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The Shapley value for directed graph games

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Abstract

The Shapley value for directed graph (digraph) games, TU games with limited cooperation introduced by an arbitrary digraph prescribing the dominance relation among the players, is introduced. It is defined as the average of marginal contribution vectors corresponding to all permutations that do not violate the subordination of players. We assume that in order to cooperate players may join only coalitions containing no players dominating them. Properties of this solution are studied and a convexity type condition is provided that guarantees its stability with respect to an appropriately defined core concept. An axiomatization for cycle digraph games for which the digraphs are directed cycles is obtained.

Keywords: TU game, Shapley value, directed graph, dominance structure, core, convexity

JEL Classification Number: C71

1 Introduction

In classical cooperative game theory it is assumed that any coalition of players may form. However, in many practical situations the set of feasible coalitions is limited by some social, economical, hierarchical, or technical structure. One of the most famous singleton solution concepts for cooperative games with transferable utility (TU games) is the Shapley value [7] which is defined as the average of the marginal contribution vectors corresponding to all permutations on the player set. Several adaptations of the Shapley value for different models of TU games with limited cooperation among the players are well known in the literature, see for example Aumann and Drèze [1] and Owen [6] for games with coalition structure, Myerson [5] for undirected graph games in which the connected subsets of players form the feasible coalitions, and Faigle and Kern [2] for games with precedence constraints.

In this paper it is assumed that restricted cooperation among the players is determined by an arbitrary directed graph (digraph) on the player set. The directed links of the digraph prescribe the subordination among the players. In this setting, the players could be jobs in a multi-stage machinery process, where the links of the digraph determine the order in which the jobs can be processed. If at each moment only one job can be performed, then when a job is completed, the next job to be performed can be any of the jobs that are immediate

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successors in the digraph or one of those the performance of which is independent of the job just completed. On the class of digraph games, which are TU games with restricted cooperation prescribed by the dominance relation on the set of players determined by a digraph, we introduce the so-called Shapley value for digraph games as the average of marginal contribution vectors corresponding to all permutations not violating the subordination of players. Contrary to the Myerson model, the feasible coalitions are not necessarily connected subsets of the digraph. We show that the Shapley value for digraph games meets efficiency, linearity, the restricted null player property, the restricted equal treatment property, and is independent of inessential links. Moreover, under a convexity type condition which is weaker than the usual convexity guaranteeing the core stability of the Shapley value for TU games, we prove that the introduced value is stable with respect to the appropriate core concept. On the subclass of directed cycle graph games an axiomatization is provided.

Since precedence constraints are determined by a partial ordering on the player set which can be represented by a cycle-free digraph, the TU games under precedence constraints form a subclass of the class of digraph games. On the subclass of cycle-free digraph games the Shapley value for digraph games coincides with the Shapley value for TU games under precedence constraints introduced in Faigle and Kern [2].

The structure of the paper is as follows. Section 2 contains preliminaries. In Section 3 we introduce the Shapley value for digraph games and discuss its properties and stability. An axiomatization on the subclass of cycle digraph games for which the digraphs are directed cycles is obtained in Section 4.

2 Preliminaries

A cooperative game with transferable utility, or *TU game*, is a pair (N, v), where $N = \{1, \ldots, n\}$ is a finite set of $n \geq 2$ players and $v\colon 2^N \to \mathbb{R}$ is a *characteristic function* with $v(\emptyset) = 0$, assigning to any *coalition* $S \subseteq N$ its *worth* $v(S)$. The set of TU games with fixed player set N is denoted by $\mathcal{G}_N$. For simplicity of notation and if no ambiguity appears we write v when we refer to a game (N, v). It is well known (cf. Shapley [7]) that *unanimity games* $\{u_T\}_{\substack{T\subseteq N\\ T\neq\emptyset}}$, defined as $u_T(S) = 1$ if $T \subseteq S$, and $u_T(S) = 0$ otherwise, form a basis in $\mathcal{G}_N$, i.e., every $v \in \mathcal{G}_N$ can be uniquely presented in the linear form $v = \sum\limits_{T\subseteq N, T\neq\emptyset} \lambda^v_T\, u_T$. A *value* on a subset $\mathcal{G}$ of $\mathcal{G}_N$ is a function $\xi\colon \mathcal{G} \to \mathbb{R}^N$ that assigns to every game $v \in \mathcal{G}$ a vector $\xi(v) \in \mathbb{R}^N$ where $\xi_i(v)$ is the *payoff* to player $i \in N$ in game v. The *marginal contribution of player $i \in N$ to coalition $S \subseteq N\backslash\{i\}$ in game $v \in \mathcal{G}_N$* is given by $m^v_i(S) = v(S \cup \{i\}) - v(S)$. In the sequel we use standard notation $x(S) = \sum\limits_{i\in S} x_i$ for any $x \in \mathbb{R}^N$ and $S \subseteq N$.

For a permutation $\pi\colon N \to N$ and any $i \in N$, $\pi(i)$ is the position of player i in π, $P_\pi(i) = \{j \in N \mid \pi(j) < \pi(i)\}$ is the set of predecessors of i in π, and $\bar{P}_\pi(i) = P_\pi(i) \cup \{i\}$. For a TU game $v \in \mathcal{G}_N$ and a permutation π on N the *marginal contribution vector* $\bar{m}^v(\pi) \in \mathbb{R}^N$ is given by $\bar{m}^v_i(\pi) = m^v_i(P_\pi(i)) = v(\bar{P}_\pi(i)) - v(P_\pi(i))$ for all $i \in N$. The *Shapley value* of a TU game $v \in \mathcal{G}^N$ is given by $Sh(v) = \sum\limits_{\pi\in\Pi} \bar{m}^v(\pi)/n!$, where Π is the set of all permutations on N.

