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Abstract

In this paper we define feedback Nash equilibria in linear quadratic differential games on an infinite time horizon in both a deterministic and stochastic environment. The relationship between existence of such equilibria and solutions of sets of algebraic Riccati equations is investigated.
We assume that the players have memoryless perfect state information and that they restrict themselves to stationary linear stabilizing feedback strategies.
We consider linear systems in as well a certain as an uncertain environment. In the latter case we treat uncertainty both in a stochastic and a deterministic way. In the stochastic setting we consider average, discounted and exponential performance criteria, respectively. In the deterministic setting, we consider an H^∞ type performance criterium. A relationship between the stochastic exponential and the deterministic H^∞ case is established.

1 Introduction

In this paper we consider the problem where N parties (henceforth called players) try to minimize their individual quadratic performance criterion. Each player controls a different set of inputs to a single system. The system is described by the following differential equation

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{N} B_i u_i(t) + Gw(t),\ x(0) = x_0. \quad (1)$$

Here, x is the n-dimensional state of the system, u_i contains the m_i (control) variables player i can manipulate, w is a q-dimensional vector of noise corrupting the system, x_0 is the arbitrarily chosen initial state of the system, A, B_i and G are constant system parameters, and $\dot{x}$ denotes the time derivative of x, for $i \in \mathbf{N} := \{1, \cdots, N\}$.

The performance criterion player $i \in \mathbf{N}$ aims to minimize is:
$J_i(u_1, \cdots, u_N, x_0, w) :=$

$$\int_0^\infty \{x(t)^T Q_i x(t) + \sum_{j \in \mathbf{N}} u_j(t)^T R_{ij} u_j(t)\} dt. \quad (2)$$

We assume that Q_i is symmetric and R_{ii} is positive definite for all $i \in \mathbf{N}$. Note that we do not make any definiteness assumptions on Q_i. So, this formulation includes the 2-person zero-sum game.
Without specification of the information the players have, this problem is ill-defined. In applications (see e.g. Engwerda et al. (1999) for a number of references in economics) it is often assumed that the players have either memoryless perfect state or closed-loop perfect state information. Given this information structure, it is then assumed that in a non-cooperative world ultimately the controls that will be played are those which constitute a Nash equilibrium. This naturally leads to the definition of the feedback Nash equilibrium concept as defined in e.g. Başar and Olsder (1999, pp.321) in case the players have a finite planning horizon. If the players have an infinite planning horizon, things are however much less clear and usually one restricts to the consideration of limiting stationary feedback Nash equilibria. In this paper we will consider feedback information structure and additionally assume that the strategies of the players in the infinite horizon framework are linear in the state and are such that the state converges to zero. To be more precisely, we shall only consider controls u_i of the type $u_i = F_i x$, with $F_i \in \mathbb{R}^{m_i \times n}$, and where $(F_1, \cdots, F_N)$ belong to the set

$$\mathcal{F} := \left\{ F = (F_1, \cdots, F_N) \,\middle|\, A + \sum_{i \in \mathbf{N}} B_i F_i \text{ is stable} \right\}$$

Obviously, this last assumption is made in order to obtain a stable system. A necessary and sufficient condition for $\mathcal{F}$ to be non-empty is stabilizability of $(A, [B_1 \cdots B_N])$. Therefore, we will assume throughout this paper that this matrix pair satisfies this stabilizability condition. Note that with this restriction on the strategies, the performance J_i is now a function of the matrices F_i, x_0 and w.

The rest of the paper is organized as follows. The next section contains some well-known results and notation that is used throughout the paper. In section 3 we will characterize all feedback Nash equilibria of (1),(2) in terms of solutions to a set of algebraic Riccati equations if the system is not corrupted by noise (i.e. $G = 0$). In section 4 we consider the case that w in system (1) represents white noise. In that case it does not make sense to consider the expected value of (2) as performance criterion since the value of it will in general be infinite. Therefore we consider three different performance criteria in this section and determine the Nash solutions for each of these criteria again in terms of solutions to some sets of algebraic Riccati equations. Section 5 considers the case that w in system (1) represents unknown deterministic noise and each of the players is looking for a worst case design. The paper ends with some concluding remarks.

2 Preliminaries

In this section on the one hand we recall some well-known concepts from literature and on the other hand introduce some notation that is used throughout the paper.

First, we recall that a matrix A is called stable if each eigenvalue of A has a strict negative real part. Furthermore, we will use the notation $A > 0$ (≥ 0) to indicate that A is positive (semi-) definite. The trace of a matrix A will be denoted by $tr(A)$.

Next, for an N-tuple $\gamma = (\gamma_1, \cdots, \gamma_N) \in \Gamma_1 \times \cdots \times \Gamma_N$, for given sets Γ_i, we introduce the notation $\gamma_{-i}(\alpha) = (\gamma_1, \cdots, \gamma_{i-1}, \alpha, \gamma_{i+1}, \cdots, \gamma_N)$, with $\alpha \in \Gamma_i$.

Furthermore, we use the matrix abbreviations $S_{ij} := B_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_j^T$, $S_i := S_{ii}$, and for given square $n \times n$ matrices X_i, $A_{cl} := A - \sum_{j \in \mathbf{N}} S_j X_j$, $A_{cl,i} := A_{cl} + S_i X_i$, and $\bar{Q}_i := \sum_{j \in \mathbf{N}, j \neq i} X_j S_{ij} X_j + Q_i$. In a one-player context we will mostly drop the indices in the above mentioned matrices. A set of matrices X_i is called stabilizing if A_{cl} becomes stable.

