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Open-loop von Stackelberg equilibrium in the cartel-vs.-fringe model

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Abstract

In this paper the binding-contracts open-loop von Stackelberg equilibrium in the cartel-vs.-fringe model of the supply side of a market for a raw material from an exhaustible natural resource is reconsidered. It is shown that the equilibrium for this model differs from what the previous literature on this model suggests. In particular, the equilibrium price trajectory can display discontinuities. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

The treatment of duopolistic markets for raw materials from an exhaustible natural resource dates back to Hotelling's (Hotelling, 1931) seminal paper. Because of the evident relevance to real world phenomena such as the market for crude oil, this theory was revived recently and was extended to oligopolistic markets with

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more than two suppliers. Lewis and Schmalensee (1980a) considered the case of similarly placed oligopolists. It was recognized by Salant (1976) that some of the markets under consideration can be characterized on the supply side by a coherent cartel and a large number of small producers called the fringe. The cartel-vs.-fringe model was further explored by Salant et al. (1979) and Lewis and Schmalensee (1980b). Pindyck (1978) did interesting empirical work on this subject. One of the important questions is whether the fringe benefits from cartelization or not. This problem is most clearly dealt with by Ulph and Folie (1980), who show that the answer can be both yes and no and who derive conditions under which each case occurs. All these authors use the Nash–Cournot equilibrium concept. Gilbert (1978) was the first to put forward that this equilibrium concept might not do enough justice to the market power of the cartel. Instead he studied the von Stackelberg equilibrium. In both models the fringe takes prices as given and chooses an extraction path so as to maximize discounted profits. In the Nash–Cournot model the cartel takes the extraction path of the fringe as given, knows the demand function and chooses either an extraction path or a price path so as to maximize discounted profits. In the von Stackelberg equilibrium the cartel announces optimal price and extraction paths taking explicitly into account the behaviour of the fringe as a price taker. It is assumed here that the fringe exactly meets the lacking supply for market equilibrium. Ulph and Folie (1981) stress that in such a von Stackelberg equilibrium dynamic inconsistency may arise. This means that the announced price and extraction paths become suboptimal when the equilibrium is reconsidered after some time has elapsed. Ulph (1982) further elaborates on the issue of dynamic inconsistency as well as Newbery (1981), who considers the cases when discount rates differ and when the demand schedule is non-linear. The basic point is that an equilibrium concept which displays dynamic inconsistency should be rejected in a framework of rational agents unless market transactions take place according to binding contracts. This last assumption is not very realistic. Newbery (1981) introduces the concept of a rational expectations von Stackelberg equilibrium. The underlying idea is that the equilibrium should have the property that none of the actors has an incentive to deviate from the equilibrium strategies at any point in time. Ulph (1982) points out that, in game theory terms, the actual problem is to find the feedback von Stackelberg equilibrium for the cartel-vs.-fringe model as an alternative for the binding-contracts open-loop von Stackelberg equilibrium. However, this problem proves to be very difficult.

In this paper the binding-contracts open-loop von Stackelberg equilibrium is reconsidered. It is shown that the results of Ulph and Folie (1981), Newbery (1981) and Ulph (1982) are not altogether correct. The equilibrium proves to be different from these results in two respects. Firstly, the specification of the marginal costs for the cartel should be corrected. As a result of this the timing of the different production stages changes somewhat. Secondly, for empirically plausible parameter values the resulting equilibrium price path may be discontinuous. This implies that not only the timing but also the order of the different production stages may differ from what was suggested in the previous literature. The possibility of the occurrence of a discontinuous price trajectory is already discussed in Groot et al. (1992).

They show that a discontinuous price trajectory can yield higher cartel profits than along the solution proposed in the previous literature, but they do not provide a full proof of the equilibrium. In spite of the fact that we shall still find dynamic inconsistency in the correct von Stackelberg equilibrium, we think it is worthwhile to present a formal derivation. Not only is this not present in the literature but our method may also prove useful in other fields as well. The mathematical difficulty is that in the control problem that has to be solved the constraint qualification does not hold. We show how to deal with this in a way that is applicable to similar control problems. In the corrected equilibrium dynamic inconsistency still occurs for more or less the same parameter values as were found in the previous literature.

