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SUPERMODULAR NTU-GAMES

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Supermodular NTU-games$^1$

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Abstract

A cooperative game with non-transferable utility (NTU-game) consists of a collection of payoff sets for the subsets of a finite set of players, for which it has to be determined how much payoff each player must receive. The core of an NTU-game consists of all payoff vectors that are in the payoff set of the coalition of all players and cannot be improved upon by any coalition of players. For cooperative games with transferable utility (TU-games) the notion of convexity was introduced to guarantee that the Shapley value, being the average of all marginal vectors of the game, is an element of the core. Convexity of a TU-game is equivalent to supermodularity of the characteristic function underlying the game.

In this paper we introduce the concept of supermodularity for NTU-games. Supermodularity for NTU-games is weaker than other existing types of convexity. Under supermodularity of an NTU-game it is shown that all appropriately defined marginal vectors of the game are elements of the core. As solution concept for NTU-games we propose a set of solutions that is determined by the average of all marginal vectors of the game. For TU-games the solution set coincides with the Shapley value of the game. Also conditions are stated under which the solution set is a subset of the core and is the set of bargaining solutions of a corresponding bargaining problem.

Key words: Core, Shapley value, convexity, supermodularity, marginal vector

AMS subject classification: 47H10, 49J40, 52C40, 90C30, 91B50.
JEL code: C71.
1 Introduction

A cooperative game with non-transferable utility (NTU-game) consists of a finite number of players and for every subset of players, a coalition, a feasible set of payoff vectors. The problem is to determine how much payoff each player must receive. The most well-known solution is the core consisting of all payoff vectors that are efficient for the grand coalition of all players and cannot be blocked by any coalition. If the payoffs of any coalition are transferable among its players an NTU-game is a game with transferable utility (TU-game).

For TU-games Shapley [13] introduces as single-valued solution the average of all marginal vectors of the game. All those marginal vectors are elements of the core of the TU-game if and only if the game is convex, which means that the characteristic function underlying the game is supermodular, see [4].

In this paper we first introduce the concept of supermodularity for NTU-games. For TU-games this concept is equivalent to convexity. If an NTU-game is supermodular all the appropriately defined marginal vectors are elements of the core of the game. The class of supermodular NTU-games contains existing classes of individual merge convex games ([3]) and strong ordinal convex games ([8]) that also guarantee the core stability of all marginal vectors. Ordinal convexity ([16]) and cardinal convexity ([15]) are introduced in relation with the von-Neumann-Morgenstern solution of NTU-games. These two concepts also coincide with convexity for TU-games but in general the two concepts do not guarantee that all marginal vectors are elements of the core of an NTU-game.

As solution concept for NTU-games we propose a set of solutions that is determined by the average of all marginal vectors of the game. The solution set is never empty. In case the NTU-game is induced by a TU-game, the solution set coincides with this average and is the Shapley value of the TU-game. In general, if the average of all marginal vectors is an efficient allocation for the grand coalition, as is always the case if the set of efficient payoff vectors for the grand coalition is a hyperplane, the solution set is a singleton, being this average. If the average of all marginal vectors is a feasible but not efficient allocation for the grand coalition, then the payoffs at the average are increased in any strictly positive direction until an efficient allocation for the grand coalition is obtained. And, if the average is not a feasible allocation for the grand coalition the payoffs are decreased in any strictly negative direction until an efficient allocation for the grand coalition is reached.

Being determined by the average of all marginal vectors of the game, the solution takes a similar approach as the marginal based compromise value, or MC-value ([11]). It turns out that the MC-value lies in the solution set if all payoff sets satisfy non-levelness. There are several other solution concepts on the class of NTU-games that generalize the Nash bargaining solution of a pure bargaining problem ([10]), such as the Harsanyi value ([1]), the Shapley NTU-value ([14]), and the consistent Shapley value ([6], [7]). From
a viewpoint of bargaining problems, the solution set we propose can be seen as the set of bargaining solutions of an induced bargaining problem in which the average of the marginal vectors is the disagreement point. In order for the allocation at the average of the marginal vectors to be the disagreement point, the allocation should be feasible for the grand coalition and not be blocked by any proper subcoalition. On the other hand, if the average of marginal vectors is not feasible, then one may see the average as a utopia point. We also discuss a specific solution in the solution set, an egalitarian type of solution.

For TU-games supermodularity of the characteristic function does not only guarantee that all marginal vectors of the game are elements of the core, but also that their average, the Shapley value, is in the core. As mentioned, this is because the core of a TU-game is a convex set. However, even for supermodular NTU-games the core is typically not a convex set and therefore the average of the marginal vectors may not be an efficient allocation or not be a feasible allocation for the grand coalition, and, moreover, it may be blocked by a proper subcoalition, although the marginal vectors are not. To guarantee that the solution set is a subset of the core, we introduce a convexity condition that roughly says that the payoff set of the grand coalition is not less convex-shaped or its complement is not more convex-shaped than for the payoff sets of any subcoalition holds.

The paper is organized as follows. In Section 2 the concept of supermodularity is introduced for NTU-games. In Section 3 the solution set is introduced and core stability is studied. Also a comparison with other solutions is made in that section.

