. Here, p_{ij}^k is the accumulated payment of agent $i \in N$ to agent $j \in N$ up to step $k \in \mathbb{N}$ in ISAP.

Before we start, we show three general characteristics of ISAP which are stated in (3.6), (3.7) and (3.8). In combination with the composition principle, this leads to (3.9) which is used frequently later on. Fix some step $k \in \mathbb{N}$ and some $i \in N$. First, the estate of agent i at step k is equal to his initial estate plus his net payments in steps $1, 2, \ldots, k-1$, i.e.,

$$e_i^k = e_i^{k-1} + \sum_{m\in N} \Phi_{mi}^{k-1} - \sum_{m\in N} \Phi_{im}^{k-1},$$

which implies that

$$e_i^k = e_i^1 + \sum_{m\in N} \sum_{\ell=1}^{k-1} \Phi_{mi}^{\ell} - \sum_{m\in N} \sum_{\ell=1}^{k-1} \Phi_{im}^{\ell}. \tag{3.6}$$

Second, in each step of ISAP the payments adhere to the limited liability requirement. In particular, the accumulated outgoing payments of agent i up to step k are at most his initial estate plus accumulated incoming payments up to step $k-1$, i.e.,

$$\sum_{m\in N} \sum_{\ell=1}^{k} \Phi_{im}^{\ell} \leq e_i^1 + \sum_{m\in N} \sum_{\ell=1}^{k-1} \Phi_{mi}^{\ell}. \tag{3.7}$$

Inequality (3.7) follows from combining equation (3.6) and the fact that the total outgoing payment of agent i at step k is at most his estate at step k, i.e.,

$$\sum_{m\in N} \Phi_{im}^k = \sum_{m\in N} \varphi_m^i(e_i^k, \bar{c}_i^k) = \min\{e_i^k, \sum_{j\in N} c_{ij}^k\} \leq e_i^k,$$

where the first equality follows from the definition of ISAP and the second equality follows from the definition of a claims rule. Additionally, estate monotonicity of φ^i and (3.7) imply that

$$\varphi^i_j\left(\sum_{m\in N}\sum_{\ell=1}^{k}\Phi^\ell_{im},\bar{c}^1_i\right)\leq\varphi^i_j(e^1_i+\sum_{m\in N}\sum_{\ell=1}^{k-1}\Phi^\ell_{mi},\bar{c}^1_i) \tag{3.8}$$

for all $j\in N$.

Now, we show that composition of φ^i implies that the left-hand side of (3.8) is in fact equal to the accumulated payment of agent i to agent $j\in N$ up to step k, i.e., for all $j\in N$, we have

$$p^k_{ij}=\varphi^i_j\left(\sum_{m\in N}\sum_{\ell=1}^{k}\Phi^\ell_{im},\bar{c}^1_i\right). \tag{3.9}$$

To this end, we first show that, for all $\ell\in\mathbb{N}$, we have

$$\varphi^i(\sum_{m\in N}\Phi^\ell_{im},\bar{c}^\ell_i)=\varphi^i(e^\ell_i,\bar{c}^\ell_i). \tag{3.10}$$

Let $\ell\in\mathbb{N}$ and recall that $\sum_{m\in N}\Phi^\ell_{im}=\min\{e^\ell_i,\sum_{j\in N}c^\ell_{ij}\}$. If $e^\ell_i\leq\sum_{j\in N}c^\ell_{ij}$, then (3.10) is immediate. Otherwise $e^\ell_i>\sum_{j\in N}c^\ell_{ij}$, which means that the estate at step ℓ can cover all claims of agent i at step ℓ. Therefore,

$$\varphi^i(\sum_{m\in N}\Phi^\ell_{im},\bar{c}^\ell_i)=\varphi^i(\sum_{j\in N}c^\ell_{ij},\bar{c}^\ell_i)=\bar{c}^\ell_i=\varphi^i(e^\ell_i,\bar{c}^\ell_i).$$

Using composition of φ^i and (3.10), we find that, for all $j\in N$, the left-hand side of (3.8) can be rewritten as

$$\begin{aligned}\varphi^i_j\left(\sum_{m\in N}\sum_{\ell=1}^{k}\Phi^\ell_{im},\bar{c}^1_i\right)&=\varphi^i_j\left(\sum_{m\in N}\Phi^1_{im},\bar{c}^1_i\right)+\varphi^i_j\left(\sum_{m\in N}\sum_{\ell=2}^{k}\Phi^\ell_{im},\bar{c}^1_i-\varphi^i\left(\sum_{m\in N}\Phi^1_{im},\bar{c}^1_i\right)\right)\\&=\varphi^i_j\left(e^1_i,\bar{c}^1_i\right)+\varphi^i_j\left(\sum_{m\in N}\sum_{\ell=2}^{k}\Phi^\ell_{im},\bar{c}^1_i-\varphi^i\left(e^1_i,\bar{c}^1_i\right)\right)\\&=\varphi^i_j\left(e^1_i,\bar{c}^1_i\right)+\varphi^i_j\left(\sum_{m\in N}\sum_{\ell=2}^{k}\Phi^\ell_{im},\bar{c}^2_i\right)\\&=\sum_{\ell=1}^{k}\varphi^i_j(e^\ell_i,\bar{c}^\ell_i)\end{aligned}$$

$$= \sum_{\ell=1}^{k} \Phi_{ij}^{\ell}$$
$$= p_{ij}^{k}.$$

The first equality follows from composition of φ^i; the second equality follows from (3.10); the third equality follows from the definition of ISAP; the fourth equality from follows from repeatedly applying the previous arguments; the fifth equality follows from the definition of ISAP. This finishes our preparations.

Let $\underline{P}^{\phi} = (p_{ij}^{\phi})_{i,j\in N}$. We will show by induction on k that $P^k \leq \underline{P}^{\phi}$ for all $k \in \mathbb{N}$. Clearly, this implies $P = \lim_{k\to\infty}\{P^k\} \leq \underline{P}^{\phi}$. Let $k = 1$. Then, for all $i, j \in N$, it holds that

$$p_{ij}^1 = \varphi_j^i(e_i^1, \bar{c}_i^1) \leq \varphi_j^i(e_i^1 + \sum_{m\in N} p_{mi}^{\phi}, \bar{c}_i^1) = p_{ij}^{\phi}.$$

The inequality follows from estate monotonicity of φ^i and the second equality follows from $\underline{P}^{\phi} \in \mathcal{P}^{\phi}(E, C)$ (see (3.2)). Next, let $k \in \mathbb{N}$ and assume that $P^k \leq \underline{P}^{\phi}$. Now consider $k + 1$. Then, for all $i, j \in N$, it holds that

$$\begin{aligned} p_{ij}^{k+1} &= \varphi_j^i(\sum_{m\in N}\sum_{\ell=1}^{k+1} \Phi_{im}^{\ell}, \bar{c}_i^1) \\ &\leq \varphi_j^i(e_i^1 + \sum_{m\in N}\sum_{\ell=1}^{k} \Phi_{mi}^{\ell}, \bar{c}_i^1) \\ &= \varphi_j^i(e_i^1 + \sum_{m\in N} p_{mi}^{k}, \bar{c}_i^1) \\ &\leq \varphi_j^i(e_i^1 + \sum_{m\in N} p_{mi}^{\phi}, \bar{c}_i^1) \\ &= p_{ij}^{\phi}. \end{aligned}$$

The first equality follows from (3.9); the first inequality follows from (3.8); the second inequality follows from the induction hypothesis and estate monotonicity of φ^i; the last equality follows from $\underline{P}^{\phi} \in \mathcal{P}^{\phi}(E, C)$ (see (3.2)).