A *graph* on N consists of N as the set of nodes and for a *directed graph*, or *digraph*, a collection of ordered pairs $\Gamma \subseteq \{(i, j) \,|\, i, j \in N,\ i \neq j\}$ as the set of *directed links* (*arcs*) from one player to another player in N, and for an *undirected graph* a collection of unordered pairs $\Gamma \subseteq \{\{i, j\} \,|\, i, j \in N,\ i \neq j\}$ as the set of *links* (*edges*) between two players in N. Observe that an undirected graph can be considered as a directed graph for which $(i, j) \in \Gamma$ if and only if $(j, i) \in \Gamma$. For a digraph Γ on N and coalition $S \subseteq N$,

$\Gamma|_S = \{(i,j) \in \Gamma \mid i,j \in S\}$ is the *subgraph* of Γ on S. Given a digraph Γ on N, a sequence of different players $(i_1,\ldots,i_r)$, $r \geq 2$, is a *path* in Γ between players i_1 and i_r if $\{(i_h,i_{h+1}),(i_{h+1},i_h)\} \cap \Gamma \neq \emptyset$ for $h=1,\ldots,r-1$, and a *directed path* in Γ from player i_1 to player i_r if $(i_h,i_{h+1}) \in \Gamma$ for $h=1,\ldots,r-1$. A directed path $(i_1,\ldots,i_r)$ in Γ is a *directed cycle* if $(i_r,i_1) \in \Gamma$. Digraph Γ is *cycle-free* if it contains no directed cycles. Players i and j in N are *connected* in Γ if there exists a path in Γ between i and j. Γ is *connected* if any two different nodes in N are connected in Γ. A coalition $S \subseteq N$ is *connected* in Γ if the subgraph $\Gamma|_S$ is connected. For $S \subseteq N$, $C^\Gamma(S)$ denotes the collection of subsets of S being connected in Γ, S/Γ is the collection of maximal connected subsets, called *components*, of S in Γ. If there exists a directed path in Γ from player $i \in N$ to player $j \in N$, then j is a *successor* of i and i is a *predecessor* of j in Γ. If $(i,j) \in \Gamma$, then j is an *immediate successor* of i and i is an *immediate predecessor* of j in Γ. For $i \in N$, $S^\Gamma(i)$ denotes the set of successors of i in Γ and $\bar{S}^\Gamma(i) = S^\Gamma(i) \cup \{i\}$. A *line-graph* (*chain*) on N is a connected cycle-free digraph on N in which each player has at most one immediate successor and at most one immediate predecessor.

For digraph Γ on N and coalition $S \subseteq N$, player $i \in S$ *dominates* player $j \in S$ in $\Gamma|_S$, denoted $i \succ_{\Gamma|_S} j$, if $j \in S^{\Gamma|_S}(i)$ and $i \notin S^{\Gamma|_S}(j)$. Observe that the dominance relation between two players may differ between different coalitions they both belong to. Player $i \in S$ is an *undominated* player in $\Gamma|_S$ if no player in S dominates i in $\Gamma|_S$, i.e., $i \in S^{\Gamma|_S}(j)$ implies $j \in S^{\Gamma|_S}(i)$. Notice that a player undominated in $\Gamma|_S$ either has no predecessor in $\Gamma|_S$ or lies on a directed cycle in $\Gamma|_S$. The set of players undominated in $\Gamma|_S$ is denoted by $U^\Gamma(S)$. Since N is finite, $U^\Gamma(S) \neq \emptyset$ for any nonempty $S \subseteq N$.

A pair (v,Γ) of a TU game $v \in \mathcal{G}_N$ and a digraph Γ on N constitutes a *directed graph game* or *digraph game*. The set of digraph games on a fixed player set N is denoted $\mathcal{G}_N^\Gamma$. A *value* on $\mathcal{G} \subseteq \mathcal{G}_N^\Gamma$ is a function $\xi\colon \mathcal{G} \to \mathbb{R}^N$ that assigns to every $(v,\Gamma) \in \mathcal{G}$ a payoff vector $\xi(v,\Gamma)$.

3 The Shapley value for digraph games

In a digraph game the digraph prescribes a dominance relation between the players that puts restrictions on the feasibility of coalitions. Assuming that in order to cooperate players may join only coalitions containing no players dominating them, the set of feasible coalitions of a digraph game consists of so-called hierarchical coalitions.

Given a digraph Γ on N, a coalition $S \subseteq N$ is *hierarchical* in Γ if $i \in S$, $(i,j) \in \Gamma$ and $i \notin S^\Gamma(j)$ imply $\bar{S}^\Gamma(j) \subset S$.