## 2 Supermodularity

We consider cooperative games without side-payment. A non-transferable utility game (or NTU-game) consists of a finite set \( [n] = \{1, \ldots, n \} \) of \( n \geq 2 \) players and a mapping \( V(\cdot) \) assigning to every subset \( S \) of \( [n] \) a subset \( V(S) \) of payoff vectors in \( \mathbb{R}^S \) with \( V(\emptyset) = \{0\} \).

In what follows we identify an NTU-game \( ([n], V) \) on player set \( [n] \) with \( V \). An element \( x = (x_i)_{i \in S} \) in \( V(S) \) is an allocation for coalition \( S \) that can be realized by the players within \( S \) and at which player \( i \in S \) receives payoff \( x_i \). For a vector \( x \in \mathbb{R}^T \) and \( S \subseteq T, T \in 2^{[n]} \), \( x_S \) denotes the vector \((x_i)_{i \in S}\) in \( \mathbb{R}^S \), with \( x_S = 0 \) if \( S = \emptyset \). We often write \((x_i, x_S \setminus \{i\})\) for \( x \in \mathbb{R}^S \) and \( i \in S \). For \( S \in 2^{[n]} \), we denote \( \mathbb{R}_S^+ = \{x \in \mathbb{R}^S \mid x_i \geq 0 \text{ for all } i \in S\} \) and \( \mathbb{R}_S^{++} = \{x \in \mathbb{R}^S \mid x_i > 0 \text{ for all } i \in S\} \).

We make the standard assumptions on \( V \) that for all \( S \in 2^{[n]}, S \neq \emptyset \), the set \( V(S) \) is closed and comprehensive in \( \mathbb{R}^S \) and that the game is zero-normalized, i.e., \( V(\{i\}) = (-\infty, 0] \) for all \( i \in [n] \). We also assume that for all \( S \in 2^{[n]}, S \neq \emptyset \), and \( b \in \mathbb{R}^S \) it holds that the set \( \{x \in V(S) \mid x_i \geq b_i \text{ for all } i \in S\} \) is either empty or, if not empty, a bounded set. Moreover, we assume that \( V \) is monotone, i.e., for any \( S \subseteq T, T \in 2^{[n]} \), and \( x \in V(S) \)
there exists $y \in V(T)$ such that $y_S \geq x$.

Given an NTU-game $V$ and coalition $S \in 2^{[n]}$, let $D^V(S) = \{x \in \mathbb{R}^S \mid \nexists y \in V(S), y \gg x\}$, $E^V(S) = V(S) \cap D^V(S)$, and $I^V(S) = V(S) \setminus E^V(S)$. Then a payoff vector $x \in \mathbb{R}^T$, with $S \subseteq T$, is blocked by coalition $S$ if $x_S \in I^V(S)$, i.e., there exists $y \in V(S)$ such that $y \gg x_S$, $x$ is not blocked by coalition $S$ if $x_S \in D^V(S)$, i.e., there exists no $y \in V(S)$ such that $y \gg x_S$, and $x$ is weakly Pareto-optimal, or efficient, for coalition $S$ if $x_S \in E^V(S)$, i.e., $x_S \in V(S)$ and there is no $y \in V(S)$ such that $y \gg x_S$. The core of an NTU-game $V$, denoted $C(V)$, is the set of weakly Pareto-optimal allocations for the grand coalition $[n]$ of all players, that are not blocked by any coalition, i.e.,

$$C(V) = \{x \in V([n]) \mid x_S \notin I^V(S), \forall S \in 2^{[n]} \setminus \{\emptyset\}\}.$$ 

Let $\Pi$ be the set of linear orders on $[n]$. Given an NTU-game $V$ and a linear order $\pi \in \Pi$, the marginal vector $m^\pi(V) \in \mathbb{R}^{[n]}$ is defined by

$$m^\pi_{\pi(k)}(V) = \max\{y_{\pi(k)} \mid y \in V(\{\pi(1), \ldots, \pi(k)\}), \ y_{\pi(i)} = m^\pi_{\pi(i)}(V), \forall i \in [k-1]\},$$

for $k = 1, 2, \ldots, n$. Notice that $m^\pi(V)$ always exists and is uniquely defined.

Next we introduce a condition that guarantees that all marginal vectors of an NTU-game are elements of the core.

**Definition 2.1** An NTU-game $V$ is supermodular if for any $A \in 2^{[n]}$, $j \in A$ and $x \in E^V(A)$ satisfying $x_{A \setminus \{j\}} \in E^V(A \setminus \{j\})$ and $x_j = \max\{y \mid (y, x_{A \setminus \{j\}}) \in V(A)\}$ it holds that for all $B \subset A$ such that $j \in B$

$$x_{B \setminus \{j\}} \in D^V(B \setminus \{j\}) \Rightarrow x_B \in D^V(B).$$

Supermodularity of an NTU-game means that if for a player in a coalition it holds that a payoff vector is efficient with and without him and a subcoalition without him cannot block this payoff vector, then this payoff vector can also not be blocked by this subcoalition together with him. Notice that if $B = \{j\}$ for some $j \in [n]$ then $D^V(B \setminus \{j\}) = \{0\}$ and $x_B \in D^V(B)$ means $x_j \geq 0$. In the following theorem we show that all marginal vectors of a supermodular NTU-game are in the core of the game.

