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Heemels, W.P.M.H.; Schumacher, J.M.; Weiland, S.

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LINEAR COMPLEMENTARITY SYSTEMS*

W. P. M. H. HEEMELS†, J. M. SCHUMACHER‡, AND S. WEILAND§

Abstract. We introduce a new class of dynamical systems called "linear complementarity systems." The time evolution of these systems consists of a series of continuous phases separated by "events" which cause a change in dynamics and possibly a jump in the state vector. The occurrence of events is governed by certain inequalities similar to those appearing in the linear complementarity problem of mathematical programming. The framework we describe is suitable for certain situations in which both differential equations and inequalities play a role; for instance, in mechanics, electrical networks, piecewise linear systems, and dynamic optimization. We present a precise definition of the solution concept of linear complementarity systems and give sufficient conditions for existence and uniqueness of solutions.

Key words. hybrid systems, differential/algebraic equations, inequality constraints, complementarity problem

AMS subject classifications. 34A12, 68U20, 70F35, 93B12

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1. Introduction.

In many technical and economic applications one encounters systems of differential equations and inequalities. For a quick roundup of examples, one may think of the following: motion of rigid bodies subject to unilateral constraints; electrical networks with ideal diodes; optimal control problems with inequality constraints in the states and/or controls; dynamical systems with piecewise linear characteristics, such as saturation functions, deadzones, relays, Coulomb friction, and one-sided springs; projected dynamical systems; dynamic versions of linear and nonlinear programming problems; and dynamic Walrasian economies. It has to be noted that there is considerable inherent complexity in systems of differential equations and inequalities, since nonsmooth trajectories and possibly jumps have to be taken into account. As a result of this, even basic issues such as existence and uniqueness of solutions are difficult to settle. Given the wealth of possible applications, however, it is of interest to overcome these difficulties.

In the literature one can find many strands of research dealing with dynamics subject to inequality constraints, some mainly motivated by problems in mechanics, others more closely connected to operations research and economics. The framework of *differential inclusions* (see, for instance, [2]) gives a general setting for the study of systems in which both differential equations and inequalities play a role. In this paper, however, we shall be interested in more specific dynamical systems for which uniqueness of solutions holds. Although, of course, one can get unique solutions from a differential inclusion by imposing suitable side constraints, we prefer to think of the systems considered in this paper as systems that switch between modes on the basis of certain inequality constraints and that behave within each mode as ordinary differen-

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†Department of Electrical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands (w.p.m.h.heemels@tue.nl).

‡Department of Econometrics, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands (jms@kub.nl).

§Department of Electrical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands (s.weiland@tue.nl).

tial systems rather than as differential inclusions. This "multimodal" way of thinking is natural in a number of applications: in the study of Coulomb friction, one has the transition between stick mode and slip mode; in the study of electrical networks with ideal diodes, there is the transition between the conducting and the blocking mode of each diode; and in the context of dynamic optimization, one has mode transitions when an inactive constraint becomes active, or vice versa. A similar point of view may be found in the literature on the so-called "hybrid systems" encompassing both continuous and discrete dynamics, which have recently been a popular subject of study both for computer scientists and for control theorists (see, for instance, [1, 27]).

Among the studies that have been made of dynamical systems exhibiting some sort of switching behavior, one may mention a number that have been inspired by applications in mechanics [6, 21, 22, 23, 25, 30, 26, 31, 32, 33], in electrical engineering [4, 20], and in operations research [11, 24], as well as general studies such as [12]. The work in this paper is more general than most of the cited studies in the sense that we do not a priori impose conditions on the "index" of the constraints. (The index measures the number of actual constraints following from a given algebraic constraint within the context of a given set of differential equations; the term comes from numerical analysis; see, for instance, [5].) Our treatment is also general in that we allow an arbitrary finite number of state variables and an arbitrary finite number of constraints. On the other hand, our work is more restricted, since we consider only linear differential equations; in conjunction with the switching rules, the systems that we study are therefore piecewise linear dynamical systems.

As a consequence of the fact that we are looking at systems of arbitrary index, we have to take into account the possibility of solutions containing *impulses*. The occurrence of such impulses is state-dependent and in this sense our situation is different from the one in [3] where impulses are externally imposed rather than generated by the system itself. One of the main reasons for restricting the development in this paper to linear dynamics within each mode is the fact that this allows us to treat impulses within a standard distributional framework. Earlier works in the research program that have led to the current paper [28, 29] have used a nonlinear framework which made it difficult to treat impulses, so that a *complete* specification of dynamics on a general level could in fact not be given. Without a complete solution concept, issues of existence and uniqueness of solutions can be studied only partially. The contribution of this paper is as follows: (i) It gives a complete definition of what is to be understood by a solution of a linear complementarity system; (ii) it gives sufficient conditions for well-posedness of linear complementarity systems, in the sense of existence and uniqueness of solutions; (iii) it presents an effective procedure for generating solutions to linear complementarity systems. In addition to this, we establish an explicit connection to the literature on mechanical systems that are subject to mode-switching by showing that our formulation agrees with the one of Moreau [23] (see also [6, 22]) for the class of systems covered by both formulations, namely, linear mechanical systems.

The paper is organized as follows. We start with an example to motivate the ingredients needed for defining a solution concept for complementarity systems. To introduce the notion of solution, some mathematical preliminaries as presented in section 3 are required. A definition of the class of linear complementarity systems with its solution concept is given in section 4. The definition relies on a mapping which assigns a "next mode" to each continuous state; several alternative ways of constructing this mapping are discussed in section 5. Sufficient conditions for local existence and uniqueness of solutions follow in section 6. After that, we present a computational example to illustrate the construction of solutions from the definition. In section 8, we establish the connection with the sweeping process formulation of Moreau. Finally, conclusions follow in section 9.

In this paper, the following notational conventions will be in force. $\mathbb{R}$ denotes the real numbers, $\mathbb{R}_+$ the nonnegative real numbers, and $\mathbb{N} := \{0, 1, 2, \ldots\}$. For a positive integer l, $\bar{l}$ denotes the set $\{1, 2, \ldots, l\}$. If a is a (column) vector with k real components, we write $a \in \mathbb{R}^k$ and denote the ith component by a_i. For two vectors a, $b \in \mathbb{R}^k$, the notation $a \perp b$ means that for all $i \in \bar{k}$ either $a_i = 0$ or $b_i = 0$. Given two vectors $a \in \mathbb{R}^k$ and $b \in \mathbb{R}^l$, then $\operatorname{col}(a, b)$ denotes the vector in $\mathbb{R}^{k+l}$ that arises from stacking a over b. $M \in \mathbb{R}^{m \times n}$ means that M is a real matrix with dimensions $m \times n$. $M^\top$ is the transpose of the matrix M. The kernel of M is denoted by Ker M and the image by Im M. Given $M \in \mathbb{R}^{k \times l}$ and two subsets $I \subseteq \bar{k}$ and $J \subseteq \bar{l}$, the (I, J)-submatrix of M is defined as $M_{IJ} := (m_{ij})_{i \in I, j \in J}$. In case $J = \bar{l}$, we also write $M_{I\bullet}$ and if $I = \bar{k}$, we write $M_{\bullet J}$. For a vector a, $a_I := (a_i)_{i \in I}$. The diagonal matrix with diagonal entries $a_1, \ldots, a_k$ is denoted by $\operatorname{diag}(a_1, \ldots, a_k)$.

The field of rational functions in one indeterminate is denoted by $\mathbb{R}(s)$. Rational vector functions with k components and rational matrices with dimensions $m \times n$ are denoted by $\mathbb{R}^k(s)$ and $\mathbb{R}^{m \times n}(s)$, respectively. For reasons of clarity, we shall systematically use a notation in which vectors over $\mathbb{R}(s)$ are written with an argument s to distinguish between the vector $u \in \mathbb{R}^k$ and the rational vector $u(s) \in \mathbb{R}^k(s)$. A rational matrix is called proper if for all entries the degree of the numerator is smaller than or equal to the degree of the denominator. A rational matrix is called biproper if it is square, proper, and has a proper inverse. If two rational vectors $u(s)$, $y(s) \in \mathbb{R}^k(s)$ satisfy that for all $i \in \bar{k}$ either $u_i(s) = 0$ or $y_i(s) = 0$, we write $u(s) \perp y(s)$.

The set $C^\infty(\mathbb{R}, \mathbb{R})$ denotes the set of smooth functions, i.e., all functions from $\mathbb{R}$ to $\mathbb{R}$ that are arbitrarily often differentiable. For a smooth function u the ith derivative is denoted by $u^{(i)}$.

A vector $u \in \mathbb{R}^k$ is called nonnegative, and we write $u \geq 0$ if $u_i \geq 0, i \in \bar{k}$, and positive ($u > 0$) if $u_i > 0, i \in \bar{k}$. If a vector u is not nonnegative, we write $u \not\geq 0$. A sequence of scalars $(u^1, u^2, \ldots, u^r)$ is called lexicographically nonnegative, written as $(u^1, u^2, \ldots, u^r) \succeq 0$, if $(u^1, u^2, \ldots, u^r) = (0, 0, \ldots, 0)$ or $u^j > 0$, where $j := \min\{p \in \bar{r} \mid u^p \neq 0\}$. A sequence of scalars is called lexicographically positive, denoted by $(u^1, u^2, \ldots, u^r) \succ 0$, if $(u^1, u^2, \ldots, u^r) \succeq 0$ and $(u^1, u^2, \ldots, u^r) \neq (0, 0, \ldots, 0)$. For a sequence of vectors $(u^1, u^2, \ldots, u^r)$ with $u^i \in \mathbb{R}^k$, we write $(u^1, u^2, \ldots, u^r) \succeq 0$ when $(u_i^1, u_i^2, \ldots, u_i^r) \succeq 0$ for all $i \in \bar{k}$. Likewise, we write $(u^1, u^2, \ldots, u^r) \succ 0$ when $(u_i^1, u_i^2, \ldots, u_i^r) \succ 0$ for all $i \in \bar{k}$.

For sets $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \setminus \mathcal{B} := \{x \in \mathcal{A} \mid x \notin \mathcal{B}\}$ and $\mathcal{P}(\mathcal{A})$ denotes the power set of $\mathcal{A}$, i.e., the collection of all subsets of $\mathcal{A}$. For two subspaces V, T of $\mathbb{R}^n$, the notation $V \oplus T = \mathbb{R}^n$ means that V and T form a direct sum decomposition of $\mathbb{R}^n$, i.e., $V + T := \{v + t \mid v \in V,\ t \in T\} = \mathbb{R}^n$ and $V \cap T = \{0\}$.

2. Example. Before specifying the class of linear complementarity systems, we illustrate some of the aspects that play a role in the evolution of such systems with an example of two carts connected by a spring (used also in [28]). The left cart is attached to a wall by a spring. The motion of the left cart is constrained by a completely inelastic stop. The system is depicted in Figure 2.1.

For simplicity, the masses of the carts and the spring constants are set equal to 1. The stop is placed at the equilibrium position of the left cart. By x_1, x_2 we denote the deviations of the left and right cart, respectively, from their equilibrium positions

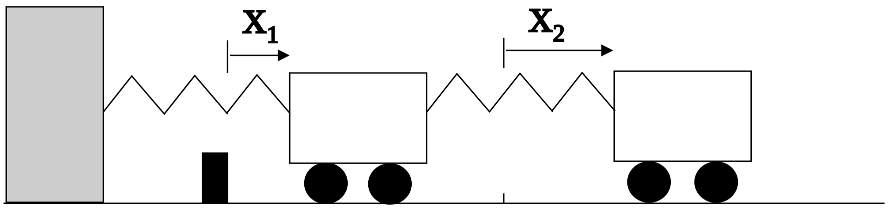

FIG. 2.1. *Two-carts system.*

and x_3, x_4 are the velocities of the left and right cart, respectively. By u, we denote the reaction force exerted by the stop. Furthermore, the variable y is set equal to x_1. Simple mechanical laws lead to the dynamical relations

$$\begin{aligned} \dot{x}_1(t) &= x_3(t), \\ \dot{x}_2(t) &= x_4(t), \\ \dot{x}_3(t) &= -2x_1(t) + x_2(t) + u(t), \\ \dot{x}_4(t) &= x_1(t) - x_2(t), \\ y(t) &:= x_1(t). \end{aligned} \tag{2.1}$$

To model the stop in this setting, the following reasoning applies. The variable $y(t) = x_1(t)$ should be nonnegative because it is the position of the left cart with respect to the stop. The force exerted by the stop can act only in the positive direction implying that $u(t)$ should be nonnegative. If the left cart is not at the stop at time t ($y(t) > 0$), the reaction force vanishes at time t, i.e., $u(t) = 0$. Similarly, if $u(t) > 0$, the cart must necessarily be at the stop, i.e., $y(t) = 0$. This is expressed by the conditions

$$0 \leq y(t) \perp u(t) \geq 0. \tag{2.2}$$

The system can be represented by two modes, depending on whether or not the stop is active. We distinguish between the unconstrained mode ($u(t) = 0$) and the constrained mode ($y(t) = 0$). The dynamics of these modes are given by the following differential and algebraic equations (DAEs):

$$\begin{array}{rl} \underline{\text{unconstrained}} & \underline{\text{constrained}} \\ \dot{x}_1(t) = x_3(t) & \dot{x}_1(t) = x_3(t) \\ \dot{x}_2(t) = x_4(t) & \dot{x}_2(t) = x_4(t) \\ \dot{x}_3(t) = -2x_1(t) + x_2(t) & \dot{x}_3(t) = -2x_1(t) + x_2(t) + u(t) \\ \dot{x}_4(t) = x_1(t) + x_2(t) & \dot{x}_4(t) = x_1(t) + x_2(t) \\ u(t) = 0 & y(t) = x_1(t) = 0. \end{array}$$

When the system is represented by either of these modes, the triple (u, x, y) is given by the corresponding dynamics as long as the following inequalities in (2.2):

$$\begin{array}{rl} \underline{\text{unconstrained}} & \underline{\text{constrained}} \\ y(t) \geq 0 & u(t) \geq 0 \end{array}$$

are satisfied. A mode change is triggered by violation of one of these inequalities. The mode transitions that are possible for the two-carts systems are described below.

- **Unconstrained → constrained.** The inequality $y(t) \geq 0$ tends to be violated at a time instant $t = \tau$. The left cart hits the stop and stays there. The velocity of the left cart is reduced to zero instantaneously at the time of impact: the kinetic energy of the left cart is totally absorbed by the stop due to a purely inelastic collision. A state for which this happens is, for instance, $x(\tau) = (0, -1, -1, 0)^\top$.
- **Constrained → unconstrained.** The inequality $u(t) \geq 0$ tends to be violated at $t = \tau$. The right cart is located at or moving to the right of its equilibrium position, so the spring between the carts is stretched and pulls the left cart away from the stop. This happens, for example, if $x(\tau) = (0, 0, 0, 1)^\top$.
- **Unconstrained → unconstrained with re-initialization according to constrained mode.** The inequality $y(t) \geq 0$ tends to be violated at $t = \tau$. As an example, consider $x(\tau) = (0, 1, -1, 0)^\top$. At the time of impact, the velocity of the left cart is reduced to zero just as in the first case. Hence, a state jump (re-initialization) to $(0, 1, 0, 0)^\top$ occurs. The right cart is at the right of its equilibrium position and pulls the left cart away from the stop. Stated differently, from $(0, 1, 0, 0)^\top$ smooth continuation in the unconstrained mode is possible.

This last transition is a special one in the sense that, first, the constrained mode is active, causing the corresponding state jump. After the jump, no smooth continuation is possible in the constrained mode resulting in a second mode change back to the unconstrained mode.

From state $x(\tau) = (0, -1, -1, 0)^\top$, we can enter the constrained mode by starting with an instantaneous jump to $x(\tau+) = (0, -1, 0, 0)^\top$. This jump can be modeled as the result of a (Dirac) pulse δ exerted by the stop. In fact, $u = \delta$ results in the state jump $x(\tau+) - x(\tau) = (0, 0, 1, 0)^\top$. This motivates the use of distributional theory as a suitable mathematical framework for describing physical phenomena such as collisions with discontinuities in the state vector.

To summarize, the motion of the carts is governed by two systems of DAEs called the constrained mode and the unconstrained mode. A change of mode is triggered by violation of certain inequalities corresponding to the current mode. The time instants at which this occurs are called “event times.” At an event time, the system will switch to a new mode. A mode transition often calls for a state jump or re-initialization. In the example, velocity jumps occur when the left cart arrives at the stop with negative velocity. In this paper, the above dynamics will be formalized for the complete class of linear complementarity systems, and special attention will be paid to the mode selection problem and well-posedness issues. However, first we recall some facts concerning systems of linear differential and algebraic equations, such as those appearing in the constrained and unconstrained mode descriptions.

3. Mathematical preliminaries. We consider a linear differential/algebraic system of the form

$$\dot{x}(t) = Kx(t) + Lu(t), \tag{3.1a}$$

$$0 = Mx(t) + Nu(t). \tag{3.1b}$$

The time arguments will often be suppressed for brevity. Throughout this section, $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$. The system parameters K, L, M, and N are constant matrices of dimensions $n \times n$, $n \times m$, $r \times n$, and $r \times m$, respectively.