If a player in a hierarchical coalition dominates an immediate successor in the digraph, then the coalition also contains this latter player and all his successors in the digraph. Every hierarchical coalition preserves the subordination of players and therefore is feasible. For a cycle-free digraph Γ, a coalition $S \subseteq N$ is hierarchical if and only if every successor of any player $i \in S$ in Γ belongs to S, i.e., $\bar{S}^\Gamma(i) \subseteq S$ for all $i \in S$. So, for a cycle-free digraph the set of hierarchical coalitions coincides with the set of feasible coalitions in Faigle and Kern [2] when the precedence constraints are induced by the same digraph. Notice that both the empty coalition and the grand coalition of all players are always hierarchical. A hierarchical coalition does not need to be connected in the underlying digraph. Moreover, in an undirected graph, in particular in the empty graph, every coalition is hierarchical. For a digraph Γ on N, the set of all coalitions hierarchical in Γ is denoted by $H(\Gamma)$ and its subset consisting of all connected hierarchical coalitions by $H^c(\Gamma)$. Observe that $S,T \in H(\Gamma)$ implies $S \cup T, S \cap T \in H(\Gamma)$.

Given a digraph Γ on N, a permutation $\pi \in \Pi$ is *consistent* with Γ if it preserves the subordination of players determined by Γ, i.e., $\pi(j) < \pi(i)$ only if $j \not\succ_{\Gamma|_{\bar{P}_\pi(i)}} i$.

Remark 3.1 In a permutation π consistent with Γ every $i \in N$ is an undominated player in the subgraph of Γ on the set composed by i together with all his predecessors in π, i.e., $i \in U^\Gamma(\bar{P}_\pi(i))$ for all $i \in N$.

The set of permutations on N consistent with Γ is denoted by Π^Γ. Since N is finite, for any digraph Γ on N, $\Pi^\Gamma \neq \emptyset$. The next proposition shows that for every consistent permutation both sets of predecessors of any player, with and without this player, form a hierarchical coalition.

Proposition 3.1 *Given a digraph* Γ *on* N, *if* $\pi \in \Pi^\Gamma$, *then* $\bar{P}_\pi(i), P_\pi(i) \in H(\Gamma)$ *for all* $i \in N$.

Proof. Since $N \in \Pi^\Gamma$, $N = \bar{P}_\pi(k)$ for some $k \in N$, and for each $i \in N$ it holds that $P_\pi(i) = \bar{P}_\pi(j)$ for $j \in P_\pi(i)$ such that $\pi(j) = \max\limits_{k \in P_\pi(i)} \pi(k)$, it suffices to show that if $\bar{P}_\pi(k) \in \Pi^\Gamma$ for some $k \in N$, then $P_\pi(k) \in \Pi^\Gamma$ as well. Observe that if $\bar{P}_\pi(k) \in \Pi^\Gamma$, then $i \in P_\pi(k)$, $(i,j) \in \Gamma$ and $i \notin S^\Gamma(j)$ imply $\bar{S}^\Gamma(j) \subset \bar{P}_\pi(k)$. To prove that $P_\pi(k) \in \Pi^\Gamma$ we need to show that $\bar{S}^\Gamma(j) \subset P_\pi(k)$. Suppose $k \in \bar{S}^\Gamma(j)$. Then $(i,j) \in \Gamma$ implies $k \in S^\Gamma(i)$. From $i \in P_\pi(j)$ and $\bar{S}^\Gamma(j) \subset \bar{P}_\pi(k)$ it follows that $k \in S^{\Gamma|_{\bar{P}_\pi(k)}}(i)$. Since $\bar{P}_\pi(k) \in \Pi^\Gamma$, $k \in U^\Gamma(\bar{P}_\pi(k))$, which implies $i \in S^{\Gamma|_{\bar{P}_\pi(k)}}(k)$, and therefore, $i \in S^\Gamma(k)$. Then $k \in \bar{S}^\Gamma(j)$ implies $i \in S^\Gamma(j)$, which contradicts $i \notin S^\Gamma(j)$. ∎

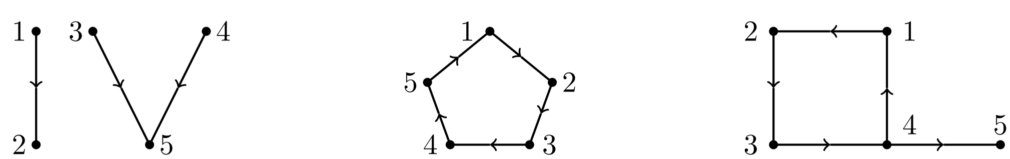
Remark 3.2 If Γ is a directed cycle on N, then for all $\pi \in \Pi^\Gamma$ and $i \in N$ both coalitions $\bar{P}_\pi(i)$ and $P_\pi(i)$ are connected in Γ. Moreover, $U^\Gamma(N) = N$ and $U^\Gamma(\bar{P}_\pi(i)) = \{i\}$ if $\bar{P}_\pi(i) \neq N$.

Proposition 3.1 implies that every consistent permutation generates a sequence of feasible coalitions consisting of a player and his predecessors in the permutation. This player is undominated in the subgraph on such a coalition and receives his marginal contribution for joining.