The paper is organized as follows. In Section 2 the cartel-vs.-fringe model is described. The model is basically the same as the one Ulph and Folie (1981) and Ulph (1982) used. The only difference is that in this paper the fringe is explicitly modelled as a large number of small resource owners having identical extraction costs and an identical initial stock. The fringe members are oligopolists, which implies that they all know the demand function and that each of them takes the extraction rate of all other resource owners as given. The cartel has only one instrument, its own extraction rate. The advantage of this approach is that it is not necessary to assume that the cartel controls the price path. Furthermore, the extraction rate of each fringe member is uniquely determined, so that no additional assumptions are needed here. In the limiting case of infinitely many fringe members the behaviour of the fringe as a group is price-taking, so that the rational reaction of the fringe is the same as in the previous literature. A qualitative characterization of the equilibrium order of exploitation and of the resulting dynamic equilibrium price path is presented in Section 3. Section 4 deals with the calculation of an equilibrium. Section 5 concludes.

2. The cartel-vs.-fringe model: problem statement

The cartel-vs.-fringe model is a model of the supply side of a market for an exhaustible natural resource. The demand schedule is assumed to be given. On the supply side there is a large coherent cartel and a group of many small resource owners, called the fringe. All producers are endowed with a given initial stock of the resource and choose their extraction path so as to maximize their discounted profits.

In the binding-contracts open-loop von Stackelberg equilibrium all fringe members are followers and the cartel is the leader. Each producer has one instrument: its own extraction rate. It is assumed that each producer knows the demand function. The cartel chooses its optimal extraction path, explicitly taking into account the optimal reactions of all other producers. Each fringe member takes the extraction path of the cartel and all other fringe members as given.

This approach is slightly different from what is done in the previous literature on this model (e.g. Ulph and Folie, 1981 or Ulph, 1982). In that literature the cartel is assumed to have two instruments, the price path and its own extraction rate. The fringe is treated as a homogeneous group, which takes the price path as given. The assumption that the cartel controls the entire price path, even after its resource is exhausted, is not realistic. Furthermore, it is not possible to derive at what rate the fringe will extract its resource. It can only be determined whether the fringe is willing to produce or not. There is no mechanism forcing the fringe to produce exactly the amount required for a market equilibrium. By assuming that each fringe member is an oligopolist, its extraction rates are fully determined. As shown by Lewis and Schmalensee (1980a), the behaviour of the fringe as a group will approach the competitive behaviour, if the number of fringe members grows to infinity and if they are all assumed to have identical costs and endowments. In this way it is ensured that the fringe as a group will exactly produce what is required for a market equilibrium.

Let the market demand function for the resource be given by

$$p(t) = \bar{p} - y(t), \tag{1}$$

where $t \in [0,\infty)$ denotes time, $p(t)$ is the price, $y(t)$ is demand and $\bar{p}$ is a positive constant. A linear specification is chosen for computational convenience, but it can be shown that many of the qualitative results remain valid for more general demand schedules.

Demand is met by supply E^c of the cartel and E_i^f of fringe member i, $i = 1,\dots,N$, where N denotes the total number of fringe members. Per unit extraction costs are constant: k^c for the cartel and k^f for each fringe member. In order to avoid trivialities it is assumed that $\bar{p} > \max(k^c,k^f)$. The initial stocks of the cartel and fringe member i are given and denoted by S_0^c and S_{0i}^f, respectively. All producers maximize their discounted profits and they all have the same constant discount rate r.