Definition 3.1. *A state x_0 is said to be consistent for (K, L, M, N) if there exist smooth functions u and x such that $x(0) = x_0$ and (3.1) is satisfied. The set of* *all consistent states for (K, L, M, N) is denoted by $V(K, L, M, N)$ and is called the consistent subspace.*

The following sequence of subspaces converges in at most n (dimension of state) steps to $V = V(K, L, M, N)$ (for a proof, see [14]):

$$\begin{aligned} V_0 &= \mathbb{R}^n, \\ V_{i+1} &= \{x \in \mathbb{R}^n \mid \exists u \in \mathbb{R}^m \text{ such that } Kx + Lu \in V_i,\ Mx + Nu = 0\}. \end{aligned} \tag{3.2}$$

DEFINITION 3.2. *The quadruple (K, L, M, N) is called autonomous if for every consistent state x_0 the system* (3.1) *has a unique solution (x, u).*

The system (3.1) is autonomous if the full-column-rank condition

$$\operatorname{Ker} \begin{bmatrix} L \\ N \end{bmatrix} = \{0\} \tag{3.3}$$

holds together with

$$V(K, L, M, N) \cap T(K, L, M, N) = \{0\}, \tag{3.4}$$

where $T(K, L, M, N)$ is the subspace that is obtained as the limit of the sequence

$$\begin{aligned} T_0 \quad &= \{0\}, \\ T_{i+1} &= \{x \in \mathbb{R}^n \mid \exists u \in \mathbb{R}^m,\ \exists \bar{x} \in T_i \text{ such that } x = K\bar{x} + Lu, M\bar{x} + Nu = 0\}. \end{aligned} \tag{3.5}$$

This sequence converges in maximally n (dimension of state) steps (proof can be found in [14]). The subspace $T = T(K, L, M, N)$ can be interpreted as the *jump space* associated to (K, L, M, N), i.e., the space along which fast motions will occur that take an inconsistent initial state instantaneously to a point in the consistent subspace V.

To formalize the interpretation of T as a jump space, we introduce the class of impulsive-smooth distributions as studied by Hautus and Silverman [14]. The general form of an impulsive-smooth distribution $\mathtt{u}$ (note the different font used for distributions) is

$$\mathtt{u} = \underbrace{\sum_{i=0}^{l} u^{-i} \delta^{(i)}}_{\mathtt{u}_{imp}} + \mathtt{u}_{reg}, \tag{3.6}$$

where $\delta = \delta^{(0)}$ denotes the delta distribution with support at zero, $\delta^{(r)}$ is its rth distributional derivative, u^0, $u^{-1}, \ldots,$ u^{-l} are coefficients in $\mathbb{R}$, and $\mathtt{u}_{reg}$ is a distribution that can be identified with the restriction to $[0, \infty)$ of some smooth function. The regular part of an impulsive-smooth distribution $\mathtt{u}$ is denoted by $\mathtt{u}_{reg}$ and its impulsive part by $\mathtt{u}_{imp}$. The class of impulsive-smooth distributions will be denoted by C_{imp}. For an element $\mathtt{u}$ of C_{imp} of the form (3.6), we write $\mathtt{u}(0+)$ for the limit value $\lim_{t \downarrow 0} \mathtt{u}_{reg}(t)$. Having introduced the class C_{imp}, we can replace the system of equations (3.1) by its distributional version

$$\begin{aligned} \dot{\mathtt{x}} &= K\mathtt{x} + L\mathtt{u} + x_0 \delta, \\ 0 &= M\mathtt{x} + N\mathtt{u} \end{aligned} \tag{3.7}$$

in which the initial condition x_0 appears explicitly, and we can look for a solution of (3.7) in the class of vector-valued impulsive-smooth distributions. In [14] it is shown that under the conditions (3.3) and (3.4) there exists a unique solution $(\mathtt{u}, \mathtt{x}) \in C_{imp}^{m+n}$ to (3.7) for all $x_0 \in V + T$; moreover, the solution is such that $\mathtt{x}(0+)$ is equal to $P_V^T x_0$, the projection of x_0 onto V along the jump space T. In fact, $\mathtt{x}(0+)$ depends only on the impulsive part of $\mathtt{u}$: if $\mathtt{u}_{imp} = \sum_{i=0}^{l} u^{-i}\delta^{(i)}$, then

$$\mathtt{x}(0+) = x_0 + \sum_{i=0}^{l} K^i L u^{-i}. \tag{3.8}$$

Lemma 3.3. *Consider the system* (3.1) *and suppose that the number of inputs (m) equals the number of constraints (r). Then the following statements are equivalent:*

(1) (K, L, M, N) *is autonomous.*
(2) *The system* (3.7) *admits a unique impulsive-smooth distribution for each initial condition.*
(3) $V(K, L, M, N) \oplus T(K, L, M, N) = \mathbb{R}^n$ *and Ker* $\begin{bmatrix} L \\ N \end{bmatrix} = \{0\}$.
(4) $G(s) := M(sI - K)^{-1}L + N$ *is invertible as a rational matrix.*

Proof. The implication $2 \Rightarrow 1$ follows from the definition of an autonomous system. The quadruple (K, L, M, N) is autonomous if and only if the system $\Sigma : \dot{x} = Kx + Lu, y = Mx + Nu$ is left invertible in the sense of [14]. In [14], it is proven that the statements

- the system Σ is left invertible,
- $V(K, L, M, N) \cap T(K, L, M, N) = \{0\}$ and Ker $\begin{bmatrix} L \\ N \end{bmatrix} = \{0\}$,
- $G(s)$ is left invertible

are equivalent. Since $G(s)$ is assumed to be square $(m = r)$, left invertibility is the same as invertibility. Hence, $1 \Rightarrow 4$. According to [14, Thm. 3.24], invertibility of $G(s)$ implies additionally that $V(K, L, M, N) \oplus T(K, L, M, N) = \mathbb{R}^n$. This proves $4 \Rightarrow 3$. Finally, $3 \Rightarrow 2$ is a consequence of the fact that the assumptions (3.3)–(3.4) imply that there is a unique solution $(\mathtt{u}, \mathtt{x}) \in C_{imp}^{m+n}$ to (3.7) for all $x_0 \in V + T$, as mentioned earlier. Since $V + T$ is equal to $\mathbb{R}^n$, this implies 2. $\square$

The systems studied in this paper are described by standard state space equations of linear systems together with complementarity conditions, as in the complementarity problems of mathematical programming. Therefore some concepts from complementarity theory will be recalled briefly. The linear complementarity problem (LCP) [7] is defined as follows.

Given a matrix $M \in \mathbb{R}^{k \times k}$ and $q \in \mathbb{R}^k$, find u, $y \in \mathbb{R}^k$ such that

$$y = q + Mu, \tag{3.9}$$
$$0 \leq y \perp u \geq 0. \tag{3.10}$$

This problem is denoted by LCP(q, M).

Let a matrix M of size $k \times k$ and two subsets I and J of $\bar{k}$ of the same cardinality be given. The (I, J)-*minor* of M is the determinant of the square matrix $M_{IJ} := (m_{ij})_{i \in I, j \in J}$. The (I, I)-minors are also known as the principal minors. M is called a *P-matrix* if all principal minors are positive. A square matrix M is said to be *positive definite* if $x^\top M x > 0$ for all nonzero $x \in \mathbb{R}^n$. Note that a positive definite matrix is not necessarily symmetric according to this definition.

We state the following results.

Theorem 3.4. *For given M, the problem LCP(q, M) has a unique solution for all vectors q if and only if M is a P-matrix.*

Proof. For the proof see [7, Thm. 3.3.7]. $\square$

THEOREM 3.5. *A positive definite matrix is a P-matrix.*

Proof. For the proof see [7, Thms. 3.1.6 and 3.3.7]. ∎

4. Linear complementarity systems. In this section, we introduce linear complementarity systems (LCS) and formulate the notion of a solution for such systems.

An LCS is governed by the simultaneous equations

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{4.1a}$$

$$y(t) = Cx(t) + Du(t), \tag{4.1b}$$

$$0 \leq y(t) \perp u(t) \geq 0. \tag{4.1c}$$

The notation in (4.1c) is consistent with the notation used in complementarity problems in mathematical programming (see the formulation of the LCP in section 3). In this section, we will describe how the relations above have to be interpreted to arrive at a notion of a solution to such a complementarity system. The functions u, x, and y take values in $\mathbb{R}^k$, $\mathbb{R}^n$, and $\mathbb{R}^k$, respectively; A, B, C, and D are constant matrices of appropriate dimensions. Note that the dimensions of the variables $y(t)$ and $u(t)$ are the same. Equation (4.1c) states that for every component $i = 1, \ldots, k$ either $u_i(t) = 0$ or $y_i(t) = 0$. The set of indices for which $y_i(t) = 0$, called the *mode* or *active index set*, may change during the time evolution of the system. The system may therefore switch from one "operation mode" to another. To define the dynamics of (4.1) completely, one has to specify when the mode switches occur, what their effect will be on the state variables, and how a new mode will be selected. We will do this below, extending earlier treatments in [28] (where only systems with a single constraint were considered ($k = 1$); see also Example 8.3 for a comparison of the mode selection criteria) and [29], which treated only existence and uniqueness of *smooth* continuations while impulsive motions and re-initialization rules were left out of consideration and only a limited discussion of mode selection criteria could be given. A generalization from smooth to impulsive-smooth continuations is not straightforward. The interpretation of the inequalities for impulsive motions is not obvious. A requirement of such an interpretation will be that it must comply with physical laws for "real-life" systems included in the class of complementarity systems. In this section, we will formalize a distributional interpretation of the inequalities that agrees with Moreau's re-initialization rules for linear mechanical systems (see section 8).

The system has 2^k modes. Each mode is characterized by the active index set $I \subseteq \bar{k}$, which indicates that $y_i = 0$, $i \in I$, and $u_i = 0$, $i \in I^c$, where $I^c := \bar{k} \setminus I = \{i \in \bar{k} \mid i \notin I\}$. For each such mode the laws of motion are given by systems of DAEs. Specifically, in mode I they are given by

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \\ y_i(t) &= 0, \ i \in I, \\ u_i(t) &= 0, \ i \in I^c, \end{cases} \tag{4.2}$$

or equivalently,

$$\begin{cases} \dot{x}(t) &= Ax(t) + B_{\bullet I} u_I(t), \\ 0 &= C_{I\bullet} x(t) + D_{II} u_I(t), \\ y_{I^c}(t) &= C_{I^c\bullet} x(t) + D_{I^c I} u_I(t), \\ u_{I^c}(t) &= 0, \\ y_I(t) &= 0. \end{cases} \tag{4.3}$$

The set of consistent states for mode I, denoted by V_I, equals $V(A, B_{\bullet I}, C_{I\bullet}, D_{II})$. The jump space is given by $T_I := T(A, B_{\bullet I}, C_{I\bullet}, D_{II})$. We call mode I *autonomous* if the quadruple $(A, B_{\bullet I}, C_{I\bullet}, D_{II})$ is autonomous. A standing assumption in the rest of this paper will be the following.

ASSUMPTION 4.1. *All modes of the LCS* (4.1) *are autonomous.*

By Lemma 3.3 this is equivalent to saying that $G_{II}(s) := C_{I\bullet}(sI - A)^{-1}B_{\bullet I} + D_{II}$ is invertible for each index set $I \subseteq \bar{k}$. Note that the notation $G_{II}(s)$ is consistent in the sense that $G_{II}(s)$ is the (I, I)-submatrix of the rational matrix $G(s) := C(sI - A)^{-1}B + D$. Again by Lemma 3.3, Assumption 4.1 implies that $V_I \oplus T_I = \mathbb{R}^n$ for all $I \subseteq \bar{k}$ and that (4.2) has a unique impulsive-smooth solution for all individual modes given an arbitrary initial state.

4.1. Continuous phase.

DEFINITION 4.2. *Given $x_0 \in \mathbb{R}^n$ and $I \subseteq \bar{k}$, we denote the unique distributional solution to* (4.2) *for mode I and initial state x_0 by $(\mathtt{u}^{x_0,I}, \mathtt{x}^{x_0,I}, \mathtt{y}^{x_0,I}) \in C_{imp}^{k+n+k}$.*

According to [14, Thm. 3.10], there exists a linear mapping F_I such that (4.2) is satisfied for $x_0 \in V_I$ by taking $u(t) = F_I x(t)$. Substituting this feedback in (4.2) transforms the DAE into an ordinary differential equation (ODE). Hence, the regular part of an impulsive-smooth solution $\mathtt{u}$ satisfying (4.2) for a given initial state is a *Bohl function*, i.e., $\mathtt{u}_{reg}$ is of the form

$$\mathtt{u}_{reg}(t) = \begin{cases} 0 & (t < 0), \\ Ee^{Gt}v & (t \geq 0) \end{cases} \tag{4.4}$$

for real matrices E, G, and a vector v depending on the initial state and the specific mode I.

4.2. Re-initialization. If initial states of (4.2) are not consistent, i.e., if $x_0 \notin V_I$, then a re-initialization of the initial state will be necessary as pointed out in section 3. Indeed, if $x_0 \notin V_I$, then the solution to (4.2) will contain a nontrivial impulsive part resulting in an instantaneous jump or re-initialization of the state variable.

As discussed in section 3, the re-initialized vector $\mathtt{x}^{x_0,I}(0+)$ is equal to the projection of x_0 onto the consistent subspace V_I along the jump space T_I. That is, $\mathtt{x}^{x_0,I}(0+) := P_I x_0$, where P_I is the projection operator $P_{V_I}^{T_I}$.

4.3. Event detection. Suppose that the current time, state, and mode are $\tau = 0$, x_0, and I, respectively. Note that due to the time-invariance of the system description (4.1), the assumption $\tau = 0$ is just a normalization. The system (4.1) will be represented by (4.2) for mode I as long as the inequalities in (4.1c),

$$\mathtt{u}_{reg}^{x_0,I}(t) \geq 0 \text{ and } \mathtt{y}_{reg}^{x_0,I}(t) \geq 0, \tag{4.5}$$

are satisfied for $t \geq \tau$. The function $\theta : \mathbb{R}^n \times \mathcal{P}(\bar{k}) \to \mathbb{R}_+$ gives the length of the time interval during which the system evolves in mode I from initial state x_0. Note that we consider only the regular part here. In formal terms, θ is defined as follows.

DEFINITION 4.3. *The time-to-next-event function $\theta : \mathbb{R}^n \times \mathcal{P}(\bar{k}) \to \mathbb{R}_+$ is defined as*

$$\theta(x_0, I) := \inf\{t > 0 \mid \mathtt{u}_{reg}^{x_0,I}(t) \not\geq 0 \text{ or } \mathtt{y}_{reg}^{x_0,I}(t) \not\geq 0\},$$

with the convention $\inf \emptyset = \infty$.

The next event time after time τ will be $\tau + \theta(x(\tau), I)$ (by time-invariance) when the mode and the state at time τ are equal to I and $x(\tau)$, respectively. Since smooth continuation is not possible in mode I after the event time $\tau+\theta(x(\tau),I)$, a transition to another mode must occur. An important aspect of the solution concept will be how to select the new mode.

To illustrate the definition of θ, consider Examples 4.4 and 4.5 of the two-carts system in the next subsection. In these cases,

$$\theta((0,-1,0,0)^\top,\{1\})=\frac{\pi}{2}\quad\text{and}\quad\theta((0,1,-1,0)^\top,\{1\})=0.$$

4.4. Mode selection. The mode selection procedure that we propose is based on the concept of *initial solution*. Loosely speaking, an initial solution with initial state x_0 is a triple $(\mathtt{u},\mathtt{x},\mathtt{y})\in C_{imp}^{k+n+k}$ satisfying (4.2) for some mode I and satisfying (4.5) either on a time interval of positive length or on a time instant at which delta distributions are active. The idea is that an initial solution is a starting trajectory for the "global" solution to (4.1).

Example 4.4. Consider the two-carts system with initial state $(0,-1,0,0)^\top$. The solution to the constrained mode is $u(t)=\cos t$ and $y(t)=0$. Hence, it satisfies (4.2) for $I=\{1\}$ on $[0,\infty)$ and (4.5) on $[0,\frac{\pi}{2})$. Thus, this solution satisfies (4.1) on $[0,\frac{\pi}{2})$. Therefore, we admit selection of the constrained mode ($I=\{1\}$) as smooth continuation in this mode is possible.

Example 4.5. From the initial state $x_0=(0,1,-1,0)^\top$, first a state jump occurs to $P_{\{1\}}x_0=(0,1,0,0)^\top$ governed by the laws of the constrained mode, but no smooth continuation is possible in the constrained mode. Solving the dynamics corresponding to the constrained mode, i.e., (4.2) with $I=\{1\}$, gives $(\mathtt{u},\mathtt{x},\mathtt{y})$ with $\mathtt{u}=\delta+\mathtt{u}_{reg}$, where $\mathtt{u}_{reg}(t)=-\cos t$. Although (4.5) is not satisfied on a positive time interval, incorporation of this solution in the definition of initial solutions seems well motivated on physical grounds. We admit selection of $I=\{1\}$.