**Theorem 2.2** Let $V$ be a supermodular NTU-game, then $m^\pi(V) \in C(V)$ for all $\pi \in \Pi$.

**Proof.** We prove the result by induction. The theorem clearly holds for $n = 2$. Suppose for $n = k$, $k \geq 2$, the theorem is true. For $n = k + 1$ we may assume without loss of generality that $\pi(k + 1) = k + 1$. From the construction of $m^\pi(V)$ and the induction
argument it follows that \( m^\pi(N, V) \) cannot be blocked by any coalition \( S \subseteq [k] \). Take any coalition \( S \cup \{k + 1\} \) where \( S \subseteq [k] \). Then for \( A = [k + 1], B = S \cup \{k + 1\} \) and \( j = k + 1 \), we have that \( m^\pi(V) \in E^V(A), m^\pi_j(V) = \max \{ y \mid (y, m^\pi_{A \setminus \{j\}}(V)) \in V(A) \} \) and \( m^\pi_{A \setminus \{j\}}(V) \in E^V(A \setminus \{j\}) \) since \( m^\pi(V) \) is a marginal vector and \( A \setminus \{j\} = [k] \). If \( S = \emptyset \), then supermodularity of \( V \) implies that \( m^\pi_j(V) \geq 0 \) and therefore singleton coalition \( \{k + 1\} \) can not block \( m^\pi(V) \). Suppose \( S \neq \emptyset \). Then it follows from the induction argument that \( m^\pi_{B \setminus \{j\}}(V) \in D^V(B \setminus \{j\}) \). Since \( V \) is supermodular and \( S \neq \emptyset \), this implies that \( m^\pi_B(V) \in D^V(B) \), i.e., \( m^\pi(V) \) is not blocked by coalition \( S \cup \{k + 1\} \). \( \Box \)

Next, we show that for a transferable utility game convexity is equivalent to supermodularity of the corresponding NTU-game. A (zero-normalized) transferable utility game (TU-game) \( v \) is described by a characteristic function \( v : 2^{[n]} \to \mathbb{R} \), with \( v(\emptyset) = 0 \) and \( v(\{\} \}) = 0 \) for all \( i \in [n] \), assigning to coalition \( S \in 2^{[n]} \) its worth \( v(S) \) that can be freely distributed amongst its members. A TU-game \( v \) induces an NTU-game \( V \), where \( V(S) = \{ x \in \mathbb{R}^S \mid \sum_{i \in S} x_i \leq v(S) \} \) for all \( S \in 2^{[n]} \). A TU-game \( v \) is convex if the function \( v \) is supermodular, i.e., for all \( S \subseteq T, T \subseteq 2^{[n]} \), and \( i \in S \) it holds that \( v(T) - v(T \setminus \{i\}) \geq v(S) - v(S \setminus \{i\}) \).

**Proposition 2.3** A TU-game \( v \) is convex if and only if the induced NTU-game \( V \) is supermodular.

**Proof.** Suppose that the NTU-game \( V \) induced from a TU-game \( v \) is supermodular. Given \( T \in 2^{[n]}, S \subseteq T \) and \( i \in S \), take any \( x \in \mathbb{R}^T \) such that \( \sum_{h \in S \setminus \{i\}} x_h = v(S \setminus \{i\}) \), \( \sum_{h \in T \setminus \{i\}} x_h = v(T \setminus \{i\}) \) and \( \sum_{h \in T} x_h = v(T) \). It holds for \( V \) that \( x \in E^V(T), x_{T \setminus \{i\}} \in E^V(T \setminus \{i\}), x_i = v(T) - v(T \setminus \{i\}) = \max \{ y \mid (y, x_{T \setminus \{i\}}) \in V(T) \} \) and \( x_{S \setminus \{i\}} \in E^V(S \setminus \{i\}) \) \( \subseteq D^V(S \setminus \{i\}) \) if \( S \setminus \{i\} \neq \emptyset \). Since \( V \) is supermodular, it follows that \( x_i \geq 0 = v(\{i\}) \). Thus if \( S \setminus \{i\} = \emptyset \), then \( v(T) - v(T \setminus \{i\}) \geq v(\{i\}) \) is satisfied. If \( S \setminus \{i\} \neq \emptyset \) then it follows from supermodularity that \( x_S \in D^V(S) \), i.e., \( \sum_{h \in S} x_h \geq v(S) \), and therefore

\[
v(T) - v(T \setminus \{i\}) = x_i = \sum_{h \in S} x_h - \sum_{h \in S \setminus \{i\}} x_h \geq v(S) - v(S \setminus \{i\}) .
\]

Next, suppose that a TU-game \( v \) is convex and let \( V \) be the NTU-game induced by \( v \). Given \( A \in 2^{[n]}, B \subseteq A \) and \( j \in B \), take any \( x \in \mathbb{R}^{[n]} \) such that \( x_A \in E^V(A), x_{A \setminus \{j\}} \in E^V(A \setminus \{j\}) \) and \( x_{B \setminus \{j\}} \in D^V(B \setminus \{j\}) \). Then \( \sum_{h \in A} x_h = v(A) \) and \( \sum_{h \in A \setminus \{j\}} x_h = v(A \setminus \{j\}) \), which implies \( x_j = v(A) - v(A \setminus \{j\}) = \max \{ y \mid (y, x_{A \setminus \{j\}}) \in V(A) \} \), and \( \sum_{h \in B \setminus \{j\}} x_h \geq v(B \setminus \{j\}) \). Since \( v \) is convex and zero-normalized, it follows that

\[
\sum_{h \in B} x_h = \sum_{h \in B \setminus \{j\}} x_h + x_j \geq v(B \setminus \{j\}) + v(A) - v(A \setminus \{j\}) \geq v(B) ,
\]
which implies \( x_B \in D^V(B) \).