We use $\parallel x \parallel$ to denote the Euclidean norm of a vector.

Finally we note that usually we will drop in our notation the dependency of the state on the specific initial state, input and noise that is applied to the system and thus denote the state of system (1) by $x(t)$. If any of these arguments is unclear within the context we will include them in our notation and then use e.g. $x(t; x_0, F)$ instead of just $x(t)$.

3 The undisturbed case

In this section we consider the case $G = 0$. Consequently, J_i does not depend on w anymore. We define the concept of a feedback Nash equilibrium in this noise-free context as follows

Definition 1:
An N-tuple $\hat{F} = (\hat{F}_1, \cdots, \hat{F}_N) \in \mathcal{F}$ is called a *feedback Nash equilibrium* if for all i the following holds:

$$J_i(\hat{F}, x_0) \leq J_i(\hat{F}_{-i}(\alpha), x_0), \tag{3}$$

for all $x_0 \in \mathbb{R}^n$ and for all $\alpha \in \mathbb{R}^{m_i \times n}$ such that $\hat{F}_{-i}(\alpha) \in \mathcal{F}$. □

Next, we characterize all feedback Nash equilibria for system (1), with $G = 0$, in case the performance criterion of each player is given by (2). To that end we first establish the next relationship between existence of a stabilizing solution of an algebraic Riccati equation and existence of a solution to a regulator problem. A proof can be found in the appendix. The only if part of the theorem is proved by using the calculus of matrix differentials.

Theorem 2:
Consider the one-player case.
Then, there exists an $\hat{F} \in \mathcal{F}$ such that for all $F \in \mathcal{F}$ for all x_0, $J(\hat{F}, x_0) \leq J(F, x_0)$ if and only if the algebraic Riccati equation

$$-XSX + XA + A^T X + Q = 0 \tag{4}$$

has a stabilizing solution X (i.e. a solution X such that $A - SX$ is stable).
Moreover, this minimum equals $x_0^T X x_0$ and is uniquely attained by the feedback $F = -R^{-1} B^T X$. □

Remark 3:
1) Willems showed a similar result in (1971) under the assumption that the system is controllable.
2) It is well-known, see e.g. Kimura (1997, pp.43), that (4) has at most one stabilizing solution.
3) Using the notion of the matrix sign Lancaster and Rodman derive in (1995, theorem 22.4.1) both necessary and sufficient conditions for existence of the stabilizing solution of (4). □

Using theorem 2 we are able to prove the next result, establishing the link between solvability of a set of coupled algebraic Riccati equations and existence of feedback Nash equilibria for the differential game under consideration. The relevant set of algebraic Riccati equations is given by (see also Başar and Olsder (1999, pp.337))

$$\sum_{j \in \mathbf{N}} X_j S_{ij} X_j + X_i A_{cl} + A_{cl}^T X_i + Q_i = 0,\ i \in \mathbf{N}, \quad (ARE)$$

Theorem 4:
Let there exist an N-tuple of symmetric matrices $X_i, i \in \mathbf{N}$ satisfying (ARE) such that A_{cl} is stable. Define the N-tuple $\hat{F} := (\hat{F}_1, \cdots, \hat{F}_N)$ by $\hat{F}_i := -R_{ii}^{-1} B_i^T X_i$. Then $(\hat{F}_1, \cdots, \hat{F}_N)$ is a feedback Nash equilibrium and $J_i(\hat{F}, x_0) = x_0^T X_i x_0$.
Conversely, if $(\hat{F}_1, \cdots, \hat{F}_N)$ is a feedback Nash equilibrium then the set of algebraic Riccati equations (ARE) has a stabilizing solution.

Proof:
" $\Rightarrow$ " Let $X_i, i \in \mathbf{N}$ satisfy (ARE) such that A_{cl} is stable. Furthermore, denote $\hat{F}_i := -R_{ii}^{-1}B_i^T X_i$. Then, player i is confronted with the optimization problem to minimize $J_i(\hat{F}_{-i}(F_i), x_0) = \int_0^\infty \{x(t)^T \bar{Q}_i x(t) + x^T(t) F_i^T R_{ii} F_i x(t)\} dt$, subject to the constraint $\dot{x}(t) = (A_{cl,i} + B_i F_i)x(t),\ x(0) = x_0$. According theorem 2 this problem has a unique solution if and only if the algebraic Riccati equation

$$-XS_iX + XA_{cl,i} + A_{cl,i}^T X + \bar{Q}_i = 0$$

has a stabilizing solution. Substitution of $X := X_i$ into the above equation shows after some rewriting that we obtain the equation

$$\sum_{j \in \mathbf{N}} X_j S_{ij} X_j + X_i A_{cl} + A_{cl}^T X_i + Q_i = 0, \qquad (5)$$