We now make the notion of initial solution more precise. Given an impulsive-smooth distribution $\mathtt{v}\in C_{imp}$, we define the leading coefficient of its impulsive part by

$$\text{lead}(\mathtt{v}):=\begin{cases}0 & \text{if } \mathtt{v}_{imp}=0,\\ v^{-l} & \text{if } \mathtt{v}_{imp}=\sum_{i=0}^{l}v^{-i}\delta^{(i)} \text{ with } v^{-l}\neq 0.\end{cases}\tag{4.6}$$

DEFINITION 4.6. *We call a scalar-valued impulsive-smooth distribution* $\mathtt{v}\in C_{imp}$ *initially nonnegative if*

$$\begin{cases}\textit{lead}(\mathtt{v})>0 & \textit{in case } \mathtt{v}_{imp}\neq 0,\\ \textit{there exists an } \varepsilon>0 \textit{ such that for all } t\in[0,\varepsilon)\ \mathtt{v}_{reg}(t)\geq 0 & \textit{otherwise.}\end{cases}$$

A vector-valued impulsive-smooth distribution in C_{imp}^k *is called initially nonnegative if each of its components is initially nonnegative. We call an impulsive-smooth distribution u initially positive if* $\mathtt{u}$ *is initially nonnegative and additionally if* $\mathtt{u}_i$ *is regular, then for some* $\varepsilon>0$ $\mathtt{u}_i(t)>0$, $t\in(0,\varepsilon)$.

DEFINITION 4.7. *We call* $(\mathtt{u},\mathtt{x},\mathtt{y})\in C_{imp}^{k+n+k}$ *an initial solution to* (4.1) *with initial state* x_0 *if*

(1) *there exists an* $I\subseteq\bar{k}$ *such that* $(\mathtt{u},\mathtt{x},\mathtt{y})$ *satisfies* (4.2) *with initial state* x_0 *in the distributional sense; and*

(2) $\mathtt{u},\mathtt{y}$ *are initially nonnegative.*

Given a state x_0, define the set $\mathcal{S}(x_0)$ by

$$\mathcal{S}(x_0):=\{J\subseteq\bar{k}\ |\text{there exists an initial solution } (\mathtt{u},\mathtt{x},\mathtt{y}) \text{ to (4.1) with initial state } x_0 \text{ such that } \mathtt{u}_i=0,\ i\in J^c, \text{ and } \mathtt{y}_i=0,\ i\in J\}.\tag{4.7}$$

The set $\mathcal{S}(x_0)$ denotes the set of all possible modes in which an initial solution exists with initial state x_0.

Remark 4.8. There may be more than one mode corresponding to a given initial solution $(\mathtt{u}, \mathtt{x}, \mathtt{y})$ to (4.1). With the index set I defined by

$$J := \{i \in \bar{k} \mid \mathtt{u}_i \neq 0\}, \tag{4.8}$$

the complementarity conditions require $\mathtt{y}_i = 0$ for $i \in J$. Hence, $(\mathtt{u}, \mathtt{x}, \mathtt{y})$ is an initial solution in mode J. Consider now the "undetermined index set"

$$K := \{i \in \bar{k} \mid \mathtt{u}_i = 0 \text{ and } \mathtt{y}_i = 0\}.$$

Any mode $J \subseteq I \subseteq J \cup K$ may also be selected and the initial solution $(\mathtt{u}, \mathtt{x}, \mathtt{y})$ satisfies (4.2) for $I = J$ with initial state x_0 as well. As an example consider $x_0 = 0$. In this case, $(\mathtt{u}, \mathtt{x}, \mathtt{y}) = 0$ is a possible initial solution. J and K as defined above are equal to $\emptyset$ and $\bar{k}$, respectively. Consequently, mode I can be chosen arbitrarily, which means that this initial solution satisfies the mode dynamics for each mode. For a given initial solution, the freedom in the choice of the mode corresponding to this solution is exactly characterized by the undetermined index set.

Remark 4.9. If an initial solution $(\mathtt{u}, \mathtt{x}, \mathtt{y})$ has a nontrivial impulsive part, it can be the case that the corresponding mode is valid only for the time instant 0 itself. This happens when the smooth part $(\mathtt{u}_{reg}, \mathtt{y}_{reg})$ is not initially nonnegative. An example is provided by Example 4.5, which explains also the special mode transition as mentioned in section 2. The constrained mode $(\mathcal{S}((0,1,-1,0)^\top) = \{\{1\}\})$ is selected only for the re-initialization of the state $(\theta((0,1,-1,0)^\top, \{1\}) = 0)$. From the re-initialized state $P_{\{1\}}(0,1,-1,0)^\top = (0,1,0,0)^\top$ (see also section 7) a new mode is selected $(\mathcal{S}((0,1,0,0)^\top) = \{\emptyset\})$. In the unconstrained mode a smooth initial solution exists with the re-initialized state $(0,1,0,0)^\top$ as initial state.

4.5. Solution concept. We are now in a position to define a solution concept for (4.1). A point $\tau \in \mathcal{E} \subset \mathbb{R}$ is called a right-accumulation point of $\mathcal{E}$ if there exists a sequence $\{\tau_i\}_{i\in\mathbb{N}}$ such that $\tau_i \in \mathcal{E}$ and $\tau_i < \tau$ for all i and furthermore, $\lim_{i\to\infty} \tau_i = \tau$. A left-accumulation point is defined similarly by interchanging "<" with ">." A set $\mathcal{E} \subset \mathbb{R}$ is called right-isolated if it contains no left-accumulation points. We call $\tau \in \mathcal{E}$ isolated if it is not an accumulation point of $\mathcal{E}$.

DEFINITION 4.10. *A solution to* (4.1) *on* $[0, T_e)$, $T_e > 0$, *with initial state* x_0 *is a quadruple* $(\mathcal{E}, x_c, u_c, y_c)$, *where* $\mathcal{E}$, *the set of event times, is a right-isolated closed subset of* $[0, T_e)$, *with empty interior and*

$$\begin{aligned} x_c &: (0, T_e) \setminus \mathcal{E} \to \mathbb{R}^n, \\ u_c &: (0, T_e) \setminus \mathcal{E} \to \mathbb{R}^k, \\ y_c &: (0, T_e) \setminus \mathcal{E} \to \mathbb{R}^k \end{aligned}$$

being arbitrarily often differentiable, that satisfies the following:

(1) $0 \in \mathcal{E}$.

(2) *For* $\tau \in \mathcal{E}$, $x_c(\tau+) := \lim_{t\downarrow\tau, t\notin\mathcal{E}} x_c(t) = \lim_{i\to\infty} z_i$, *where* $\{z_i\}_{i\in\mathbb{N}}$ *satisfies*

$$\left\{ \begin{array}{l} z_{i+1} = P_{I_{i+1}} z_i, \\ I_{i+1} \in \mathcal{S}(z_i) \end{array} \right. \tag{4.9}$$

and

$$z_0 := \left\{ \begin{array}{ll} x_c(\tau-) := \lim_{t\uparrow\tau, t\notin\mathcal{E}} x_c(t) & \text{if } \tau > 0, \\ x_0 & \text{if } \tau = 0. \end{array} \right. \tag{4.10}$$

(3) *For isolated* $\tau \in \mathcal{E}$ *there exists an* $I \in \mathcal{S}(x_c(\tau+))$ *such that*

$$\tau^* := \min\{t > \tau \mid t \in \mathcal{E}\} = \tau + \theta(x_c(\tau+), I) \tag{4.11}$$

and $(u_c(t), x_c(t), y_c(t))$ *satisfies* (4.2) *for mode* I *and for* $t \in (\tau, \tau^*)$.

$P_{I_{i+1}}$ denotes the projection operator corresponding to mode I_{i+1} as introduced in subsection 4.2. The definition requires that the limits in item 2 above and in the first case of (4.10) exist.

The set $\mathcal{E}$ specifies the event times, i.e., the times at which there is a change of mode. Two successive isolated event times (τ and τ^*) are related by item 3 above in terms of the time-to-next-event function θ (Definition 4.3). This requirement is included in the solution concept to exclude redundant event times. The triple (x_c, u_c, y_c) denotes the trajectories in the continuous phases of the complementarity system (as imposed by item 3 above). Item 2 links the continuous phases at the event times by a series of mode selections and re-initializations. The *multiplicity* $m(\tau)$ of the event time $\tau \in \mathcal{E}$ is defined as the $\min\{i \in \mathbb{N} \mid z_i = x_c(\tau+)\}$, i.e., the number of re-initializations needed before smooth continuation (a continuous phase) is possible. In case $m(\tau) = \infty$, one needs a limiting operation to determine the state just after the event, $x_c(\tau+)$. If $m(\tau)$ is finite, then only a finite number of mode selections and re-initializations (projections) in (4.9) are needed. Item 2 specifies also the initial conditions.

Remark 4.11. In the literature of hybrid dynamical systems it is often assumed that only a finite number of events exists in a finite time interval. Solutions with this property are sometimes called *non-Zeno* solutions. The relaxation of our solution concept is twofold. First, we allow that there are infinitely many mode switchings and re-initializations at one time instant. Second, right-accumulation points of event times are included. We incorporate solutions that could be called *right-Zeno* to be consistent with the literature on hybrid systems. As an example of a right-Zeno solution, consider the example of a bouncing ball with elastic impacts (with restitution coefficient smaller than one). This system has a right-accumulation point, because the ball is at rest within a finite time span but after infinitely many bounces. Since our solution concept complies with mechanical systems with inelastic impacts (see section 8), the bouncing ball example does not fit in the class of systems that we study, but it indicates that there exist models of physical systems that require right-Zeno solutions. An example of a complementarity system allowing right-Zeno solutions is provided by a time reversed version of a system studied by Filippov [12, p. 116], i.e.,

$$\dot{x}_1 = -\mathrm{sgn}(x_1) + 2\mathrm{sgn}(x_2), \tag{4.12a}$$

$$\dot{x}_2 = -2\mathrm{sgn}(x_1) - \mathrm{sgn}(x_2), \tag{4.12b}$$

where "sgn" denotes the signum-function given by $\mathrm{sgn}(x) = 1$ if $x > 0$ and $\mathrm{sgn}(x) = -1$ if $x < 0$. Because this system consists of two relay characteristics, it can be modeled as a linear complementarity system [16]. Solutions of this piecewise constant systems are spiraling toward the origin, which is an equilibrium point. Since $\frac{\mathrm{d}}{\mathrm{d}t}(\mid x_1(t) \mid + \mid x_2(t) \mid) = -2$, solutions reach the origin in finite time. However, solutions cannot arrive at the origin without going through an infinite number of mode transitions; since these mode switches occur in a finite time interval, the event times contain a right-accumulation point (i.e., the time that the solution reaches the origin) after which the solution stays at zero. Left-accumulation points are excluded from Definition 4.10 due to the requirement that the event set $\mathcal{E}$ is right-isolated. However, note that the time-reverse of the system (4.12) (which is the original example in [12]) has (infinitely many) left-Zeno solutions corresponding to initial state $x_0 = 0$ in a generalized solution concept that admits left-accumulation points. Such a generalized solution concept results in a nondeterministic system and nonuniqueness of solutions, which is undesirable from the point of view of modeling and simulation. In the solution concept of Definition 4.10 the only solution emanating from the origin in Filippov's original example is the zero solution.

Before we present conditions on the complementarity system to guarantee the existence and uniqueness of solutions, two algebraic mode selection procedures will be introduced.

5. Mode selection methods.

An essential problem in the definition of the solution concept and in the time simulation of complementarity systems is to find the set of possible continuation modes $\mathcal{S}(x_0)$ for a given state x_0. In fact, this is the construction of a (possibly multivalued) map from the continuous state space $\mathbb{R}^n$ to the discrete space $\mathcal{P}(\bar{k})$. The determination of $\mathcal{S}(x_0)$ in the previous section is based on finding all initial solutions and the corresponding modes. In this section, we obtain two alternative representations of $\mathcal{S}(x_0)$ that do not require the solution of differential equations.

5.1. Rational complementarity problem.

As noticed in section 4, the solutions to (4.2) are impulsive-smooth distributions whose regular parts are Bohl functions. Such "Bohl distributions" have rational Laplace transforms. Specifically, the Laplace transform $\hat{\mathtt{u}}(s)$ of $\mathtt{u} = \sum_{i=0}^{l} u^{-i}\delta^{(i)} + \mathtt{u}_{reg}$ with $\mathtt{u}_{reg}$ as in (4.4) equals [13]

$$\hat{\mathtt{u}}(s) = \sum_{i=0}^{l} u^{-i}s^{i} + E(sI - G)^{-1}v \ .$$

Observe that the polynomial part of the Laplace transform corresponds to the impulsive part and the strictly proper part to the regular part of the Bohl distribution.

Lemma 5.1. *Let $\mathtt{v} = \sum_{i=0}^{l} v^{-l}\delta^{(i)} + \mathtt{v}_{reg} \in C_{imp}$ be a Bohl distribution. The following statements are equivalent:*

(1) $\mathtt{v}$ *is initially nonnegative.*
(2) *There exists a $\sigma_0 \in \mathbb{R}$ such that the Laplace transform $\hat{\mathtt{v}}(s)$ satisfies $\hat{\mathtt{v}}(\sigma) \geq 0$ for all $\sigma \in \mathbb{R}, \sigma \geq \sigma_0$.*
(3) *The sequence $(v^{-l}, v^{-l+1}, \ldots, v^{0}, \mathtt{v}_{reg}(0), \mathtt{v}_{reg}^{(1)}(0), \mathtt{v}_{reg}^{(2)}(0), \ldots)$ is lexicographically nonnegative.*

Also the following statements are equivalent:

(1) $\mathtt{v}$ *is the zero distribution.*
(2) *The Laplace transform $\hat{\mathtt{v}}(s)$ is the zero function.*
(3) *The sequence $(v^{-l}, v^{-l+1}, \ldots, v^{0}, \mathtt{v}_{reg}(0), \mathtt{v}_{reg}^{(1)}(0), \mathtt{v}_{reg}^{(2)}(0), \ldots)$ is the zero sequence.*

Proof. The proof is evident. □

Let $(\mathtt{u}, \mathtt{x}, \mathtt{y})$ be an initial solution to (4.1) with initial state x_0. The Laplace transforms of $\mathtt{u}$, $\mathtt{y}$, denoted by $\hat{\mathtt{u}}(s)$, $\hat{\mathtt{y}}(s)$, are rational and satisfy

$$\hat{\mathtt{y}}(s) = C(sI - A)^{-1}x_0 + [C(sI - A)^{-1}B + D]\hat{\mathtt{u}}(s) \text{ and } \hat{\mathtt{y}}(s) \perp \hat{\mathtt{u}}(s) \tag{5.1}$$

for all $i \in \bar{k}$; moreover, there exists a $\sigma_0 \in \mathbb{R}$ such that

$$\hat{\mathtt{y}}(\sigma) \geq 0, \ \hat{\mathtt{u}}(\sigma) \geq 0 \tag{5.2}$$

for all $\sigma \in \mathbb{R}$, $\sigma \geq \sigma_0$. The converse is true as well, so the Laplace transforms are rational and satisfy (5.1)–(5.2) if and only if the corresponding time functions define an initial solution to (4.1).

The above observations result in the formulation of the *rational complementarity problem* (RCP) (terminology introduced in [29]). Note that the formulation of the RCP here is a relaxation of the one in [29] because we allow general rational solutions.

Rational complementarity problem (RCP(x_0)). Let a system description (A, B, C, D) and initial state x_0 be given. Find rational vector functions $y(s)$ and $u(s)$ such that the equalities

$$y(s) = C(sI - A)^{-1}x_0 + [C(sI - A)^{-1}B + D]u(s) \text{ and } y(s) \perp u(s) \tag{5.3}$$

hold for all $i \in \bar{k}$, and there exists a $\sigma_0 \in \mathbb{R}$ such that for all $\sigma \geq \sigma_0$ we have

$$y(\sigma) \geq 0, \;\; u(\sigma) \geq 0. \tag{5.4}$$

If $(u(s), y(s))$ is a solution to RCP(x_0), any index set $J \subseteq \bar{k}$ satisfying $u_{J^c}(s) = 0$ and $y_J(s) = 0$ represents a mode J in which an initial solution exists. Hence, it is easily observed that due to the one-to-one relation between initial solutions and solutions to the corresponding RCP the set of possible continuation modes $\mathcal{S}(x_0)$ must be equal to $\mathcal{S}_{\mathrm{RCP}}(x_0)$, where

$$\begin{aligned} \mathcal{S}_{\mathrm{RCP}}(x_0) = \{I \subseteq \bar{k} \mid & \exists (u(s), y(s)) \text{ solution to RCP}(x_0) \\ & \text{such that } u_{I^c}(s) = 0 \text{ and } y_I(s) = 0\}. \end{aligned} \tag{5.5}$$

A second algebraic mode selection method can be derived by using the power series expansion of the solutions to RCP(x_0). This is described next.