Finally, to show that $P \in \mathcal{P}^{\phi}(E, C)$, we need to show that (see (3.2)), for all $i, j \in N$,

$$p_{ij} = \varphi_j^i(e_i^1 + \sum_{m\in N} p_{mi}, \bar{c}_i^1).$$

So let $i, j \in N$. We distinguish between two mutually exclusive cases. Define the set

$$B = \{p \in N \colon e_p^k < \sum_{m \in N} c_{pm}^k \text{ for all } k \in \mathbb{N}\}$$

as the set containing agents that never have sufficient funds to pay off their remaining debts in ISAP.

First, if $i \in B$, then in each step $k \in \mathbb{N}$ of ISAP agent i allocates his full estate. That is, for all $k \in \mathbb{N}$, we have $\sum_{m \in N} \Phi_{mi}^k = e_i^k$, or equivalently, using (3.6),

$$\sum_{m \in N} \sum_{\ell=1}^{k} \Phi_{im}^{\ell} = e_i^1 + \sum_{m \in N} \sum_{\ell=1}^{k-1} \Phi_{mi}^{\ell}. \tag{3.11}$$

Consequently, we have

$$\begin{aligned} p_{ij} &= \lim_{k \to \infty} p_{ij}^k \\ &= \lim_{k \to \infty} \varphi_j^i \left(\sum_{m \in N} \sum_{\ell=1}^{k} \Phi_{im}^{\ell}, \bar{c}_i^1 \right) \\ &= \lim_{k \to \infty} \varphi_j^i (e_i^1 + \sum_{m \in N} \sum_{\ell=1}^{k-1} \Phi_{mi}^{\ell}, \bar{c}_i^1) \\ &= \varphi_j^i (e_i^1 + \sum_{m \in N} \left(\lim_{k \to \infty} p_{mi}^{k-1} \right), \bar{c}_i^1) \\ &= \varphi_j^i (e_i^1 + \sum_{m \in N} p_{mi}, \bar{c}_i^1). \end{aligned}$$

The second equality follows from (3.9); the third equality follows from (3.11); the fourth equality follows from estate continuity of φ^i.

Second, if $i \notin B$, then there exists a $K \in \mathbb{N}$ such that $e_i^K \geq \sum_{m \in N} c_{im}^K$ and, as a consequence, $\sum_{m \in N} \Phi_{im}^K = \sum_{m \in N} c_{im}^K$. In other words, agent i pays off all his remaining debts at time moment K, which means that $\Phi_{ij}^k = 0$ for all $k \in \{K+1, K+2, \dots\}$.

It suffices to prove that $p_{ij}^K = c_{ij}^1$. If $p_{ij}^K = c_{ij}^1$, then clearly $p_{ij} = c_{ij}^1$ since $\Phi_{ij}^k = 0$ for all $k \in \{K+1, K+2, \dots\}$. Moreover, if $p_{ij}^K = c_{ij}^1$, we also have

$$c_{ij}^1 = p_{ij} = p_{ij}^K$$

$$
\begin{aligned}
&= \varphi_j^i\left(\sum_{m\in N}\sum_{\ell=1}^{K}\Phi_{im}^{\ell}, \bar{c}_i^1\right)\\
&\leq \varphi_j^i(e_i^1 + \sum_{m\in N}\sum_{\ell=1}^{K-1}\Phi_{im}^{\ell}, \bar{c}_i^1)\\
&= \varphi_j^i(e_i^1 + \sum_{m\in N} p_{im}^{K-1}, \bar{c}_i^1)\\
&\leq \varphi_j^i(e_i^1 + \sum_{m\in N} p_{im}, \bar{c}_i^1)\\
&\leq c_{ij}^1,
\end{aligned}
$$

which implies that all inequalities are equalities and therefore, in particular,

$$
p_{ij} = \varphi_j^i(e_i^1 + \sum_{m\in N} p_{mi}, \bar{c}_i^1).
$$

Here, the third equality follows from (3.9); the first inequality follows from (3.8); the second inequality follows from estate monotonicity of φ^i; the last inequality follows from the definition of a claims rule.

To show that $p_{ij}^K = c_{ij}^1$, we first show that the accumulated payments of agent i up to K are in fact equal to his total claims. That is, by definition of ISAP,

$$
\begin{aligned}
\sum_{m\in N}\sum_{\ell=1}^{K}\Phi_{im}^{\ell} &= \sum_{m\in N}\sum_{\ell=1}^{K-1}\Phi_{im}^{\ell} + \sum_{m\in N}\Phi_{im}^{K}\\
&= \sum_{m\in N}\sum_{\ell=1}^{K-1}\Phi_{im}^{\ell} + \sum_{m\in N} c_{im}^{K}\\
&= \sum_{m\in N}\sum_{\ell=1}^{K-1}\Phi_{im}^{\ell} + \sum_{m\in N}\left(c_{im}^{K-1} - \Phi_{im}^{K-1}\right)\\
&= \sum_{m\in N}\sum_{\ell=1}^{K-2}\Phi_{im}^{\ell} + \sum_{m\in N} c_{im}^{K-1}\\
&= \sum_{m\in N} c_{im}^{1}.
\end{aligned}
$$

In the second equality we use the fact that $i \notin B$ which implies that

$\sum_{m \in N} \Phi_{im}^K = \sum_{m \in N} c_{im}^K$ as argued before. From this we can conclude that

$$\begin{aligned} p_{ij}^K &= \varphi_j^i \left(\sum_{m \in N} \sum_{\ell=1}^{K} \Phi_{im}^{\ell}, \bar{c}_i^1 \right) \\ &= \varphi_j^i (\sum_{m \in N} c_{im}^1, \bar{c}_i^1) \\ &= c_{ij}^1. \end{aligned}$$

□

The following corollary is a direct consequence of Proposition 3.1 and Theorem 3.4. It says that the equivalence relationship between a ϕ-based mutual claims rule ρ^ϕ and a recursive ϕ-based mutual claims rule r^ϕ also holds with agent-specific claims rules, thereby generalizing the main result of Ketelaars et al. (2020).

Corollary 3.1. *Let $(E, C) \in \mathcal{L}^N$ and let $\phi = (\varphi^i)_{i \in N}$ be a vector of claims rules where each claims rule φ^i satisfies composition. Then,*

$$\rho^\phi(E, C) = \alpha^{P^\phi} = r^\phi(E, C).$$

4 Joint characterization of the three musketeers

Moulin (2000) shows that the three musketeers PROP, CEA, and CEL are the only claims rules on $\mathcal{C}$ that meet the requirements of the following five properties: scale invariance, equal treatment of equals, composition, path independence and consistency.

Theorem 4.1 (cf. Moulin (2000))**.** *The proportional rule, constrained equal-awards rule and constrained equal-losses rule are the only three claims rules on $\mathcal{C}$ that simultaneously satisfy scale invariance, equal treatment of equals, composition, path independence and consistency.*

We will show that this result for claims rules on $\mathcal{C}$ can be extended to (recursive) ϕ-based mutual claims rules on $\mathcal{L}$. To this end, we first state the five properties for claims rules and propose adequate corresponding extensions. Next, we show that the (recursive) ϕ-based mutual claims rules where ϕ is any vector of claims rules in which each coordinate is one of the three musketeers are the only (recursive) ϕ-based mutual claims rules on $\mathcal{L}$ that satisfy the specific extensions.