which holds by assumption. Furthermore, by assumption the corresponding closed-loop system A_{cl} is stable. So, inequality (3) is satisfied.
" $\Leftarrow$ " Assume that $(\hat{F}_1, \cdots, \hat{F}_N)$ is a feedback Nash equilibrium. Then, by definition 1, $\hat{F}_i$ solves the optimization problem to minimize for all $F_i \in \mathbb{R}^{n \times m}$, $J_i(\hat{F}_{-i}(F_i), x_0) = \int_0^\infty \{x(t)^T(\sum_{j \in \mathbf{N}, j \neq i} \hat{F}_j^T R_{ij} \hat{F}_j + Q_i)x(t) + x^T(t) F_i^T R_{ii} F_i x(t)\} dt$ subject to the constraint $\dot{x}(t) = (A + \sum_{j \in \mathbf{N}, j \neq i} B_j \hat{F}_j)x(t) + B_i F_i x(t),\ x(0) = x_0$. Since this problem has a solution, we know from theorem 2 that its solution is uniquely determined by $F_i = -R_{11}^{-1} B_i^T X_i$, where X_i satisfies a corresponding algebraic Riccati equation. So, necessarily we have that $\hat{F}_i = -R_{11}^{-1} B_i^T X_i$. Since this property should hold for every $i \in \mathbf{N}$, we can substitute this into the according Riccati equations, which yields then the stated property. □

Remark 5:
The assumption that we restrict the set of admissible controls to $\mathcal{F}$ is essential here. If we drop this assumption, one can easily construct an example (see Mageirou (1976)) where the set of algebraic Riccati equations (ARE) has a stabilizing solution, and where it is possible to construct a (non-stabilizing) feedback controller which yields lower cost for one player in case the other player sticks to the stabilizing controller advocated by the (ARE). □

4 The disturbed case: stochastic uncertainty

In this section we analyze what the effect will be on the Nash equilibria if the system (1) is corrupted by stochastic white noise. That is, we assume in this section that the system is described by the following stochastic differential equation

$$\dot{x}(t) = \left(A + \sum_{i=1}^{N} B_i F_i\right) x(t) + G w(t), \qquad (6)$$

where $w(t)$ is a standard white noise stochastic process (for a formal definition see e.g. Davis (1977, pp.79)) and $x(0)$ is a random vector, uncorrelated with w, with mean x_0 and $E\{x(0)x^T(0)\} = W_0 > 0$.
Now, for the one player case it is well-known (see e.g. Davis (1977, pp.185)) that, if Q is positive definite in (2), the expectation of the cost (2) does not exist (it becomes infinite). Therefore we consider three modified cost functionals

1.) $J_i^{av}(F_1, \cdots, F_N) :=$

$$\lim_{T \to \infty} E \frac{1}{T} \{ \int_0^T \{x(t)^T (Q_i + \sum_{j \in \mathbf{N}} F_j^T R_{ij} F_j) x(t)\} dt\}, \quad (7)$$

2.) $J_i^{d}(F_1, \cdots, F_N) :=$

$$\lim_{T \to \infty} E\{ \int_0^T e^{-\theta t} \{x(t)^T (Q_i + \sum_{j \in \mathbf{N}} F_j^T R_{ij} F_j) x(t)\} dt\}, \quad (8)$$

3.) $J_i^{LEQG}(F_1, \cdots, F_N) :=$

$$\lim_{T \to \infty} \frac{1}{T} \ln E e^{\{\frac{1}{2\gamma_i^2} \int_0^T x(t)^T (Q_i + \sum_{j \in \mathbf{N}} F_j^T R_{ij} F_j) x(t) dt\}}. \quad (9)$$

Here θ and γ_i are positive numbers and all matrices satisfy the previous assumptions (see below (2)). In particular, we still assume that $Q_i, i \in \mathbf{N}$ are not necessarily positive semi-definite.
Note that since we take the expectation operation each of the cost functionals is a deterministic function, which depends on the statistics of the noise processes. For notational convenience we dropped this dependency (as usual).
These performance criteria have the following characteristics. Whereas the first, average, cost criterion measures accurately the long run performance, and neglects the behaviour of the system over any initial interval, the second, discounted, cost criterion has the opposite effect: it emphasizes the initial performance. The third performance criterion, the linear exponential-quadratic gaussian (LEQG) one, is motivated by the fact that it expresses that player i has a constant level of risk aversion $\frac{1}{2\gamma_i^2}$ concerning J_i (see Pratt (1964)). The risk aversion parameter γ_i expresses that the smaller it is, the more player i likes to steer the state of the system towards zero as fast as possible.
Since the statistics are assumed to be known a priori to all players, we can copy more or less our previous definition 1 to define the notion of feedback Nash equilibrium in a stochastic context. Viewed in this way, note that this equilibrium concept is in fact more in line with the memoryless perfect state information structure than with the feedback information structure.

Definition 6:
Assume the statistics of the noise processes corrupting the deterministic system are known to all players. Then, an N-tuple $\hat{F} = (\hat{F}_1, \cdots, \hat{F}_N) \in \mathcal{F}$ is called a *feedback Nash equilibrium* if for all i the following holds:

$$J_i(\hat{F}) \leq J_i(\hat{F}_{-i}(\alpha)), \qquad (10)$$

for all $\alpha \in R^{m_i \times n}$ such that $\hat{F}_{-i}(\alpha) \in \mathcal{F}$. $\square$

To find feedback Nash equilibria for the above sketched games, one first has to verify whether each game is well-defined in the sense that for each $F \in \mathcal{F}$ the costs of the players are finite. We first consider again the one-player case. The proof of the next lemma and theorem can be found in the appendix.