5.2. Linear dynamic complementarity problem (LDCP). If $(u(s), y(s))$ is a solution to RCP(x_0), then it necessarily has to satisfy $u_{I^c}(s) = 0$ and $y_I(s) = 0$ for some $I \subseteq \bar{k}$. Consequently,

$$\begin{aligned} 0 &= R_{I\bullet}(s)x_0 + G_{II}(s)u_I(s), \\ y_{I^c}(s) &= R_{I^c\bullet}(s)x_0 + G_{I^cI}(s)u_I(s), \end{aligned}$$

where $G(s)$ is the proper transfer function $C(sI - A)^{-1}B + D$ and $R(s)$ is the strictly proper rational matrix $C(sI - A)^{-1}$. Note that $G_{II}(s)$ is invertible by Assumption 4.1. This implies that $u_I(s) = -G_{II}^{-1}(s)R_{I\bullet}(s)x_0$ and

$$y_{I^c}(s) = [R_{I^c\bullet}(s) - G_{I^cI}(s)G_{II}^{-1}(s)R_{I\bullet}(s)]x_0.$$

It follows from the representation theory of rational matrix functions (see, for instance, [18]) that the degree of the polynomial part of $G_{II}^{-1}(s)$ is at most n. Hence, the polynomial parts of the rational functions $u(s)$ and $y(s)$ have degree at most $n - 1$. In terms of time-domain solutions, this means that only derivatives of the Dirac function up to order $n - 1$ can appear in initial solutions. Thus we can write

$$y(s) = \sum_{i=-n+1}^{\infty} y^i s^{-i} \tag{5.6}$$

and likewise for $u(s)$. To translate the nonnegativity conditions (5.4) to the coefficients of the power series expansion around infinity, we use that $y(s)$ is nonnegative for all sufficiently large real s, if and only if

$$(y^{-n+1}, y^{-n+2}, \ldots) \succeq 0 \tag{5.7}$$

and similarly for $u(s)$.

Given the system description (A, B, C, D), the *Markov parameters* of the system are defined by

$$H^i = \begin{cases} D, & \text{if } i = 0, \\ CA^{i-1}B & \text{if } i = 1, 2, \dots. \end{cases} \tag{5.8}$$

Note that

$$G(s) = \sum_{i=0}^{\infty} H^i s^{-i}. \tag{5.9}$$

Using the power series expansions of $y(s)$ and $u(s)$ and (5.9), RCP(x_0) can be reformulated as the *linear dynamic complementarity problem* (LDCP) (terminology introduced in [29]) by considering the coefficients corresponding to equal powers of s. The formulation here extends the concept of LDCP as introduced in [29], because impulsive motions are included.

Linear dynamic complementarity problem (LDCP$_\kappa(x_0)$). Let a system description (A, B, C, D), an integer $\kappa \geq -n+1$, and an initial state x_0 be given. Let H^i, $i \geq 0$, be given by (5.8). Find sequences $(y^{-n+1}, y^{-n+2}, \dots, y^\kappa)$ and $(u^{-n+1}, u^{-n+2}, \dots, u^\kappa)$ such that the equations

$$y^i = \sum_{j=-n+1}^{i} H^{i-j} u^j \text{ if } -n+1 \leq i \leq \min(0, \kappa), \tag{5.10a}$$

$$y^i = CA^{i-1}x_0 + \sum_{j=-n+1}^{i} H^{i-j} u^j \text{ if } 1 \leq i \leq \kappa \tag{5.10b}$$

are satisfied, and for all indices $i \in \bar{k}$ at least one of the following is true:

$$(y_i^{-n+1}, y_i^{-n+2}, \dots, y_i^\kappa) = 0 \text{ and } (u_i^{-n+1}, u_i^{-n+2}, \dots, u_i^\kappa) \succeq 0, \tag{5.11}$$

$$(y_i^{-n+1}, y_i^{-n+2}, \dots, y_i^\kappa) \succeq 0 \text{ and } (u_i^{-n+1}, u_i^{-n+2}, \dots, u_i^\kappa) = 0. \tag{5.12}$$

LDCP$_\infty(x_0)$ denotes the problem of finding vector sequences $(u^j)_{j=-n+1}^\infty$ and $(y^j)_{j=-n+1}^\infty$ that satisfy LDCP$_\kappa(x_0)$ for all $\kappa \geq -n+1$.

If $(u^j)_{j=-n+1}^\kappa$ and $(y^j)_{j=-n+1}^\kappa$ form a solution to LDCP$_\kappa(x_0)$, then index sets $J \subseteq \bar{k}$ satisfying (5.11), $i \in J$ and (5.12), $i \in J^c$ represent candidate modes for selection.

The complete set of candidates for selection, denoted by $\mathcal{S}_{\text{LDCP}}^\kappa(x_0)$, is defined by

$$\mathcal{S}_{\text{LDCP}}^\kappa(x_0) := \{J \subseteq \bar{k} \mid \exists (u^j)_{j=-n+1}^\kappa, (y^j)_{j=-n+1}^\kappa \text{ solution to LDCP}_\kappa(x_0) \text{ such that } (5.11) \text{ holds for } i \in J \text{ and } (5.12) \text{ holds for } i \in J^c\}.$$

THEOREM 5.2. *Let a system (A, B, C, D) be given. The following statements are equivalent when Assumption* 4.1 *holds:*

(1) *The equations* (4.1) *have an initial solution for initial state x_0.*
(2) *RCP(x_0) has a solution.*
(3) *LDCP$_\infty(x_0)$ has a solution.*

There is a one-to-one correspondence between initial solutions to (4.1)*, solutions to RCP*(x_0)*, and solutions to LDCP*$_\infty(x_0)$*. Furthermore, for all* $x_0 \in \mathbb{R}^n$*,*

$$\mathcal{S}(x_0) = \mathcal{S}_{RCP}(x_0) = \mathcal{S}^\infty_{LDCP}(x_0).$$

Proof. From the derivation of RCP, it follows that 1 and 2 are equivalent. If $(u(s), y(s))$ is a solution to RCP(x_0), then the coefficients of the power series expansion of this solution around infinity form a solution to LDCP$_\infty(x_0)$. Hence, 2 implies 3.

To see that 3 implies 1, suppose that $(y^{-n+1}, y^{-n+2}, \dots)$, $(u^{-n+1}, u^{-n+2}, \dots)$ is a solution to LDCP$_\infty(x_0)$. Take $I \subseteq \bar{k}$ such that (5.11) holds for $i \in I$ and (5.12) holds for $i \in I^c$. Define $p(0) := x_0 + \sum_{i=0}^{n-1} A^i B u^{-i}$. We first show that $p(0) \in V_I$. To this end, note that $y^i_I = 0$ and $u^i_{I^c} = 0$ for all $i \in \{-n+1, -n+2, \dots\}$. From (5.10b), it follows that $p(0)$ satisfies

$$\begin{array}{lllll}
0 = y^1_I = C_{I\bullet}p(0) & +D_{II}v(0), & & & \\
0 = y^2_I = C_{I\bullet}Ap(0) & +D_{II}v(1) & +C_{I\bullet}B_{\bullet I}v(0), & & \\
\vdots & \vdots & & & \\
0 = y^\kappa_I = C_{I\bullet}A^{\kappa-1}p(0) & +D_{II}v(\kappa-1) & +C_{I\bullet}B_{\bullet I}v(\kappa-2) + \dots + C_{I\bullet}A^{\kappa-2}B_{\bullet I}v(0), & & \\
\vdots & \vdots & \vdots & &
\end{array} \tag{5.13}$$

where $v(i) = u^{i+1}_I$, $i \geq 0$. Combining algorithm (3.2) and the equations above, it follows that for $l \geq 0$ the states $A^l p(0) + \sum_{i=0}^{l-1} A^i B_{\bullet I} v(l-1-i)$ belong to $V_j(A, B_{\bullet I}, C_{I\bullet}, D_{II})$, $j \geq 0$. In particular, for $l = 0$ this means that $p(0) \in \lim V^j(A, B_{\bullet I}, C_{I\bullet}, D_{II}) = V_I$. Hence, there exists a smooth solution $(\mathtt{u}_{reg}, \mathtt{x}_{reg}, \mathtt{y}_{reg})$ to (4.2) for mode I with initial state $x(0) = p(0)$.

By differentiating (4.2) in time and evaluating the resulting equalities at time instant 0 for the solution $(\mathtt{u}_{reg}, \mathtt{x}_{reg}, \mathtt{y}_{reg})$, we observe that $\tilde{v}(i) := \mathtt{u}^{(i)}_{reg,I}(0)$, $i = 0, 1, \dots$ satisfies (5.13) as well. To show that this implies that $\tilde{v}^i = v^i$ for all i, observe that due to (5.13) both sequences satisfy the discrete-time analogue of the first two lines of (4.3), i.e.,

$$p(i+1) = Ap(i) + B_{\bullet I}v(i), \qquad 0 = C_{I\bullet}p(i) + D_{II}v(i), \qquad i = 0, 1, 2, \dots, \tag{5.14}$$

with initial state $p(0)$. The difference $w(i) := v(i) - \tilde{v}(i)$ satisfies (5.14) with initial state 0. We introduce the formal z-transform

$$w(z) := \sum_{i=0}^{\infty} w^i z^{-i}.$$

Using the z-transform $G_{II}(z)$ of the discrete-time system (see, e.g., [19]), we get $0 = G_{II}(z)w(z)$. The invertibility of $G_{II}(z)$ implies that $w(z) = 0$ and hence, $v(i) = \tilde{v}(i)$ for all $i \geq 0$, or equivalently, $u^{i+1}_I = \mathtt{u}^{(i)}_{reg,I}(0)$, $i \geq 0$. This also implies that $y^{i+1} = \mathtt{y}^{(i)}_{reg}(0), i \geq 0$.

We define $\mathtt{u} := \sum_{i=0}^{n-1} u^{-i}\delta^{(i)} + \mathtt{u}_{reg}$, $\mathtt{y} := \sum_{i=1}^{n-1} y^i \delta^{(i-1)} + \mathtt{y}_{reg}$ and let $\mathtt{x}$ be the solution to $\dot{\mathtt{x}} = A\mathtt{x} + B\mathtt{u} + x_0\delta$. Obviously, $(\mathtt{u}, \mathtt{x}, \mathtt{y})$ satisfies 1 in Definition 4.7. We have only to show that 2 in Definition 4.7 is satisfied. Since $(y^{-n+1}, y^{-n+2}, \dots) = (y^{-n+1}, \dots, y^0, \mathtt{y}^{(0)}_{reg}(0), \mathtt{y}^{(1)}_{reg}(0), \dots)$ and $(u^{-n+1}, u^{-n+2}, \dots) = (u^{-n+1}, \dots, u^0, \mathtt{u}^{(0)}_{reg}, \mathtt{u}^{(1)}_{reg}, \dots)$ form a solution to LDCP$_\infty(x_0)$, (5.11) or (5.12) is satisfied for all $i \in \bar{k}$.

According to Lemma 5.1, this is equivalent to $\mathtt{u}$ and $\mathtt{y}$ being initially nonnegative. Consequently $(\mathtt{u}, \mathtt{x}, \mathtt{y})$ is an initial solution with initial state x_0.

The one-to-one correspondence follows easily from the above because solutions to RCP and initial solutions are related through Laplace transform and its inverse. Solutions to RCP are uniquely transformed to solutions to LDCP by taking the coefficients of a power series expansion around infinity. Moreover, a solution to LDCP is linked to an initial solution by setting the derivatives of an initial solution at zero equal to the LDCP solution as stated above (see also Remark 5.3). The final statement is a result of the one-to-one correspondence. ☐

Remark 5.3. Note that in the proof of Theorem 5.2, a direct link between initial solutions and solutions to $\mathrm{LDCP}_\infty(x_0)$ is given. If $(\mathtt{u}, \mathtt{x}, \mathtt{y})$ is an initial solution with $\mathtt{u} = \sum_{i=0}^{n-1} u^{-i}\delta^{(i)} + \mathtt{u}_{reg}$ and $\mathtt{y} = \sum_{i=0}^{n-1} y^{-i}\delta^{(i)} + \mathtt{y}_{reg}$ for initial state x_0, define $\tilde{u}^i := u^i$, $i = -n+1, \ldots, 0$, and $\tilde{u}^{i+1} = \mathtt{u}_{reg}^{(i)}(0)$, $i \geq 0$, and let $\tilde{y}^i$, $i \geq -n+1$, be defined analogously. Then $(\tilde{u}^i)_{i=-n+1}^{\infty}$, $(\tilde{y}^i)_{i=-n+1}^{\infty}$ is a solution to $\mathrm{LDCP}_\infty(x_0)$. We shall use the transformations between $\mathrm{LDCP}_\infty(x_0)$, $\mathrm{RCP}(x_0)$, and initial solutions frequently. The above proof also yields an alternative way of deriving the LDCP: differentiate the initial solution with incorporation of the impulsive part and evaluate the results at time instant zero. For smooth continuations, this method can also be used in the nonlinear case [29, 21].

In the above theorem it is shown that the infinite version of LDCP can be used to select the correct modes. However, under suitable conditions, already the finite version $\mathrm{LDCP}_n(x_0)$ selects the right modes, where n is the dimension of the state variable (see Theorem 6.10 below). In [10], it has been shown that $\mathrm{LDCP}_\kappa(x_0)$ for finite κ is a special case of the generalized linear complementarity problem (GLCP) [8] and the extended linear complementarity problem (ELCP) [9]. In [8], an algorithm is proposed to find all solutions to GLCP. Such algorithms can be used to efficiently solve the LDCP.

6. Well-posedness results.

Due to the multimodal and nonlinear behavior of LCS, basic questions like existence and uniqueness of solutions given an initial state are nontrivial. It is not difficult to find LCS for which no solution exists from certain initial conditions or for which the solution is not unique (see [28]). In this section we will derive conditions guaranteeing local well-posedness as defined below.

DEFINITION 6.1. *The complementarity system* (4.1) *is (locally) well-posed if for each initial state there exists an* $\varepsilon > 0$ *such that a unique solution on* $[0, \varepsilon)$ *in the sense of Definition* 4.10 *exists.*

An equivalent way of defining well-posedness is by requiring that for each state there exists a unique solution on an interval of positive length starting with either a finite number of jumps or an infinite number of jumps with convergence of the event states followed by smooth continuation on that interval.

DEFINITION 6.2. *Let* (A, B, C, D) *be a system with Markov parameters* H^i, $i = 0, 1, 2, \ldots$. *The leading column indices* $\eta_1, \ldots, \eta_k$ *of the linear system* (A, B, C, D) *are defined for* $j \in \bar{k}$ *as*

$$\eta_j := \inf\{i \in \mathbb{N} \mid H^i_{\bullet j} \neq 0\},$$

with the convention $\inf \emptyset = \infty$. *The leading row indices* $\rho_1, \ldots, \rho_k$ *of* (A, B, C, D) *are defined for* $j \in \bar{k}$ *as*

$$\rho_j := \inf\{i \in \mathbb{N} \mid H^i_{j \bullet} \neq 0\}.$$

Since we consider only invertible transfer functions (see Assumption 4.1 and Lemma 3.3), the leading row and column indices are all finite. Due to the Cayley–Hamilton theorem, we even have $\rho_i \leq n$ and $\eta_i \leq n$. The *leading row coefficient matrix* $\mathcal{M}(A,B,C,D)$ and *leading column coefficient matrix* $\mathcal{N}(A,B,C,D)$ for the system (A,B,C,D) are defined as

$$\mathcal{M}(A,B,C,D) := \begin{pmatrix} H_{1\bullet}^{\rho_1} \\ \vdots \\ H_{k\bullet}^{\rho_k} \end{pmatrix} \text{ and } \mathcal{N}(A,B,C,D) := (H_{\bullet 1}^{\eta_1} \dots H_{\bullet k}^{\eta_k}), \tag{6.1}$$

respectively. We omit the arguments (A,B,C,D) if they are clear from the context.

The main result of this section is stated as follows. Recall that a square matrix is a p-matrix, if all of its principal minors are strictly positive (see section 3).

THEOREM 6.3. *If the leading row coefficient matrix $\mathcal{M}$ and the leading column coefficient matrix $\mathcal{N}$ are both P-matrices, then the linear complementarity system* (4.1) *is well-posed. From each initial condition, at most one state jump occurs before smooth continuation is possible, i.e., the multiplicity of an event time is at most one.*

Remark 6.4. The definition of well-posedness often includes continuous dependence of solutions on initial conditions. Such continuous dependence is not claimed in the above theorem. An example of a linear complementarity system that displays discontinuous dependence on initial conditions will be given in section 8.

Remark 6.5. Local existence does not imply "global existence" (i.e., on an a priori specified interval $[0, T_e)$). There is a problem when the event times have a right-accumulation point $\tau^* < T_e$ and there is no limit for $x_c(t)$ as $t \uparrow \tau^*$. In fact, this is the only phenomenon that may prevent a local well-posed system from being globally well-posed. Note that local uniqueness of solutions and "global uniqueness" are equivalent using the solution concept of Definition 4.10.