We define the *Shapley value for digraph games* as the average of the marginal contribution vectors corresponding to all consistent permutations, i.e., for any $(v,\Gamma) \in \mathcal{G}_N^\Gamma$,

$$Sh(v,\Gamma) = \frac{1}{|\Pi^\Gamma|} \sum_{\pi \in \Pi^\Gamma} \bar{m}^v(\pi). \tag{1}$$

Example 3.1 Consider the 5-player digraph games (v,Γ), (v,Γ'), and (v,Γ'') with characteristic function $v(S) = |S|^2$ for all $S \subseteq N$ and digraphs $\Gamma = \{(1,2),(3,5),(4,5)\}$, $\Gamma' = \{(1,2),(2,3),(3,4),(4,5),(5,1)\}$, and $\Gamma'' = \{(1,2),(2,3),(3,4),(4,1),(4,5)\}$, as depicted in Figure 1.



a) digraph Γ b) digraph Γ' c) digraph Γ''

Figure 1

For the digraph Γ there are 20 permutations consistent with Γ: $\pi^1=(5,4,3,2,1)$, $\pi^2=(5,3,4,2,1)$, $\pi^3=(5,4,2,3,1)$, $\pi^4=(5,2,4,3,1)$, $\pi^5=(2,5,4,3,1)$, $\pi^6=(5,3,2,4,1)$, $\pi^7=(5,2,3,4,1)$, $\pi^8=(2,5,3,4,1)$, $\pi^9=(5,2,4,1,3)$, $\pi^{10}=(2,5,4,1,3)$, $\pi^{11}=(5,4,2,1,3)$, $\pi^{12}=(5,2,1,4,3)$, $\pi^{13}=(2,5,1,4,3)$, $\pi^{14}=(2,1,5,4,3)$, $\pi^{15}=(5,2,3,1,4)$, $\pi^{16}=(2,5,3,1,4)$, $\pi^{17}=(5,3,2,1,4)$, $\pi^{18}=(5,2,1,3,4)$, $\pi^{19}=(2,5,1,3,4)$, $\pi^{20}=(2,1,5,3,4)$, and $Sh(v,\Gamma)=(7,3,13/2,13/2,2)$. For the digraph Γ' there are 5 permutations consistent with Γ': $\pi^1=(5,4,3,2,1)$, $\pi^2=(4,3,2,1,5)$, $\pi^3=(3,2,1,5,4)$, $\pi^4=(2,1,5,4,3)$, $\pi^5=(1,5,4,3,2)$, and $Sh(v,\Gamma')=(5,5,5,5,5)$. For the digraph Γ'' there are 10 permutations consistent with Γ'': $\pi^1=(5,4,3,2,1)$, $\pi^2=(5,1,4,3,2)$, $\pi^3=(5,3,2,1,4)$, $\pi^4=(5,2,1,4,3)$, $\pi^5=(1,5,4,3,2)$, $\pi^6=(2,1,5,4,3)$, $\pi^7=(2,5,1,4,3)$, $\pi^8=(3,2,1,5,4)$, $\pi^9=(3,2,1,5,4)$, $\pi^{10}=(3,2,5,1,4)$, and $Sh(v,\Gamma'')=(5,2,4.6,5.2,7,3)$. To compare, to calculate the Shapley value of TU game v we need to consider 120 marginal contribution vectors determined by all permutations on the player set N and $Sh(v)=(5,5,5,5,5)$. Due to the symmetry of both the game v and the graph Γ', $Sh(v,\Gamma')=Sh(v)$.

When digraph Γ represents an undirected graph, which means that there is no subordination between the players in Γ, the Shapley value of the digraph game (v,Γ) coincides with the Shapley value of the TU game v. Both values also coincide if the TU game is symmetric and the digraph is a directed cycle, as for the game (v,Γ') in Example 3.1. In general, the Shapley value of a digraph game does not coincide with the Myerson value of the corresponding undirected graph game because the Myerson value is defined as the average of all marginal contribution vectors of the Myerson restricted game. Since a cycle-free digraph on the player set provides a partial ordering of the players and for a cycle-free digraph the set of hierarchical coalitions coincides with the set of feasible coalitions considered in Faigle and Kern [2], it holds that on the subclass of cycle-free digraph games the Shapley value for digraph games coincides with the Shapley value for cooperative games under precedence constraints defined in Faigle and Kern [2]. Moreover, if for a connected digraph game all covering trees of the digraph are line-graphs, the Shapley value for digraph games coincides with the average covering tree value introduced in Khmelnitskaya, Selcuk, and Talman [4]. In particular, this holds for cycle digraph games for which the digraph is a directed cycle.

A value ξ on $\mathcal{G}\subseteq\mathcal{G}_N^{\Gamma}$ is *efficient* (E) if for any $(v,\Gamma)\in\mathcal{G}$ it holds that $\sum_{i\in N}\xi_i(v,\Gamma)=v(N)$.

A value ξ on $\mathcal{G}\subseteq\mathcal{G}_N^{\Gamma}$ is *linear* (L) if for any $(v,\Gamma),(w,\Gamma)\in\mathcal{G}$ and $a,b\in\mathbb{R}$ it holds that $\xi(av+bw,\Gamma)=a\xi(v,\Gamma)+b\xi(w,\Gamma)$, where $(av+bw)(S)=av(S)+bw(S)$ for all $S\subseteq N$.

A value ξ on $\mathcal{G}\subseteq\mathcal{G}_N^{\Gamma}$ satisfies the *restricted equal treatment property* $(RETP)$ if for any $(v,\Gamma)\in\mathcal{G}$ and $i,j\in N$, $i\neq j$, hierarchically symmetric in (v,Γ) it holds that $\xi_i(v,\Gamma)=\xi_j(v,\Gamma)$.