4.1 The five properties

The extension of Theorem 4.1 is a characterization within the class of (recursive) ϕ-based mutual claims rules. Hence, formally speaking, we need only provide extensions of properties that are defined on the class of (recursive) ϕ-based mutual claims rules. Nevertheless, for the sake of generality, we provide general extensions of the five properties with the exception of equal treatment of equals.

Scale invariance is the first of the five main properties that characterizes the three musketeers and it relates to invariance with respect to a change in measurement units. The three musketeers, the Talmud rule and the equal sacrifice methods rule all satisfy scale invariance.

Definition 4.1. A claims rule φ on $\mathcal{C}$ satisfies *scale invariance* if, for all finite N, for all $(e, c) \in \mathcal{C}^N$ and all $\lambda > 0$, it holds that

$$\lambda\varphi(e, c) = \varphi(\lambda e, \lambda c).$$

The extension of scale invariance to the mutual claims problem setting is straightforward.

Definition 4.2. A mutual claims rule μ on $\mathcal{L}$ satisfies *scale invariance* if, for all finite N, for all $(E, C) \in \mathcal{L}^N$ and all $\lambda > 0$, it holds that

$$\lambda\mu(E, C) = \mu(\lambda E, \lambda C).$$

Equal treatment of equals requires an equal allocation to claimants that are "equal". In claims problems, claimants are considered equal if their claim on the estate is the same. Equal treatment of equals is satisfied by the three musketeers, the Talmud rule and the equal sacrifice methods rule.

Definition 4.3. A claims rule φ on $\mathcal{C}$ satisfies *equal treatment of equals* if, for all finite N, for all $(e, c) \in \mathcal{C}^N$ and all $i, j \in N$ with $c_i = c_j$, it holds that

$$\varphi_i(e, c) = \varphi_j(e, c).$$

Equality of agents in mutual claims problems is defined as them having the same estate as well as having the same claims on and debts to other agents. In regard to the class of ϕ-based mutual claims rules, we additionally impose that equal agents use the same claims rule.

Definition 4.4. A ϕ-based mutual claims rule ρ^ϕ on $\mathcal{L}$ satisfies *equal treatment of equals* if, for all finite N, for all $(E, C) \in \mathcal{L}^N$ and all $i, j \in N$ with $\varphi^i = \varphi^j$, $e_i = e_j$, $c_{im} = c_{jm}$ and $c_{mi} = c_{mj}$ for all $m \in N$, it holds that

$$\rho_i^\phi(E, C) = \rho_j^\phi(E, C).$$

Composition for claims rules, as was defined in the previous section, can in a natural way be extended to mutual claims rules. In the network setting, an allocation has been prescribed in which (some of) the agents have underestimated their true estate value. Therefore, there is still an excess amount to be distributed among the agents with respect to a *residual claims matrix*. A residual claims matrix is a matrix containing the remaining claims of agents after agents received payments that are induced by a mutual claims rule. In Definition 4.5, a residual claims matrix is given by $C - P$, where $P \in \mathcal{P}(E, C)$ is a transfer scheme corresponding to the allocation vector $\mu(E, C)$.

Definition 4.5. A mutual claims rule μ on $\mathcal{L}$ satisfies *composition* if, for all finite N, for all $(E, C) \in \mathcal{L}^N$ and $(\tilde{E}, C) \in \mathcal{L}^N$ with $E \leq \tilde{E}$ there exists a $P \in \mathcal{P}(E, C)$ such that $\mu(E, C) = \alpha^P$ and such that

$$\mu(\tilde{E}, C) = \mu(E, C) + \mu(\tilde{E} - E, C - P).$$

We arrive at the path independence principle (Moulin, 1987) if we reason conversely. Here, an estate value that is larger than expected has been allocated. Path independence allows us to allocate the actual (smaller) estate value on the basis of either actual claims or revised claims. In claims problems, revised claims are the initially prescribed unfeasible allocations to the claimants. The three musketeers and the equal sacrifice methods rule satisfy path independence, while the Talmud rule does not.

Definition 4.6. A claims rule φ on $\mathcal{C}$ satisfies *path independence* if, for all finite N, for all $(e, c) \in \mathcal{C}^N$ and $(\tilde{e}, c) \in \mathcal{C}^N$ with $\tilde{e} \leq e$ it holds that

$$\varphi(\tilde{e}, c) = \varphi(\tilde{e}, \varphi(e, c)).$$

In mutual claims problems, revised claims are the (excess) payments made by agents on the basis of estate values that are larger than expected.

Definition 4.7. A mutual claims rule μ on $\mathcal{L}$ satisfies *path independence* if, for all finite N, for all $(E, C) \in \mathcal{L}^N$ and $(\tilde{E}, C) \in \mathcal{L}^N$ with $\tilde{E} \leq E$ there exists a $P \in \mathcal{P}(E, C)$ such that $\mu(E, C) = \alpha^P$ and such that

$$\mu(\tilde{E}, C) = \mu(\tilde{E}, P).$$

Finally, consistency is an invariance property with respect to the set of claimants (agents) which requires that no subset of claimants has an incentive to reallocate the allocation that was agreed upon. In particular, a rule is consistent if a reallocation of an amount that (rightfully) belongs to a subset S — as prescribed by the same rule — coincides with the initial individual allocations to members of S. In a claims problem (e, c) on N, the amount that belongs to S is the initial estate minus the allocations to agents outside of S.[5] The associated restricted claims problem is then given by $(e - \sum_{j \in N \setminus S} \varphi_j(e, c), c^S) \in \mathcal{C}^S$, where $c^S = (c_i)_{i \in S}$. The three musketeers, the Talmud rule and the equal sacrifice methods rule all satisfy consistency.

Definition 4.8. A claims rule φ on $\mathcal{C}$ satisfies *consistency* if for all finite N, for all $(e, c) \in \mathcal{C}^N$, for all $S \subseteq N$ and for all $i \in S$ it holds that

$$\varphi_i(e, c) = \varphi_i(e - \sum_{j \in N \setminus S} \varphi_j(e, c), c^S).$$

In a mutual claims problem (E, C) on N, the associated restricted mutual claims problem with respect to payments made in accordance with a mutual claims rule μ, i.e., $P \in \mathcal{P}(E, C)$ with $\mu(E, C) = \alpha^P$, is given by $(E^{S,P}, C^S) \in \mathcal{L}^S$, where $E^{S,P} = (e_i^{S,P})_{i \in S}$ with

$$e_i^{S,P} = e_i + \sum_{j \in N \setminus S} p_{ji} - \sum_{j \in N \setminus S} p_{ij} \tag{4.1}$$

for all $i \in S$, and $C^S = (c_{ij})_{i,j \in S}$. The amount restricted to members in S is now given by (4.1) and is determined on the basis of payments from and to agents outside of S. If the total payments of some agent $i \in S$ to agents outside of S exceeds his estate value plus the total incoming payments from agents outside of S, then $e_i^{S,P} < 0$ and thus $(E^{S,P}, C^S) \notin \mathcal{L}^S$. In this case no consistency requirement is imposed. The following definition is due to Groote Schaarsberg et al. (2018).

Definition 4.9. A mutual claims rule μ on $\mathcal{L}$ satisfies *consistency* if for all finite N and for all $(E, C) \in \mathcal{L}^N$ there exists a $P \in \mathcal{P}(E, C)$ such that $\mu(E, C) = \alpha^P$ and such that for all $S \subseteq N$ with $(E^{S,P}, C^S) \in \mathcal{L}^S$ and all $i \in S$ it holds that

$$\mu_i(E, C) = \mu_i(E^{S,P}, C^S).$$

[5] Alternatively, the amount that belongs to S is the total allocation to S prescribed by φ, i.e., $\sum_{j \in S} \varphi_j(e, c)$.

