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Abstract

In this paper we analyze the set of scalar algebraic Riccati equations (ARE) that play an important role in finding feedback Nash equilibria of the scalar N-player linear-quadratic differential game. We show that in general there exist at most $2^N - 1$ solutions of the (ARE) that give rise to a Nash equilibrium. In particular we analyze the number of equilibria as a function of the autonomous growth parameter and present both necessary and sufficient conditions for the existence of a unique solution of the ARE.

Keywords: Differential games, Linear-quadratic control, Feedback Nash equilibrium, Algebraic Riccati equations

1 Introduction

During the last decade there has been an increasing interest in studying several problems in economics using a dynamic game theoretical setting. In particular in the area of environmental economics and macro-economic policy coordination this is a very natural framework for modeling problems (see e.g. Engwerda *et al.* [2] for references). In, e.g., policy coordination problems usually two basic questions arise: first, are policies coordinated and, second, which information do the participating parties have. Usually both these points are rather unclear and, therefore, strategies for different possible scenarios are calculated and compared with each other. One of these scenarios is the so-called feedback Nash scenario (see Başar and Olsder [1] for a precise definition and survey of the relevant literature).

Since according to this scenario the participating parties can react to each other's policies, its relevance is in economics usually larger than that of the open-loop Nash scenario. In particular the feedback Nash scenario is very popular in studying problems where the underlying model can be described by a (set of) linear differential equation(s) and the individual objectives pursued by the parties can be approximated by functions which quadratically penalize deviations from some (equilibrium) targets. Under the assumption that the parties have a finite planning horizon, this problem was first analyzed by Starr and Ho in [6] (see also Lukes [4] for a result on uniqueness within the class of affine memoryless strategies).

In this paper we study the infinite planning horizon case and concentrate on solving the algebraic Riccati equations associated with this problem. In

Weeren *et al.* [7] it was shown that in the two-player scalar case the number of solutions to these equations can vary between one and three (see also Engwerda [3] for a detailed study under which conditions on the system parameters these different situations occur). In this paper we study the general N-player scalar case. We show that for any number N of players there exists a positive number such that if the autonomous growth parameter is larger than this number, there exist (in general) $2^N - 1$ solutions for the (ARE) equations yielding a Nash equilibrium. Furthermore, we give both necessary and sufficient conditions under which there is exactly one solution for the (ARE) equations.

The outline of the paper is as follows. In section two we start by stating the problem analyzed in this paper. Section three analyzes the solutions of the algebraic Riccati equations. These results are used in section four to find necessary and sufficient conditions for the existence of a unique solution. The paper ends with some concluding remarks.

2 Problem statement

In this paper we consider the problem in which N parties (henceforth called players) aim at minimization of their individual quadratic performance criteria. Each player controls a different set of inputs to a single system. The system is described by the following differential equation

$$\dot{x} = ax + \sum_{i=1}^{N} b_i u_i,\ x(0) = x_0. \qquad (1)$$

Here x is the state of the system, u_i is a (control) variable player i can manipulate, x_0 is the arbitrarily chosen initial state of the system, a (the "autonomous growth" parameter) and b_i, $i \in \mathbf{N} := \{1, .., N\}$, are constant system parameters, and $\dot{x}$ denotes the time derivative of x.

The performance criterion player $i \in \mathbf{N}$ aims to minimize is:

$$J_i(u_1, \cdots, u_N) := \frac{1}{2} \int_0^\infty \{q_i x^2(t) + r_i u_i^2(t)\} dt.$$

We assume that both q_i and r_i are positive and b_i differs from zero.

We consider the existence of limiting stationary feedback Nash equilibria of this differential game.

To that end we study the following set of coupled algebraic Riccati equations (ARE):

$$2(a - \sum_{j=1}^{N} k_j s_j) k_i + q_i + s_i k_i^2 = 0, \ i \in \mathbf{N} \tag{2}$$

where $s_i := r_i^{-1} b_i^2$.

Since both q_i and r_i are positive it is obvious that $J_i > 0$, whenever $x_0 \neq 0$. Therefore, we immediately deduce from Başar and Olsder [1, proposition 6.8] that:

Theorem 1: Let $\bar{k}_i > 0$ solve the set of Riccati equations (2).

Then the stationary feedback policies

$$u_i = -r_i^{-1} b_i \bar{k}_i x, \ i \in \mathbf{N}, \tag{3}$$

provide a Nash equilibrium, yielding the cost $J_i(u_1, \cdots, u_N) := \bar{k}_i x_0^2$, for player i. Moreover, the resulting system dynamics described by $\dot{x} = a_{cl} x$, with $a_{cl} := a - \sum_{i=1}^{N} s_i \bar{k}_i$, is asymptotically stable. $\square$

In fact we conclude from Weeren *et al.* [7, corollary 3.1] that, when the players are restricted at the outset to memoryless strategies (cf. Lukes [4]), existence of a positive solution to the above scalar Riccati equations is both a necessary and a sufficient condition for existence of a feedback Nash equilibrium.