In the literature several convexity conditions of NTU-games are introduced under which every marginal vector of an NTU-game is in the core of the game and are equivalent to convexity of TU-games. One of such conditions is introduced by [9] as strong ordinal convexity.

**Definition 2.4** An NTU-game \( V \) is **strong ordinal convex** if for any \( S, T \subseteq [n] \) and \( x \in \mathbb{R}^n \) it holds that

\[
x_S \in V(S), \ x_{S \cap T} \in D^V(S \cap T) \text{ and } x_T \in V(T) \implies x_{S \cup T} \in V(S \cup T).
\]

[9] shows that every marginal vector of a strong ordinal convex NTU-game is a core element and that the NTU-game induced from a TU-game is strong ordinal convex if the TU-game itself is convex, while [8] shows that if the NTU-game induced from a TU-game is strong ordinal convex then the TU-game is convex. Therefore on the class of TU-games supermodularity and strong ordinal convexity are equivalent. The next proposition shows that strong ordinal convexity implies supermodularity. An NTU-game \( V \) is superadditive if for any \( S, T \subseteq [n] \) and \( x \in \mathbb{R}^{S \cup T} \), it holds that \( x_S \in V(S) \) and \( x_T \in V(T) \) imply \( x \in V(S \cup T) \). Note that a strong ordinal convex game is superadditive.

**Proposition 2.5** Let \( V \) be a strong ordinal convex NTU-game. Then \( V \) is superadditive.

**Proof.** Suppose \( V \) is not supermodular. For any \( A \in 2^n \), \( j \in A \) and \( x \in E^V(A) \) satisfying \( x_{A \setminus \{j\}} \in E^V(A \setminus \{j\}) \) and \( x_j = \max\{y \mid (y, x_{A \setminus \{j\}}) \in E^V(A)\} \) it follows from superadditivity of \( V \) and since \( 0 \in V(\{j\}) \) that \( x_j \geq 0 \). Then there exists \( A \in 2^n \), \( B \subseteq A \), \( j \in B \) with \( B \setminus \{j\} \neq \emptyset \) and \( x \in \mathbb{R}^n \) such that \( x_A \in E^V(A), x_{A \setminus \{j\}} \in E^V(A \setminus \{j\}) \), \( x_j = \max\{y \mid (y, x_{A \setminus \{j\}}) \in V(A)\} \), \( x_{B \setminus \{j\}} \in D^V(B \setminus \{j\}) \), and \( x_B \in I^V(B) \). Let \( S = A \setminus \{j\} \) and \( T = B \). Since \( x_T \in I^V(T) \), there exists \( x' \in \mathbb{R}^n \) such that \( x'_S = x_S \), \( x'_j > x_j \), and \( x'_T \in V(T) \). Then \( x'_j > x_j, x'_S = x_S, \) and \( x_j = \max\{y \mid (y, x_S) \in V(S \cup \{j\})\} \) imply \( x'_{S \cup \{j\}} \notin V(S \cup \{j\}) \). Therefore, \( x'_S \in V(S) \), \( x'_{S \cap T} \in D^V(S \cap T) \), and \( x'_T \in V(T) \), whereas \( x'_{S \cup T} \notin V(S \cup T) \), which contradicts that \( V \) is strong ordinal convex.

The following example shows that supermodularity is weaker than strong ordinal convexity.

**Example 2.6** Consider the 4-person NTU-game \( V \) with

\[
V(\{i\}) = (-\infty, 0] \text{ for } i = 1, 2, 3, 4,
\]

\[
V(S) = \{x \in \mathbb{R}^S \mid \sum_{i \in S} x_i \leq |S|^2\} \text{ if } |S| \geq 2 \text{ and } S \neq \{2, 3, 4\},
\]

\[
V(\{2, 3, 4\}) = \{x \in \mathbb{R}^{(2,3,4)} \mid x_2 + x_3 + x_4 \leq 9\} \cup \{x \in \mathbb{R}^{(2,3,4)} \mid x_2, x_3 \leq 0, x_4 \leq 17\}.
\]
This game is not strong ordinal convex. For example, consider the payoff vector \( x = (0, 0, 0, 17) \) and take \( S = \{1, 2\} \) and \( T = \{2, 3, 4\} \). It holds that \( x_S = (0, 0) \in V(S), x_{S\cap T} = 0 \in D^V(S \cap T) \) and \( x_T = (0, 0, 17) \in V(T) \), while \( x \notin V([4]) \). However, \( V \) is supermodular.

[3] introduces another notion of convexity, called individual merge convexity, and shows that for an NTU-game satisfying this condition all marginal vectors are core elements.

**Definition 2.7** An NTU-game \( V \) is **individual merge convex** if it is superadditive and for any \( i \in [n], T \subseteq [n] \setminus \{i\} \) and nonempty \( S \subset T \) it holds that for any \( p \in E^V(S), q \in V(T) \) and \( r \in V(S \cup \{i\}) \) such that \( r_S \geq p \) there exists \( s \in V(T \cup \{i\}) \) satisfying \( s_T \geq q \) and \( s_i \geq r_i \).