4.2 Characterization in mutual claims problems

To bridge the gap between claims problems and mutual claims problems, we will show how a claims problem can be interpreted as a mutual claims problem. To this end, we introduce an artificial agent, *agent zero*, with non-negative estate. If $(e, c) \in \mathcal{C}^N$ is a claims problem with $N = \{1, 2, \ldots, n\}$, then (e, c) can be associated with a mutual claims problem (E, C) on $\bar{N} = \{0\} \cup N$ given by

$$E = \begin{pmatrix} e \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \text{ and } C = \begin{bmatrix} 0 & c_1 & \ldots & c_n \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix}. \tag{4.2}$$

It readily follows that the allocations prescribed by the mutual claims rules ρ^ϕ and r^ϕ for (E, C) on $\bar{N}$ coincide with the allocation prescribed by the claims rule φ^0 of agent zero for the underlying claims problem (e, c) on N.

Proposition 4.1. *Let (E, C) be a mutual claims problem on $\bar{N} = \{0\} \cup N$ as defined in* (4.2) *and let $\phi = (\varphi^0, \varphi^1, \ldots, \varphi^n)$ be a vector of claims rules. Then, for all $i \in N$,*

(i) $\rho_i^\phi(E, C) = \varphi_i^0(e, c)$,

(ii) $r_i^\phi(E, C) = \varphi_i^0(e, c)$.

Before stating our characterization result, we point out a useful feature of the bottom ϕ-transfer scheme, namely its so-called *monotonicity of payments.*

Lemma 4.1. *Let $(E, C) \in \mathcal{L}^N$ and $(\tilde{E}, C) \in \mathcal{L}^N$ with $E \leq \tilde{E}$ and let $\phi = (\varphi^i)_{i \in N}$ be a vector of claims rules. Let $\underline{P}^\phi \in \mathcal{P}^\phi(E, C)$ and $\underline{\tilde{P}}^\phi \in \mathcal{P}^\phi(\tilde{E}, C)$ be the corresponding bottom ϕ-transfer schemes. Then, $\underline{P}^\phi \leq \underline{\tilde{P}}^\phi$.*

Proof. See Appendix A. □

For expositional convenience, we let $\mathcal{M}^N$ denote the set of vectors of claims rules ϕ with respect to N in which each coordinate is one of the three musketeers, i.e., $\mathcal{M}^N = \{\phi \mid \phi_i \in \{\text{PROP}, \text{CEA}, \text{CEL}\} \text{ for all } i \in N\}$. Let $\mathcal{M}$ be the set of all such vectors of claims rules with respect to arbitrary but finite N.

We now have all ingredients for our axiomatic characterization of the three musketeers in mutual claims problems.

Theorem 4.2. *The ϕ-based mutual claims rules with $\phi \in \mathcal{M}$ are the only ϕ-based mutual claims rules on $\mathcal{L}$ that simultaneously satisfy scale invariance, equal treatment of equals, composition, path independence and consistency.*

Proof. See Appendix A. □

In the proof of Theorem 4.2, we show that a ϕ-based mutual claims rule ρ^ϕ satisfies the five properties on $\mathcal{L}$ if and only if $\phi \in \mathcal{M}$. In particular, we employ the bottom ϕ-transfer scheme to assert that ρ^ϕ satisfies scale invariance, equal treatment of equals, composition, path independence and consistency. Clearly, if $\phi \in \mathcal{M}$, then $\rho^\phi = \alpha^{P^\phi} = r^\phi$ (by Corollary 3.1) so the recursive ϕ-based mutual claims rule r^ϕ satisfies scale invariance, equal treatment of equals, composition, path independence and consistency as well.[6] Using Proposition 4.1, also the arguments in the proof of Theorem 4.2 for the reverse statement can be applied to the class of recursive ϕ-based mutual claims rules. This leads to the following corollary.

Corollary 4.1. *The recursive ϕ-based mutual claims rules with $\phi \in \mathcal{M}$ are the only recursive ϕ-based mutual claims rules on $\mathcal{L}$ that simultaneously satisfy scale invariance, equal treatment of equals, composition, path independence and consistency.*

A note of warning however on the difference between the inheritance of properties with respect to ϕ-based mutual claims rules and recursive ϕ-based mutual rules. The following example illustrates that, even if ϕ consists of claims rules that all satisfy path independence on $\mathcal{C}$, the corresponding recursive ϕ-based mutual claims rule r^ϕ need not necessarily satisfy path independence on $\mathcal{L}$. On the other hand, in the proof of Theorem 4.2, we show that, for any of the five axioms, a ϕ-based mutual claims rule ρ^ϕ satisfies the axiom on $\mathcal{L}$ if all underlying claims rules in ϕ satisfy the same axiom on $\mathcal{C}$, independently of the other axioms.

Example 4.1. Consider the mutual claims problem $(E, C) \in \mathcal{L}^N$ given by $N = \{1, 2, 3\}$,

$$E = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ and } C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 4 & 1 & 0 \end{bmatrix}.$$

Assume that each agent uses the equal sacrifice methods rule ESM as his underlying payment mechanism in ϕ-based ISAP. This means that $\phi =$

[6] Recall that the equal treatment of equals axiom was only defined for a ϕ-based mutual claims rule ρ^ϕ. Its definition for a recursive ϕ-based mutual claims rule r^ϕ is similar.

(ESM, ESM, ESM). Note that the equal sacrifice methods rule ESM satisfies path independence.

In the first step of ISAP, agents 1 and 2 both pay agent 3 an amount of 1, thereby becoming debt free. Subsequently, agent 3 allocates an amount of 2 among agents 1 and 2 using ESM. ISAP then terminates because agents 1 and 2 are debt free and agent 3 has an estate of zero. We have $\mathrm{ESM}(2,(4,1,0)) = (1\frac{1}{3}, \frac{2}{3}, 0)$, which gives us

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1\frac{1}{3} & \frac{2}{3} & 0 \end{bmatrix}.$$

Hence, the transfer allocation is equal to $r^{\phi}(E, C) = (1, 1, 0) + (1\frac{1}{3}, \frac{2}{3}, 2) - (1, 1, 2) = (1\frac{1}{3}, \frac{2}{3}, 0)$.

Now, suppose that the actual estates vector is given by $\tilde{E} = (0, 1, 0)$. Then, it follows that

$$r^{\phi}(\tilde{E}, C) = (0.1883, 0.8117, 0) \neq (\frac{1}{3}, \frac{2}{3}, 0) = r^{\phi}(\tilde{E}, P),$$

which, together with the fact that P is unique, means that the recursive ϕ-based mutual claims rule r^{ϕ} does not satisfy path independence.

△

5 Sequential clearing mechanisms

In addition to the recursive ϕ-based clearing mechanisms, there exist two large classes of ϕ-based (decentralized) *sequential* clearing mechanisms for which payments converge to the bottom ϕ-transfer scheme (cf. Theorem 3.2 and Theorem 3.4). As a result, the mutual claims rule corresponding to a such a ϕ-based sequential clearing mechanism can also be uniquely characterized by the five properties provided in Theorem 4.2.