A natural question which arises is: how many solutions does the above set of algebraic Riccati equations (ARE) have. To analyze this question we introduce (for notational convenience) the following variables:

$$\sigma_i := s_i q_i; \kappa_i := s_i k_i,\ i \in \mathbf{N};\ \text{and}\ \kappa_{N+1} := -a_{cl}.$$

Using this notation, (2) can be rewritten as

$$\kappa_i^2 - 2\kappa_{N+1}\kappa_i + \sigma_i = 0,\ i \in \mathbf{N}, \tag{4}$$

where

$$\kappa_{N+1} = -a + \sum_{j=1}^{N} \kappa_j \tag{5}$$

So our problem can be reformulated as follows.

Problem statement 2: Assume $\sigma_i > 0$. Find conditions under which the N quadratic equations (4) under the equality constraint (5) have a positive solution $\kappa_i,\ i \in \mathbf{N+1}$. $\square$

In the next section we will study this problem in detail.

3 The solution set

From Bézout's theorem (see e.g. Shafarevich [5]) we know that the number of intersection points of a set of N quadratic polynomial equations will not exceed the product of the degrees of the equations (if things are appropriately defined). Consequently, our equations will have at most 2^N real solutions. We will show in theorems 6 and 9 that the number of positive solutions may range from 1 up to $2^N - 1$. We will see that this implies that there also exists always at least one negative solution to (ARE). If $N = 2$, $a = -\frac{1}{2}$, and $\sigma_1 = \sigma_2 = \frac{1}{2}$, easy calculations show that (ARE) has two real solutions. This shows that in general the number of real solutions of (ARE) can be strictly smaller than $2^N - 1$.

To simplify the analysis below we will assume, without loss of generality, that the σ_i's satisfy $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq \sigma_N$. Lemma 3 reformulates our problem statement 2 in terms of just one unknown scalar variable κ_{N+1}.

Lemma 3: Problem 2 has a solution if and only if there exist $t_i \in \{-1, 1\}$, $i \in \mathbf{N}$, such that the equation

$$(N-1)\kappa_{N+1} + t_1\sqrt{\kappa_{N+1}^2 - \sigma_1} + \cdots + t_N\sqrt{\kappa_{N+1}^2 - \sigma_N} = a. \tag{6}$$

has a solution $\kappa_{N+1} \geq \sqrt{\sigma_1}$.

Proof: " $\Rightarrow$ " Consider (4). Obviously, $\sqrt{\sigma_1} \leq \kappa_{N+1}$ must hold. Furthermore, we conclude that (4) has two positive solutions: $\kappa_i = \kappa_{N+1} + \sqrt{\kappa_{N+1}^2 - \sigma_i}$ and $\kappa_i = \kappa_{N+1} - \sqrt{\kappa_{N+1}^2 - \sigma_i}$, $i \in \mathbf{N}$. Substitution into (5) proves the claim.

" $\Leftarrow$ " Let $\kappa_{N+1} \geq \sqrt{\sigma_1}$ solve (6). Define κ_i by $\kappa_{N+1} + t_i\sqrt{\kappa_{N+1}^2 - \sigma_i}$, where t_i is as in (6). Then it is straightforward to verify that $\kappa_1 > 0, \cdots, \kappa_N > 0$ satisfy (4,5). $\square$.

To study the number of solutions $\kappa_{N+1} > \sqrt{\sigma_1}$ of (6), we define recursively the following functions for $n \in \mathbf{N-1}$:

$$f_i^{n+1}(x) := f_i^n(x) + x - \sqrt{x^2 - \sigma_{n+1}},\ i \in \mathbf{2^n} \tag{7}$$

$$f_{i+2^n}^{n+1}(x) := f_i^n(x) + x + \sqrt{x^2 - \sigma_{n+1}},\ i \in \mathbf{2^n} \tag{8}$$

with

$$f_1^1(x) := -\sqrt{x^2 - \sigma_1} \text{ and } f_2^1(x) := \sqrt{x^2 - \sigma_1}. \tag{9}$$

As a result of this construction, the functions f_i^N satisfy the monotonicity property

$$f_i^N(\sqrt{\sigma_1}) \leq f_{i+1}^N(\sqrt{\sigma_1}),\ i \in \mathbf{2^N} - \mathbf{1}. \tag{10}$$

In particular, the following three functions will play an important role in the subsequent analysis:

$$f_1^N(x) = (N-1)x - \sum_{i=1}^{N} \sqrt{x^2 - \sigma_i} \tag{11}$$

$$f_2^N(x) = (N-1)x + \sqrt{x^2 - \sigma_1} - \sum_{i=2}^{N} \sqrt{x^2 - \sigma_i} \tag{12}$$

and

$$f_3^N(x) = (N-1)x - \sqrt{x^2 - \sigma_1} + \sqrt{x^2 - \sigma_2} - \sum_{i=3}^{N} \sqrt{x^2 - \sigma_i}. \tag{13}$$

Now, each function f_i^N, $i \in \mathbf{2^N} - \mathbf{1}$, corresponds to a function obtained from the left hand side of (6) by making a specific choice of $t_j, j \in \mathbf{N}$, and substituting x for κ_{N+1}.