Two different players in N are *hierarchically symmetric* in $(v,\Gamma)\in\mathcal{G}_N^{\Gamma}$ if they are both symmetric in Γ and hierarchically symmetric in v. Given digraph Γ on N, players $i,j\in N$, $i\neq j$, are *symmetric in* Γ if they have the same sets of immediate successors and immediate predecessors in Γ, i.e., $(i,k)\in\Gamma \iff (j,k)\in\Gamma$ and $(k,i)\in\Gamma \iff (k,j)\in\Gamma$. Given digraph game $(v,\Gamma)\in\mathcal{G}_N^{\Gamma}$, players $i,j\in N$, $i\neq j$, are *hierarchically symmetric in* v if for all $S\subseteq N\backslash\{i,j\}$ such that $S, S\cup\{i\}, S\cup\{j\}, S\cup\{i,j\}\in H(\Gamma)$, it holds that $v(S\cup\{i\})=v(S\cup\{j\})$, or, equivalently, $m_i^v(S)=m_j^v(S)$.

A value ξ on $\mathcal{G}\subseteq\mathcal{G}_N^{\Gamma}$ meets the *(restricted) hierarchical null-player property* (*(R)HNP*) if for all $(v,\Gamma)\in\mathcal{G}$ it holds that $\xi_i(v,\Gamma)=0$ whenever i is a (restricted) hierarchical null-player in (v,Γ).

A player $i\in N$ is a *(restricted) hierarchical null-player* in $(v,\Gamma)\in\mathcal{G}_N^{\Gamma}$ if for every $S\subseteq N\backslash\{i\}$ such that $S, S\cup\{i\}\in H(\Gamma)$ ($S, S\cup\{i\}\in H^c(\Gamma)$) it holds that $v(S\cup\{i\})=v(S)$, or, equivalently, $m_i^v(S)=0$.

Remark 3.3 Each null-player in $v \in \mathcal{G}_N$ is a hierarchical null-player in any $(v,\Gamma) \in \mathcal{G}_N^\Gamma$, and every hierarchical null-player in $(v,\Gamma) \in \mathcal{G}_N^\Gamma$ is also a restricted hierarchical null-player in (v,Γ), i.e., RHNP implies HNP.

A value ξ on $\mathcal{G} \subseteq \mathcal{G}_N^\Gamma$ is *(restricted) hierarchically marginalist* (*(R)HM*) if for any $(v,\Gamma),(w,\Gamma) \in \mathcal{G}$ and $i \in N$ for which $m_i^v(S) = m_i^w(S)$ for all $S \subseteq N\backslash\{i\}$ such that $S, S\cup\{i\} \in H(\Gamma)$ ($S, S\cup\{i\} \in H^c(\Gamma)$) and $i \in U^\Gamma(S\cup\{i\})$ it holds that $\xi_i(v,\Gamma)=\xi_i(w,\Gamma)$.

If in a directed cycle digraph game all proper subcoalitions of players are powerless, i.e., only the full cooperation within the grand coalition is productive, then since all players are undominated and have equal power, it is natural to require that all players receive the same payoff.

A value ξ on $\mathcal{G} \subseteq \mathcal{G}_N^\Gamma$ is *strongly symmetric on directed cycles* (*SSDC*) if for any $(v,\Gamma) \in \mathcal{G}$ such that Γ is a directed cycle on N and $v(S) = 0$ for all $S \subsetneq N$, i.e., $v = \lambda u_N$ for some real λ, it holds that $\xi_i(v,\Gamma)=\xi_j(v,\Gamma)$ for all $i,j \in N$, $i \neq j$.

A value ξ on $\mathcal{G} \subseteq \mathcal{G}_N^\Gamma$ is *independent of inessential directed links* (*IIDL*) if for any $(v,\Gamma) \in \mathcal{G}$ and inessential directed link $(i,j) \in \Gamma$ it holds that $\xi(v,\Gamma) = \xi(v,\Gamma \setminus \{(i,j)\})$.

For a digraph Γ on N, a directed link $(i,j) \in \Gamma$ is *inessential* if i dominates j in Γ and there exists a directed path in Γ from i to j different from (i,j), i.e., $i \notin S^\Gamma(j)$ and there exists $i' \in N$ such that $(i,i') \in \Gamma$, $i \notin S^\Gamma(i')$, and $j \in S^\Gamma(i')$.

Proposition 3.2 *The Shapley value for digraph games on $\mathcal{G}_N^\Gamma$ meets E, L, RETP, HNP, HM, SSDC, and IIL.*

Proof. (E) Efficiency follows straightforwardly from the efficiency of all marginal contribution vectors on $\mathcal{G}_N$.

(L) Since all digraph games (v,Γ), (w,Γ) and $(av+bw,\Gamma)$ are determined by the same digraph Γ, the set Π^Γ of consistent permutations is the same for all of them. Then linearity follows from the linearity of all marginal contribution vectors on $\mathcal{G}_N$.