To prove the main result, we first need some auxiliary results.

LEMMA 6.6. *If the leading row coefficient matrix $\mathcal{M}$ has only nonzero principal minors, then Assumption* 4.1 *is satisfied, i.e., all modes are autonomous. The same holds when the leading column coefficient matrix $\mathcal{N}$ has only nonzero principal minors.*

Proof. Lemma 3.3 states that it is sufficient to show that $G_{II}(s)$ is invertible for all $I \subseteq \bar{k}$. For notational convenience, we assume $I = \bar{l}$ for some $l \in \bar{k}$. If $\mathcal{M}$ has only nonzero principal minors, then $\mathcal{M}_{II}$ is invertible. Hence, $G_{II}(s) = \operatorname{diag}(s^{-\rho_1}, \dots, s^{-\rho_l})V(s)$, where $V(s)$ is a biproper matrix, because $V(\infty) = \mathcal{M}_{II}$ is invertible [13, Thm. 4.5]. The reasoning is analogous for the case in which $\mathcal{N}$ has only nonzero minors. ◻

DEFINITION 6.7. *A state x_0 of the complementarity system* (4.1) *is called regular if there exists a smooth initial solution with initial state x_0.*

A state x_0 is regular if and only if $\mathrm{RCP}(x_0)$ has a strictly proper solution or, equivalently, x_0 is regular if and only if $\mathrm{LDCP}_\infty(x_0)$ has a solution with $u^{-n+1} = \dots = u^0 = 0$.

Under the assumption that the leading row coefficient matrix is a P-matrix, the following result characterizes the regular states. The result is an extension of a similar result in [29] which was derived under the additional assumption of "uniform relative degree" (i.e., $\rho_1 = \rho_2 = \dots = \rho_k = \rho$). In contrast to [29] we restrict ourselves here to the linear case, but an extension to the nonlinear case is straightforward.

THEOREM 6.8. *Let a system (A,B,C,D) be given. Suppose that the leading row coefficient matrix $\mathcal{M}$ is a P-matrix. Then $x_0 \in \mathbb{R}^n$ is a regular state if and only if for* *all* $i \in \bar{k}$

$$(C_{i\bullet}x_0, C_{i\bullet}Ax_0, \ldots, C_{i\bullet}A^{\rho_i-1}x_0) \succeq 0. \tag{6.2}$$

Moreover, the smooth continuation is unique.

Proof. Note that $y_i^{(j)}(0) = C_{i\bullet}A^j x_0, j = 0, \ldots, \rho_i - 1$, $i = 1, \ldots, k$, independently of the choice of a smooth input u. Hence, the above condition is necessary to guarantee $y(t) \geq 0$, $t \in [0, \varepsilon)$ for some positive ε.

To prove the converse, we will show that if for all $i \in \bar{k}$ (6.2) holds, the corresponding $\text{LDCP}_\infty(x_0)$ has a solution with $u^{-n+1} = \cdots = u^0 = 0$. This is sufficient to show that a smooth initial solution exists. The idea of the proof is to reduce the $\text{LDCP}_\infty(x_0)$ to a series of LCPs that can all be solved uniquely. This idea originates in [21].

We will show that $\text{LDCP}_\infty(x_0)$ with the additional requirement $y^{-n+1} = \cdots = y^0 = 0$, $u^{-n+1} = \cdots = u^0 = 0$ has a unique solution. From such a solution, it is immediately clear that (5.10a) is satisfied. The remaining equalities can be written as

$$y_i^j = C_{i\bullet}A^{j-1}x_0,\ j = 1, 2, \ldots, \rho_i,\ i = 1, \ldots, k, \tag{6.3}$$

and

$$\begin{pmatrix} y_1^{\rho_1+p} \\ \vdots \\ y_k^{\rho_k+p} \end{pmatrix} = \xi_p(x_0, u^1, \ldots, u^{p-1}) + \mathcal{M}u^p, \tag{6.4}$$

where ξ_1, $\xi_2, \ldots$ are certain linear functions. We denote by $L(l)$, $l \in \mathbb{N}$, the truncated problem of finding u^j, $j = 1, \ldots, l$, and y_i^j, $i \in \bar{k}$, $j = 1, \ldots, \rho_i + l$, satisfying (6.3) and (6.4), $p \in \{1, \ldots, l\}$ together with the requirement that for all indices $i \in \bar{k}$ at least one of the following statements is true:

$$(y_i^1, y_i^2, \ldots, y_i^{\rho_i+l}) = 0 \text{ and } (u_i^1, u_i^2, \ldots, u_i^l) \succeq 0, \tag{6.5}$$

$$(y_i^1, y_i^2, \ldots, y_i^{\rho_i+l}) \succeq 0 \text{ and } (u_i^1, u_i^2, \ldots, u_i^l) = 0. \tag{6.6}$$

The problem $L(l)$ is a subproblem of $\text{LDCP}_\infty(x_0)$ and if we find a solution $(y^1, y^2, \ldots)$, $(u^1, u^2, \ldots)$ satisfying $L(l)$ for all $l \geq 0$, then this solution is a solution to the corresponding $\text{LDCP}_\infty(x_0)$ with $y^{-n+1} = \cdots = y^0 = 0$, $u^{-n+1} = \cdots = u^0 = 0$.

We claim that $L(l)$ has a unique solution for all $l \geq 0$. This is obvious for $l = 0$. We will proceed by induction in the same way as in [29, 21].

We write I_l, J_l, K_l for the active (input) index set, the inactive index set, and the undecided index set, respectively, determined by $L(l)$. Formally, for $l \geq 1$, $I_l = \{i \in \bar{k} \mid (u_i^1, \ldots, u_i^l) \succ 0\}$, $J_l = \{i \in \bar{k} \mid (y_i^1, \ldots, y_i^{\rho_i+l}) \succ 0\}$, and $K_l = \bar{k} \setminus (I_l \cup J_l)$ with y_i^j, $i = 1, \ldots, k,\ j = 1, \ldots, \rho_i + l$, and u^i, $i = 1, \ldots, l$ determined (uniquely) by $L(l)$. For convenience we also define $I_0 := \emptyset, J_0 = \{i \in \bar{k} \mid (y_i^1, \ldots, y_i^{\rho_i}) \succ 0\}$ and $K_0 = \bar{k} \setminus J_0$.

Note that $L(l-1)$ is a subproblem of $L(l)$, so variables uniquely determined by $L(l-1)$ are automatically uniquely specified for $L(l)$. As a consequence, I_{l-1}, J_{l-1}, K_{l-1} are determined as well. Comparing $L(l)$ with $L(l-1)$, we observe that $L(l)$ has one additional equation: (6.4) for $p = l$. We divide this equation into the three parts given by I_{l-1}, J_{l-1}, and K_{l-1}. For notational convenience, we omit all indices depending on l and all superscripts:

$$\begin{pmatrix} y_I \\ y_J \\ y_K \end{pmatrix} = \begin{pmatrix} z_I \\ z_J \\ z_K \end{pmatrix} + \begin{pmatrix} \mathcal{M}_{II} & \mathcal{M}_{IJ} & \mathcal{M}_{IK} \\ \mathcal{M}_{JI} & \mathcal{M}_{JJ} & \mathcal{M}_{JK} \\ \mathcal{M}_{KI} & \mathcal{M}_{KJ} & \mathcal{M}_{KK} \end{pmatrix} \begin{pmatrix} u_I \\ u_J \\ u_K \end{pmatrix}, \tag{6.7}$$

with $z = \xi_l(x_0, u^1, \ldots, u^{l-1})$. From the definition of I_{l-1}, J_{l-1}, and K_{l-1}, we get $y_I = 0$ and $u_J = 0$ because (6.5) or (6.6) should hold. By substituting this result in (6.7), we obtain

$$0 = z_I + \mathcal{M}_{II} u_I + \mathcal{M}_{IK} u_K, \tag{6.8}$$

$$y_J = z_J + \mathcal{M}_{JI} u_I + \mathcal{M}_{JK} u_K, \tag{6.9}$$

$$y_K = z_K + \mathcal{M}_{KI} u_I + \mathcal{M}_{KK} u_K. \tag{6.10}$$

Since $\mathcal{M}_{II}$ is a principal submatrix of a P-matrix, it is invertible and hence we get from (6.8) that $u_I = -\mathcal{M}_{II}^{-1}(z_I + \mathcal{M}_{IK} u_k)$. Substituting this expression in (6.10) leads to

$$y_K = z_K - \mathcal{M}_{KI}\mathcal{M}_{II}^{-1} z_I + (\mathcal{M}_{KK} - \mathcal{M}_{KI}\mathcal{M}_{II}^{-1}\mathcal{M}_{IK}) u_K. \tag{6.11}$$

Due to (6.5) and (6.6) and the definition of K_{l-1}, the complementarity conditions

$$0 \leq u_K \perp y_K \geq 0 \tag{6.12}$$

hold. Thus, (6.11) and (6.12) constitute an LCP. Since $\mathcal{M}_{KK} - \mathcal{M}_{KI}\mathcal{M}_{II}^{-1}\mathcal{M}_{IK}$ is a Schur complement of a P-matrix, it is itself a P-matrix by Proposition 2.3.5 in [7]. According to Theorem 3.4, the corresponding LCP has a unique solution. From u_K we can compute u_I and y_J. Hence, the induction hypothesis has been proven for l. So we find a solution of LDCP$_\infty(x_0)$ with $u^{-n+1} = \cdots = u^0 = 0$, $y^{-n+1} = \cdots = y^0 = 0$, and hence a smooth initial solution corresponding to x_0 exists. Since the solution to LDCP$_\infty(x_0)$ with $u^{-n+1} = \cdots = u^0 = 0$ is unique, the one-to-one correspondence between initial solutions and solutions of LDCP$_\infty(x_0)$ implies that the corresponding smooth initial solution is unique. ☐

One can even prove that the initial solution corresponding to a regular initial state is unique and thus smooth. Our next result is concerned with the uniqueness of solutions emanating from a not necessarily regular initial state.

THEOREM 6.9. *Let a system (A, B, C, D) be given. If the leading column coefficient matrix $\mathcal{N}$ is a P-matrix, then for every state x_0 and every $\kappa \geq 0$, the problem LDCP$_\kappa(x_0)$ has a solution that is unique except for u_i^j, $i \in \bar{k}$, $j = \kappa - \eta_i + 1, \ldots, \kappa$, which are left undetermined. Furthermore, $u_i^{-n+1} = u_i^{-n+2} = \cdots = u_i^{-\eta_i} = 0$, $i \in \bar{k}$, and $y^{-n+1} = \cdots = y^0 = 0$.*

Proof. The proof is based on separation of the equalities (5.10) into two parts, (5.10a) and (5.10b), providing the equations for y^i, $i = -n+1, \ldots, 0$, and y^i, $i = 1, \ldots, \kappa$, respectively. For both parts we start an induction that is analogous to the one used in the previous proof: we reduce the LDCP to a series of LCPs which can be solved uniquely. This is done by selecting certain equations from (5.10) for each successive LCP in such a way that only principal submatrices of the leading column coefficient matrix $\mathcal{N}$ appear in these LCPs.

We introduce the index sets $O_j := \{i \in \bar{k} \mid \eta_i = j\}$, $j = 0, 1, \ldots, n$, and $S_j := \bigcup_{i=0}^{j} O_i$, $j = 0, 1, \ldots, n$. So, the η_jth Markov parameter is the first Markov parameter in which the jth column is nonzero. O_j is the set of indices i for which the ith column in the sequence of Markov parameters $(H^0, H^1, \ldots)$ is nonzero for the first time in H^j. S_j is the set of indices i for which the matrix $(H^0_{\bullet i}, H^1_{\bullet i}, \ldots, H^j_{\bullet i})$ is nonzero. As noted before, $\eta_i \leq n$. Hence, $S_n = \bar{k}$. By definition, $H^i_{\bullet S_j^c} = 0$, $i \leq j$, and $S_0 \subseteq S_1 \subseteq S_2 \subseteq \cdots \subseteq S_n$.

After suitable permutation of rows and columns if necessary, there are integers $k_0, \ldots, K_{n+1}$ with $0 = k_0 \leq k_1 \leq k_2 \leq \cdots \leq k_n \leq k_{n+1} = k$ such that $O_j = \{k_j + 1, \ldots, k_{j+1}\}$, $j = 0, 1, \ldots, n$. Then

$$\mathcal{N} = [H^0_{\bullet O_0}\ H^1_{\bullet O_1}, \ldots,\ H^n_{\bullet O_n}].$$

We claim that for $1 \leq r \leq n$ the problem LDCP$_{-n+r}(x_0)$ has a solution with

$$u^{-n+1}_{S_{r-1}} = u^{-n+2}_{S_{r-2}} =, \ldots, = u^{-n+r}_{S_0} = 0, \tag{6.13}$$

$$y^{-n+1} = y^{-n+2} =, \ldots, = y^{-n+r} = 0. \tag{6.14}$$

The remaining variables $u^{-n+1}_{S^c_{r-1}}, u^{-n+2}_{S^c_{r-2}}, \ldots, u^{-n+r}_{S^c_0}$ are left undetermined. This will be the induction hypothesis.

For $r = 1$, we have only the equation

$$y^{-n+1} = H^0 u^{-n+1}, \tag{6.15}$$

with the complementarity conditions $0 \leq y^{-n+1} \perp u^{-n+1} \geq 0$. The complementarity conditions follow from the fact that for each index either (5.11) or (5.12) should hold. Since $H^0_{\bullet S^c_0} = 0$, (6.15) reduces to

$$y^{-n+1} = H^0_{\bullet S_0} u^{-n+1}_{S_0}. \tag{6.16}$$

Since $u^{-n+1}_{S^c_0}$ does not appear in this equation, it is left completely undetermined (except for the condition $u^{-n+1}_{S^c_0} \geq 0$). Considering (6.16) and the complementarity conditions only for y^{-n+1}_i, $i \in S_0$, results in the LCP

$$\begin{aligned} &y^{-n+1}_{S_0} = H^0_{S_0 S_0} u^{-n+1}_{S_0} = \mathcal{N}_{S_0 S_0} u^{-n+1}_{S_0}, \\ &0 \leq y^{-n+1}_{S_0} \perp u^{-n+1}_{S_0} \geq 0. \end{aligned}$$

Since $\mathcal{N}_{S_0 S_0}$ is a principal submatrix of $\mathcal{N}$, it is a P-matrix. Theorem 3.4 then implies that the above LCP has a unique solution. Obviously, $y^{-n+1}_{S_0} = 0$, $u^{-n+1}_{S_0} = 0$ is the unique solution. From (6.16), $y^{-n+1} = 0$ follows immediately. This proves the induction hypothesis for $r = 1$.

Suppose that the induction hypothesis above holds for $r - 1$, where $2 \leq r \leq n$. Since LDCP$_{-n+r-1}(x_0)$ is a subproblem of LDCP$_{-n+r}(x_0)$, we consider only the additional equality in (5.10):

$$\begin{aligned} y^{-n+r} &= H^0 u^{-n+r} + H^1 u^{-n+r-1} + \cdots + H^{r-1} u^{-n+1} \\ &= H^0_{\bullet S_0} u^{-n+r}_{S_0} + H^1_{\bullet S_1} u^{-n+r-1}_{S_1} + \cdots + H^{r-1}_{\bullet S_{r-1}} u^{-n+1}_{S_{r-1}} \\ &= H^0_{\bullet S_0} u^{-n+r}_{S_0} + H^1_{\bullet S_1 \setminus S_0} u^{-n+r-1}_{S_1 \setminus S_0} + \cdots + H^{r-1}_{\bullet S_{r-1} \setminus S_{r-2}} u^{-n+1}_{S_{r-1} \setminus S_{r-2}} \\ &= H^0_{\bullet O_0} u^{-n+r}_{O_0} + H^1_{\bullet O_1} u^{-n+r-1}_{O_1} + \cdots + H^{r-1}_{\bullet O_{r-1}} u^{-n+1}_{O_{r-1}}. \end{aligned} \tag{6.17}$$

The second equality follows from $H^i_{\bullet S^c_i} = 0$; the third one follows from the induction hypothesis (6.13). The last equality is a consequence of $S_j \setminus S_{j-1} = O_j$. Since $u^{-n+1}_{S^c_{r-1}}, u^{-n+2}_{S^c_{r-2}}, \ldots, u^{-n+r}_{S^c_0}$ do not appear in this additional equation, these variables remain undetermined.