From lemma 3 it is obvious then that problem 2 has a solution if and only if $f_i^N = a$ has a solution $x \geq \sqrt{\sigma_1}$ for some $i \in \mathbf{2^N}$. Consequently, the number of positive solutions of (ARE) coincides with the number of solutions $x \geq \sqrt{\sigma_1}$ of the equation

$$\Pi_{i=1}^{2^N}(f_i^N - a) = 0. \tag{14}$$

We will denote the function on the left hand side of this equation, $\Pi_{i=1}^{2^N}(f_i^N - a)$, by $f(x)$ and show below that it is a polynomial of degree 2^N. To that end, we concentrate for the moment on the 2-player case. Introducing $a_0 := x - a$ and $a_i := \sqrt{x^2 - \sigma_i}$, $i = 1, 2$, it is easily verified that $f(x)$ has the following algebraic structure

$$f(a_0, a_1, a_2) := (a_0 - a_1 - a_2)(a_0 + a_1 - a_2)(a_0 - a_1 + a_2)(a_0 + a_1 + a_2). \tag{15}$$

The structure of f for the general N-player case is similar and is omitted in order to avoid unnecessary cumbersome notation. Using this structure we show next that all entries a_i in f appear quadratically.

Lemma 4: $f(a_0, \cdots, a_N)$ is a sum of terms, in which each term can be written as $\Pi_{i=0}^{N} a_i^{2k_i}$ for some nonnegative integers k_i satisfying $\sum_{i=0}^{N} 2k_i = 2^N$.

Proof: It is easily verified that $f(-a_0, a_1, \cdots, a_N) = (-1)^{2^N} f(a_0, \cdots, a_N) = f(a_0, \cdots, a_N)$ and, also, $f(a_0, \cdots, -a_i, \cdots, a_N) = f(a_0, \cdots, a_i, \cdots, a_N)$, for any $i \in \mathbf{N}$.

Now, assume that f has a term in which, e.g., a_0 has an odd exponent. Then collect all terms of f containing odd exponents in a_0. As a consequence $f = a_0 g(a_0, \cdots, a_N) + h(a_0, \cdots, a_N)$, where a_0 appears with an even exponent in all terms of both g and h. Since $f(-a_0, a_1, \cdots, a_N) = f(a_0, a_1, \cdots, a_N)$ we conclude immediately from this that g must be zero. The rest of the proof follows in a straightforward manner. $\square$

Corollary 5: $f(x)$ is a polynomial of degree 2^N.

Proof: Let $a_0 := (N-1)x - a$ and $a_i := \sqrt{x^2 - \sigma_i}$. With this notation, $f(x)$ coincides with (15). The result follows directly from lemma 4. $\square$

Using this corollary we can easily derive the following result on the number of solutions to the (ARE) equations

Theorem 6: (ARE) has at least one and at most $2^N - 1$ positive solutions.

Proof: Using the notation of corollary 5, we show that the polynomial $f(x)$ has at most $2^N - 1$ roots larger than $\sqrt{\sigma_1}$. To this end we rewrite f as

$$f = \Pi_{i=1}^{2^{N-2}}(a_0 - (a_1 + g_i))(a_0 + (a_1 + g_i))(a_0 - (a_1 - g_i))(a_0 + (a_1 - g_i)), \quad (16)$$

where g_i is a linear combination (with coefficients $+1$ or -1) of $a_2, \cdots, a_N$. From (16) we immediately have that $f = \Pi_{i=1}^{2^{N-2}}(a_0^2 - (a_1+g_i)^2)(a_0^2 - (a_1-g_i)^2)$. Now at $x = \sqrt{\sigma_1}$, $a_1 = 0$. Therefore we conclude that at $x = \sqrt{\sigma_1}$, $f = \Pi_{i=1}^{2^{N-2}}(a_0^2 - (g_i)^2)^2 > 0$. Furthermore, it is easily verified that except for the term $a_0 - \sum_{i=1}^N a_i$, all terms $a_0 \pm a_1 \pm g_i$ in (16) are positive if $x \to \infty$. Therefore, the leading term x^{2^N} of the polynomial has a negative sign. So, we conclude that the polynomial always has a root located at the left hand side of $\sqrt{\sigma_1}$; or, stated differently, (ARE) has at most $2^N - 1$ positive solutions. To see that (ARE) always has at least one positive solution, we study the equations $f_1^N(x) = a$ and $f_2^N(x) = a$ (see (11,12)). Obviously, $f_1^N(\sqrt{\sigma_1}) = f_2^N(\sqrt{\sigma_1})$. Since both functions are continuous with $\lim_{x\to\infty} f_1^N(x) = -\infty$ and $\lim_{x\to\infty} f_2^N(x) = \infty$, it is clear that either the equation $f_1^N(x) = a$ or $f_2^N(x) = a$ has a solution $x \geq \sqrt{\sigma_1}$, which completes the proof. $\square$

Remark 7: By substituting $\kappa_i = -\tau_i$ into (4,5) it is readily verified that (ARE) has a negative solution if and only if the set of equations

$$\tau_i^2 - 2\tau_{N+1}\tau_i + \sigma_i = 0,\ i \in \mathbf{N},\ \tau_{N+1} = a + \sum_{j=1}^{N} \tau_j$$

has a positive solution. So, from the previous theorem we immediately conclude that (ARE) will always have at least one negative solution. $\square$

Next, we analyze how the number of solutions of (ARE) varies with the autonomous growth parameter a. To get an impression of this relationship, we show for the three player case the curves f_i^3 for two different parameter choices in figure 1.