Lemma 7:
let $F \in \mathcal{F}$. Then
1) $|J_1^{av}(F)| < \infty$.
2) $W(t;F) := E\{x(t;F)x^T(t;F)\} \rightarrow W(F)$ exponentially fast. Furthermore, $W(t;F)$ is invertible for all t and $W(F) \geq 0$. $\square$

Theorem 8:
Consider the one-player case.
Then, $\min_{F \in \mathcal{F}} J^{av}(F)$ exists if the algebraic Riccati equation (4) has a solution X such that A_{cl} is stable. Moreover, this minimum equals $tr(G^TXG)$ and is attained by the feedback $F = -R^{-1}B^TX$. $\square$

Note that this optimal control coincides with that of the undisturbed case. Furthermore, we do not claim here that the proposed control is the unique argument that minimizes the cost functional. A sufficient condition to conclude that there is only one optimal feedback control under the assumption that (4) has a stabilizing solution is that $(A + BF, G)$ is controllable for all $F \in \mathcal{F}$. Under this condition $W(F)$ (see lemma 7) is strictly positive definite for all $F \in \mathcal{F}$ (see e.g. Kimura (1997, lemma 2.14, 2.15)).
Analogously to the first part of theorem 4 we get then for the multi-player case the following result.

Theorem 9:
Let there exist an N-tuple of symmetric matrices $X_i, i \in \mathbf{N}$ satisfying (ARE) such that A_{cl} is stable. Define the N-tuple $\hat{F} := (\hat{F}_1, \cdots, \hat{F}_N)$ by $\hat{F}_i := -R_{ii}^{-1}B_i^TX_i$. Then $(\hat{F}_1, \cdots, \hat{F}_N)$ is a feedback Nash equilibrium and $J_i^{av}(\hat{F}) = tr(G^TX_iG)$. $\square$

Next, we consider the discounted cost criterion. Basically, a similar result like theorem 9 will be derived: only matrix A has to be replaced by matrix $A_d := A - \frac{1}{2}\theta I$. Again, first we consider the one-player case. The well-posedness follows in the same way as lemma 7 and is therefore omitted.

Theorem 10:
Consider the one-player case.
Then, $\min_{F \in \mathcal{F}} J^d(F)$ exists if the algebraic Riccati equation

$$-XSX + XA_d + A_d^TX + Q = 0 \tag{11}$$

has a stabilizing solution X. In that case this minimum equals $tr(W_0X) + tr(G^TXG)$ and is attained uniquely by the feedback $F = -R^{-1}B^TX$. $\square$

The proof can be found in the appendix. Note the subtle difference with theorem 8 that the argument which yields the stated optimal control is unique here.
The next multi-player result can be derived then analogously to theorem 4. A proof is therefore omitted.

Theorem 11:
Let there exist an N-tuple of symmetric matrices $X_i, i \in \mathbf{N}$ satisfying (ARE), in which matrix A_{cl} is replaced by $A_{cl}^d := A_{cl} - \frac{1}{2}\theta I$, such that A_{cl}^d is stable. Define the N-tuple $\hat{F} := (\hat{F}_1, \cdots, \hat{F}_N)$ by $\hat{F}_i := -R_{ii}^{-1}B_i^TX_i$. Then $(\hat{F}_1, \cdots, \hat{F}_N)$ is a feedback Nash equilibrium and $J_i^d(\hat{F}) = tr(W_0X_i) + tr(G^TX_iG)$. $\square$

Finally we consider the LEQG performance criterion. ¿From Runolfsson (1994) we have the following result

Theorem 12:
Consider the one-player case. Assume that $Q \geq 0$; (A, B), (A, G) are controllable and (Q, A) is observable.
Then, $\min_{F \in \mathcal{F}} J^{LEQG}(F)$ exists if the algebraic Riccati equation

$$-X(S - \frac{1}{\gamma^2}GG^T)X + XA + A^TX + Q = 0 \tag{12}$$

has a (smallest) solution $X(> 0)$ such that both $A - (S - \frac{1}{\gamma^2}GG^T)X$ and $A - SX$ are stable and $(A - SX, G)$ is controllable.
Moreover, this minimum equals $\frac{1}{2\gamma^2}tr(G^TXG)$ and is attained by the feedback $F := -\gamma^2R^{-1}B^TX$ (and $w(t) = G^TXx(t)$). $\square$

To formulate the multi-player case we introduce the following set of algebraic Riccati equations

$$\sum_{j \in \mathbf{N}} X_jS_{ij}X_j + \frac{1}{\gamma_i^2}X_iGG^TX_i + X_iA_{cl} + A_{cl}^TX_i + Q_i$$

$$= 0, \; i \in \mathbf{N}. \tag{13}$$

Then, using the previous result one can show that:

Theorem 13:
Let there exist an N-tuple of symmetric matrices $X_i > 0, i \in \mathbf{N}$ satisfying (13). Assume that with these solutions A_{cl} and $A_{cl} + \frac{1}{\gamma_i^2}GG^TX_i$ are stable; $\bar{Q}_i \geq 0$; $(A_{cl,i}, B_i)$, (A_{cl}, G) and $(A_{cl,i}, G)$ are controllable; and $(\bar{Q}_i, A_{cl,i})$ are observable, $i \in \mathbf{N}$.
Define the N-tuple $\hat{F} := (\hat{F}_1, \cdots, \hat{F}_N)$ by $\hat{F}_i := -R_{ii}^{-1}B_i^TX_i$. Then, $(\hat{F}_1, \cdots, \hat{F}_N)$ is a feedback Nash equilibrium and $J_i^d(\hat{F}, x_0) = tr(G^TX_iG)$. $\square$

5 The disturbed case: deterministic uncertainty

In the previous framework, the disturbances, though unpredictable, are not completely unknown. One needs *a priori* knowledge of their statistics. If these statistics are not adequately described then this theory is not useful. In fact, it is assumed that all players have the same intuition on the uncertainty that prevails in the system, and that they cope with this uncertainty by taking the expectation operation.