(RETP) Let players $i,j \in N$ be hierarchically symmetric in $(v,\Gamma) \in \mathcal{G}_N^\Gamma$. Then $\pi \in \Pi^\Gamma$ if and only if $\pi' \in \Pi^\Gamma$, where $\pi'(i) = \pi(j)$, $\pi'(j) = \pi(i)$ and $\pi'(k) = \pi(k)$ for all $k \in N \setminus \{i,j\}$. So, to prove RETP it suffices to show that $\bar{m}_i^v(\pi) = \bar{m}_j^v(\pi')$ and $\bar{m}_j^v(\pi) = \bar{m}_i^v(\pi')$ for any $\pi,\pi' \in \Pi^\Gamma$ such that $\pi'(i) = \pi(j)$, $\pi'(j) = \pi(i)$, and $\pi'(k) = \pi(k)$ for all $k \in N \setminus \{i,j\}$. Without loss of generality assume that $\pi(i) > \pi(j)$. To show $\bar{m}_i^v(\pi) = \bar{m}_j^v(\pi')$ notice that $\pi'(i) = \pi(j)$ and $\pi'(k) = \pi(k)$ for all $k \in N\backslash\{i,j\}$ implies $\bar{P}_\pi(i) = \bar{P}_{\pi'}(j)$ and $P_\pi(i)\backslash\{j\} = P_{\pi'}(j)\backslash\{i\}$. Let $S = P_\pi(i)\backslash\{j\} = P_{\pi'}(j)\backslash\{i\}$. By Proposition 3.1, $S\cup\{i\}, S\cup\{j\}, S\cup\{i,j\} \in H(\Gamma)$. Since i and j are hierarchically symmetric in v, $v(S\cup\{i\}) = v(S\cup\{j\})$, which means $v(P_\pi(i)) = v(P_{\pi'}(j))$. Together with $\bar{P}_\pi(i) = \bar{P}_{\pi'}(j)$, we obtain $\bar{m}_i^v(\pi) = v(\bar{P}_\pi(i)) - v(P_\pi(i)) = v(\bar{P}_{\pi'}(j)) - v(P_{\pi'}(j)) = \bar{m}_j^v(\pi')$. In order to show $\bar{m}_j^v(\pi) = \bar{m}_i^v(\pi')$ observe that $P_\pi(j) = P_{\pi'}(i)$. Let $S = P_\pi(j) = P_{\pi'}(i)$. By Proposition 3.1, $S\cup\{i\}, S\cup\{j\}, S \in H(\Gamma)$. Since i and j are hierarchically symmetric in v, $v(S\cup\{i\}) = v(S\cup\{j\})$, which means $v(\bar{P}_\pi(j))=v(\bar{P}_{\pi'}(i))$. So, $\bar{m}_j^v(\pi)=v(\bar{P}_\pi(j))-v(P_\pi(j))=v(\bar{P}_{\pi'}(i))-v(P_{\pi'}(i))=\bar{m}_i^v(\pi')$.

(HNP) Let $i \in N$ be a hierarchical null player in $(v,\Gamma) \in \mathcal{G}_N^\Gamma$ and $\pi \in \Pi^\Gamma$. By Proposition 3.1, $\bar{P}_\pi(i), P_\pi(i) \in H(\Gamma)$ and since i is a hierarchical null-player in (v,Γ), $\bar{m}_i^v(\pi) = v(\bar{P}_\pi(i)) - v(P_\pi(i)) = 0$. Hence, $Sh_i(v,\Gamma) = 0$.

(HM) The hierarchical marginality of the Shapley value for digraph games follows straightforwardly from (1), Remark 3.1, and Proposition 3.1.

(SSDC) SSDC results directly from the coincidence of the Shapley value for digraph games determined by symmetric TU games on directed cycles with the Shapley value of the underlying TU games because on the subclass of symmetric TU games the Shapley value coincides with the egalitarian solution.

(IIDL) Let $(v,\Gamma) \in \mathcal{G}_N^{\Gamma}$ for which $(i,j) \in \Gamma$ is inessential. Then there exists $i' \in N$ such that $(i,i') \in \Gamma$, $i \notin S^{\Gamma}(i')$, and $j \in S^{\Gamma}(i')$. Let $\Gamma' = \Gamma \setminus \{(i,j)\}$. We show that $\Pi^{\Gamma} = \Pi^{\Gamma'}$, which implies $Sh(v,\Gamma) = Sh(v,\Gamma')$. Take any $\pi \in \Pi^{\Gamma}$. Since $(i,i') \in \Gamma$ and $i \notin S^{\Gamma}(i')$, Proposition 3.1 implies $\bar{P}_{\pi}(i') \subseteq P_{\pi}(i)$. Hence, for all $S \supseteq P_{\pi}(i)$, $U^{\Gamma}(S) = U^{\Gamma'}(S)$. Moreover, $\Gamma|_{P_{\pi}(i)} = \Gamma'|_{P_{\pi}(i)}$. This implies that $\pi \in \Pi^{\Gamma'}$. Conversely, take any $\pi' \in \Pi^{\Gamma'}$. Since $\Gamma' = \Gamma \setminus \{(i,j)\}$, it holds that $(i,i') \in \Gamma'$ and $i \notin S^{\Gamma'}(i')$. From Proposition 3.1 it follows that $\bar{P}_{\pi'}(i') \subseteq P_{\pi'}(i)$. Hence, for all $S \supseteq P_{\pi'}(i)$, $U^{\Gamma'}(S) = U^{\Gamma}(S)$. Moreover, $\Gamma'|_{P_{\pi'}(i)} = \Gamma|_{P_{\pi'}(i)}$. This implies $\pi' \in \Pi^{\Gamma}$. ∎

Under the assumption that in a digraph game the digraph represents the dominance structure on the player set, only the hierarchical coalitions are feasible. So, we define the *dominance core* $C^D(v,\Gamma)$ of a digraph game $(v,\Gamma) \in \mathcal{G}_N^{\Gamma}$ as the set of efficient payoff vectors that cannot be blocked by any hierarchical coalition, i.e., $C^D(v,\Gamma) = \{x \in \mathbb{R}^N \mid x(N) = v(N),\ x(S) \geq v(S) \text{ for all } S \in H(\Gamma)\}$.