From the first plot we see, by counting the number of points of the different curves f_i^3 which have level a, that the number of solutions of (ARE) increases monotonically from 1 to 7 as a function of a. That this monotonicity does not always hold is illustrated by the second plot, where we illustrated for different parameter values f_2^3 and f_3^3. Since f_1^3 is a monotonically decreasing function and $f_i^3(x) \geq f_3^3(x)$ for $i > 3$ (as we will show later on (see lemma 11)), we see from this second plot that the number of solutions first increases from 1 to 3 and then drops back to 1 before it increases again. In particular

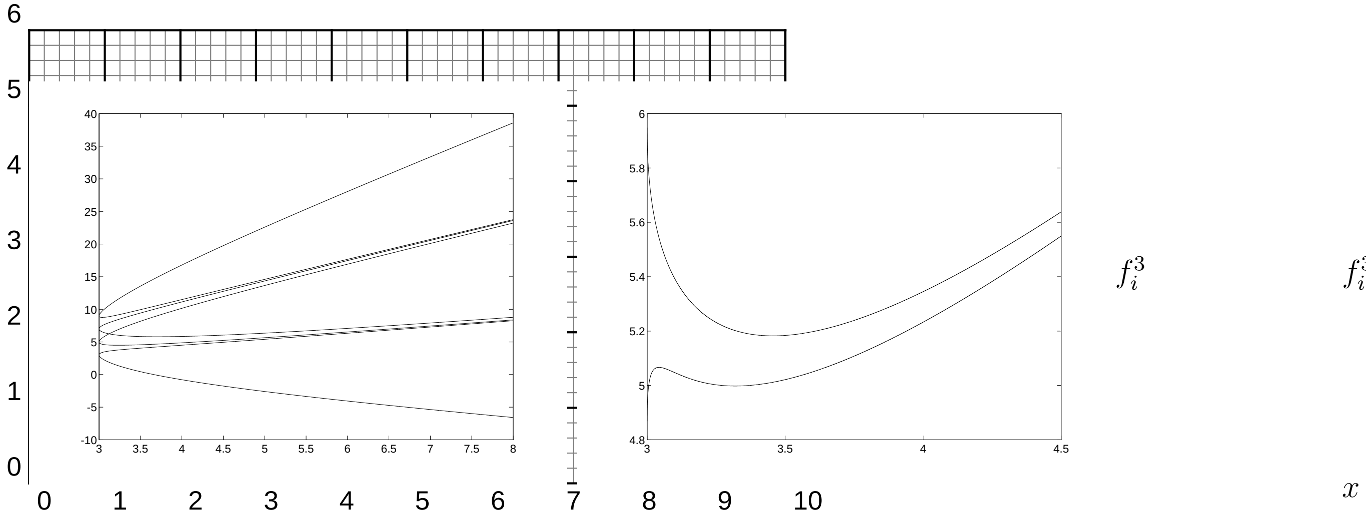

Figure 1: The curves $f_i^3,\ i = 1, \cdots, 8$ for $\sigma_1 = 9;\ \sigma_2 = 8;\ \sigma_3 = 5$, with $f_i^3(3.01) < f_{i+1}^3(3.01)\ i = 1, \cdots, 7$ and f_2^3 (lower graph), f_3^3 (upper graph) for $\sigma_1 = 9;\ \sigma_2 = 8.7;\ \sigma_3 = 8.65$.

note from these examples that an even number of solutions occurs only for isolated values of a, whereas an odd number of solutions occurs for values of a in certain ranges. We will not elaborate this subject further here, but it seems that this property holds in general.

Next, we show that the graphs of the functions $f_i^N(x)$ do not intersect if x becomes large. To prove this property we first concentrate on the case that all σ_i differ. So, we assume from now on that $\sigma_1 > \sigma_2 > \cdots > \sigma_N$.

The next lemma is a preliminary result and will be used in the proof of theorem 9.

Lemma 8: Assume that all σ_i differ. Then, there exists a constant x_1 such that the functions $f_i^N(x),\ i = 2, \cdots, 2^N$ do not intersect on the interval (x_1, ∞).

Proof: We show that any two functions f_i^N and f_j^N only have a finite number of intersection points, from which the conclusion is obvious.

So, assume $f_i^N(x) = f_j^N(x)$. Since all σ_i's differ, the equation $f_i^N - f_j^N = 0$ can be rewritten as

$$2 * (t_1\sqrt{x^2 - \sigma_1} + \cdots + t_N\sqrt{x^2 - \sigma_N}) = 0, \tag{17}$$

where $t_i \in \{-1, 0, 1\}$ and not all t_i are simultaneously zero. Now, denote $t_i\sqrt{x^2 - \sigma_i}$ by $a_i(x)$. Then the question whether (17) has a finite number of zeros can be rephrased as whether $\sum_{i=1}^{N} a_i(x) = 0$ has a finite number of zeros. We will prove this property for $N = 3$. The general case can be proved similarly.

So, we have to prove that $a_1 + a_2 + a_3 = 0$ has only a finite number of zeros. As in (15) we consider the following function

$$f(a_1, a_2, a_3) := (a_1 - a_2 - a_3)(a_1 + a_2 - a_3)(a_1 - a_2 + a_3)(a_1 + a_2 + a_3).$$

Obviously, $a_1 + a_2 + a_3 = 0$ has a finite number of zeros, if f has a finite number of zeros. However, using lemma 4, it is easily seen that f is a polynomial whose degree is at most 8. So, f has at most 8 zeros, which proves the claim. $\square$

The next theorem states, roughly speaking, that if the uncontrolled system is very unstable then there will be $2^N - 1$ equilibria.