A possible alternative approach is to assume that the disturbance entering the system is completely unknown, though square integrable, and that each player has his own intuition on how strong this disturbance will be. That is, we assume in this section that the system (1) is now described by the following differential equation

$$\dot{x}(t) = (A + \sum_{i=1}^{N} B_i F_i)x(t) + Gw(t), x(0) = x_0 \quad (14)$$

where $w \in L_2^q(0,\infty)$ is an unknown deterministic disturbance. We assume that all players have feedback information and act non-cooperative.

Then, like in the stochastic case, we have to model how the players cope with this uncertain dynamics. We replace the statistics by a measure of the magnitude of the disturbance the individual player expects. This is done by replacing J_i in (2) by

$$J_i^{det,w} := J_i - \int_0^\infty w^T(t)V_i w(t)dt. \quad (15)$$

Here $V_i > 0$ measures the expectation of player i concerning the magnitude of the disturbance. Furthermore, we will assume in this section that Q_i is positive semi-definite.

The intuition for introducing V_i into this performance criterion is that the larger V_i is the smaller player i expects that the influence of the disturbance on the system will be. As such this approach can be viewed as an attempt to incorporate some probabilistics concerning the magnitude of the disturbance entering into the system. The fact that we are looking for a worst case design is expressed by replacing the expectation operator by the supremum operator. That is, we assume that each player is interested to implement a control which yields a guaranteed maximum level for his costs, whatever the disturbance will be, and where this level is as small as possible. Summarizing, we assume that in this deterministic model each player likes to minimize

$$J_i^{det} := \sup_{w \in L_2^q(0,\infty)} J_i^{det,w} \quad (16)$$

subject to (14). Definition 1 can then be used again to define the concept of a feedback Nash equilibrium.

Obviously this approach is inspired by the H^∞ control theory. This theory is used to derive the next characterization for feedback Nash equilibria in terms of Riccati equations.

Theorem 14:

Consider system (14) for the one player case.

Assume there exists a matrix $X \geq 0$ satisfying the algebraic Riccati equation

$$-XSX + A^T X + XA + XGV^{-1}G^T X + Q = 0 \quad (17)$$

such that $A - SX + GV^{-1}G^T X$ is stable.

Let $\hat{F} := -R^{-1}B^T X$. Then, $\hat{F} \in \mathcal{F}$ and

$$\forall F \in \mathcal{F},\ \forall x_0,\ J^{det}(\hat{F}, x_0) \leq J^{det}(F, x_0). \quad (18)$$

□

In the multi-player case the next set of algebraic Riccati equations play an important role

$$\sum_{j \in \mathbf{N}} X_j S_{ij} X_j + X_i G V_i^{-1} G^T X_i + X_i A_{cl} + A_{cl}^T X_i + Q_i = 0,\ i \in \mathbf{N}. \quad (19)$$

Then we have:

Theorem 15:

Consider system (14).

If there exists $X_i \geq 0$ satisfying (19) such that $A_{cl} + GV_i^{-1}G^T X_i,\ i \in \mathbf{N}$ are stable, then $\hat{F} := (\hat{F}_1, \cdots, \hat{F}_N)$ with $\hat{F}_i := -R_{ii}^{-1} B_i^T X_i$ is a feedback Nash equilibrium. □

We conclude this section by pointing out a relationship between the equilibria we obtained in this deterministic noise setting and the LEQG feedback Nash equilibria. Comparing the formulae (13) and (19) we see that both coincide if we make the assumption that $V_i = \gamma_i^2,\ i \in \mathbf{N}$. ¿From this we conclude that, at least generically, the solution to the LEQG problem will also provide an admissible equilibrium in the deterministic noise case and, vice versa. This kind of relationship has already been noticed by a number of people in literature, dating back to Jacobson (1973).

6 Conclusion

The aim of this paper is to analyze the relationship between existence of feedback Nash equilibria in infinite-time horizon linear quadratic games and solutions for sets of algebraic Riccati equations. Furthermore, it provides a framework to deal with robust optimal control problems in a multi-player context.

By restricting the strategy space to the set of stabilizing feedback controllers, we showed that the notion of feedback Nash equilibrium can be defined straightforwardly. We formulated both necessary and sufficient conditions for existence of such equilibria in terms of existence of stabilizing solutions to algebraic Riccati equations.

Engwerda showed in (2000) that already in the scalar case (for $Q_i > 0$ and $R_{ij} = 0$ for $i \neq j$) the above set of algebraic Riccati equations (ARE) has in general more than one stabilizing solution. Moreover, both necessary and sufficient conditions were presented under which this set has a unique solution. It will be clear that an extension of these results to the multivariable case would enhance a better understanding of these games. Another point of research is whether some assumptions can be further relaxed, and particularly for the uncertain context, whether here also both necessary and sufficient conditions are possible in terms of existence of solutions to algebraic Riccati equations.

Appendix

In this appendix we present a number of proofs to the theorems. We start with a lemma which is used in the proof of theorem 2. Both results require some notation used in the theory of matrix differential calculation. Therefore, we start of with giving these definitions (see e.g. Magnus et al. 1999).