[3] proves that for a TU-game individual merge convexity of the induced NTU-game is equivalent to convexity. The next proposition shows that individual merge convexity implies supermodularity.

**Proposition 2.8** Let \( V \) be an individual merge convex NTU-game. Then \( V \) is supermodular.

**Proof.** Suppose \( V \) is not supermodular. For any \( A \in 2^{[n]}, j \in A \) and \( x \in E^V(A) \) satisfying \( x_{A \setminus \{j\}} \in E^V(A \setminus \{j\}) \) and \( x_j = \max \{ y \mid (y, x_{A \setminus \{j\}}) \in V(A) \} \) it holds from superadditivity of \( V \) and \( 0 \in V(\{j\}) \) that \( x_j \geq 0 \). Then there exist \( A \in 2^{[n]}, B \subset A, j \in B \) with \( B \setminus \{j\} \neq \emptyset \) and \( x \in \mathbb{R}^{[n]} \) such that \( x_A \in E^V(A), x_{A \setminus \{j\}} \in E^V(A \setminus \{j\}) \), \( x_j = \max \{ y \mid (y, x_{A \setminus \{j\}}) \in V(A) \}, x_{B \setminus \{j\}} \in D^V(B \setminus \{j\}) \), and \( x_B \in I^V(B) \). Let \( S = B \setminus \{j\}, T = A \setminus \{j\}, \) and \( i = j \). Since \( x_{S \cup \{i\}} \in I^V(S \cup \{i\}) \), there exists \( x' \in \mathbb{R}^{[n]} \) such that \( x'_T = x_T, x'_i > x_i, \) and \( x'_{S \cup \{i\}} \in V(S \cup \{i\}) \). Then \( x'_i > x_i, x'_T = x_T, \) and \( x_i = \max \{ y \mid (y, x_T) \in V(T \cup \{i\}) \} \) imply \( x'_{T \cup \{i\}} \notin V(T \cup \{i\}) \). Since \( x'_S = x_S \) and \( x_S \in D^V(S) \), there exists \( z \in x'_S \) such that \( z \in E^V(S) \). Take \( p = z, q = x_T, \) and \( r = x'_{S \cup \{i\}} \), then \( p \in E^V(S), q \in V(T), \) and \( r \in V(S \cup \{i\}) \). Moreover, \( r_S \geq p \), but there exists no \( s \in V(T \cup \{i\}) \) such that \( s_T \geq q \) and \( s_i \geq r_i \), since \( x'_{T \cup \{i\}} \notin V(T \cup \{i\}) \), which contradicts that \( V \) is individual merge convex. □

The next example shows that supermodularity is weaker than individual merge convexity.
Example 2.9 (Example 4.5 in [2]) Consider the 3-person NTU-game \( V \) with

\[
V\{i\} = (-\infty, 0] \text{ for } i = 1, 2, 3,
\]
\[
V\{1, 2\} = \{ x \in \mathbb{R}^{\{1,2\}} \mid x_1 + x_2 \leq 3 \},
\]
\[
V\{1, 3\} = \{ x \in \mathbb{R}^{\{1,3\}} \mid x_1 + x_3 \leq 2 \},
\]
\[
V\{2, 3\} = \{ x \in \mathbb{R}^{\{2,3\}} \mid x_2 + x_3 \leq 6 \},
\]
\[
V\{3\} = \{ x \in \mathbb{R}^{N} \mid \frac{x_1}{6} + \frac{x_2}{10} + \frac{x_3}{14} \leq 1 \}.
\]

To show that \( V \) is not individual-merge convex, take \( p = 0 \in E^V\{2\} \), \( q = (6, 0) \in V\{2, 3\} \) and \( r = (3, 0) \in V\{1, 2\} \), then \( r_2 \geq 0 \), but there exists no \( s \in V\{1, 2, 3\} \) such that \( s_{2,3} \geq (6, 0) \) and \( s_1 \geq 3 \). However, \( V \) is supermodular. Take for example \( A = \{1, 2, 3\} \) and \( j = 3 \). Then \( x = (x_1, 3 - x_1, \frac{49}{5} - \frac{14}{15} x_1) \) satisfies \( x \in E^V(A) \), \( x_{A\setminus\{j\}} \in E^V(A \setminus \{j\}) \) and \( x_j = \max\{y \mid (y, x_{A\setminus\{j\}}) \in V(A)\} \). When \( B = \{1, 3\} \) and \( x_{B\setminus\{j\}} \in D^V(B \setminus \{j\}) \), i.e., \( x_1 \geq 0, \) then \( x_1 + x_3 = \frac{49}{5} + \frac{14}{15} x_1 > 2 \) and therefore \( x_B \in D^V(B) \). When \( B = \{2, 3\} \) and \( x_{B\setminus\{j\}} \in D^V(B \setminus \{j\}) \), i.e., \( x_3 \geq 0 \) and therefore \( x_1 \leq 3 \), then \( x_2 + x_3 = 3 - x_1 + \frac{49}{5} - \frac{14}{15} x_1 = \frac{64}{5} - \frac{29}{15} x_1 > 6 \) and therefore \( x_B \in D^V(B) \), and so on.