Theorem 9: Assume that the σ_i differ. Then, there exists a positive number $\hat{a}$ such that for every autonomous growth parameter $a \geq \hat{a}$ the set of algebraic Riccati equations (2) has $2^N - 1$ positive solutions.

Proof: By differentiating $f_2^N(x)$ it is easily verified that $f_2^N(x)$ is monotonically increasing for all $x \geq x_1^*$ for some number $x_1^* > \sqrt{\sigma_1}$. Furthermore, since $\lim_{x\to\infty} f_2^N(x) = \infty$ and $f_2^N(x)$ is bounded from above on the interval $(\sqrt{\sigma_1}, x_1^*)$, it follows that there exists a positive number a_1^{**} such that for all $a \geq a_1^{**}$ the equation $f_2^N(x) = a$ has exactly one solution. A similar reasoning holds for all other $f_i^N(x)$, $i \in \mathbf{2^N}$ (see also figure 1 for a rendering in case $N = 3$). Next, take the maximum over all a_i^{**}. According to lemma 8, for a fixed a the solutions for $f_i^N(x) = a$ differ for all i if a is chosen sufficiently large. Therefore it is easily verified that the corresponding solutions $(\kappa_1, \cdots, \kappa_N)$ to (4,5) will also differ. $\square$

Remark 10:

In case the σ_i do not differ, it is easily verified from the above analysis that a similar conclusion holds. That is, there exists a number $\hat{a}$ such that for all $a > \hat{a}$ the number of solutions to (ARE) remains constant. This constant equals the number of distinct (ultimately) monotonically increasing functions f_i^N. Without providing a formal proof we note that, if one denotes by s the number of σ_i's that coincide, careful counting shows that the number of solutions is

$$2^N - 2^{N-s} \sum_{i=1}^{s-1} \begin{pmatrix} s \\ s-i \end{pmatrix} + (s-1)2^{N-s} - 1,$$

for $N > s$. Here the term $2^N - 2^{N-s} \sum_{i=1}^{s-1} \begin{pmatrix} s \\ s-i \end{pmatrix}$ counts the number of solutions that do not coincide with any other solution; $(s-1)2^{N-s}$ counts the number of solutions that occur with multiplicity > 1 and -1 comes from the number of monotonically decreasing functions. Furthermore, it is easily verified that, if $N = s$, the number of solutions equals $\lfloor \frac{N}{2} \rfloor + 1$. Here $\lfloor \frac{N}{2} \rfloor$ denotes the largest integer smaller than $\frac{N}{2}$ (e.g. $\lfloor \frac{3}{2} \rfloor = 1$). So, e.g. if $N = 5$ and $\sigma_1 = \sigma_2 = \sigma_3 > \sigma_4 > \sigma_5$, $s = 3$ and the maximum number of solutions will be 15. $\square$

4 Uniqueness conditions

In this section we will give in theorem 13 both necessary and sufficient conditions under which (ARE) has a unique positive solution. To solve this problem, we study the functions f_i^N as defined in (7,8) in some more detail. First we note that the functions f_1^N, f_2^N and f_3^N satisfy a monotonicity property.

Lemma 11: For every $N \geq 2$ the following inequalities hold: $f_1^N \leq f_2^N \leq f_3^N \leq f_i^N$ for any $i \geq 4$.

Proof: The proof is by induction.

For $N = 2$, $f_1^2(x) = x - \sqrt{x^2 - \sigma_1} - \sqrt{x^2 - \sigma_2}$, $f_2^2(x) = x + \sqrt{x^2 - \sigma_1} - \sqrt{x^2 - \sigma_2}$, $f_3^2(x) = x - \sqrt{x^2 - \sigma_1} + \sqrt{x^2 - \sigma_2}$ and $f_4^2(x) = x + \sqrt{x^2 - \sigma_1} + \sqrt{x^2 - \sigma_2}$. Since by assumption $\sigma_1 \geq \sigma_2$, the correctness of all inequalities follows by straightforward verification.

Now, assume the inequalities hold for $N = k$. Then, by definition, for $i = 1, 2, 3$ we have $f_i^{k+1}(x) = f_i^k(x) + x - \sqrt{x^2 - \sigma_{k+1}} \leq f_{i+1}^k(x) + x - \sqrt{x^2 - \sigma_{k+1}} = f_{i+1}^{k+1}(x)$. In a similar way we have for $i = 5, \cdots, 2^k$ that $f_i^{k+1}(x) = f_i^k(x) + x - \sqrt{x^2 - \sigma_{k+1}} \geq f_4^k(x) + x - \sqrt{x^2 - \sigma_{k+1}} = f_4^{k+1}(x)$, and for $i = 2^k + 1, \cdots, 2^{k+1}$ $f_{i+2^k}^{k+1}(x) = f_i^k(x) + x + \sqrt{x^2 - \sigma_{k+1}} \leq f_4^k(x) + x - \sqrt{x^2 - \sigma_{k+1}} = f_4^{k+1}(x)$. $\square$

Next, we introduce a convention w.r.t. local versus global extrema. By a local extremum we mean an extremum which occurs somewhere on the open interval $(\sqrt{\sigma_1}, \infty)$; whereas for the definition of a global extremum we take the whole domain of definition $[\sqrt{\sigma_1}, \infty)$.