The vec operator will be used to transform a matrix into a vector by stacking the columns of the matrix one underneath the other. The notation $v(P)$ will be used to denote the $\frac{1}{2}n(n+1)$ vector that is obtained from vec P by eliminating all supradiagonal elements of P; and D_n will denote the duplication matrix which transforms, for symmetric P, $v(A)$ into vec P, i.e. $D_n v(P) = \text{vec } P$. Finally, $\otimes$ will denote the Kronecker product and D^+ the Moore-Penrose inverse of matrix D.

Lemma A.1:
Let $b \in \mathbb{R}^{n(n+1)/2}$. Assume that $(a^T \otimes a^T)D_n b = 0$ for all $a \in \mathbb{R}^n$. Then, $b = 0$. **Proof:**
Let B be the $n \times n$ symmetric matrix defined by $v(B) = b$. Then, by definition of the duplication matrix it follows that $D_n b = D_n v(B) = \text{vec } B$. Hence, we have

$$\begin{aligned} 0 &= (a^T \otimes a^T)D_n b = (a^T \otimes a^T)\text{vec } B = \text{vec } a^T B a \\ &= a^T B a, \end{aligned}$$

for all $a \in \mathbb{R}^n$. This clearly implies that $B = 0$, or equivalently, $b = 0$. $\square$

Proof of theorem 2:
" $\Rightarrow$ " Consider an arbitrary $F \in \mathcal{F}$. It is well-known that then $J(F, x_0) = x_0^T P x_0$, where P is the unique symmetric solution of the Lyapunov equation

$$(A + BF)^T P + P(A + BF) = -(Q + FRF) \qquad (20)$$

Note that the parameters of this Lyapunov equation, i.e. the matrices $A + BF$ and $-(C + DF)^T(C + DF)$ are differentiable functions of $F \in \mathcal{F}$. Hence P is also a differentiable function of $F \in \mathcal{F}$ (see e.g. Lancaster et al. (1995, section 5.4)). So, J is differentiable with respect to F.

Now assume that the minimum of $J(F, x_0)$ is attained at $F = \hat{F}$. Since $\hat{F} \in \mathcal{F}$ it is easily verified that there is an open environment $\mathcal{B} \subset \mathcal{F}$ of $\hat{F} \in \mathbb{R}^{n \times m}$ (with e.g. the topology induced by the Euclidean matrix norm). Since for all $F \in \mathcal{B}$ $J(\hat{F}, x_0) \leq J(F, x_0)$, we conclude that

$$\frac{\partial J(F, x_0)}{\partial (\text{vec} F)^T} = 0, \forall x_0 \text{ at } F = \hat{F} \qquad (21)$$

Using matrix differential calculus, see e.g. Magnus et al. (1999), it is easily verified that

$$\frac{\partial J(F, x_0)}{\partial (\text{vec} F)^T} = (x_0^T \otimes x_0^T) D_n \frac{\partial v(P)}{\partial (\text{vec} F)^T}. \qquad (22)$$

Since x_0 is arbitrarily we conclude from lemma A.1 that

$$\frac{\partial v(P)}{\partial (\text{vec} F)^T} = 0 \text{ at } F = \hat{F}. \qquad (23)$$

On the other hand we see that by differentiation of (20) w.r.t. F we get

$$D_n^+(I_n \otimes PB) + D_n^+(I_n \otimes (A + BF)^T) D_n \frac{\partial v(P)}{\partial (\text{vec} F)^T} = -D_n^+(I_n \otimes (F^T R)),$$

Using (23) we see that consequently at $F = \hat{F}$

$$D_n^+(I_n \otimes (PB + F^T R)) = 0.$$

Therefore, also $D_n^T(I_n \otimes (PB + F^T R)) = 0$.
Since $D_n = \begin{pmatrix} I_n & 0_{\frac{1}{2}n(n-1)} \\ * & * \end{pmatrix}$, we have that $D_n^T(I_n \otimes (PB + \hat{F}^T R)) = \begin{pmatrix} PB + \hat{F}^T R & * \\ * & * \end{pmatrix}$. So $PB + \hat{F}^T R = 0$ or, stated differently, $\hat{F}^T = -R^{-1}B^T P$. Substitution of this last equality into (20) yields then the advertised result.
" $\Leftarrow$ " Let X be the stabilizing solution of (4). Consider

$$J(u_1, x_0, T) := \int_0^T \{x(t)^T Q x(t) + u_1(t)^T R u_1(t)\} dt. \qquad (24)$$

Then (see e.g. Willems (1971, lemma 6)), for all $x_0 \in \mathbb{R}^n$, for all $F \in \mathbb{R}^{m \times n}$, and for all $T > 0$, the following identity holds

$$J(F, x_0, T) = \int_0^T \| (F + R^{-1}B^T X) x(t; F, x_0) \|^2 \, dt + x_0^T X x_0 - x(T; F, x_0)^T X x(T; F, x_0).$$

Therefore, for all $F \in \mathcal{F}$, and for all x_0,

$$J(F, x_0) := \lim_{T \to \infty} J(F, x_0, T) =$$

$$\int_0^\infty \| (F + R^{-1}B^T X) x(t; F, x_0) \|^2 \, dt + x_0^T X x_0 \geq x_0^T X x_0.$$

Hence, $\{J(F, x_0) | F \in \mathcal{F}\}$ is bounded from below for all x_0. So, the infimum exists and in fact, the infimum is a minimum and attained by $F = -B^T R^{-1} X$. $\square$