Fringe member $i (i = 1,\dots,N)$ has to solve the following maximization problem:

$$\max_{E_i^f} \int_0^\infty e^{-rt}[\bar{p} - k^f - E^c(t) - E^f(t)]E_i^f(t)\mathrm{d}t \tag{2}$$

subject to

$$\int_0^\infty E_i^f(t)\mathrm{d}t \leq S_{0i}^f \tag{3}$$

$$E_i^f(t) \geq 0, \tag{4}$$

where $E^f(t) = \sum_{i=1}^{N} E_i^f(t)$. According to Pontryagin's maximum principle[1] there exists a constant $\lambda_i^f \geq 0$ such that

[1]These conditions are both necessary and sufficient since the profit function is concave and the state equation is linear.

$$E^c(t) + E^f(t) + E_i^f(t) + k^f + \lambda_i^f e^{rt} - \bar{p} \geq 0 \tag{5}$$

$$E_i^f(t)\left[E^c(t) + E^f(t) + E_i^f(t) + k^f + \lambda_i^f e^{rt} - p\right] = 0 \tag{6}$$

Under the assumption that all fringe members are endowed with the same initial stock, it follows that the constants λ_i^f, $i = 1,\ldots,N$, have the same value λ^f, and each fringe member will produce the same amount, $(1/N)E^f(t)$, at each point in time. For the fringe as a group total production, $E^f(t)$, can then be found by solving the following system:

$$\int_0^\infty E^f(t)\mathrm{d}t \leq S_0^f := \sum_{i=1}^{N} S_{0i}^f \tag{7}$$

$$E^f(t) \geq 0 \tag{8}$$

$$E^c(t) + \frac{N+1}{N}E^f(t) + k^f + \lambda^f e^{rt} - \bar{p} \geq 0 \tag{9}$$

$$E^f(t)\left[E^c(t) + \frac{N+1}{N}E^f(t) + k^f + \lambda^f e^{rt} - \bar{p}\right] = 0 \tag{10}$$

If the number of fringe members, N, goes to infinity, the expression $(N+1)/N$ approaches unity and the outcome will be the same as in the case where the fringe acts as a group of price takers. From now on the limiting case of the fringe consisting of infinitely many very small producers will be considered.[2]

The constant λ^f is strictly positive. To see this suppose $\lambda^f = 0$. Inequality [Eq. (9)] then implies $E^c(t) + E^f(t) \geq \bar{p} - k^f$. Total production at each point in time is larger than some positive constant. This implies that total production over time is infinite. But that violates the limited availability of the resource. Hence, $\lambda^f > 0$. As a result the stock of the fringe will be exhausted and Eq. (7) holds with equality.

The cartel maximizes its discounted profits taking explicitly into account the reaction of the fringe to its announced extraction path. The optimal reaction of the fringe is the solution of the system [Eqs. (7)–(10)]. For the cartel this system is an additional constraint. The maximization problem of the cartel becomes

$$\max \int_0^\infty e^{-rt}[\bar{p} - k^c - E^c(t) - E^f(t)]E^c(t)\mathrm{d}t \tag{11}$$

[2] The assumption that each fringe member has the same initial stock can be relaxed by assuming that, when $N \to \infty$, the initial stock of each fringe member relative to the total stock of the fringe, S_{0i}^f/S_0^f goes to zero.

subject to

$$\int_0^\infty E^c(t)\mathrm{d}t \leq S_0^c, \quad \int_0^\infty E^f(t)\mathrm{d}t \leq S_0^f \tag{12}$$

$$E^c(t) \geq 0, \quad E^f(t) \geq 0 \tag{13}$$

$$E^c(t) + E^f(t) + k^f + \lambda^f e^{rt} - \bar{p} \geq 0 \tag{14}$$

$$E^f(t)\left[E^c(t) + E^f(t) + k^f + \lambda^f e^{rt} - \bar{p}\right] = 0, \tag{15}$$

where the maximization takes place with respect to E^c, E^f and λ^f. The next section deals with the characterization of the solution of the problem stated above.