The following technical results will be used in the proof of theorem 13.

Lemma 12:

i) For all $i = 2, \cdots, 2^N$, there exists an x_i such that $f_i^N(x)$ is strictly monotonically increasing for all $x \geq x_i$. $f_1^N(x)$ is strictly monotonically decreasing.

ii) If $\sigma_1 > \sigma_2$, $f_3^N(x)$ has exactly one local minimum.

iii) $f_2^N(x)$ has at most two local extrema.

iv) If $f_2^N(x)$ assumes a local minimum at x_0, then $x_0 \leq \arg\min f_3^N(x)$.

Proof: i) This is verified by straightforward differentiation of $f_i^N(x)$.

ii) The first derivative of $f_3^N(x)$ (see (13)) is $N - 1 - \sum_{i \neq 2}^N \frac{x}{\sqrt{x^2-\sigma_i}} + \frac{x}{\sqrt{x^2-\sigma_2}}$. So, if $\sigma_1 > \sigma_2$, $\lim_{x \downarrow \sqrt{\sigma_1}} f_3^{N'}(x) = -\infty$ and $\lim_{x \to \infty} f_3^{N'}(x) = 1$. Furthermore, the second derivative $f_3^{N"}(x) = \sum_{i \neq 2}^N \frac{\sigma_i}{(x^2-\sigma_i)^{3/2}} - \frac{\sigma_2}{(x^2-\sigma_2)^{3/2}}$. Since $\sigma_1 \geq \sigma_2$ it is clear that $f_3^{N"}(x) > 0$. So, $f_3^{N'}(x)$ has exactly one zero, from which the conclusion is obvious.

iii) Differentiation of $f_2^N(x)$ (see (12)) yields $f_2^{N'}(x) = N-1-\sum_{i\neq 1}^{N} \frac{x}{\sqrt{x^2-\sigma_i}} + \frac{x}{\sqrt{x^2-\sigma_1}}$ and $f_2^{N"}(x) = \sum_{i\neq 1}^{N} \frac{\sigma_i}{(x^2-\sigma_i)^{3/2}} - \frac{\sigma_1}{(x^2-\sigma_1)^{3/2}}$. Now, assume $f_2^{N"}(x)$ has a zero at p. Some rewriting of $f_2^{N"}(p) = 0$ shows then that $\sigma_1 = (p^2 - \sigma_1)^{3/2} \sum_{i\neq 1}^{N} \frac{\sigma_i}{(p^2-\sigma_i)^{3/2}}$. Substitution of this expression into $f_2^{N"}(x)$ yields:

$$
\begin{aligned}
f_2^{N"}(x) &= \sum_{i\neq 1}^{N} \frac{\sigma_i}{(x^2-\sigma_i)^{3/2}} - \frac{(p^2-\sigma_1)^{3/2}}{(x^2-\sigma_1)^{3/2}} \sum_{i\neq 1}^{N} \frac{\sigma_i}{(p^2-\sigma_i)^{3/2}} \\
&= \sum_{i\neq 1}^{N} \left(\frac{\sigma_i}{(x^2-\sigma_i)^{3/2}} - \frac{(p^2-\sigma_1)^{3/2}}{(x^2-\sigma_1)^{3/2}} \frac{\sigma_i}{(p^2-\sigma_i)^{3/2}}\right) \\
&= \sum_{i\neq 1}^{N} \sigma_i \left(\frac{(x^2-\sigma_1)^{3/2}(p^2-\sigma_i)^{3/2} - (x^2-\sigma_i)^{3/2}(p^2-\sigma_1)^{3/2}}{(x^2-\sigma_1)^{3/2}(p^2-\sigma_i)^{3/2}(x^2-\sigma_i)^{3/2}}\right).
\end{aligned}
$$

Now, $\sqrt{(x^2-\sigma_1)(p^2-\sigma_i)} - \sqrt{(x^2-\sigma_i)(p^2-\sigma_1)} = \frac{(x^2-\sigma_1)(p^2-\sigma_i)-(x^2-\sigma_i)(p^2-\sigma_1)}{\sqrt{(x^2-\sigma_1)(p^2-\sigma_i)}+\sqrt{(x^2-\sigma_i)(p^2-\sigma_1)}} = \frac{(\sigma_1-\sigma_i)(x^2-p^2)}{\sqrt{(x^2-\sigma_1)(p^2-\sigma_i)}+\sqrt{(x^2-\sigma_i)(p^2-\sigma_1)}} > 0$, if and only if $x > p$. From this it follows easily that $f_2^{N"}(x)$ has only one root and that $f_2^{N'}(x)$ has a local minimum at p. The stated result follows directly.