In each step of a sequential clearing mechanism, exactly one agent is selected that makes a payment to the other agents on the basis of a claims rule φ. Like the individual settlement allocation procedure, sequential clearing mechanisms need not terminate in a finite number of iterations. Therefore, we require that the selection process is such that each agent is in principle selected an infinite number of times.[7] The selection process need not be

[7] An agent may otherwise never be selected despite receiving payments and being able to make a positive payment. Consequently, he has not allocated all his incoming payments which implies that the resulting payment matrix is not a ϕ-transfer scheme.

deterministic. In general, the selection process can be history dependent, stochastic or both. We represent a *realization* of a selection process by an ordering of the agents $\sigma\colon \mathbb{N} \to N$, where $\sigma(k) \in N$ is the agent that is selected to pay at step $k \in \mathbb{N}$. The set of all realized orderings σ of N that are the direct consequence of all possible selection processes with the property that $|\{k \in \mathbb{N}\colon \sigma(k) = i\}| = \infty$ for all $i \in N$ is denoted by $\Pi(N)$.

We will now outline two large classes of sequential clearing mechanisms, both of which are defined for any $\sigma \in \Pi(N)$. The first ϕ-based sequential clearing mechanism is a variation on the ϕ-based individual settlement allocation procedure; a procedure in which agents pay simultaneously in each step. To accommodate for sequential payments based on a realized ordering $\sigma \in \Pi(N)$, we need only change the second component of ISAP of Definition 3.6 in the following way:

2. The payment of agent $\sigma(k-1) = i$ to agent $j \in N$ in step $k-1$ is equal to
$$\Phi_{ij}^{k-1} = \varphi_j^i(e_i^{k-1}, \bar{c}_i^{k-1}),$$
where $\bar{c}_i^{k-1} \in \mathbb{R}_+^N$ is the i-th row of claims matrix C^{k-1}; for all $i \neq \sigma(k-1)$, it holds that $\Phi_{ij}^{k-1} = 0$ for all $j \in N$.

The other steps in Definition 3.6 remain the same. Correspondingly, given a selection procedure and a corresponding realized ordering $\sigma \in \Pi(N)$, we call such a ϕ-based sequential clearing mechanism a *ϕ-based asynchronous ISAP*.

Proposition 5.1. *If each coordinate of ϕ is a claims rule that satisfies composition, then the resulting payment matrix of a ϕ-based asynchronous ISAP is the bottom ϕ-transfer scheme, irrespective of the realized ordering $\sigma \in \Pi(N)$.*

This statement follows from a direct modification of the proof of Theorem 3.4; the crux of the proof is in establishing (3.9).

The second ϕ-based sequential clearing mechanism is a variation on the recursive procedure provided in Definition 3.5 of the bottom ϕ-transfer scheme. Given a realized ordering $\sigma \in \Pi(N)$, the clearing mechanism generates a sequence of payment matrices $\{P^k\}_{k\in\mathbb{N}}$ with $P^k = (p_{ij}^k)_{i,j\in N}$ as follows. Initially, set $P^0 = 0$. Define, for all $i \in N$ and $k \in \mathbb{N}$,

$$\delta_i(k+1) = e_i + \sum_{j\in N} p_{ji}^k, \tag{5.1}$$

with $\delta_i(1) = e_i$, where, for all $j \in N$,

$$p_{ji}^k = \begin{cases} \varphi_i^j(\delta_j(k), \bar{c}_j) & \text{if } j = \sigma(k), \\ p_{ji}^{k-1} & \text{if } j \neq \sigma(k). \end{cases} \tag{5.2}$$

Then, define $P = (p_{ij})_{i,j \in N}$ by setting, for all $i, j \in N$,

$$p_{ij} = \lim_{k \to \infty} p_{ij}^k. \tag{5.3}$$

In fact, the above ϕ-based sequential clearing mechanism is the analogue of a decentralized clearing process introduced by Csóka and Herings (2018) for the discrete setup.[8]

Correspondingly, given a selection procedure and a corresponding realized ordering $\sigma \in \Pi(N)$, we call such a ϕ-based sequential clearing mechanism a *ϕ-based decentralized clearing process*. The process can be interpreted as follows. Each agent keeps track of the amount to his disposal, i.e., his initial estate plus the payments he has received so far, as represented by (5.1). Each time an agent is selected, he makes a (possible) incremental payment according to his claims rule by allocating what he currently has to his disposal.

Proposition 5.2. *The resulting payment matrix P given in* (5.3) *of a ϕ-based decentralized clearing process is the bottom ϕ-transfer scheme, irrespective of the realized ordering $\sigma \in \Pi(N)$.*

To see this, we first argue that P is a ϕ-transfer scheme. Since each claims rule in ϕ is estate monotone, (5.1) constitutes a monotonically increasing sequence $\{\delta(k)\}_{k \in \mathbb{N}}$ that is bounded from above. Hence, its limit exists as a result of the monotone convergence theorem. Consequently, estate continuity of claims rules in ϕ implies that (5.2) constitutes a monotonically increasing sequence

$$P^1 \leq P^2 \leq P^3 \leq \dots,$$

which has a limit as well. By construction of a ϕ-based decentralized clearing process, it follows that the resulting payment matrix P given in (5.3) is a ϕ-transfer scheme, irrespective of the realized ordering $\sigma \in \Pi(N)$. The

[8]By using a claims rule φ as an underlying payment mechanism, we implicitly require that payments are maximal in the sense that each agent either allocates all of his available estate or pays off all his debts. This need not always be the case in a decentralized clearing process as defined by Csóka and Herings (2018) for the discrete setup.

assertion that P is the bottom ϕ-transfer scheme then follows directly from the proof of Theorem 3.2.

The following example illustrates both types of ϕ-based sequential clearing mechanisms.

Example 5.1. Reconsider the mutual claims problem $(E, C) \in \mathcal{L}^N$ of Example 3.1 given by $N = \{1, 2, 3\}$,

$$E = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \text{ and } C = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 5 & 2 & 0 \end{bmatrix}.$$

Again, let $\phi = (\text{CEA}, \text{CEL}, \text{TAL})$. Consider the following realized ordering of the agents $\sigma = (1, 3, 2, 1, 3, 2, \dots)$. We will illustrate the corresponding ϕ-based asynchronous ISAP and ϕ-based decentralized clearing process in conjunction.

Let, for all $k \in \mathbb{N}$, $\hat{P}^k = \sum_{\ell=1}^{k} \Phi^\ell$ denote the accumulated payments at step k under ϕ-based asynchronous ISAP. Furthermore, in regard to the ϕ-based decentralized clearing process, let P^k be as defined in (5.2) for all $k \in \mathbb{N}$.

Initially, set $E^1 = E = \delta(1)$ and $P^0 = 0$. Agent 1 is first selected to pay. We have $\text{CEA}(2, (0, 1, 2)) = (0, 1, 1)$ in both clearing mechanisms, so

$$\hat{P}^1 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } P^1 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence,

$$\begin{aligned} E^2 &= (2, 1, 1) + (0, 1, 1) - (2, 0, 0) = (0, 2, 2), \\ \text{and } \delta(2) &= (2, 1, 1) + (0, 1, 1) = (2, 2, 2). \end{aligned}$$

Agent 3 is selected next and pays $\text{TAL}(2, (5, 2, 0)) = (1, 1, 0)$ in both mechanisms because $E_3^2 = \delta_3(2) = 2$, so

$$\hat{P}^2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \text{ and } P^2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Hence,

$$E^3 = (2, 1, 1) + (1, 2, 1) - (2, 0, 2) = (1, 3, 0),$$

and $\delta(3) = (2,1,1) + (1,2,1) = (3,3,2)$.