3. Characterization of the open-loop solution

The optimal control problem of Section 2 does not allow for a standard application of Pontryagin's maximum principle. To see this consider Eqs. (13)–(15). Assume that the optimal level of λ^f has already been determined. Then we have a so-called Hestenes problem (see e.g. Takayama, 1974, p. 657) or a Bolza problem (see e.g. Cesari, 1983, p. 196). In the particular case at hand, with equality and inequality constraints, a rather general constraint qualification would read that the rank of the matrix

$$\begin{pmatrix} 1 & 0 & E^c & 0 & 0 & 0 \\ 0 & 1 & 0 & E^f & 0 & 0 \\ 1 & 1 & 0 & 0 & E^c + E^f + k^f + \lambda^f e^{rt} - \bar{p} & 0 \\ E^f & E^c + 2E^f + k^f + \lambda^f e^{rt} - \bar{p} & 0 & 0 & 0 & 0 \end{pmatrix}$$

equals 4 when the matrix is evaluated at the optimum (see e.g. Takayama (o.c.) p. 658). It is easily seen that when $E^f = 0$, $E^c > 0$ and $E^c + E^f + k^f + \lambda^f e^{rt} - \bar{p} = 0$, this condition is not satisfied whereas it can be shown that the proposed solution is indeed optimal in certain circumstances. The difficulty can also be illustrated in the more usual Kuhn–Tucker setting. For a fixed t the feasible set can be drawn as in Fig. 1.

If we consider the point $(E^c,E^f) = (\hat{E}^c,0)$ and draw line l originating from this point, then it is clear that we cannot find a differentiable curve in the feasible set which, at $(\hat{E}^c,0)$, is tangent at the line. For more details on constraint qualifications we refer to Bazaraa and Shetty (1976).

A consequence of this difficulty is that one cannot be sure that the multiplier associated with the objective function when the Hamiltonian is written down, is necessarily non-zero. Seierstad and Sydsaeter (1987) provide necessary conditions

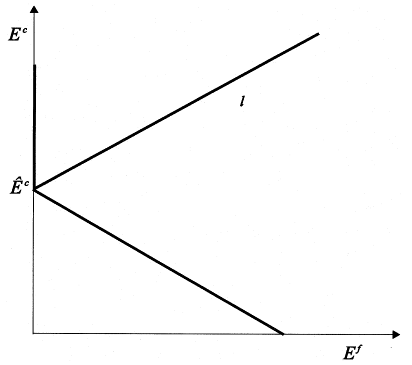

Fig. 1. Feasible set.

of optimality for cases where the constraint qualification is not satisfied. (These conditions are due to Neustadt, 1976.) Indeed, it turns out to be impossible to show that this multiplier is non-zero. And, if it is zero, the necessary conditions hardly provide any information. Nevertheless, Groot et al. (1992) were able to show that the results in the literature on the equilibrium can not be correct, assuming that this multiplier equals unity and then deriving from the necessary conditions trajectories which are superior to the ones proposed in that literature. Moreover, the intuition behind these results is convincing. However, because it was assumed that the multiplier is non-zero, it was not proven that the cartel could not even do better.

Here we shall proceed along an alternative route, avoiding the difficulty posed by the absence of the constraint qualification. Attention will be restricted to the case where $1/2\,(\bar{p} + k^c) > k^f > k^c$, which for obvious reasons is economically the most appealing one. All other cases can be treated in a similar way.

Let (λ^f, E^c, E^f) constitute a solution of the problem posed above. Eq. (1) and market equilibrium yield:

$$p(t) = \bar{p} - E^c(t) - E^f(t) \quad t \in [0,\infty)$$

Define

$$F := \{t \in [0,\infty) | E^f(t) > 0\}$$

$$C^c := \{t \in [0,\infty) | E^c(t) > 0, p(t) = k^f + \lambda^f e^{rt}\}$$

$$C^m := \{t \in [0,\infty) | E^c(t) > 0, p(t) < k^f + \lambda^f e^{rt}\}$$

$$O := \{t \in [0,\infty) | E^c(t) = E^f(t) = 0\}.$$

So, F denotes the instants of time where the fringe is supplying. C^c and C^m denote the instants of time where the cartel is supplying at what shall be called the competitive price and at a price below the competitive price, respectively. O is the set of times where there is no supply. The following three lemmata give the basis for the derivation of the optimal trajectory.

Lemma 1 states that the fringe supplies only at the competitive price. It also says that there is never simultaneous supply of the cartel and the fringe. Finally, it introduces the monopoly price, which rules if the cartel supplies at a price below the competitive one.