3 Solution concept

In this section we propose a solution concept for NTU-games and discuss its core stability. For an NTU-game \( V \), let \( a(V) \) be the average of the marginal vectors over all linear orders, i.e.,

\[
a(V) = \frac{1}{n!} \sum_{\pi \in \Pi} m^\pi(V).
\]

If \( V \) is induced by a TU-game \( v \), then \( a(V) \) is the Shapley value of \( v \). In general, the vector \( a(V) \) may not be an allocation for the grand coalition and if it is an allocation for the grand coalition, it may not be efficient. For this reason we propose as solution concept the following set.

Definition 3.1 The solution set of an NTU-game \( V \) is the closure, \( S(V) \), of the set

\[
S^\circ(V) = \{ x \in E^V([n]) \mid x = a(V) + \lambda d \text{ for some } \lambda \in \mathbb{R} \text{ and } d \in \mathbb{R}^n_{++} \}.
\]

If \( V \) satisfies the non-levelness condition\(^1\), then \( S(V) \) can also be defined by taking in the definition of \( S^\circ(V) \) any \( d \in \mathbb{R}^n_{++} \) instead of taking the closure of \( S^\circ(V) \). In case the

\(^1\) An NTU-game \( V \) satisfies the non-levelness condition if for all \( S \in 2^{[n]} \), \( S \neq \emptyset, x, y \in E^V(S) \) and \( y \geq x \) imply \( y = x \).
NTU-game $V$ is induced by a TU-game $v$, then the solution set $S(V)$ of $V$ is a singleton, the point $a(V)$, being the Shapley value of $v$. For an arbitrary NTU-game $V$, the solution set $S(V)$ is a nonempty set and contains all payoff vectors that are efficient allocations for the grand coalition and are obtained by if necessary either increasing or decreasing the payoffs of the average $a(V)$ of all marginal vectors. More precisely, if $a(V)$ is an efficient allocation for the grand coalition, as is always the case for an NTU-game induced by a TU-game or more general for an NTU-game $V$ for which the set $E^V([n])$ of efficient allocations for the grand coalition is a hyperplane, the solution set $S(V)$ consists of only the singleton $a(V)$.

If the average of all marginal vectors $a(V)$ of an NTU-game $V$ is an inefficient allocation for the grand coalition, e.g., when $V([n])$ is a strictly convex set, then the solution set $S(V)$ of $V$ is obtained by increasing the payoffs of $a(V)$ in any strictly positive direction until the payoffs become efficient for the grand coalition. When in this case $a(V)$ is not blocked by any proper subcoalition, $S(V)$ is precisely equal to the set of bargaining solutions of the bargaining problem $B(V([n]),a(V))$ with disagreement point $a(V)$ and bargaining set $V([n])$. If the average $a(V)$ of marginal vectors is not a feasible allocation for the grand coalition, e.g., when $D^V([n])$ is a strictly convex set, then the solution set $S(V)$ is obtained by decreasing the payoffs of $a(V)$ in any strictly negative direction until the payoffs become efficient for the grand coalition. For this case one may see $a(V)$ as a utopia point. In general, the solution set $S(V)$ of an NTU-game $V$ consists of all efficient allocations for the grand coalition that are “close” to $a(V)$.

In [11] the marginal based compromise value is introduced. The marginal based compromise value, or MC-value, of an NTU-game $V$ is given by

$$MC(V) = a(V) \max \{ \lambda \in \mathbb{R} \mid \lambda a(V) \in V([n]) \}.$$

In case the non-levelness condition is satisfied or if $a(V)$ is a strictly positive payoff vector, then the MC-value of an NTU-game $V$ belongs to the solution set $S(V)$.

**Example 3.2** Consider the 2-person NTU-game $V$ with $V(\{i\}) = (-\infty, 0]$, $i = 1, 2$, and $V(\{2\}) = \{ x \in \mathbb{R}^2 \mid x_2 \leq \epsilon - \epsilon x_1, x_1 \leq 1, \text{ or } x_1 \leq 1 - \epsilon x_2, x_1 \geq 1 \}$ for some $\epsilon \geq 0$. For $\epsilon > 0$, $V$ satisfies the non-levelness condition and $S(V)$ consists of the average $a(V) = (\frac{1}{2}, \frac{1}{2}\epsilon)$ of the two marginal vectors $m^1(V) = (1, 0)$ and $m^2(V) = (0, \epsilon)$. For $\epsilon = 0$, $V$ does not satisfy the non-levelness condition and $S(V)$ is still a singleton, being the average $a(V) = (\frac{1}{2}, 0)$ of the two marginal vectors $m^1(V) = (1, 0)$ and $m^2(V) = (0, 0)$. For $\epsilon > 0$ it holds that $MC(V) = (\frac{1}{2}, \frac{1}{2}\epsilon) = a(V)$, while for $\epsilon = 0$, $MC(V) = (1, 0)$, being the marginal vector $m^1(V)$. For any $\epsilon \geq 0$, both vectors $a(V)$ and $MC(V)$ are elements of the core $C(V)$ of $V$, also for $\epsilon = 0$. Notice that the MC-value is not continuous in the parameter $\epsilon$ when $\epsilon$ converges to zero.
Example 3.3 Consider the 2-person NTU-game $V$ with $V(\{i\}) = (-\infty, 0]$, $i = 1, 2$, and $V(\{2\}) = \{x \in \mathbb{R}^2 \mid x_1 \leq 1, x_2 \leq \epsilon\}$ for some $\epsilon \geq 0$. $V$ does not satisfy the non-levelness condition for any $\epsilon \geq 0$. For $\epsilon > 0$, $S(V) = \{x \in \mathbb{R}^2 \mid \max\{\epsilon x_1, x_2\} = \epsilon, x_1 \geq \frac{1}{2}, x_2 \geq \frac{1}{2}\epsilon\}$. This set contains the MC-value of $V$, $MC(V) = (1, \epsilon)$. As in the previous example, for $\epsilon = 0$, $S(V)$ consists of the singleton payoff vector $a(V) = (\frac{1}{2}, 0)$, being the average of the two marginal vectors $m^1(V) = (1, 0)$ and $m^2(V) = (0, 0)$, and $MC(V) = (1, 0)$, being the marginal vector $m^1(V)$. For any $\epsilon \geq 0$, every payoff vector in $S(V)$ and also $MC(V)$ are elements of the core of $V$, also for $\epsilon = 0$. Notice that in this case the MC-value is continuous in the parameter $\epsilon$.