In the third step agent 2 is selected and pays $\text{CEL}(3,(1,0,1)) = (1,0,1)$ in both mechanisms because $E_2^3 = \delta_2(3) = 3$, so

$$\hat{P}^3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{ and } P^3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Hence, $E^4 = (2,1,1)$ and $\delta(4) = (4,3,3)$. Moreover, the claims matrix in step 4 for ϕ-based asynchronous ISAP is equal to

$$C^4 = C - \hat{P}^3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 4 & 1 & 0 \end{bmatrix}.$$

In subsequent steps, agent 2 will not make any incremental payments since he is debt free. In the fourth step agent 1 is selected and becomes debt free by paying $\text{CEA}(E_1^4, \bar{c}_1^4) = \text{CEA}(2,(0,0,1)) = (0,0,1)$ under ϕ-based asynchronous ISAP and by paying $\text{CEA}(\delta_1(4), \bar{c}_1^1) = \text{CEA}(4,(0,1,2)) = (0,1,2)$ under the ϕ-based decentralized clearing process. Consequently,

$$\hat{P}^4 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \text{ and } P^4 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

so that $E^5 = (1,1,2)$ and $\delta(5) = (4,3,4)$.

Note that the matrices $\hat{P}^4$ and P^4 are still equal, however they will become different in the next step. Since agents 1 and 2 have become debt free, there will be one more payment by agent 3 after which both procedures essentially terminate. Agent 3 is selected in the next step and pays $\text{TAL}(E_3^5, \bar{c}_3^5) = \text{TAL}(2,(4,1,0)) = (1\frac{1}{2}, \frac{1}{2}, 0)$ under ϕ-based asynchronous ISAP and pays $\text{TAL}(\delta_3(5), \bar{c}_3^1) = \text{TAL}(4,(5,2,0)) = (3,1,0)$ under the ϕ-based decentralized clearing process. Therefore, the limiting payment matrices $\hat{P}^5 = \hat{P}^6 = \cdots = \hat{P}$ and $P^5 = P^6 = \cdots = P$ are given by

$$\hat{P} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2\frac{1}{2} & 1\frac{1}{2} & 0 \end{bmatrix} \text{ and } P = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix},$$

respectively. The payment matrix P equals the bottom ϕ-transfer scheme obtained in Example 3.1. Note that the Talmud rule does not satisfy composition and thus the payment matrix $\hat{P}$ may differ from the bottom ϕ-transfer scheme as is the case here. Correspondingly, the transfer allocations are unequal:

$$\alpha^{\hat{P}} = (2\frac{1}{2}, 1\frac{1}{2}, 0) \neq (3, 1, 0) = \alpha^{P}.$$

△

The previous example illustrates how ϕ-based (asynchronous) individual settlement allocation procedures and ϕ-based decentralized clearing processes differ in interpretation. In each step of a ϕ-based decentralized clearing process, agents essentially *communicate* what they will pay to each other. The only transfers between agents that take place are those with respect to the resulting limiting payment matrix, i.e., the bottom ϕ-transfer scheme. This contrasts with ϕ-based (asynchronous) individual settlement allocation procedures in which *actual transfers* between agents take place in each step.

6 Concluding remarks

In this paper, we show that composition of the underlying claims rules is the cornerstone of the *unification* of the centralized clearing mechanism based on ϕ-transfer schemes, the ϕ-based individual settlement allocation procedure and ϕ-based (decentralized) sequential clearing mechanisms since, for each mutual claims problem, all of them lead to the bottom ϕ-transfer scheme and give rise to the same transfer allocation. Moreover, we provide an extension of the axiomatic characterization for claims rules as given by Moulin (2000). The extension to ϕ-based mutual claims rules is based on newly defined adequate extensions of the corresponding five axioms: scale invariance, equal treatment of equals, composition, path independence and consistency.

In Remark 3.1, we note that another ϕ-transfer scheme can be obtained by replacing the starting point of the recursive centralized procedure. If we replace the starting point of a ϕ-based decentralized clearing process by the same starting point, then we obtain another ϕ-based (decentralized) clearing mechanism that leads to the same ϕ-transfer scheme. In fact, both of these resulting ϕ-transfer schemes are the top ϕ-transfer scheme. The corresponding transfer allocation is nonetheless the same as the one following from the bottom ϕ-transfer scheme as a result of Theorem 3.1.

Example 4.1 shows that a recursive ϕ-based mutual claims rule r^{ϕ} need not satisfy path independence despite that all of its underlying claims rules in ϕ satisfy path independence. In fact, a recursive ϕ-based mutual claims rule r^{ϕ} need not satisfy the extension of estate monotonicity; a condition we imposed on all claims rules from the outset.

Definition 6.1. A mutual claims rule μ on $\mathcal{L}$ satisfies *estate monotonicity* if, for all finite N, for all $(E, C) \in \mathcal{L}^N$ and $(\tilde{E}, C) \in \mathcal{L}^N$ with $E \leq \tilde{E}$ it holds that $\mu(E, C) \leq \mu(\tilde{E}, C)$.

For a recursive ϕ-based mutual claims rule r^ϕ it may happen that some agents end up receiving strictly more if it turns out that (some of the) agents have a smaller estate value. We can observe this in Example 4.1, where $\tilde{E} \leq E$ with $E = (1, 1, 0)$ and $\tilde{E} = (0, 1, 0)$, but $r^\phi_2(\tilde{E}, C) = 0.8117 > \frac{2}{3} = r^\phi_2(E, C)$. On the other hand, one can readily verify that a ϕ-based mutual claims rule ρ^ϕ satisfies estate monotonicity if all claims rules in ϕ satisfy estate monotonicity by using Lemma 4.1.

A Proofs

Proof of Lemma 4.1.
Let $\underline{P}^{\phi} = (p_{ij})_{i,j \in N}$ and $\tilde{\underline{P}}^{\phi} = (\tilde{p}_{ij})_{i,j \in N}$ be given by

$$p_{ij} = \lim_{k \to \infty} \varphi^i_j(\gamma_i(k), \bar{c}_i),$$
$$\tilde{p}_{ij} = \lim_{k \to \infty} \varphi^i_j(\tilde{\gamma}_i(k), \bar{c}_i),$$

respectively, where, for all $i \in N$ and $k \in \mathbb{N}$, $\gamma_i(k)$ and $\tilde{\gamma}_i(k)$ are defined according to Definition 3.5.

We first show that $\gamma(k) \leq \tilde{\gamma}(k)$ for all $k \in \mathbb{N}$. For $k = 1$ and all $i \in N$, we have $\gamma_i(1) = e_i \leq \tilde{e}_i = \tilde{\gamma}(1)$. Proceeding by induction, assume that for some $k \in \mathbb{N}$, we have $\gamma(k) \leq \tilde{\gamma}(k)$. Let $i \in N$. Estate monotonicity of the claims rules in ϕ and $e_i \leq \tilde{e}_i$ imply that

$$\gamma_i(k+1) = e_i + \sum_{j \in N} \varphi^j_i(\gamma_j(k), \bar{c}_j) \leq \tilde{e}_i + \sum_{fj \in N} \varphi^j_i(\tilde{\gamma}_j(k), \bar{c}_j) = \tilde{\gamma}_i(k+1).$$

Hence, estate continuity of φ^i implies that $p_{ij} \leq \tilde{p}_{ij}$ for all $j \in N$. □

Proof of Theorem 4.2.
The proof comprises two parts. First, we show that ρ^{ϕ} satisfies the five properties on $\mathcal{L}$ if $\phi \in \mathcal{M}$. Second, we show that $\phi \in \mathcal{M}$ if ρ^{ϕ} satisfies the five properties on $\mathcal{L}$.