A value ξ on $\mathcal{G} \subseteq \mathcal{G}_N^{\Gamma}$ is *D-stable* if for any digraph game $(v,\Gamma) \in \mathcal{G}$ it holds that $\xi(v,\Gamma) \in C^D(v,\Gamma)$.

A digraph game $(v,\Gamma) \in \mathcal{G}_N^{\Gamma}$ is *hierarchically convex* if for any $S, T \in H(\Gamma)$ it holds that $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$.

Recall that for any two hierarchical coalitions their union and intersection are also hierarchical. Remark also that the notion of hierarchical convexity for a digraph game (v,Γ) is weaker than convexity for the underlying TU game v where the inequality is required to hold for all $S, T \subseteq N$.

Theorem 3.1 *The Shapley value for digraph games is D-stable on the class of hierarchically convex digraph games.*

Proof. Let $(v,\Gamma) \in \mathcal{G}_N^{\Gamma}$ be hierarchically convex. By definition of the Shapley value for digraph games and due to its efficiency, it suffices to show that $\sum_{i \in S} \bar{m}_i^v(\pi) \geq v(S)$ for any $S \in H(\Gamma)$ and $\pi \in \Pi^{\Gamma}$. Let $S_1, \ldots, S_k$ be the unique maximal partition of S such that $S_h = \{i \in S \mid b_h \leq \pi(i) \leq a_h\}$, $h = 1, \ldots, k$, where a_h and b_h, $h = 1, \ldots, k$, satisfy $a_{h-1} < b_h \leq a_h$, with $a_0 = 0$. We define $\bar{P}_{\pi}(0) = \emptyset$. For any given $h \in \{1, \ldots, k\}$ consider the sets $S \cup \bar{P}_{\pi}(a_{h-1})$ and $P_{\pi}(b_h)$. By Proposition 3.1 and since S is hierarchical, both sets are hierarchical coalitions. Moreover, their intersection is equal to $\bar{P}_{\pi}(a_{h-1})$ and their union is equal to $S \cup \bar{P}_{\pi}(a_h)$. From hierarchical convexity it then follows that

$$v(S \cup \bar{P}_{\pi}(a_h)) + v(\bar{P}_{\pi}(a_{h-1})) \geq v(S \cup \bar{P}_{\pi}(a_{h-1})) + v(P_{\pi}(b_h)).$$

By repeated application of this inequality for $h = 1, \ldots k$, we obtain

$$v(S \cup \bar{P}_{\pi}(a_k)) + \sum_{h=1}^{k} v(\bar{P}_{\pi}(a_{h-1})) \geq v(S \cup \bar{P}_{\pi}(a_0)) + \sum_{h=1}^{k} v(P_{\pi}(b_h)).$$

Because $\bar{P}_{\pi}(a_0) = \emptyset$ and $S \cup \bar{P}_{\pi}(a_k) = \bar{P}_{\pi}(a_k)$, it follows that

$$\sum_{h=1}^{k} v(\bar{P}_{\pi}(a_h)) \geq v(S) + \sum_{h=1}^{k} v(P_{\pi}(b_h)).$$

Since for $h = 1, \ldots, k$ it holds that

$$\sum_{i \in S_h} \bar{m}_i^v(\pi) = v(\bar{P}_\pi(a_h) - v(P_\pi(b_h))$$

and $\sum_{i \in S} \bar{m}_i^v(\pi) = \sum_{h=1}^k \sum_{i \in S_h} \bar{m}_i^v(\pi)$, we obtain $\sum_{i \in S} \bar{m}_i^v(\pi) \geq v(S)$. ∎

4 Axiomatization for cycle digraph games

On the subclass of cycle-free digraph games the Shapley value for digraph games coincides with the Shapley value for TU games with precedence constraints of Faigle and Kern [2]. Thus, the axiomatization of the latter value obtained in [2] serves also as an axiomatization of the Shapley value for cycle-free digraph games. Now we introduce an axiomatization of the Shapley value on a distinct subclass of digraph games, the subclass $\mathcal{G}_N^{\Gamma c}$ of cycle digraph games. From Remark 3.2 it follows that a directed cycle on a player set is a connected digraph every node of which is an undominated player.

Theorem 4.1 *The Shapley value for digraph games is the unique value on* $\mathcal{G}_N^{\Gamma c}$ *that meets E, L, RHM, and SSDC.*

Proof. I [Existence]. The proof that the Shapley value for digraph games on $\mathcal{G}_N^{\Gamma c}$ meets E, L, RHM, and SSDC is similar to the proof of Proposition 3.2 concerning E, L, HM, and SSDC on $\mathcal{G}_N^{\Gamma}$. For the proof of RHM we only need to add that due to Remark 3.2 all hierarchical coalitions involved are connected.