Equation (6.17) consists of k scalar equations. Considering only the equalities for y_i^{-n+r}, $i \in S_{r-1}$, we find

$$
\begin{aligned}
y_{S_{r-1}}^{-n+r} &= \begin{pmatrix} H_{S_{r-1}O_0}^0 & H_{S_{r-1}O_1}^1, \dots, H_{S_{r-1}O_{r-1}}^{r-1} \end{pmatrix} \begin{pmatrix} u_{O_0}^{-n+r} \\ u_{O_1}^{-n+r-1} \\ \vdots \\ u_{O_{r-1}}^{-n+1} \end{pmatrix} \\
&= \mathcal{N}_{S_{r-1}S_{r-1}} \underbrace{\begin{pmatrix} u_{O_0}^{-n+r} \\ u_{O_1}^{-n+r-1} \\ \vdots \\ u_{O_{r-1}}^{-n+1} \end{pmatrix}}_{=:v_{-r}}.
\end{aligned}
$$

Since (5.11) or (5.12) should hold for all i, it follows that

$$0 \le y_{S_{r-1}}^{-n+r} \perp v_{-r} \ge 0.$$

This is the LCP we are looking for. Since $\mathcal{N}_{S_{r-1}S_{r-1}}$ (as a submatrix of $\mathcal{N}$) is also a P-matrix, the above LCP has a unique solution (Theorem 3.4). Hence, this solution must be $v_{-r} = y_{S_{r-1}}^{-n+r} = 0$. Using this in (6.17) shows that $y^{-n+r} = 0$. In combination with the induction hypothesis for $r-1$, this yields the hypothesis for r. This completes our induction step and hence the proof of our first claim.

To complete the proof, we start a second induction with hypothesis as stated in the formulation of the theorem. Note that this is equivalent to saying that $\mathrm{LDCP}_\kappa(x_0)$ has a unique solution for every state x_0; only $u_{S_0^c}^\kappa, u_{S_1^c}^{\kappa-1}, \dots, u_{S_{n-1}^c}^{\kappa-n+1}$ are left undetermined. For $\kappa = 0$ this hypothesis is true, for it follows from the previous induction by taking $r = n$. Suppose the hypothesis is true for $\kappa - 1, \kappa \ge 1$. Since $\mathrm{LDCP}_{\kappa-1}(x_0)$ is a subproblem of $\mathrm{LDCP}_\kappa(x_0)$, the variables $u_{S_0}^{\kappa-1}, \dots, u_{S_{n-1}}^{\kappa-n}, u^{\kappa-n-1}, \dots, u^{-n+1}$ are already uniquely determined. We set

$$
\begin{aligned}
I &:= \{i \in \bar{k} \mid (u_i^{-n+1}, u_i^{-n+2}, \dots, u_i^{\kappa-\eta_i-1}) \succ 0\}, \\
J &:= \{i \in \bar{k} \mid (y_i^{-n+1}, y_i^{-n+2}, \dots, y_i^{\kappa-1}) \succ 0\}, \text{ and} \\
K &:= \bar{k} \setminus (I \cup J).
\end{aligned}
$$

In comparison with $\mathrm{LDCP}_{\kappa-1}(x_0)$, $\mathrm{LDCP}_\kappa(x_0)$ has the additional equality

$$y^\kappa = \sigma(x_0, u_{S_0}^{\kappa-1}, u_{S_1}^{\kappa-2}, \dots, u_{S_{n-1}}^{\kappa-n}, u^{\kappa-n-1}, \dots, u^{-n+1}) + \mathcal{N} \begin{pmatrix} u_{O_0}^\kappa \\ u_{O_1}^{\kappa-1} \\ \vdots \\ u_{O_{n-1}}^{\kappa-n+1} \end{pmatrix}$$

for some function σ. Splitting this equation into three parts according to the index sets I, J, K, we can follow the same reasoning as in the proof of Theorem 6.8 to conclude that $y^\kappa, u_{O_0}^\kappa, u_{O_1}^{\kappa-1}, \dots, u_{O_{n-1}}^{\kappa-n+1}$ are uniquely determined and thus we prove the induction hypothesis for κ. ∎

We are now in a position to prove Theorem 6.3.

Proof of Theorem 6.3. Lemma 6.6 implies that all modes are autonomous. Take an arbitrary initial state x_0. It follows from Theorem 6.9 that $\mathrm{LDCP}_\infty(x_0)$ has a unique solution which satisfies $u_i^{-n+1} = u_i^{-n+2} = \cdots = u_i^{-\eta_i} = 0$, $i \in \bar{k}$, and $y^{-n+1} = \cdots = y^0 = 0$. Due to the one-to-one correspondence between initial solutions and solutions to $\mathrm{LDCP}_\infty(x_0)$, an initial solution $(\mathtt{u}, \mathtt{x}, \mathtt{y})$ exists and the solution must be unique as well. In case the initial condition is regular, the initial solution is smooth. In other cases, we have to prove that after the state jump corresponding to $(\mathtt{u}, \mathtt{x}, \mathtt{y})$ smooth continuation is possible. Stated otherwise, we have to show that the re-initialized state $\mathtt{x}(0+)$ is regular. The re-initialization is given by the impulsive part $\mathtt{u}_{imp} = \sum_{i=0}^{n-1} u^{-i}\delta^{(i)}$, where the coefficients u^{-i} follow from $\mathrm{LDCP}_\infty(x_0)$. Since the impulsive part is unique, the re-initialization is unique; it results in $\mathtt{x}(0+) := x_0 + \sum_{i=0}^{n-1} A^i B u^{-i}$ (see (3.8)). The complementarity conditions (5.11) and (5.12) imply that $(y^1, y^2, \ldots, y^n) \succeq 0$. The right-hand side of (5.10) contains for $y_i^1, \ldots, y_i^{\rho_i}$, $i \in \bar{k}$, only coefficients corresponding to the impulsive part, i.e., only $u^0, \ldots, u^{-n+1}$. Hence, observe that $(C_{i\bullet}x(0+), \ldots, C_{i\bullet}A^{\rho_i - 1}x(0+)) = (y_i^1, \ldots, y_i^{\rho_i}) \succeq 0$, $i \in \bar{k}$. According to Theorem 6.8, $\mathtt{x}(0+)$ is a regular state. So after at most one re-initialization, (unique) smooth continuation is guaranteed. ∎

The next theorem states that in case $\mathcal{N}$ is a P-matrix, it is sufficient to consider $\mathrm{LDCP}_n(x_0)$ (instead of $\mathrm{LDCP}_\infty(x_0)$) for selection of a mode. Hence, only an algebraic problem with a finite number of constraints has to be solved.

THEOREM 6.10. *Let a system (A, B, C, D) be given. If the leading column coefficient matrix $\mathcal{N}$ is a P-matrix, then from every initial state there exists a unique initial solution to* (4.1). *This solution evolves in mode I, where $I := \{i \in \bar{k} \mid (u_i^{-n+1}, u_i^{-n+2}, \ldots, u_i^{n-\eta_i}) \succ 0\}$, where $(u^j)_{j=-n+1}^n$, $(y^j)_{j=-n+1}^n$ constitutes a solution to $LDCP_n(x_0)$.*

Proof. Let $(y^{-n+1}, y^{-n+2}, \ldots, y^n)$ and $(u^{-n+1}, u^{-n+2}, \ldots, u^n)$ be a solution to $\mathrm{LDCP}_n(x_0)$ and let I be defined as in the formulation of the theorem. Define $p(0) := x_0 + \sum_{i=0}^{n-1} A^i B u^{-i}$. Note that this is the state after the jump induced by the impulsive distribution $\sum_{i=0}^{n-1} u^{-i}\delta^{(i)}$ starting from x_0. It follows from the definition of I that $(u_i^{-n+1}, \ldots, u_i^{n-\eta_i}) = 0$, $i \in I^c$, and in combination with (5.11), (5.12) the same definition yields $(y_i^{-n+1}, \ldots, y_i^n) = 0$, $i \in I$. Using (5.10b), we conclude that $p(0)$ satisfies

$$\begin{array}{llll} 0 = y_I^1 = C_{I\bullet}p(0) & +D_{II}v(1), & & \\ 0 = y_I^2 = C_{I\bullet}Ap(0) & +D_{II}v(2) & +C_{I\bullet}B_{\bullet I}v(1), & \\ \vdots & \vdots & & \ddots \\ 0 = y_I^n = C_{I\bullet}A^{n-1}p(0) & +D_{II}v(n) & +C_{I\bullet}B_{\bullet I}v(n-1) + \cdots + C_{I\bullet}A^{n-2}B_{\bullet I}v(1), & \end{array} \tag{6.18}$$

with $v(i) = u_I^i$. By using (3.2) and the equations above, we can show that for all $j = 0, 1, \ldots, n$ the vector $p(0) \in V_j(A, B_{\bullet I}, C_{I\bullet}, D_{II})$. So $p(0) \in \lim V_j(A, B_{\bullet I}, C_{I\bullet}, D_{II}) = V_n(A, B_{\bullet I}, C_{I\bullet}, D_{II}) = V_I$, for the algorithm converges within n steps (similarly as in the proof of Theorem 5.2). Hence, there exists a regular solution $(\mathtt{u}_{reg}, \mathtt{x}_{reg}, \mathtt{y}_{reg})$ to (4.2) in mode I with initial state $p(0)$. We define

$$\tilde{\mathtt{u}} := \sum_{i=0}^{n-1} u^{-i}\delta^{(i)} + \mathtt{u}_{reg},$$
$$\tilde{y} := \mathtt{y}_{reg}.$$

Furthermore, $\tilde{\mathtt{x}}$ denotes the solution to (4.2) in mode I corresponding to $\tilde{\mathtt{u}}$ and initial state x_0. Note that according to Theorem 6.9 $y^{-n+1} = \cdots = y^0 = 0$. Obviously, this is a solution to (4.2) in mode I; so it remains only to show that $\tilde{\mathtt{u}}$, $\tilde{\mathtt{y}}$ are initially nonnegative. We shall do this by proving that $\mathtt{u}_{reg}^{(i)}(0) = u^{i+1}$ for all $i = 0, 1, \ldots, n - \eta_i - 1$ and consequently, $\mathtt{y}_{reg}^{(i)}(0) = y^{i+1}$.

Notice that both $v(i) = u_{reg,I}^{(i-1)}(0)$, $i = 1, \ldots, n$, and $v(i) = u_I^i$, $i = 1, \ldots, n$, satisfy (6.18). We extend the solution of $\mathrm{LDCP}_n(x_0)$ with zeros to get an infinite sequence $(u^{-n+1}, \ldots, u^n, 0, 0, \ldots)$. The difference $w(i) = u_{reg,I}^{(i)}(0) - u_I^{i+1}$, $i \geq 0$, can be taken as an input to the discrete-time system

$$\begin{aligned} q(i+1) &= Aq(i) + B_{\bullet I} w(i), \quad q(0) = 0, \\ \bar{y}(i) &= C_{I\bullet} q(i) + D_{II} w(i) \end{aligned} \tag{6.19}$$

satisfying $\bar{y}(0) = \cdots = \bar{y}(n-1) = 0$. Taking the z-transform of the discrete-time system (6.19) (see, e.g., [19]) with input $w(i)$ gives (with some abuse of notation, the z-transform of w is denoted by $w(z)$)

$$G_{II}(z) w(z) = \sum_{i=0}^{\infty} \bar{y}(i) z^{-i} = z^{-n} p(z) \tag{6.20}$$

for some proper rational vector function $p(z)$. For notational simplicity, we set $I = \bar{l}$, $l \in \bar{k}$. Since $\mathcal{N}_{II}$ is a P-matrix (and hence invertible), $G_{II}(z)$ can be written as

$$G_{II}(z) = V_2(z) \mathrm{diag}(z^{-\eta_1}, \ldots, z^{-\eta_l}), \tag{6.21}$$

where V_2 is biproper (i.e., proper rational with proper rational inverse), because $V_2(\infty) = \mathcal{N}_{II}$ is invertible (see Theorem 4.5 in [13]). Hence, (6.20) yields

$$w(z) = G_{II}^{-1}(z) p(z) = \mathrm{diag}(z^{-\eta_1 - n}, \ldots, z^{-\eta_l - n}) \tilde{p}(z),$$

where $\tilde{p}(z) = V_2^{-1}(z) p(z)$ is proper. The definition of $w(i)$ now implies that $\mathtt{u}_{reg,j}^{(i)}(0) = u_j^{i+1}$, $j \in I$, $i = 0, 1, \ldots, n - \eta_j - 1$.

Since for $j \in I$,

$$(u_j^{-n+1}, \ldots, u_j^0, \mathtt{u}_{reg,j}^{(0)}(0), \ldots, \mathtt{u}_{reg,j}^{(n-\eta_i-1)}(0)) = (u_j^{-n+1}, \ldots, u_j^{n-\eta_i}) \succ 0$$

the distribution $\tilde{\mathtt{u}}_j \in C_{imp}$ is initially positive for $j \in I$. Note that $\tilde{\mathtt{y}}_I = 0$ by construction of $\tilde{\mathtt{y}}$: $\tilde{\mathtt{y}} = \mathtt{y}_{reg}$ satisfies, together with $\mathtt{u}_{reg}$, the condition (4.2) for mode I and initial state $p(0)$. Similarly, for $j \in I^c$, $\tilde{\mathtt{u}}_j = 0$. Note that

$$(y^{-n+1}, \ldots, y^0, \mathtt{y}_{reg}^{(0)}, \ldots, \mathtt{y}_{reg}^{(n-1)}) = (y^{-n+1}, \ldots, y^n) \succeq 0$$

because $\mathtt{u}_{reg,j}^{(i)} = u_j^{i+1}$ for $j \in I$ and $i = 0, 1, \ldots, n - \eta_j - 1$. Hence, if $(y_j^{-n+1}, \ldots, y_j^n) \succ 0$, then $\tilde{\mathtt{y}}_j \in C_{imp}$ is initially positive. For $j \in I^c$, it may happen that $(y_j^{-n+1}, \ldots, y_j^n) = 0$; however, this implies that $\tilde{\mathtt{y}}_j$ is identically zero. To see this, note that $\mathtt{y}_{reg,I^c}$ can be written as the output of the system

$$\begin{aligned} \dot{x} &= (A + BF_I)x, \\ \mathtt{y}_{reg,I^c} &= (C_{I^c} + D_{I^c\bullet} F_I)x \end{aligned}$$

because the input u satisfying (4.2) can be given in feedback form by $u(t) = F_I x(t)$ (see section 4). By the Cayley–Hamilton theorem and because the state space dimension of the system is equal to n, $(y_j^{-n+1}, \ldots, y_j^0, \mathtt{y}_{reg,j}(0), \mathtt{y}_{reg,j}^{(1)}(0), \ldots, \mathtt{y}_{reg,j}^{(n-1)}(0)) = 0$ implies

$$(y_j^{-n+1}, y_j^{-n+2}, \ldots, y_j^0, \mathtt{y}_{reg,j}(0), \mathtt{y}_{reg,j}^{(1)}(0), \ldots) = 0.$$

Since $\mathtt{y}_{reg,j}$ is a Bohl function, $\tilde{\mathtt{y}}_j = \mathtt{y}_{reg,j} \in C_{imp}$ is identically zero (see Lemma 5.1). Hence, $(\tilde{\mathtt{u}}, \tilde{\mathtt{x}}, \tilde{\mathtt{y}})$ is an initial solution to (4.1).

Uniqueness follows from the fact that $\mathrm{LDCP}_\infty(x_0)$ has a unique solution (Theorem 6.9). Indeed, the one-to-one correspondence between initial solutions and solutions to $\mathrm{LDCP}_\infty(x_0)$ implies that there is only one initial solution, which must evolve in the above mode. □

Remark 6.11. Since $\mathrm{LDCP}_\infty(x_0)$ has a unique solution, the mode I as defined in the previous theorem (selected by $\mathrm{LDCP}_n(x_0)$) is obviously contained in $\mathcal{S}^\infty_{\mathrm{LDCP}}(x_0) = \mathcal{S}_{\mathrm{RCP}}(x_0)$. Since there is only one corresponding initial solution, it evolves in all the modes contained in $\mathcal{S}^\infty_{\mathrm{LDCP}}(x_0)$. Hence, all selected index sets in $\mathcal{S}^\infty_{\mathrm{LDCP}}(x_0)$ are appropriate. Of course, the additional modes contained in $\mathcal{S}^\infty_{\mathrm{LDCP}}(x_0)$ are characterized by the undetermined index set K as in Remark 4.8.

Remark 6.12. Solving $\mathrm{LDCP}_n(x_0)$ can be simplified by using Theorem 6.9. This theorem states that the variables $y^{-n+1}, y^{-n+2}, \ldots, y^0$ and $u_i^{-n+1}, u_i^{-n+2}, \ldots, u_i^{-\eta_i}$, $i \in \bar{k}$, can immediately be set to zero, which reduces the number of equations to be solved.

In the section below, we illustrate the above theory by means of the two-carts example.

7. Algorithm for constructing solutions. In this section, a method will be proposed to construct analytical solutions to LCS. The method will be illustrated by applying it to the two-carts example of section 2. We emphasize that it is not the purpose of this paper to give a *numerical* scheme for the simulation of complementarity systems, although the analytical algorithm may be used as a guideline for the development of such a scheme.

The algorithm is described by the following procedure.

ALGORITHM 7.1. Let x_0 be the initial state and T_e the final time.

Initialization: Set $z := x_0$, $\mathcal{E} := \{0\}$, and $t' := 0$ as the initial state and time.

Step one: Select for initial state z a mode $I \in \mathcal{S}(z)$.