Let N be an arbitrary but finite set of agents and let $\phi = (\varphi^i)_{i \in N} \in \mathcal{M}^N$. By Theorem 4.1, the three musketeers satisfy the five properties on $\mathcal{C}^N$. In particular, φ^i satisfies the five properties for all $i \in N$. Let $(E, C) \in \mathcal{L}^N$ and consider the bottom ϕ-transfer scheme $\underline{P}^{\phi} = (p_{ij})_{i,j \in N} \in \mathcal{P}^{\phi}(E, C)$.

Let $\lambda > 0$. To show that ρ^{ϕ} satisfies scale invariance, it suffices to show that $\lambda \rho^{\phi}(E, C) = \rho^{\phi}(\lambda E, \lambda C)$. We first show that $\lambda \underline{P}^{\phi} \in \mathcal{P}^{\phi}(\lambda E, \lambda C)$. Let $i \in N$. Scale invariance of φ^i implies that, for all $j \in N$,

$$\lambda p_{ij} = \lambda \varphi^i_j(e_i + \sum_{m \in N} p_{mi}, \bar{c}_i) = \varphi^i_j(\lambda e_i + \sum_{m \in N} \lambda p_{mi}, \lambda \bar{c}_i).$$

Hence, $\lambda \underline{P}^{\phi} \in \mathcal{P}^{\phi}(\lambda E, \lambda C)$. Consequently,

$$\lambda \rho^{\phi}(E, C) = \lambda \alpha^{\underline{P}^{\phi}} \overset{(3.1)}{=} \alpha^{\lambda \underline{P}^{\phi}} = \rho^{\phi}(\lambda E, \lambda C).$$

Let $i, j \in N$ be such that $\varphi^i = \varphi^j$, $e_i = e_j$, $c_{im} = c_{jm}$ and $c_{mi} = c_{mj}$ for all $m \in N$. To show that ρ^ϕ satisfies equal treatment of equals, it suffices to show that $\rho_i^\phi(E, C) = \rho_j^\phi(E, C)$. We first show that $p_{ki} = p_{kj}$ and $p_{ik} = p_{jk}$ for all $k \in N$. First, we have $c_{ij} = c_{jj} = 0$ and $c_{ji} = c_{ii} = 0$ and, by assumptions (i) and (ii) of a transfer scheme as given in Definition 3.1, this implies that $p_{ij} = 0 = p_{ji}$. Second, let $k \in N \setminus \{i, j\}$. Then, by Definition 3.3 of a ϕ-transfer scheme, we have

$$p_{ki} = \varphi_i^k(e_k + \sum_{m \in N} p_{mk}, \bar{c}_k) = \varphi_j^k(e_k + \sum_{m \in N} p_{mk}, \bar{c}_k) = p_{kj}$$

where the second equality follows from the fact that $c_{ki} = c_{kj}$ and because φ^k satisfies equal treatment of equals. As a consequence, $e_i + \sum_{m \in N} p_{mi} = e_j + \sum_{m \in N} p_{mj}$, and thus also

$$p_{ik} = \varphi_k^i(e_i + \sum_{m \in N} p_{mi}, \bar{c}_i) = \varphi_k^j(e_j + \sum_{m \in N} p_{mj}, \bar{c}_j) = p_{jk}.$$

Here, the second equality follows from the fact that $c_{ik} = c_{jk}$, $\varphi^i = \varphi^j$ and the fact that both φ^i and φ^j satisfy equal treatment of equals. Hence, $\rho_i^\phi(E, C) = \alpha_i^{\underline{P}^\phi} \overset{(3.1)}{=} \alpha_j^{\underline{P}^\phi} = \rho_j^\phi(E, C)$.

Let $(\tilde{E}, C) \in \mathcal{L}^N$ with $E \leq \tilde{E}$ and consider the corresponding bottom ϕ-transfer scheme $\underline{\tilde{P}}^\phi = (\tilde{p}_{ij})_{i,j \in N} \in \mathcal{P}^\phi(\tilde{E}, C)$. To show composition, it suffices to show that $\rho^\phi(\tilde{E}, C) = \rho^\phi(E, C) + \rho^\phi(\tilde{E} - E, C - \underline{P}^\phi)$. First, we show that $(\underline{\tilde{P}}^\phi - \underline{P}^\phi)$ is a ϕ-transfer scheme for $(\tilde{E} - E, C - \underline{P}^\phi)$. Let $i, j \in N$. Since $E \leq \tilde{E}$ and $\underline{P}^\phi \leq \underline{\tilde{P}}^\phi$ (by Lemma 4.1), we have

$$e_i + \sum_{m \in N} p_{mi} \leq \tilde{e}_i + \sum_{m \in N} \tilde{p}_{mi}.$$

Hence,

$$\begin{aligned}
(\tilde{p}_{ij} - p_{ij}) &= \varphi_j^i(\tilde{e}_i + \sum_{m \in N} \tilde{p}_{mi}, \bar{c}_i) - \varphi_j^i(e_i + \sum_{m \in N} p_{mi}, \bar{c}_i) \\
&= \varphi_j^i\left((\tilde{e}_i - e_i) + \sum_{m \in N} (\tilde{p}_{mi} - p_{mi}), \bar{c}_i - \varphi^i(e_i + \sum_{m \in N} p_{mi}, \bar{c}_i)\right) \\
&= \varphi_j^i((\tilde{e}_i - e_i) + \sum_{m \in N} (\tilde{p}_{mi} - p_{mi}), \bar{c}_i - \bar{p}_i),
\end{aligned}$$

where $\bar{p}_i$ is the i-th row of $\underline{P}^\phi$. The first equality and third equality follow from condition (3.2) of a ϕ-transfer scheme; the second equality follows from composition of φ^i. Therefore, $(\underline{\tilde{P}}^\phi - \underline{P}^\phi) \in \mathcal{P}^\phi(\tilde{E} - E, C - \underline{P}^\phi)$ (see (3.2)) and thus

$$\rho^\phi(\tilde{E} - E, C - \underline{P}^\phi) = \alpha^{(\underline{\tilde{P}}^\phi - \underline{P}^\phi)} \overset{(3.1)}{=} \alpha^{\underline{\tilde{P}}^\phi} - \alpha^{\underline{P}^\phi} = \rho^\phi(\tilde{E}, C) - \rho^\phi(E, C).$$

Let $(\tilde{E}, C) \in \mathcal{L}^N$ with $\tilde{E} \leq E$ and consider the corresponding bottom ϕ-transfer scheme $\underline{\tilde{P}}^\phi = (\tilde{p}_{ij})_{i,j \in N} \in \mathcal{P}^\phi(\tilde{E}, C)$. To show path independence, it suffices to show that $\rho^\phi(\tilde{E}, C) = \rho^\phi(\tilde{E}, \underline{P}^\phi)$. We first show that $\underline{\tilde{P}}^\phi \in \mathcal{P}^\phi(\tilde{E}, \underline{P}^\phi)$. Let $i, j \in N$. Since $\tilde{E} \leq E$ and $\underline{\tilde{P}}^\phi \leq \underline{P}^\phi$ (by Lemma 4.1), we have

$$\tilde{e}_i + \sum_{m \in N} \tilde{p}_{mi} \leq e_i + \sum_{m \in N} p_{mi}.$$

Hence,

$$\begin{aligned}
\tilde{p}_{ij} = \varphi^i_j(\tilde{e}_i + \sum_{m \in N} \tilde{p}_{mi}, \bar{c}_i) &= \varphi^i_j(\tilde{e}_i + \sum_{m \in N} \tilde{p}_{mi}, \varphi^i(e_i + \sum_{m \in N} p_{mi}, \bar{c}_i)) \\
&= \varphi^i_j(\tilde{e}_i + \sum_{m \in N} \tilde{p}_{mi}, \bar{p}_i),
\end{aligned}$$

where $\bar{p}_i$ is the i-th row of $\underline{P}^\phi$. The first equality and third equality follow from condition (3.2) of a ϕ-transfer scheme; the second equality follows from path independence of φ^i. Therefore, $\underline{\tilde{P}}^\phi \in \mathcal{P}^\phi(\tilde{E}, \underline{P}^\phi)$ and thus $\rho^\phi(\tilde{E}, C) = \alpha^{\underline{\tilde{P}}^\phi} = \rho^\phi(\tilde{E}, \underline{P}^\phi)$.