Lemma 1.

(i) $p(t) = k^f + \lambda^f e^{rt}$ for $t \in F$;
(ii) $F \cap C^m = F \cap C^c = C^c \cap C^m = 0$;
(iii) there is a constant λ^c such that $p(t) = 1/2\,(\bar{p} + k^c) + 1/2\lambda^c e^{rt}$ for $t \in C^m$.

Proof:

(i) This follows from $p(t) = \bar{p} - E^c(t) - E^f(t)$ and Eq. (15).
(ii) The property $F \cap C^m = C^c \cap C^m = 0$ is obvious from the definition of the sets involved. The fact that $k^c < k^f$ implies that it is not optimal that $F \cap C^c$ has a positive measure, because it would then be more profitable for the cartel to supply before the fringe. Moreover, E^c and E^f are right-continuous.
(iii) For any $t \in C^m$ we have $E^f(t) = 0$ and

$$\begin{aligned}\partial e^{-rt}[\bar{p} - k^c - E^c(t) - E^f(t)]E^c(t)/\partial E^c(t) &= e^{-rt}(\bar{p} - k^c - 2E^c(t)) \\ &= 2e^{-rt}(p(t) - \tfrac{1}{2}(\bar{p} + k^c)).\end{aligned}$$

Necessary for optimality is that this is a constant along C^m. This constant is denoted by λ^c.

Clearly, λ^c is to be interpreted as the price the cartel is willing to pay to acquire an additional unit of the stock S_0^c.

The second lemma gives marginal discounted profits for the cartel when its own supply is increased.

Lemma 2.

$$\partial e^{-rt}[\bar{p} - k^c - E^c(t) - E^f(t)]E^c(t)/\partial E^c(t)$$
$$= \begin{cases} \lambda^f + e^{-rt}(k^f - k^c) & \text{if } t \in F \\ \lambda^c & \text{if } t \in C^m \\ 2\lambda^f - e^{-rt}(\bar{p} - k^c - 2k^f) & \text{if } t \in C^c. \end{cases}$$

Proof:
This is straightforward, taking into account the previous lemma.
Define, for given λ^f and λ^c,

$$P^c(t) := k^f + \lambda^f e^{rt}$$

$$P^m(t) := \tfrac{1}{2}(\bar{p} + k^c) + \tfrac{1}{2}\lambda^c e^{rt}.$$

So, P^c and P^m are the competitive price and the monopoly price, respectively.

The next lemma characterizes the sequential order of the different supply regimes. It says that supply by the fringe will always occur later than supply by the cartel at the competitive price (i); that the cartel will never supply at the competitive price after it has been supplying at the monopoly price (ii); that, initially, the cartel supplies at the competitive price (iii and iv); and that, if there is ever supply by the cartel at the monopoly price, at least part of this supply will occur only after exhaustion of the fringe (v and vi).

Lemma 3.

(i) if $t_1 \in C^c$ and $t_2 \in F$ then $t_2 > t_1$;
(ii) if $t_1 \in C^c$ and $t_2 \in C^m$ then $t_2 > t_1$;
(iii) $0 \notin C^m$
(iv) $0 \notin F$
(v) if $C^m \neq 0$ then there is $t_1 \in C^m$ such that $t_1 > t$ for all $t \in F$;
(vi) if $t_1 \in F$ and $t_2 \in F$ then $[t_1,t_2] \in F$.

Proof:

(i) This follows from $k^f > k^c$, as argued in the proof of lemma 1 (ii).
(ii) Suppose the statement of the lemma is false. Then there exist $0 \leq t_1 < t_2$ such that $t_2 \in C^c$ and $[t_1,t_2) \subset C^m$ because in view of (i) the fringe will not supply before any instant of time where the cartel supplies at the competitive price.
First, note that $P^c(t_2) > P^m(t_2)$. This is so because $P^c(t_1) > P^m(t_1)$, using the definition of P^c and P^m, lemma 1 and $1/2(\bar{p} + k^c) > k^f$. Second, we must have

$$\left[e^{-rt_2}(P^c(t_2) - k^c) - \lambda^c\right](\bar{p} - P^c(t_2))$$
$$= \left[e^{-rt_2}(P^m(t_2) - k^c) - \lambda^c\right](\bar{p} - P^m(t_2))$$

because otherwise the cartel's profit can be increased by having $E^c(t) > 0$ and $p(t) = P^c(t)$ just before t_2 or by having $E^c(t) > 0$ and $p(t) = P^m(t)$ just after t_2 (see also lemma 2). However, the above equality holds if and only if $P^c(t_2) = P^m(t_2)$, contradicting that $P^c(t_2) > P^m(t_2)$.

(iii) Suppose $0 \in C^m$. It then follows from the previous part of this lemma that $C^c = 0$. Since $\bar{p} > k^f$, it must be the case that $F \neq 0$. Hence, there exist $0 < t_1 < t_2$ such that $[0,t_1) \subset C^m$ and $[t_1,t_2) \subset F$. Since $P^c(0) > P^m(0)$ we have $P^c(t_1) > P^m(t_1)$, as in (ii). Suppose that $k^c + \lambda^c e^{rt_1} \geq k^f + \lambda^f e^{rt_1}$. It then follows that $P^m(t_1) \geq \bar{p}$ and $P^c(t_1) > \bar{p}$, which contradict $t_1 \in F$. So, $k^c + \lambda^c e^{rt_1} < k^f + \lambda^f e^{rt_1}$ and hence $\lambda^f + e^{-rt_1}(k^f - k^c) > \lambda^c$. So, marginal profits for the cartel are higher along F than along C^m. The cartel should therefore decrease λ^f and supply itself at the competitive price, rather than at the monopoly price.

(iv) Suppose $0 \in F$. Then $C^c = 0$. Now essentially the same argument as in (iii) can be used to obtain a contradiction.

(v) Suppose that the statement of this part of the lemma is false. C^c cannot come after F (by i), nor after C^m (by ii), so there exist $0 < t_1 < t_2 < t_3$ such that $[t_1,t_2) \subset C^m$, $[t_2,t_3) \subset F$ and $[t_3,\infty) = O$. It follows that $P^m(t_3) < \bar{p}$ and

$$\lambda^f + e^{-rt_3}(k^f - k^c) > \lambda^c$$

so that, from lemma 2, it is profitable for the cartel to transfer sales from $[t_1,t_2)$ to t_3.

(vi) Suppose, on the contrary, and taking into account the previous results of this lemma, that there exist $0 < t_1 < t_2 < t_3 < t_4 < t_5$ such that

$$[t_1,t_2) \subset F,\ [t_2,t_3) \subset C^m,\quad [t_3,t_4) \subset F,\ \text{and}\ [t_4,t_5) \subset C^m.$$

Define $v_i (i = 2, 3, 4)$ by

$$\left[e^{-rt_i}(P^m(t_i) - k^c) - \lambda^c\right](\bar{p} - P^m(t_i)) = v_i(\bar{p} - P^c(t_i)).$$

The left-hand side of this expression gives the discounted profits of having $E^c(t_i) > 0$ with $p(t_i) = P^m(t_i)$ where the costs of acquiring E^c (λ^c per unit) is taken into account. The right-hand side gives the costs of reducing E^f (v_i per unit) at t_i. Along an optimum one should have $v_2 = v_3 = v_4$. But this is ruled out since the expression is quadratic. So we have obtained a contradiction.

It follows from the previous lemmata that the solution of the problem stated above can be characterized as follows.

Theorem 1. There exist $0 < t_1 \leq t_2 < t_3 \leq t_4$ such that

(i) $t_3 = t_4 \Rightarrow t_1 = t_2$;
(ii) $C^c = [0, t_1)$;
(iii) $C_m = [t_1,t_2) \cup [t_3,t_4)$;
(iv) $F = [t_2,t_3)$;
(v) $O = [t_4,\infty)$.