iv) Assume $f_3^N(x)$ has a local minimum at p, so $f_3^{N'}(p) = 0$. From this we have that $N-1 = \sum_{i\neq 2}^{N} \frac{p}{\sqrt{p^2-\sigma_i}} - \frac{p}{\sqrt{p^2-\sigma_2}}$. Substitution of this expression into $f_2^{N'}(p+\delta)$ yields for positive δ

$$
\begin{aligned}
f_2^{N'}(p+\delta) &= N-1-\sum_{i\neq 1}^{N} \frac{p+\delta}{\sqrt{(p+\delta)^2-\sigma_i}} + \frac{p+\delta}{\sqrt{(p+\delta)^2-\sigma_1}} \\
&= \sum_{i\neq 2}^{N} \frac{p}{\sqrt{p^2-\sigma_i}} - \frac{p}{\sqrt{p^2-\sigma_2}} - \sum_{i\neq 1}^{N} \frac{p+\delta}{\sqrt{(p+\delta)^2-\sigma_i}} + \frac{p+\delta}{\sqrt{(p+\delta)^2-\sigma_1}} \\
&= \frac{p+\delta}{\sqrt{(p+\delta)^2-\sigma_1}} - \frac{p+\delta}{\sqrt{(p+\delta)^2-\sigma_2}} + \frac{p}{\sqrt{p^2-\sigma_1}} - \frac{p}{\sqrt{p^2-\sigma_2}} + \\
&\quad \sum_{i\neq 3}^{N} \left(\frac{p}{\sqrt{p^2-\sigma_i}} - \frac{p+\delta}{\sqrt{(p+\delta)^2-\sigma_i}}\right) \\
&> 0,
\end{aligned}
$$

where the last inequality follows from the facts that $\sigma_1 \geq \sigma_2$ and, according to the mean value theorem, $\frac{p}{\sqrt{p^2-\sigma_i}} - \frac{p+\delta}{\sqrt{(p+\delta)^2-\sigma_i}} = \delta\frac{\sigma_i}{(\xi^2-\sigma_i)^{3/2}}$, for some $p < \xi < p + \delta$.

So, the derivative of $f_2^{N'}(x)$ is always positive at the right hand side of the local minimum of $f_2^{N'}(x)$, which proves the claim. $\square$

Theorem 13: Assume that $\sigma_1 > \sigma_2$. Then, (ARE) has exactly one positive solution if and only if either one of the following conditions is satisfied:

i) f_2^N is monotonically increasing and $a < \min f_3^N$;

ii) f_2^N is not monotonically increasing and a satisfies either I. $a <$ local minimum $f_2^N(x)$ or II. local maximum $f_2^N(x) < a < \min f_3^N(x)$.

Proof:

First consider the case that $f_2^N(x)$ is monotonically increasing. From the facts that $f_1^N(x)$ is strictly monotonically decreasing (lemma 12.i), $f_1^N(\sqrt{\sigma_1}) = f_2^N(\sqrt{\sigma_1})$ and $f_i^N(x) \geq f_3^N(x), i = 4, \cdots, 2^N$ (lemma 11) it is obvious that for a fixed a there will be only one intersection point with the functions $f_i^N(x)$ if and only if a is smaller than the global minimum of $f_3^N(x)$ (see also figure 1).

Next, consider the case that $f_2^N(x)$ is not monotonically increasing. According to lemma 12.iii, $f_2^N(x)$ has then a local maximum and a local minimum. Furthermore (lemma 12.ii and iv), this local minimum is located at the left hand side of the local minimum of $f_3^N(x)$ (see the second plot of figure 1 for an illustration of this situation). Since $f_3^N(x) \geq f_2^N(x)$ it is clear that for all a smaller than the local minimum of $f_2^N(x)$, there is only one intersection point with the different f_i^N. Obviously, when a is located between the local minimum and the local maximum of $f_2^N(x)$ there are three solutions. In case the local minimum of $f_3^N(x)$ is larger than the local maximum value of $f_2^N(x)$, the number of solutions drops, again, to 1. If a is larger than this local minimum of $f_3^N(x)$, there will always be at least one intersection point with $f_2^N(x)$ and one with $f_3^N(x)$, which concludes the proof. $\square$

Remark 14: In case $\sigma_1 = \sigma_2$, $f_2^N(x)$ and $f_3^N(x)$ coincide. Moreover, at $\sqrt{\sigma_1}$, $f_i^N(x), i = 1, \cdots, 4$ coincide. From this it is easily seen that there will be exactly one intersection point of a with all these functions if and only if a is smaller than the global minimum of $f_2^N(x)$. In fact this inequality has to be strict in case $f_2^N(x)$ has a local minimum, which is then also the global one. $\square$

In figure 2 below we illustrate, for fixed σ_i, the two possibilities that can occur for the set of parameters a for which there is a unique equilibrium.

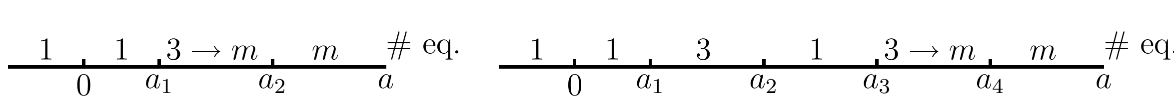

Figure 2: Structure of sets where (ARE) has a unique positive solution.

Here $m \leq 2^N - 1$ denotes the maximum number of solutions.

We conclude this section with three related issues.

First we mention that under the condition that $\sigma_1 \geq \sigma_2 + \cdots + \sigma_N$ the set of a-parameters for which (ARE) has a unique positive solution is given by a halfline.