Let $S \subseteq N$ be such that $(E^{S, \underline{P}^\phi}, C^S) \in \mathcal{L}^S$. Denote the i-th row of claims matrix C^S by $\bar{c}^S_i$. To show consistency, it suffices to show that $\rho^\phi_i(E, C) = \rho^\phi_i(E^{S, \underline{P}^\phi}, C^S)$ for all $i \in S$. We first show that the bottom ϕ-transfer scheme $\underline{P}^\phi$ restricted to agents in S, given by $P^{S,\phi} = (p_{ij})_{i,j \in S}$, belongs to $\mathcal{P}^\phi(E^{S, \underline{P}^\phi}, C^S)$. Let $i, j \in S$. Then,

$$\begin{aligned}
p_{ij} &= \varphi^i_j(e_i + \sum_{m \in N} p_{mi}, \bar{c}_i) \\
&= \varphi^i_j(e_i + \sum_{m \in N} p_{mi} - \sum_{k \in N \setminus S} \varphi^i_k(e_i + \sum_{m \in N} p_{mi}, \bar{c}_i), \bar{c}^S_i) \\
&= \varphi^i_j(e_i + \sum_{m \in N} p_{mi} - \sum_{k \in N \setminus S} p_{ik}, \bar{c}^S_i)
\end{aligned}$$

$$= \varphi_j^i(e_i + \sum_{m \in N \setminus S} p_{mi} - \sum_{m \in N \setminus S} p_{im} + \sum_{m \in S} p_{mi}, \bar{c}_i^S)$$
$$= \varphi_j^i(e_i^{S,\underline{P}^\phi} + \sum_{m \in S} p_{mi}, \bar{c}_i^S).$$

The first and third equality follow from $\underline{P}^\phi \in \mathcal{P}^\phi(E, C)$; the second equality follows from consistency of φ^i; the last equality follows from (4.1). Therefore, $P^{S,\phi} \in \mathcal{P}^\phi(E^{S,\underline{P}^\phi}, C^S)$ and thus $\rho_i^\phi(E, C) = \alpha_i^{\underline{P}^\phi} \overset{(3.1)}{=} \alpha_i^{P^{S,\phi}} = \rho_i^\phi(E^{S,\underline{P}^\phi}, C^S)$ for all $i \in S$.

Next, assume that ρ^ϕ, where ϕ is an arbitrary vector of claims rules, satisfies the five properties on $\mathcal{L}$. We will argue that $\phi \in \mathcal{M}$, that is, ϕ can can only consist of the three musketeers.

Without loss of generality, let $N = \{1, 2, \ldots, n\}$ and $\phi = (\varphi^1, \varphi^2, \ldots, \varphi^n)$. Since $\mathcal{L}$ contains all mutual claims problems of the form

$$E = \begin{matrix} 1 \\ 2 \\ \vdots \\ n \end{matrix} \begin{pmatrix} e \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad C = \begin{matrix} \\ 1 \\ 2 \\ \vdots \\ n \end{matrix} \begin{matrix} \begin{matrix} 1 & 2 & \ldots & n \end{matrix} \\ \begin{bmatrix} 0 & c_2 & \ldots & c_n \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix} \end{matrix},$$

it follows from Proposition 4.1 that φ^1 satisfies the five properties on $\mathcal{C}$. Hence, φ^1 must be one of the three musketeers due to Theorem 4.1. Using similar appropriate forms, it follows that, for all $i \in N \setminus \{1\}$, φ^i also must be one of the three musketeers.

□

References

Aumann, R., & Maschler, M. (1985). Game theoretic analysis of a bankruptcy problem from the talmud. *Journal of Economic Theory*, *36*(2), 195-213.

Csóka, P., & Herings, P. J.-J. (2018). Decentralized clearing in financial networks. *Management Science*, *64*(10), 4681-4699.

Csóka, P., & Herings, P. J.-J. (2021). An axiomatization of the proportional rule in financial networks. *Management Science*, *67*(5), 2799-2812.

Eisenberg, L., & Noe, T. H. (2001). Systemic risk in financial systems. *Management Science*, *47*(2), 236-249.

Glasserman, P., & Young, H. P. (2016). Contagion in financial networks. *Journal of Economic Literature*, *54*(3), 779-831.

Groote Schaarsberg, M., Reijnierse, H., & Borm, P. (2018). On solving mutual liability problems. *Mathematical Methods of Operations Research*, *87*(3), 383–409.

Herrero, C., & Villar, A. (2001). The three musketeers: four classical solutions to bankruptcy problems. *Mathematical Social Sciences*, *42*(3), 307-328.

Kaminski, M. M. (2000). 'Hydraulic' rationing. *Mathematical Social Sciences*, *40*(2), 131-155.

Ketelaars, M. (2020). *The three musketeers and their five unifying properties.* (Research Master thesis). CentER Graduate School for Economics and Business, Tilburg University, Tilburg, The Netherlands.

Ketelaars, M., Borm, P., & Quant, M. (2020). Decentralization and mutual liability rules. *Mathematical Methods of Operations Research*, *92*(3), 577–599.

Moulin, H. (1987). Equal or proportional division of a surplus, and other methods. *International Journal of Game Theory*, *16*(3), 161-186.

Moulin, H. (2000). Priority rules and other asymmetric rationing methods. *Econometrica*, *68*(3), 643-684.

O'Neill, B. (1982). A problem of rights arbitration from the talmud. *Mathematical Social Sciences*, *2*(4), 345-371.

Rogers, L. C. G., & Veraart, L. A. M. (2013). Failure and rescue in an interbank network. *Management Science*, *59*(4), 882-898.

Tarski, A. (1955). A lattice-theoretical fixpoint theorem and its applications. *Pacific Journal of Mathematics*, *5*(2), 285-309.

Thomson, W. (2003). Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey. *Mathematical Social Sciences*, *45*(3), 249 - 297.

Thomson, W. (2013). Game-theoretic analysis of bankruptcy and taxation problems: Recent advances. *International Game Theory Review*, *15*(3), 1-14.

Thomson, W. (2015). Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: An update. *Mathematical Social Sciences*, *74*, 41 - 59.

Wessels, B., & Madaus, S. (2017). *Rescue of business in insolvency law.* (ELI Business Rescue Report). European Law Institute, Vienna, Austria.

Young, H. P. (1988). Distributive justice in taxation. *Journal of Economic Theory*, *44*(2), 321 - 335.