Step two: Consider the following two possibilities:

(1) From the state z smooth continuation is possible in mode I, i.e., $z \in V_I$. Go to Step four.

(2) No smooth continuation is possible in mode I from z, i.e., $z \notin V_I$. Go to Step three.

Step three: Compute the projection P_I of z along T_I onto V_I (see subsection 4.2). Set $z := P_I z$. Go to Step one.

Step four: Compute the solution $(\mathtt{u}^{z,I}, \mathtt{x}^{z,I}, \mathtt{y}^{z,I})$ (see subsection 4.1).

Step five: Determine the next event time $\theta(z, I)$. Define $(u_c(t), x_c(t), y_c(t)) := (\mathtt{u}^{z,I}(t - t'), \mathtt{x}^{z,I}(t - t'), \mathtt{y}^{z,I}(t - t'))$ for $t \in (t', t' + \theta(z, I))$. Set $t' := t' + \theta(z, I)$, $\mathcal{E} := \mathcal{E} \cup \{t'\}$ and $z := x_c(t'-)$. If $t' \geq T_e$ the algorithm terminates. Otherwise, go to Step one.

The algorithm can be visualized by the flow diagram as given by Figure 7.1.

Remark 7.2. Algorithm 7.1 produces a solution on $[0, T_e)$ if the following conditions are satisfied:

(1) The algorithm does not get into a situation with $t' < T_e$ and $\mathcal{S}(z) = \emptyset$. Such a situation is called "deadlock."

(2) All encountered event times have a finite multiplicity. Stated otherwise, the algorithm does not end up in an infinite loop consisting of only re-initializations and mode selections, where a limiting operation is required.

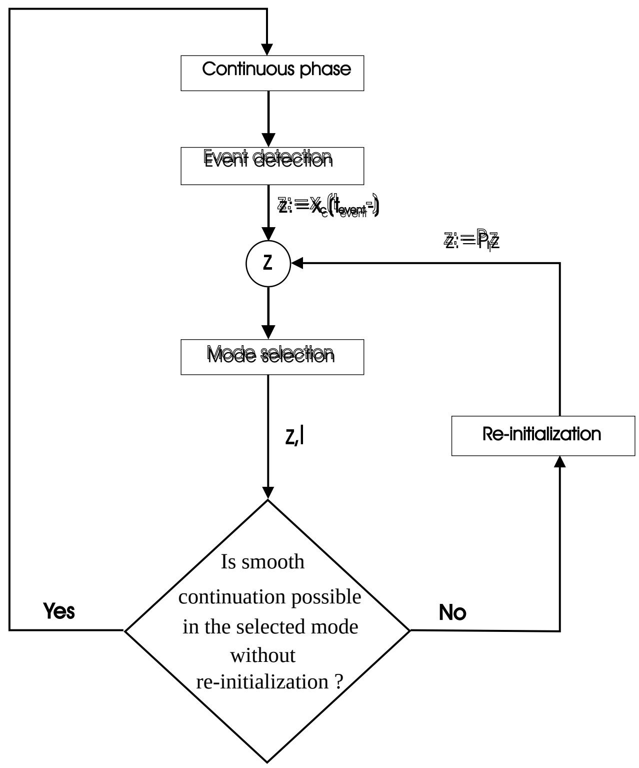

FIG. 7.1. *Schematic description of complete dynamics.*

(3) The event times do not have a finite accumulation point strictly smaller than T_e.

THEOREM 7.3. *Let a system* (A, B, C, D) *be given satisfying the conditions of Theorem* 6.3. *Algorithm* 7.1 *produces a solution on* $[0, \infty)$ *if and only if accumulation of events does not occur.*

Proof. By Theorem 6.3 the first two conditions mentioned in Remark 7.2 are satisfied (deadlock cannot occur and the maximal multiplicity of an event time is one). Therefore the result follows. ∎

Returning to the two-carts system of section 2, we suppose that the initial state equals

$$x_0 = e^{-A}(0 \ \ -1 \ \ -1 \ \ 0)^\top \approx (0.3202, -0.4335, 0.3716, -1.0915)^\top$$

and $T_e = 3$. Note that for this system the Markov parameters are given by $H^0 = H^1 = 0$ and $H^2 = \mathcal{M} = \mathcal{N} = 1$. Hence, the two-carts system satisfies the sufficient conditions for local well-posedness presented in this paper. Consequently, Algorithm 7.1 can fail only if the set of event times contains a finite accumulation point $\tau < 3$. According to Algorithm 7.1, we start by setting $\mathcal{E} := \{0\}$, $z := x_0$, and $t' := 0$:

Step one: This step selects the unconstrained mode ($I = \emptyset \in \mathcal{S}(z)$) because the only initial solution for initial state z is $(\mathtt{u}, \mathtt{x}, \mathtt{y})$ given by $(0, e^{At}z, Ce^{At}z)$. Note that $\mathtt{y}$ is initially nonnegative because $\mathtt{y}(0+) = x_{01} \approx 0.3202$ is equal to the distance of the cart from the stop which is strictly positive.

Step two: This step leads to the decision that smooth continuation in the selected mode is possible because $z \in V_\emptyset = \mathbb{R}^4$ (every state is consistent for the unconstrained mode).

Step four: The unconstrained dynamics is specified by a linear ordinary differential equation; the solution is equal to $\mathtt{u}^{z,I}(t) = 0$, $\mathtt{x}^{z,I}(t) = e^{At}z$, $\mathtt{y}^{z,I}(t) = Ce^{At}z$.

Step five: Determining the zero crossing of $\mathtt{y}^{z,I}$ gives $\theta(z, I) := 1$. The corresponding state is equal to $(0, -1, -1, 0)^\top$, which is not regular for the unconstrained mode. Note that $\mathtt{y}^{z,I}_{reg}(1) = 0$, $\dot{\mathtt{y}}^{z,I}_{reg}(1) < 0$, so continuing in the unconstrained mode would violate the inequality constraint $y(t) \geq 0$. Hence, $u_c(t) = 0$, $x_c(t) = e^{A(t-1)}(0\ \ -1\ \ -1\ 0)^\top$, $y_c(t) = Ce^{At}(0\ \ -1\ \ -1\ 0)^\top$ for $t \in (0, 1)$, $\mathcal{E} = \{0, 1\}$, $t' := 1$, and $z := (0\ \ -1\ \ -1\ 0)^\top$. Since $t' < T_e$, we go to Step one.

Step one: For the purpose of illustrating mode selection by RCP, the dynamical system is transformed to the Laplace domain:

$$(s^4 + 3s^2 + 1)y(s) = \begin{pmatrix} s(s^2+1), & s, & s^2+1, & 1 \end{pmatrix} \begin{pmatrix} x_{10} \\ x_{20} \\ x_{30} \\ x_{40} \end{pmatrix} + (s^2+1)u(s). \tag{7.1}$$

Substituting z for $(x_{10}, x_{20}, x_{30}, x_{40})^\top$ results in

$$(s^4 + 3s^2 + 1)y(s) = -s - s^2 - 1 + (s^2+1)u(s).$$

Since $y(s)$ or $u(s)$ should be zero, there are only two possibilities:

$$\begin{aligned} \text{unconstrained mode: } & u(s) = 0\ ;\ y(s) = \frac{-s^2 - s - 1}{s^4 + 3s^2 + 1}, \\ \text{constrained mode: } & y(s) = 0\ ;\ u(s) = 1 + \frac{s}{s^2+1}. \end{aligned}$$

Since the RCP requires nonnegativity for sufficiently large values of the indeterminate s, the combination $y(s) = 0$, $u(s) = 1 + \frac{s}{s^2+1}$ is the unique solution to RCP(z); thus $\mathcal{S}(z) = \mathcal{S}_{\mathrm{RCP}}(z) = \{\{1\}\}$. Hence, the constrained mode must be selected ($I := \{1\}$).

Step two: Since the solution to RCP(z) is not strictly proper, the answer to the question in the decision block in Figure 7.1 is negative, so we have to re-initialize.

Step three: Using (3.2) and (3.5), we can compute the consistent states and the jump space

$$T_{\{1\}} = \mathrm{Im} \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix};\ V_{\{1\}} = \mathrm{Ker} \begin{pmatrix} 1\,0\,0\,0 \\ 0\,0\,1\,0 \end{pmatrix} = \mathrm{Im} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

To re-initialize we have to project z onto $V_{\{1\}}$ along $T_{\{1\}}$, which results in

$$z := P_{\{1\}}z = P^{T_{\{1\}}}_{V_{\{1\}}} z = (0, -1, 0, 0)^\top.$$

Step one: We have to solve RCP(z):

$$(s^4 + 3s^2 + 1)y(s) = -s + (s^2+1)u(s)$$

together with the complementarity conditions. The only solution is $y(s) = 0$, $u(s) = \frac{s}{s^2+1}$ resulting in $I := \{1\}$.

Step two: Since the solution to RCP(z) is strictly proper, smooth continuation in the selected mode is possible. The physical interpretation is clear: the left cart hits the stop. Instantaneously, the velocity is put to zero and the right cart keeps the left cart pushed against the stop.

Step four: The dynamics of the constrained mode is given by a set of DAEs. However, these can easily be translated into an ODE (note that there must exist a linear mapping $F_{\{1\}}$ such that $u(t) = F_I x(t)$ satisfies the mode dynamics; see subsection 4.1). The input u must be chosen in such a way that it keeps y identically zero. Since $y = x_1$, $\dot{y} = x_3$, $\ddot{y} = 2x_1 + x_2 + u$, u should equal $-2x_1 - x_2$. (Note that $F_I = (-2\ -1\ 0\ 0)$ is a possible choice, but is not the only choice. $F_I = (\alpha\ -1\ \beta\ 0)$ is an alternative for every α and β, because $x_1 = x_3 = 0$ for consistent states.) Hence, the dynamics in the constrained mode is given by $x_1 = x_3 = 0$, $\ddot{x}_2 = -x_2$, $u = -x_2$. Solving this set of equations for initial state z gives $\mathtt{u}^{z,I}(t) = \cos t$, $\mathtt{x}_1^{z,I}(t) = 0$, $\mathtt{x}_2^{z,I}(t) = -\cos t$, and $\mathtt{y}^{z,I}(t) = 0$. Note that we could have also concluded this by taking the inverse Laplace transform of the solution $(u(s), y(s))$ to the RCP in the last mode selection.

Step five: An event is detected at $\theta(z, I) = \inf\{t \geq 0 \mid \cos(t) < 0\} = \frac{\pi}{2}$. The piece of $(u_c(t), x_c(t), y_c(t))$ on $(1, 1 + \frac{\pi}{2})$ is given by the initial solution above as described in Algorithm 7.1. $\mathcal{E} := \{0, 1, 1 + \frac{\pi}{2}\}$, $t' := 1 + \frac{\pi}{2}$, and $z := (0, 0, 0, 1)^\top$. Since $t' < 3 = T_e$, we proceed with Step one.

Step one: This time LDCP will be demonstrated as a mode selection tool. Since the conditions of Theorem 6.10 are satisfied, a finite version of the LDCP can be used for mode selection: LDCP$_4(z)$ reads

$$
\begin{aligned}
y^{-3} &= 0,\\
y^{-2} &= 0,\\
y^{-1} &= u^{-3},\\
y^{0} &= u^{-2},\\
y^{1} &= u^{-1} - 2u^{-3},\\
y^{2} &= u^{0} - 2u^{-2} + u^{-3},\\
y^{3} &= u^{1} - 2u^{-1} + u^{-2} + 3u^{-3},\\
y^{4} &= 1 + u^{2} - 2u^{0} + u^{-1} + 3u^{-2} - 3u^{-3},
\end{aligned}
$$

together with complementarity conditions (5.11) and (5.12). Setting $y^i = 0$, $i \in \{-3, \dots, 4\}$, leads to $(u^{-3}, \dots, u^1, u^2) = (0, \dots, 0, -1) \prec 0$. Hence, (5.11) does not hold. It is obvious that setting $u^i = 0$, $i \in \{-3, \dots, 4\}$, leads to $(y^{-3}, \dots, y^3, y^4) = (0, \dots, 0, 1) \succeq 0$ so that (5.12) holds. Hence, $\mathcal{S}^4_{\mathrm{LDCP}}(z) = \{\emptyset\}$ and the unconstrained mode must be selected ($I := \emptyset$).

Step two: Since the impulsive part of u is zero, i.e., $u^{-3} = u^{-2} = u^{-1} = u^0 = 0$, smooth continuation is possible. This can also be observed from the fact that $(0, 0, 0, 1)^\top$ is a consistent state for the unconstrained mode. In terms of the physical system, the right cart is on the right of its equilibrium and pulls the left cart away from the stop.

Steps four and five: Determining a new piece of $(u_c(t), x_c(t), y_c(t))$ leads to $u_c(t) = 0$, $x_c(t) = e^{A(t-1-\frac{\pi}{2})}(0, 0, 0, 1)^\top$, and $y_c(t) = Ce^{A(t-1-\frac{\pi}{2})}(0, 0, 0, 1)^\top$ in the same way as before. The next event time $1 + \frac{\pi}{2} + \theta(z, I)$ is strictly larger than $T_e = 3$ so that the algorithm halts with a complete solution on $[0, 3)$.

The computed trajectory is plotted in Figure 7.2. Note the complementarity between u and x_1 and the discontinuity in the derivative of x_1 at time $t = 1$.

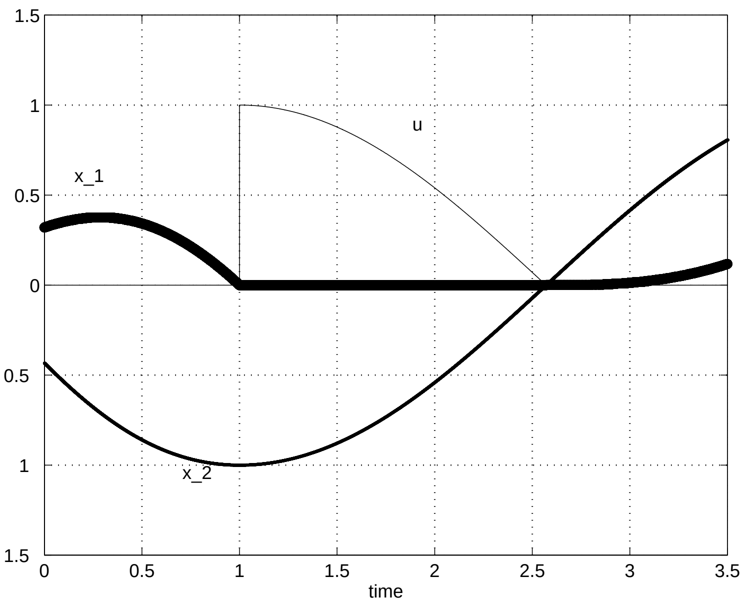

FIG. 7.2. *Solution trajectory of two-carts system.*

To show that the particular mode transition mentioned in section 2 can be handled properly by the proposed algorithm, we take the initial state $z_0 = x_0 = (0, 1, -1, 0)^\top$ (labeling of z_0 as in (4.9)). Substituting this initial condition in (7.1) results in

$$(s^4 + 3s^2 + 1)y(s) = s - s^2 - 1 + (s^2 + 1)u(s).$$

Solving RCP(z_0) (Step one) leads to $y(s) = 0$ and $u(s) = 1 - \frac{s}{s^2+1}$ and so $\mathcal{S}_{\mathrm{RCP}}(z_0) = \{\{1\}\}$. We select the constrained mode ($I_1 = \{1\}$). Smooth continuation is not possible in the selected mode (Step two), because the solution to RCP is not strictly proper. Re-initialization (Step three) leads to $z_1 := P_{\{1\}} z_0 = (0, 1, 0, 0)^\top$. RCP$(z_1)$ has to be considered (Step one):

$$(s^4 + 3s^2 + 1)y(s) = s + (s^2 + 1)u(s).$$

Notice that setting $y(s)$ equal to zero results in $u(s) = -\frac{s}{s^2+1}$, the strictly proper part of the solution of RCP(x_0). This is not a valid choice. The only solution is $u(s) = 0$ and $y(s) = \frac{s}{(s^4+3s^2+1)}$, which corresponds to the unconstrained mode, i.e., $I_2 = \emptyset$. Since the solution of RCP(z_1) is strictly proper, smooth continuation is possible in the unconstrained mode (Step two) and we can go to Steps four and five to compute the smooth continuation.

8. Mechanical systems. In this section, it will be shown that the proposed mode selection rule coincides with the one of Moreau [22, 23] when these rules are applied to the class of systems that are covered by both frameworks; to wit, linear mechanical systems.

We will focus on linear mechanical systems whose dynamics in free motion are given by the differential equations

$$M\ddot{q}(t) + D\dot{q}(t) + Kq(t) = 0, \tag{8.1}$$

where q denotes the vector of generalized coordinates. Furthermore, M denotes the generalized mass matrix, which is assumed to be positive definite, D denotes the damping matrix, and K is the elasticity matrix. The system is subject to unilateral constraints given by

$$Eq(t) \geq 0, \tag{8.2}$$

where E has full row rank. Furthermore, we assume that impacts are purely inelastic.