II [Uniqueness]. First prove that on $\mathcal{G}_N^{\Gamma c}$ E, RHM, and SSDC imply RHNP. Take any $(v, \Gamma) \in \mathcal{G}_N^{\Gamma c}$ with restricted hierarchical null-player i and let v_0 be the zero game on N, i.e., $v_0(S) = 0$ for all $S \subseteq N$. Hence, $m_i^v(S) = 0 = m_i^{v_0}(S)$ for all $S \subseteq N \backslash \{i\}$ such that $S, S \cup \{i\} \in H^c(\Gamma)$ and $i \in U^\Gamma(S \cup \{i\})$. Then, RHM implies that $\xi_i(v, \Gamma) = \xi_i(v_0, \Gamma)$. From E and SSDC it follows that $\xi_j(v_0, \Gamma) = 0$ for all $j \in N$. Whence, $\xi_i(v, \Gamma) = 0$.

Since unanimity games u_T, $T \subseteq N$, $T \neq \emptyset$, form a basis in $\mathcal{G}_N$, due to L it suffices to show that $\xi(u_T, \Gamma)$ is uniquely determined for all $(u_T, \Gamma) \in \mathcal{G}_N^{\Gamma c}$.

If $T = N$, then E and SSDC imply $\xi_i(u_N, \Gamma) = \frac{1}{n}$ for all $i \in N$.

If $T \in C^\Gamma(N)$, $T \neq N$, then due to Remark 3.2 $U^\Gamma(T) = \{r\}$ for some $r \in T$. For all $i \in T \backslash \{r\}$ and $S \subseteq N \backslash \{i\}$ such that $S, S \cup \{i\} \in H^c(\Gamma)$ and $i \in U^\Gamma(S \cup \{i\})$, $m_i^{u_T}(S) = m_i^{u_N}(S)$. From RHM it follows that $\xi_i(u_T, \Gamma) = \xi_i(u_N, \Gamma) = \frac{1}{n}$ for all $i \in T \backslash \{r\}$. Since every $i \in N \backslash T$ is a null-player in u_T, then by Remark 3.3 each $i \in N \backslash T$ is a restricted hierarchical null-player in (u_T, Γ) and by RHNP $\xi_i(u_T, \Gamma) = 0$. Then E implies $\xi_r(u_T, \Gamma) = 1 - \frac{|T|-1}{n}$.

Finally, take any $T \notin C^\Gamma(N)$. Let $T/\Gamma = \{T_1, \ldots, T_k\}$ and $U^\Gamma(T_h) = \{r_h\}$ for some $r_h \in T_h$, $h = 1, \ldots, k$. Each $i \in N \backslash T$ is a restricted hierarchical null-player in (u_T, Γ) and for all $i \in T \backslash \{r_1, \ldots, r_k\}$ and $S \subseteq N \backslash \{i\}$ such that $S, S \cup \{i\} \in H^c(\Gamma)$ and $i \in U^\Gamma(S \cup \{i\})$, $m_i^{u_T}(S) = m_i^{u_N}(S)$. Hence, from RHNP it follows that $\xi_i(u_T, \Gamma) = 0$ for all $i \in N \backslash T$ and from RHM it follows that $\xi_i(u_T, \Gamma) = \frac{1}{n}$ for all $i \in T \backslash \{r_1, \ldots, r_k\}$. For any given $h \in \{1, \ldots, k\}$, let $T^h \in C^\Gamma(N)$ be the unique smallest connected set containing T such that $U^\Gamma(T^h) = \{r_h\}$. Then each $i \in N \backslash T^h$ is a restricted hierarchical null player in (u_{T^h}, Γ) and for all $i \in T^h \backslash \{r_h\}$ and $S \subseteq N \backslash \{i\}$ such that $S, S \cup \{i\} \in H^c(\Gamma)$ and $i \in U^\Gamma(S \cup \{i\})$, $m_i^{u_{T^h}}(S) = m_i^{u_N}(S)$. Hence, from RHNP it follows that $\xi_i(u_{T^h}, \Gamma) = 0$ for all $i \in N \backslash T^h$ and from RHM it follows that $\xi_i(u_{T^h}, \Gamma) = \frac{1}{n}$ for all $i \in T^h \backslash \{r_h\}$. Then by E, $\xi_{r_h}(u_{T^h}, \Gamma) = 1 - \frac{|T^h|-1}{n}$. Since for all $S \subseteq N \backslash \{r_h\}$ satisfying $S, S \cup \{r_h\} \in H^c(\Gamma)$ and $r_h \in U^\Gamma(S \cup \{r_h\})$ it holds that $m_{r_h}^{u_{T^h}}(S) = m_{r_h}^{u_T}(S)$, from RHM it follows that $\xi_{r_h}(u_T, \Gamma) = \xi_{r_h}(u_{T^h}, \Gamma) = 1 - \frac{|T^h|-1}{n}$. ∎

Remark 4.1 Unlike Young's axiomatization in [8] of the classical Shapley value by efficiency, equal treatment property and marginality without a priori requirement of additivity, for the axiomatization of the Shapley value for digraph games on the subclass of cycle digraph games we need both linearity and restricted marginality. The reason why the induction argument of Young does not work in the latter case is that while the decomposition of a TU game is considered via the unanimity basis determined by all possible coalitions, restricted marginality as opposed to marginality considers only the hierarchical coalitions that when the digraph is a directed cycle form a proper subset of the set of all coalitions.

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