Proof of lemma 7:
Let $F \in \mathcal{F}$. Then, $W(t;F)$ is the unique solution of the differential equation: $\dot{W}(t;F) = (A+BF)W(t;F) + W(t;F)(A+BF)^T + GG^T$; $W(0) = W_0$ (see e.g. Davis (1977, pp.111)). So, $W(t;F) = e^{(A+BF)t}W_0e^{(A+BF)^Tt} + \int_0^t e^{(A+BF)\tau}GG^Te^{(A+BF)^T\tau}d\tau$. From this it is obvious that $W(t;F) > 0$ for all t and that $W(t;F) \to W(F)$ exponentially fast if $t \to \infty$, where $W(F)$ is the unique positive semi-definite solution to the Lyapunov equation

$$(A+BF)W(F) + W(F)(A+BF)^T + GG^T = 0. \quad (25)$$

Consequently,

$$J_1^{av}(F) = \lim_{T\to\infty} \frac{1}{T} tr\{\int_0^T \{W(t;F)(Q_1 + F^TR_{11}F)\}dt\}. \quad (26)$$

Since $W(t;F)$ converges exponentially fast, it is clear that $J_1^{av}(F)$ exists for all $F \in \mathcal{F}$, which shows that the optimization problem is well-defined.
Finally, note that in (25) $W(F) > 0$ if and only if $(A+BF,G)$ is controllable (see e.g. Kimura (1997, lemma 2.14, 2.15)). $\square$

Proof of theorem 8:
Let X be the symmetric solution of (4) such that $A - SX$ is stable. Then:

$$\begin{aligned}
&tr\{W(t;F)(Q+F^TRF)\} \\
&= tr\{W(t;F)(XSX - XA - A^TX + F^TRF)\} \\
&= tr\{W(t;F)(XSX - X(A+BF) - \\
&\quad (A+BF)^TX + XBF + F^TB^TX + F^TRF)\} \\
&= tr\{W(t;F)(F+R^{-1}B^TX)^TR(F+R^{-1}B^TX)\} \\
&\quad -tr\{((A+BF)W(t;F) + W(t;F)(A+BF)^T)X\} \\
&= tr\{W(t;F)(F+R^{-1}B^TX)^TR(F+R^{-1}B^TX)\} \\
&\quad -tr\{(\dot{W}(t;F) - GG^T)X\}.
\end{aligned}$$

So,

$$\begin{aligned}
&J^{av}(F) = \\
&tr(G^TXG) + \lim_{T\to\infty}\{\frac{1}{T}tr\,(W_0X - W(T;F)X) \\
&+\frac{1}{T}\int_0^T tr\{W^{\frac{1}{2}}(t;F)(F+R^{-1}B^TX)^TR \\
&(F+R^{-1}B^TX)W^{\frac{1}{2}}(t;F)\}dt\}
\end{aligned}$$

Since $W(t;F)$ is invertible and converges to a positive semi-definite solution, and R is positive definite the above integrand is always semi-positive. So,

$$J^{av}(F) \geq tr(G^TXG).$$

Furthermore, we see that by choosing F such that $F + R^{-1}B^TX = 0$, the above inequality becomes an equality. ¿From this we immediately derive the stated conclusion. $\square$

Proof of theorem 10:
Analogously to the proof of lemma 7 we first conclude (using the same notation) that the optimization problem is well defined in the sense that

$$\begin{aligned}
&J^d(F,x_0) = \\
&\lim_{T\to\infty} tr\{\int_0^T e^{-\theta t}\{W(t;F)(Q_1 + F^TRF)\}dt\}, \quad (27)
\end{aligned}$$

exists for all $F \in \mathcal{F}$.
Similarly as in the proof of theorem 8 one can show next that

$$\begin{aligned}
&tr\{W(t;F)(Q+F^TRF)\} = \\
&tr\{W(t;F)(F+R^{-1}B^TX)^TR(F+R^{-1}B^TX)\} \\
&\quad -tr\{\left(\dot{W}(t;F) - \theta W(t;F) - GG^T\right)X\}.
\end{aligned}$$

So,

$$\begin{aligned}
&J^d(F,x_0) = \\
&\lim_{T\to\infty}\{tr(GG^TX) + tr(W(0;F)X - e^{-\theta T}W(T;F)X) \\
&+ \int_0^T e^{-\theta t}tr\{W^{\frac{1}{2}}(t;F)(F+R^{-1}B^TX)^TR \\
&(F+R^{-1}B^TX)W^{\frac{1}{2}}(t;F)\}dt\}
\end{aligned}$$

Since both $W(t;F)$ and R are positive definite the above integrand is always semi-positive. In particular we see that at time $t = 0$ the integrand will be zero if and only if $F + R^{-1}B^TX = 0$. ¿From this we immediately derive the stated conclusion. $\square$

Proof of theorem 14:
Assume that X is a solution of

$$-XSX + A^TX + XA + XGV^{-1}G^TX + Q = 0 \quad (28)$$

such that $A - SX + GV^{-1}G^TX$ is stable.
First we note that due to our assumptions we also immediately have that $A - SX$ is stable (see e.g. Kimura (1997, lemma 9.1)). Then, for every $F \in \mathcal{F}$ we have

$$\begin{aligned}
&-x_0^TXx_0 \\
&= \int_0^\infty \frac{d}{dt}(x^T(t)Xx(t))dt \\
&= \int_0^\infty \{x^T(t)(A^TX + XA + F^TB^TX + XBF)x(t) \\
&\quad +w^T(t)G^TXx(t) + x^T(t)XGw(t)\}dt,
\end{aligned}$$

