Some properties of a generalized two-error components matrix (problem 01.5.1)

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SOLUTIONS

01.5.1. *Some Properties of a Generalized Two-Error Components Matrix*—Solution 1, proposed, by Franc J.G.M. Klaassen and Jan R. Magnus.

(a) Denote the submatrix $\alpha_1 J_1 + \beta_1 (I_{T_1} - J_1)$ by Ω_{11} and define Ω_{22} accordingly. Assume first that Ω_{11} is nonsingular. Then, using a standard result on the determinant of a partitioned matrix (Magnus and Neudecker, 1999, p. 25),

$$|\Omega| = |\Omega_{11}|\,|\Omega_{22} - \gamma^2 \iota_2 \iota_1' \Omega_{11}^{-1} \iota_1 \iota_2'| = \alpha_1 \beta_1^{T_1 - 1} \left| \Omega_{22} - \frac{\gamma^2 T_1 T_2}{\alpha_1} J_2 \right|$$

$$= \beta_1^{T_1 - 1} \beta_2^{T_2 - 1} \Delta.$$

Next, let $|\Omega_{11}| = 0$. Then, $\Omega_{11} + \epsilon I_{T_1} = (\alpha_1 + \epsilon) J_1 + (\beta_1 + \epsilon)(I_{T_1} - J_1)$ will be nonsingular for all ϵ sufficiently small, and the result follows by the continuity of the determinant, letting $\epsilon \to 0$.

(b) To find the eigenvalues of Ω, we notice that the matrix $\Omega - \lambda I_T$ has the same structure as Ω but with parameters $\alpha_i - \lambda$ and $\beta_i - \lambda$ instead of α_i and β_i $(i = 1,2)$. Hence,

$$|\Omega - \lambda I_T| = (\beta_1 - \lambda)^{T_1 - 1}(\beta_2 - \lambda)^{T_2 - 1}((\alpha_1 - \lambda)(\alpha_2 - \lambda) - \gamma^2 T_1 T_2).$$

The eigenvalues are therefore β_1 ($T_1 - 1$ times), β_2 ($T_2 - 1$ times), and ξ_1 and ξ_2, where $\xi_1 + \xi_2 = \alpha_1 + \alpha_2$ and $\xi_1 \xi_2 = \Delta$. Because Ω is positive definite if and only if all its eigenvalues are positive, the result follows.

(c) Let S_1 be a $T_1 \times (T_1 - 1)$ matrix such that $S_1' S_1 = I_{T_1 - 1}$ and $S_1' \imath_1 = 0$. Then $S_1 S_1' = I_{T_1} - J_1$. Let S_2 be defined similarly. Let Λ denote the diagonal $T \times T$ matrix of eigenvalues β_1 ($T_1 - 1$ times), β_2 ($T_2 - 1$ times), ξ_1 and ξ_2 and define the $T \times T$ matrix

$$V = \begin{pmatrix} S_1 & 0 & \theta_1 \imath_1 & \omega_1 \imath_1 \\ 0 & S_2 & \theta_2 \imath_2 & \omega_2 \imath_2 \end{pmatrix}.$$

Then one verifies that $\Omega V = V\Lambda$ and $V'V = I_T$ for suitable choices of θ_1, ω_1, θ_2, and ω_2. Hence, $\Omega^p = V\Lambda^p V'$, and this has the same form as Ω.

(d) For $p = -1$, we find

$$\phi_1 = \alpha_2/\Delta, \qquad \phi_2 = \alpha_1/\Delta, \qquad \delta = -\gamma/\Delta,$$

and for $p = -\frac{1}{2}$ we obtain

$$\phi_1 = \frac{\alpha_2 + \Delta^{1/2}}{\Delta^{1/2}\theta}, \qquad \phi_2 = \frac{\alpha_1 + \Delta^{1/2}}{\Delta^{1/2}\theta}, \qquad \delta = -\frac{\gamma}{\Delta^{1/2}\theta},$$

where $\theta = \sqrt{\alpha_1 + \alpha_2 + 2\Delta^{1/2}}$.

(e) Finally, part (e) follows from (c) by direct calculation.

REFERENCE

Magnus, J.R. & H. Neudecker (1999) *Matrix Differential Calculus with Applications in Statistics and Econometrics*, rev. ed. New York: Wiley.