This result follows directly from the following lemma which is proved in the appendix

Lemma 15: If $\sigma_1 \geq \sigma_2 + \cdots + \sigma_N$ then $f_2^N(x)$ is monotonically increasing. $\square$

A second related issue is that for all $a < \sqrt{\sigma_1} - \sqrt{\sigma_1 - \sigma_2}$ there will always be a unique positive solution too. To show this, first note from theorem 13 that, whenever $a <$ minimum f_2^N, (ARE) has a unique positive solution. It is easily verified that f_2^2 is monotonically increasing and therefore its minimum is given by $f_2^2(\sqrt{\sigma_1}) = \sqrt{\sigma_1} - \sqrt{\sigma_1 - \sigma_2}$. Since $f_2^N(x) \leq f_2^{N+1}(x)$, the rest of the argument follows by induction.

Finally, the third issue we like to address is the following. In Engwerda [2000] it was shown, for the two-player case, that the additional requirement that amongst all (ARE) solutions we look for a solution that minimizes aggregate performance always gives rise to a unique solution. This property does not hold for the general case, as we can see from the first plot of figure 1. In this figure we see that the curves f_4^3 and f_5^3 intersect at some point (κ_4^*, a^*) (approximately (3.2,6.5)). From (5) we therefore conclude that at this point for both solutions we have $\kappa_1+\kappa_2+\kappa_3 = \kappa_4^*+a^*$. Now, choose the parameters b_i and r_i such that $s_1 = s_2 = s_3 = 1$ (and consequently, $q_1 = 9; q_2 = 8$ and $q_3 = 5$). Then $k_i = \kappa_i$ and consequently the cost player i has at this equilibrium is $x_0^2\kappa_i$. So, the aggregate cost is $x_0^2(\kappa_1+\kappa_2+\kappa_3)$. Consequently, at $a = a^*$ two different solutions yield the same aggregate cost, which is obviously (see figure 1 again) also the minimum attainable aggregate cost in this case.

5 Concluding remarks

In this paper we have studied the positive solutions of the algebraic Riccati equations that play an important role in the study of limiting stationary feedback Nash equilibria in the N-player linear quadratic scalar differential game. We showed that this set of equations always has a finite number of different positive solutions and that this number is bounded by $2^N - 1$. In particular we analyzed the set of autonomous growth parameters for which (ARE) has a unique positive solution. Fixing all other system parameters, we saw that this set is either a halfline or the union of a halfline and an open (bounded) interval. We showed how this set can be determined from the analysis of two scalar functions. It has turned out that for all stable systems there is a unique solution to the (ARE) equations. In this respect it is interesting to recall from the two-player case (see Engwerda [2000]) that whenever the system is not stable, there always exist combinations of the remaining system parameters such that (ARE) has more than one positive solution.

On the other hand we have shown that there is a threshold such that if the autonomous growth parameter exceeds this threshold (assuming all other system parameters are fixed), the number of positive solutions does not increase. In general this number of positive solutions is $2^N - 1$.

In between these two limiting cases, the number of solutions gradually builds up from 1 to the maximum number if the autonomous growth parameter increases. However, this growth is (in general) not monotonic. So, roughly speaking, the conclusion is that the larger the instability of the system is, the more positive solutions the (ARE) equations will have.

The above outcomes raise a couple of new questions. One of them is whether aggregate efficiency can be used as an additional constraint to determine a unique equilibrium amongst all solutions of the (ARE). We showed in an example that this is not the case. The main remaining topic is of course how things generalize for the multivariable case. In view of the above analysis presented for the scalar case it seems a good idea to treat first the case of a system with a multivariable state and scalar controls. We hope that the obtained above results may be helpful in analyzing this problem.

Appendix

Proof of Lemma 15: This can be shown by a direct evaluation of its derivative. We have

$$\begin{aligned} f_2^{N'}(x) &= N - 1 + \frac{x}{\sqrt{x^2-\sigma_1}} - \sum_{i=2}^{N} \frac{x}{\sqrt{x^2-\sigma_i}} \\ &= \frac{x}{\sqrt{x^2-\sigma_1}} - \sum_{i=2}^{N} \frac{\sigma_i}{\sqrt{x^2-\sigma_i}(x+\sqrt{x^2-\sigma_i})} \end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{\sqrt{x^2-\sigma_1}}(x - \sum_{i=2}^{N} \frac{\sigma_i}{x+\sqrt{x^2-\sigma_i}}) \\
&= \frac{1}{\sqrt{x^2-\sigma_1}} \frac{1}{x+\sqrt{x^2-\sigma_2}}(x^2 + x\sqrt{x^2-\sigma_2} - \sigma_2 - \sum_{i=3}^{N} \sigma_i \frac{x+\sqrt{x^2-\sigma_2}}{x+\sqrt{x^2-\sigma_i}}) \\
&\geq \frac{1}{\sqrt{x^2-\sigma_1}} \frac{1}{x+\sqrt{x^2-\sigma_2}}(x^2 + x\sqrt{x^2-\sigma_2} - \sum_{i=2}^{N} \sigma_i) \\
&\geq \frac{1}{\sqrt{x^2-\sigma_1}} \frac{1}{x+\sqrt{x^2-\sigma_2}}(\sigma_1 + x\sqrt{x^2-\sigma_2} - \sum_{i=2}^{N} \sigma_i) > 0.
\end{aligned}
$$

$\square$

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