To obtain a complementarity formulation, we introduce the constraint forces u needed to satisfy the unilateral constraints and introduce the state vector $x = \mathrm{col}(q, \dot{q})$. According to the rules of classical mechanics, the system can then be written as follows (with omission of all time arguments):

$$\dot{x} = \underbrace{\begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{pmatrix}}_{A} x + \underbrace{\begin{pmatrix} 0 \\ M^{-1}E^{\top} \end{pmatrix}}_{B} u, \tag{8.3a}$$

$$y = \underbrace{(E \quad 0)}_{C} x, \tag{8.3b}$$

$$0 \leq y \perp u \geq 0 \tag{8.3c}$$

for all i. This systems satisfies $\rho_i = \eta_i = 2$, $i \in \bar{k}$; note that $\mathcal{M} = \mathcal{N} = EM^{-1}E^{\top}$ is positive definite and hence a P-matrix (Theorem 3.5). Hence, the system is well-posed (Theorem 6.3).

We consider only initial states $x_0 = \mathrm{col}(q_0, \dot{q}_0)$ with $Eq_0 \geq 0$. We call these points feasible. In the two-carts system, this means that we do not consider initial states for which the left cart starts on the left of the stop. In Moreau's sweeping process (see [23, 22]) no jumps occur in q itself, but jumps can occur in the velocities $\dot{q}$. These jumps are governed by the following minimization problem, where $J := \{i \in \bar{k} \mid E_{i\bullet} q_0 = 0\}$.

MINIMIZATION PROBLEM 8.1. *Let an initial state $x_0 = col(q_0, \dot{q}_0)$ be given. The new state after re-initialization, denoted by $x(0+) = col(q(0+), \dot{q}(0+))$, is determined by*

$$q(0+) = q_0,$$
$$\dot{q}(0+) = \arg \min_{\{w \mid E_{J\bullet} w \geq 0\}} \frac{1}{2}(w - \dot{q}_0)^{\top} M (w - \dot{q}_0).$$

The notation "arg min" denotes the set of vectors in the constrained set that minimize the criterion over the constrained set. Note that the minimization problem has a unique solution. The problem reflects a kind of "principle of economy": among the kinematically admissible right velocities, the one is chosen that is nearest in the kinetic metric [22, p. 75]. Observe that if we prove that jumps in our formulation correspond to the above minimization problem, then it follows that the feasible set $\{x \in \mathbb{R}^n \mid Cx \geq 0\}$ is invariant under the dynamics as introduced in section 4, since the smooth dynamics do not take the solution outside this set.

The Kuhn–Tucker conditions [17] for the minimization problem give necessary conditions for optimality. The vector $\dot{q}(0+)$ is the minimizing argument only if there exists a Lagrange multiplier λ such that

$$M(\dot{q}(0+) - \dot{q}_0) - E_{J\bullet}^{\top}\lambda = 0, \tag{8.4}$$

$$0 \leq \lambda \perp E_{J\bullet}\dot{q}(0+) \geq 0. \tag{8.5}$$

The equality (8.4) is equivalent to

$$\dot{q}(0+) = \dot{q}_0 + M^{-1}E_{J\bullet}^\top \lambda, \tag{8.6}$$

and therefore $\dot{y}(0+) = E\dot{q}(0+)$ and λ satisfy the following LCP with $\dot{y}_0 := E\dot{q}_0$:

$$\dot{y}_J(0+) = \dot{y}_0 + E_{J\bullet}M^{-1}E_{J\bullet}^\top \lambda, \tag{8.7}$$

$$0 \leq \dot{y}_J(0+) \perp \lambda \geq 0. \tag{8.8}$$

According to Theorem 3.4, this LCP has a unique solution, because $E_{J\bullet}M^{-1}E_{J\bullet}^\top$ is a P-matrix. Since Minimization Problem 8.1 is convex, the Kuhn–Tucker conditions are even sufficient for optimality. Hence, the LCP (8.7)–(8.8) is equivalent to the minimization problem for determining the jumps. Notice that once this LCP is solved, the required jumps are known, because $\dot{q}(0+)$ then follows from (8.6).

We will prove now that LDCP$_n(x_0)$ (and hence also LDCP$_\infty(x_0)$ and RCP(x_0)) are equivalent to the optimization problem in the sense that both methods produce the same state jumps and select the same mode.

THEOREM 8.2. *For linear mechanical systems of the form* (8.3) *with M positive definite and E of full row rank, the re-initialization by means of LDCP$_n(x_0)$ (or LDCP$_\infty(x_0)$ or RCP(x_0)) agrees with Moreau's sweeping process* [22, 23] *for feasible initial states. Linear mechanical complementarity systems are well-posed.*

Proof. Since the row coefficient matrix and the column coefficient matrix are P-matrices, well-posedness follows from Theorem 6.3. Furthermore, Theorem 6.9 states that $u^{-2} = u^{-3} = \cdots = u^{-n} = 0$. Because we start from a feasible state x_0, it follows also that $u^{-1} = 0$. Indeed, the first relevant LCP in the LDCP$_n(x_0)$ (as in the proof of Theorem 6.8) is given by

$$y^1 = Cx_0 + CABu^{-1}$$

with the corresponding complementarity conditions. Since this LCP has a unique solution, the solution must satisfy $u^{-1} = 0$, because $Cx_0 \geq 0$. Hence, $y^{-n+1} = y^{-n+2} = \cdots = y^0 = 0$ and $y^1 = Cx_0$. The next relevant equality in (5.10) is

$$y^2 = CAx_0 + CABu^0. \tag{8.9}$$

We define J again as $\{i \in \bar{k} \mid C_i x_0 = 0\}$. Since one of the expressions (5.11) or (5.12) has to be satisfied for $i \in J$, the conditions

$$y_i^2 \geq 0,\ u_i^0 \geq 0,\ y_i^2 \perp u_i^0,\ i \in J$$

have to hold. Because $y_i^1 > 0$ for elements $i \in J^c$, $0 = u_i^0 = u_i^1 = \cdots = u_i^n$ must hold to satisfy (5.12). Considering only $i \in J$, we can write the LCP following from (8.9) and the above complementarity conditions:

$$y_J^2 = C_{J\bullet}Ax_0 + C_{J\bullet}AB_{\bullet J}u_J^0, \tag{8.10}$$

$$0 \leq y_J^2 \perp u_J^0 \geq 0. \tag{8.11}$$

This LCP is identical to the LCP (8.7)–(8.8). This shows that the re-initialization by means of LDCP$_n(x_0)$ leads to the same result as Minimization Problem 8.1. □

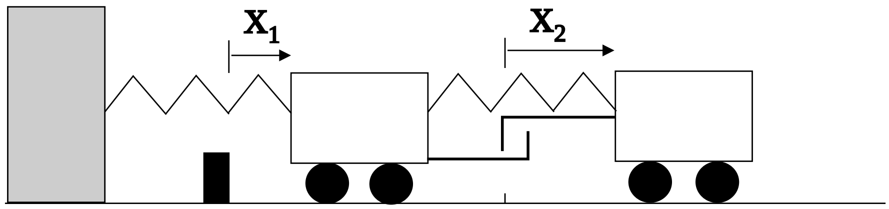

FIG. 8.1. *Two-carts system with hook.*

From this proof, we see that for feasible initial states only proper rational solutions to RCP occur; i.e., jumps take place only along Im B.

Example 8.3. To illustrate the equivalence of Moreau's rule and the complementarity rule, consider the two-carts system of section 2 extended with a hook. See Figure 8.1.

The complementarity description is given by

$$\begin{aligned}
\dot{x}_1(t) &= x_3(t),\\
\dot{x}_2(t) &= x_4(t),\\
\dot{x}_3(t) &= -2x_1(t) + x_2(t) + u_1(t) + u_2(t),\\
\dot{x}_4(t) &= x_1(t) - x_2(t) - u_2(t),\\
y_1(t) &:= x_1(t),\\
y_2(t) &:= x_1(t) - x_2(t),
\end{aligned}$$

where u_1, u_2 denote the reaction forces exerted by the stop and hook, respectively. These equations are completed by the complementarity conditions (4.1c). Taking

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};\ D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix};\ K = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix};\ E = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \tag{8.12}$$

leads to a description as in the beginning of this section.

Using the minimization problem to determine the re-initialization and mode selection in case of an initial state $(x_{10}, x_{20}, x_{30}, x_{40})^\top$ with $x_{10} = x_{20} = 0$ results in the alternatives shown in Figure 8.2. Note that the minimization problem consists of finding the minimal distance to the feasible set (area indicated by "unconstrained"). The arrows denote the re-initialization directions.

To illustrate that RCP(x_0) gives the same results, the equations corresponding to (5.1) are given below:

$$\begin{aligned}
(s^4 + 3s^2 + 1)y_1(s) &= (s^2 + 1)x_{30} + x_{40} + (s^2 + 1)u_1(s) + s^2 u_2(s),\\
(s^4 + 3s^2 + 1)y_2(s) &= s^2 x_{30} - (s^2 + 1)x_{40} + s^2 u_1(s) + (2s^2 + 1)u_2(s).
\end{aligned}$$

To determine the continuous states x_0 for which the stop-constrained mode ($I = \{1\}$) is selected, $y_1(s) \equiv 0$ and $u_2(s) \equiv 0$ are inserted in the equations above. Next we solve for $u_1(s)$ and $y_2(s)$, which leads to

$$\begin{aligned}
u_1(s) &= -x_{30} - \frac{1}{s^2 + 1} x_{40},\\
y_2(s) &= \frac{1}{s^4 + 3s^2 + 1} \left[-s^2 - 1 - \frac{-s^2}{s^2 + 1} \right] x_{40}.
\end{aligned}$$

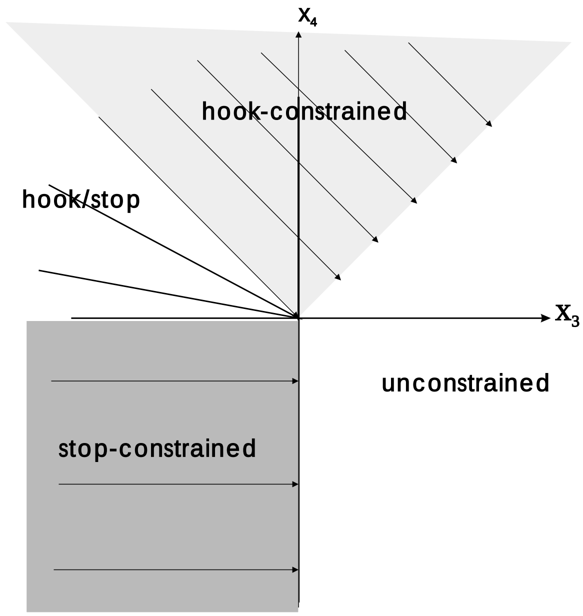

FIG. 8.2. *Re-initialization scheme.*

Entering the stop-constrained mode is allowed only if for sufficiently large values of the indeterminate s the above two expressions are nonnegative (see (5.2)). This requires $x_{30} \leq 0$ and $x_{40} \leq 0$. This indeed corresponds to the indicated area for the stop-constrained mode in Figure 8.2. Note that the polynomial parts of u_1 and u_2 equal $-x_{30}$ and 0, respectively. Hence, $\mathtt{u}_{imp} = (-x_{30},\ 0)^\top \delta$ for the corresponding initial solution $(\mathtt{u}, \mathtt{x}, \mathtt{y})$. According to (3.8), the state jump equals $B(-x_{30},\ 0)^\top = (0,\ 0,\ -x_{30},\ 0)^\top$. This agrees with the direction of the arrows in Figure 8.2. Similarly, the other modes and re-initialization directions can be verified.

This example shows also that the mode selection procedure that was suggested in [28] does not always agree with Moreau's sweeping process. It is proposed there that if I is the current mode and violation of (4.5) occurs at time τ in state $x(\tau)$, the new mode is given by

$$J := (I \setminus \Gamma_2) \cup \Gamma_1,$$

where

$$\Gamma_1 := \{i \in I^c \mid \mathtt{y}_{reg,i}^{x(\tau),I} < 0,\ t \in (\tau, \tau + \varepsilon) \text{ for some } \varepsilon > 0\},$$
$$\Gamma_2 := \{i \in I \mid \mathtt{u}_{reg,i}^{x(\tau),I} < 0,\ t \in (\tau, \tau + \varepsilon) \text{ for some } \varepsilon > 0\}.$$

In other words, this means that constraints that are active or inactive according to mode I will become inactive or active, respectively, if their corresponding inequalities would be violated by continuation of the solution in mode I. In the example, this means that if we are in the unconstrained mode ($I = \emptyset$) and we arrive in $x(\tau) = (0, 0, -1, 2)^\top$, the selected mode should be $J = \{1, 2\}$, the hook/stop constrained mode. This does not agree with the minimization problem illustrated in Figure 8.2, which indicates the hook-constrained mode. A physical argument against the choice in [28] in the indicated situation might be that removing the stop does not lead to violation of $y_1(t) \geq 0$.

The above example also illustrates the fact that the solutions of linear complementarity systems do not always depend continuously on the initial state. The discontinuous dependence is caused by the sensitivity of solutions to the order in which constraints become active. Consider the initial states $x_0(\varepsilon) = (\varepsilon, \varepsilon, -2, 1)^\top$, $\varepsilon \geq 0$. For $\varepsilon = 0$ the solution is a jump to $(0, 0, 0, 0)^\top$, after which the system stays in its equilibrium position. For $\varepsilon > 0$, first the hook becomes active, resulting in a jump to $(\varepsilon, \varepsilon, -\frac{1}{2}, -\frac{1}{2})^\top$. This is followed by a regular continuation in the hook-constrained mode until the left cart hits the stop. The state just before the impact is $(0, 0, -\frac{1}{2} + g(\varepsilon), -\frac{1}{2} + g(\varepsilon))^\top$ for some continuous function $g(\varepsilon)$ with $g(0) = 0$. Reinitialization yields the new state $(0, 0, 0, -\frac{1}{2}+g(\varepsilon))^\top$, which converges to $(0, 0, 0, -\frac{1}{2})^\top$ if $\varepsilon \downarrow 0$. Obviously, the system has a discontinuity in $(0, 0, -2, 1)^\top$. One may also note that the sequence of initial states $x_0(\varepsilon) = (0, -\varepsilon, -2, 1)$, $\varepsilon \geq 0$, leads after two reinitializations for $\varepsilon \downarrow 0$ to the limit state $(0, 0, \frac{1}{2}, \frac{1}{2})$. This alternative limit corresponds to a situation in which first the stop-constrained and then the hook-constrained mode is active.

9. Conclusions. The main purpose of this paper has been to define a new class of dynamical systems called "linear complementarity systems" (LCS). The definition builds on ideas from linear system theory and from mathematical programming and is motivated in part by systems of differential equations and algebraic inequalities that have been studied in mechanics and in electrical network theory. Applications are envisaged, for instance, in the modeling of power converters and other electrical networks that depend on controlled switching; in linear-quadratic control problems subject to linear inequality constraints; and in the study of piecewise linear systems.

An LCS can be viewed as a dynamical system that switches between several operating modes and behaves as a linear system within each mode. The state spaces corresponding to different modes are in general not all of the same dimension, although they are naturally embedded in one encompassing space; in relation to this, state trajectories may exhibit discontinuities when a mode switch takes place. To give a precise definition of what is to be understood by a solution of a complementarity system, one has to be precise about the conditions under which a transition from one given mode to another given mode can take place, and one has to specify the associated jumps of the state variable. For mode selection, we have used ideas from mathematical programming, in particular from the theory of the LCP [7]; for the determination of jumps we have relied on linear system theory, more specifically the geometric theory of linear systems [14].

When a class of dynamical systems is introduced, a first concern should be to give conditions for existence and uniqueness of solutions. We have given such conditions in terms of leading row and column coefficient matrices. Several methods for mode selection have been discussed, and a method for generating solutions has been presented. Also, we have shown that our notion of solution agrees with the one proposed by Moreau [22, 23] for the class of systems to which both solution concepts apply.

In spite of the length of this paper, it is clear that many issues remain to be investigated. The method that we have shown for constructing solutions allows us only to establish existence of solutions on intervals that do not contain accumulation points of the set of event times. To overcome this problem it seems necessary to work with sequences of approximating solutions, which may be generated, for instance, by time-stepping methods; compare the work by Stewart and Trinkle [31, 32]. A related issue is to provide conditions under which numerical solution methods for piecewise linear systems (see, for instance, [20]) can be shown to be consistent. The rational complementarity problem (RCP) that has been discussed only briefly here is expected to play a crucial role in such investigations; see [15] for a more extensive treatment of the RCP.

Of course, all of the well-known topics of interest in dynamical systems theory can also be addressed in the context of complementarity systems: conditions for stability, existence of limit cycles, occurrence of chaos, and so on. Control of mechanical systems with unilateral constraints is discussed by Brogliato [6]. Perhaps the main challenge is to effectuate the interaction between the various fields of research that find a common meeting ground in complementarity systems.

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