Above we show that under supermodularity it holds that all marginal vectors of an NTU-game belong to the core and therefore are feasible allocations for the grand coalition and cannot be blocked by any coalition. However, because for an NTU-game the core may not be a convex set, it is not guaranteed that under supermodularity the average of all marginal vectors is also feasible for the grand coalition or cannot be blocked by any coalition.

For an NTU-game $V$, let

$$V^*(\{n\}) := \{x \in V(\{n\}) \mid x_S \notin I^V(S) \forall S \in 2^{[n]} \setminus \{\{n\}\}\}$$

be the set of allocations for the grand coalition that cannot be blocked by any proper subcoalition. If $V$ is supermodular, then every marginal vector $m^\pi(V)$ belongs to $V^*(\{n\})$.

Assumption 3.4 For an NTU-game $V$, it holds that for every $x, y \in V^*(\{n\})$, $\alpha \in [0, 1]$, and $d \in \mathbb{R}_{++}^{[n]}$, there exists $\lambda \in \mathbb{R}$ such that $\alpha x + (1 - \alpha)y + \lambda d \in V^*(\{n\})$.

Notice that this assumption is automatically satisfied if the NTU-game $V$ is induced by a TU-game $v$. Assumption 3.4 means that if a convex combination of two allocations for the grand coalition that cannot be blocked by any proper subcoalition, is also an allocation for the grand coalition and is blocked by some proper subcoalition, then the payoffs can be increased in any strictly positive direction to obtain an allocation for the grand coalition that cannot be blocked by any proper subcoalition. And, if the convex combination is not an allocation for the grand coalition, then the payoffs can be decreased in any strictly negative direction to obtain an allocation for the grand coalition that cannot be blocked by any proper subcoalition.

Roughly the assumption requires that the payoff set $V(\{n\})$ for the grand coalition is “not less” convex or its complement $\mathbb{R}^{[n]} \setminus V(\{n\})$ is “not more” convex than the payoff set $V(S)$ (or its complement) of any proper subcoalition $S$. 


Lemma 3.5 Suppose that for an NTU-game \( V \) it holds that \( V([n]) \) is a convex set and that for every \( S \in 2^n \setminus \{[n]\} \) the complement of \( V(S) \), the set \( \mathbb{R}^S \setminus V(S) \), is a convex set. Then \( V \) satisfies Assumption 3.4.

Proof. Take any \( x \) and \( y \) in \( V^*([n]) \) and \( \alpha \), \( 0 < \alpha < 1 \). Since \( x \) and \( y \) are in \( V([n]) \) and \( V([n]) \) is convex, it holds that \( z = \alpha x + (1 - \alpha)y \) is also in \( V([n]) \). Take any \( S \in 2^n \setminus \{[n]\} \). Both \( x_S \) and \( y_S \) belong to \( D^V(S) \). Since \( D^V(S) \) is the closure of \( \mathbb{R}^S \setminus V(S) \) and the latter set is convex, it holds that the vector \( z_S \) belongs also to \( D^V(S) \), and therefore \( z_S \notin I^V(S) \). Consequently, \( z \in V^*([n]) \).

Notice that the condition in the lemma is automatically satisfied if the NTU-game \( V \) is induced by a TU-game \( v \). In fact, in the lemma we prove that under the condition the set \( V^*([n]) \) of an NTU-game \( V \) is a convex set. If \( V \) is also supermodular, then this implies that the payoff vector \( a(V) \) belongs to \( V^*([n]) \). Then \( a(V) \) is either an element of the core or an inefficient allocation for the grand coalition that is not blocked by any proper subcoalition. In the latter case there exists into any strictly positive direction from \( a(V) \) a unique allocation which is efficient for the grand coalition and is therefore an element of the core.

An example of the condition in Lemma 3.5 is when there exists negative externality between players except for the grand coalition, which is reflected by the convexity of the complements of the payoff sets of the proper subcoalitions and the convexity of the payoff set of the grand coalition.