Using this, we can rewrite the cost functional as follows

$$\int_0^\infty \{x^T(t)(Q+F^TRF)x(t) - w^T(t)Vw(t)\}dt$$

$$\begin{aligned}
&= x_0^TXx_0 + \int_0^\infty \{x^T(t)(A^TX + XA + F^TB^TX + \\
&\quad XBF + Q + F^TRF)x(t) + w^T(t)G^TXx(t) + \\
&\quad x^T(t)XGw(t) - w^T(t)Vw(t)\}dt \\
&= x_0^TXx_0 + \int_0^\infty H(x_0,F,w,t)dt
\end{aligned}$$

where

$$\begin{aligned}
&H(x_0,F,w,t) := \\
&\quad \{x^T(t)(F+R^{-1}B^TX)^TR(F+R^{-1}B^TX)x(t) - \\
&\quad \left(w(t)-V^{-1}G^TXx(t)\right)^T V \left(w(t)-V^{-1}G^TXx(t)\right)
\end{aligned}$$

Now, choose $F = F_1 := -R^{-1}B^TX$ (note that by assumption $F_1 \in \mathcal{F}$). Then, using the above equality, we have

$$\begin{aligned}
&\inf_F \sup_{w(.)} J(F,w,x_0) \\
&\quad \leq \sup_{w(.)} J(F_1,w,x_0) \\
&\quad = x_0^TXx_0 + \sup_{w(.)} \int_0^\infty \{-\left(w(t)-V^{-1}G^TXx(t)\right)^T \\
&\qquad V\left(w(t)-V^{-1}G^TXx(t)\right)\}dt \\
&\quad \leq x_0^TXx_0.
\end{aligned}$$

Next we show that by choosing $w(t) := \bar{w}(t)$ (as outlined below), we have that $\inf_F J(F,\bar{w},x_0) = x_0^TXx_0$, from which the claim is obvious then.
To that end introduce $F_2 := V^{-1}G^TX$, and $v(t) := R^{\frac{1}{2}}(F - F_1)x(t)$. Now, choose $\bar{w}(t) = F_2p(t)$, where $p(t)$ satisfies $\dot{p}(t) = (A + BF_1 + GF_2)p(t)$; $p(0) = x_0$ (Note that by assumption $A + BF_1 + GF_2$ is stable and therefore $\bar{w} \in L_2^q[0,\infty)$). Let $x(t;\bar{w})$ be the corresponding solution of the system equation. Then, with $\tilde{x}(t) := p(t) - x(t;\bar{w})$ we have that

$$\dot{\tilde{x}}(t) = (A+BF_1)\tilde{x}(t) - BR^{-\frac{1}{2}}v(t); \tilde{x}(0) = 0, \quad (29)$$

and

$$\begin{aligned}
&H(x_0,F,\bar{w},t) \\
&\quad = x(t;\bar{w})^T(F-F_1)^TR(F-F_1)x(t;\bar{w}) - \\
&\qquad (\bar{w}(t) - F_2x(t;\bar{w}))^T V (\bar{w}(t) - F_2x(t;\bar{w})) \\
&\quad = -\tilde{x}^T(t)F_2^TVF_2\tilde{x}(t) + v^T(t)v(t) \\
&\quad \geq -\tilde{x}^T(t)(Q + F_2^TVF_2)\tilde{x}(t) + v^T(t)v(t).
\end{aligned}$$

The last inequality holds since by assumption $Q \geq 0$. Consequently,

$$\begin{aligned}
&\inf_F \sup_w \int_0^\infty H(x_0,F,w,t)dt \geq \\
&\qquad \inf_F \int_0^\infty H(x_0,F,\bar{w},t)dt \geq
\end{aligned}$$

$$\inf_{v \in L_2^q(0,\infty)} \int_0^\infty -\tilde{x}^T(t)F_2^TVF_2\tilde{x}(t) + v^T(t)v(t)dt \quad (30)$$

Now, using (28) and the assumption that its stabilizing solution X is positive semi-definite, we can rewrite the integral in (30) as

$$\begin{aligned}
&\int_0^\infty \{\frac{d}{dt}(\tilde{x}^T(t)X\tilde{x}(t)) + \\
&(v(t) + R^{-\frac{1}{2}}B^TX\tilde{x}(t))^T(v(t) + R^{-\frac{1}{2}}B^TX\tilde{x}(t)) + \\
&\tilde{x}^T(t)Q\tilde{x}(t)\}dt \\
= &\lim_{T\to\infty} \{\tilde{x}^T(T)X\tilde{x}(T) + \\
&\int_0^T \{(v(t) + R^{-\frac{1}{2}}B^TX\tilde{x}(t))^T(v(t) + R^{-\frac{1}{2}}B^TX\tilde{x}(t)) + \\
&\tilde{x}^T(t)Q\tilde{x}(t)\}dt\}.
\end{aligned}$$

Due to our assumptions that $Q \geq 0$ and $X \geq 0$ it is clear from above that (30) is positive. So,

$$\inf_F \sup_{w(.)} J(F,w,x_0) \geq x_0^TXx_0,$$

from which the claim is obvious now. $\square$.

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