Assumption 3.4 implies that the projection of the core on any hyperplane with strictly positive normal vector is a convex set. To show this, define for any vector \( d \in \mathbb{R}^n \), and real number \( c \) the projection mapping \( P_{d,c} \) on the hyperplane \( H_{d,c} = \{ x \in \mathbb{R}^n | \sum_{i=1}^n x_id_i = c \} \), i.e., for any \( S \subseteq \mathbb{R}^n \)

\[
P_{d,c}(S) = \{ x \in H_{d,c} | x = y + \lambda d \text{ for some } \lambda \in \mathbb{R} \text{ and } y \in S \}.
\]

Lemma 3.6 For any NTU-game \( V \) which satisfies Assumption 3.4 it holds that \( P_{d,c}(C(V)) \) is a convex set for all \( d \in \mathbb{R}^{[n]}_++ \) and \( c \in \mathbb{R} \).

Proof. Take any \( d \in \mathbb{R}^{[n]}_+ \), \( c \in \mathbb{R} \), and \( u, w \in P_{d,c}(C(V)) \). It is to show that for all \( \alpha \in [0,1] \) it holds that \( z = \alpha u + (1 - \alpha)w \in P_{d,c}(C(V)) \). It follows that \( z \in H_{d,c} \) because

\[
\sum_{i=1}^n z_id_i = \alpha(\sum_{i=1}^n u_id_i) + (1 - \alpha)(\sum_{i=1}^n w_id_i) = \alpha c + (1 - \alpha)c = c.
\]

Since \( u, w \in P_{d,c}(C(V)) \), there exist \( x, y \in C(V) \) and \( \mu, \nu \in \mathbb{R} \) such that \( u = x + \mu d \) and \( w = y + \nu d \). Since \( x, y \in V^*([n]) \) and \( d \in \mathbb{R}^{[n]}_+ \), by Assumption 3.4 there exists \( \lambda \in \mathbb{R} \) such that \( s = \alpha x + (1 - \alpha)y + \lambda d \in V^*([n]) \). Therefore, there exists \( \beta \geq \lambda \) such that
\[ t = \alpha x + (1 - \alpha) y + \beta d \in C(V). \] Hence, \( t = z + \gamma d \), where \( \gamma = \beta - \alpha \mu - (1 - \alpha) \nu \). This implies that \( z \in P_{d,c}(C(V)) \).

**Theorem 3.7** For any supermodular NTU-game \( V \) satisfying Assumption 3.4, the solution set \( S(V) \) is a nonempty subset of the core \( C(V) \).

**Proof.** For any NTU-game \( V \) it holds that \( S(V) \) is nonempty. Take first any \( x \in S^o(V) \). Then \( x \in E^V([n]) \) and \( x = a(V) + \lambda d \) for some \( \lambda \in \mathbb{R} \) and \( d \in \mathbb{R}_+^n \). Take \( c = d^T a(V) \), then \( a(V) \) is the projection of \( x \) on \( H_{d,c} \). For \( \pi \in \Pi \), let \( p^\pi(V) = m^\pi(V) - \lambda^\pi d \) be the projection of \( m^\pi(V) \) on \( H_{d,c} \), then

\[ a(V) = \frac{1}{n!} \sum_{\pi \in \Pi} (p^\pi(V) + \lambda^\pi d) = \frac{1}{n!} \sum_{\pi \in \Pi} p^\pi(V) + \frac{1}{n!} \sum_{\pi \in \Pi} \lambda^\pi d. \]

Pre-multiplying by the vector \( d \) yields \( \sum_{\pi \in \Pi} \lambda^\pi = 0 \) because it holds that \( d^T d > 0 \), \( d^T a(V) = c \), and \( d^T p^\pi(V) = c \) for all \( \pi \in \Pi \). This implies that

\[ a(V) = \frac{1}{n!} \sum_{\pi \in \Pi} p^\pi(V). \]

Since for all \( \pi \in \Pi \) it holds that \( p^\pi(V) \in P_{d,c}(C(V)) \), it follows from Lemma 3.6 that \( a(V) \) is an element of \( P_{d,c}(C(V)) \). Hence, there exists \( y \in C(V) \) such that \( y = a(V) + \mu d \) for some \( \mu \in \mathbb{R} \). Since \( d \) is a strictly positive vector and \( x = a(V) + \lambda d \in E^V([n]) \), it holds that \( \mu = \lambda \) and therefore \( x \in C(V) \). This implies \( S^o(V) \subseteq C(V) \). Since \( S(V) \) is the closure of \( S^o(V) \) and the core \( C(V) \) is a closed set, it follows that \( S(V) \subseteq C(V) \).

A particular element of the solution set \( S(V) \) of an NTU-game \( V \) is the payoff vector \( e(V) = a(V) + \lambda(1, \ldots, 1)^\top \), where \( \lambda \) is the unique real number such that \( a(V) + \lambda(1, \ldots, 1)^\top \) is an element of \( E^V([n]) \). This vector \( e(V) \) is equal to the average \( a(V) \) of all marginal vectors if \( a(V) \) is an efficient allocation for the grand coalition, as is the case when \( V \) is induced by a TU-game \( v \). If \( a(V) \) is an inefficient allocation for the grand coalition and is not blocked by any proper subcoalition, i.e., \( a(V) \in V^*([n]) \), one may consider an element of the solution set as a bargaining outcome of the bargaining problem \( B(V([n]), a(V)) \). In this case, \( e(V) \) is the egalitarian outcome of the problem. Other bargaining solutions, such as the Nash bargaining solution ([10]) and the Raiffa-Kalai-Smorodinsky bargaining solution ([12] and [5]) are also for this case elements of the solution set and are therefore elements of the core under the same conditions.

**References**