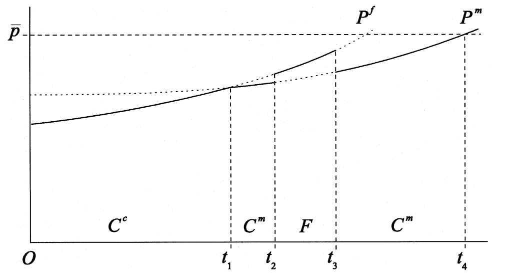

Fig. 2. Equilibrium price trajectory.

A graphical illustration is given in Fig. 2.

It follows from lemma 3 (iii and iv) that $0 \in C^c$. There must be fringe supply F somewhere after competitive cartel supply (C^c) since $\bar{p} > k^f$. If there is a monopoly phase, at least part of it should be after the fringe supply phase, but it might be that *in addition to that* there is a monopoly phase just before the fringe phase. It can be shown that with a relatively small initial size of the cartel's resource $t_3 = t_4$ and $t_1 = t_2$. As the stock increases we have $t_4 > t_3$ with $t_1 = t_2$ and with further increase the solution requires $t_4 > t_3$ and $t_2 > t_1$.

In any case there is an initial interval of time along which the cartel supplies at the competitive price. Therefore, if for example in Fig. 2 real time equals t_1 and the cartel would recalculate its optimal strategy, it would choose again for an initial interval, from t_1 on, with supply at the competitive price. So, in the case at hand we have dynamic inconsistency. In the previous literature, where it was assumed that the price trajectory is continuous, dynamic inconsistency was also found for the range of parameter values that is considered in this paper ($k^c < k^f < 1/2(\bar{p} + k^c)$). The phenomenon of inconsistency can also occur for other parameter values (see e.g. Ulph and Folie (o.c.)).

One peculiar feature of the equilibrium price trajectory are the discontinuities at t_2 and t_3 where there occur switches from the cartel producing at the monopoly price to a period of fringe supply and vice versa. In a world with arbitrators such discontinuities are ruled out because, for example, just before t_2 arbitrators would buy a large stock of oil and sell it just after t_2 and make arbitrarily large profits. In the model under consideration arbitrage possibilities are ruled out by the assumption that the instantaneous demand function is given, depending only on the current price and not on expected future prices. Given that this is the case it is still remarkable at first sight that discontinuities can occur (we consider the case where $t_2 > t_1$ and $t_4 > t_3$). The intuition behind this result can be found in a closer examination of the formula

$$\left[e^{-rt_j}\left(P^m(t_j) - k^c\right) - \lambda^c\right]\left[\bar{p} - P^m(t_j]\right. = v\left(\bar{p} - P^c(t_j)\right) \quad \text{for } j = 2{,}3 \tag{16}$$

This equation was already derived in the proof of part (vi) lemma 3 but here we shall discuss it in more detail. At t_2 there is a transition from the cartel supplying at the monopoly price to the fringe supplying. At t_3 there is a transition just the other way around. Suppose the cartel contemplates to extend the first monopoly phase. So, it would supply $\bar{p} - P^m(t_2+)$ at t_2+, making a profit per unit of $e^{-rt_2}(P^m(t_2+) - k^c) - \lambda^c$, where the opportunity costs λ^c per unit are taken into account. On the other hand, it should compensate the fringe for *not* supplying at t_2. The compensation per unit is denoted by v. This is the cost alluded to in Section 1: the cartel incurs a cost to refrain the fringe from supplying. This is not taken into account in earlier studies except in Groot et al. (o.c.). The same argument applies to instant of time t_3. Now, if there would not occur price jumps Eq. (16) cannot hold. This is easily seen by eliminating v from the equations and assuming the absence of discontinuous prices, yielding a contradiction.

4. Existence and calculation of the solution

Theorem 1 gives a full qualitative characterization of the open-loop von Stackelberg equilibrium. In this section we consider the question of how to actually calculate the equilibrium trajectory.

Let us start from Fig. 2 and assume that $t_1 < t_2$ (and hence $t_3 < t_4$). The equilibrium is fully determined by six variables, namely t_1, t_2, t_3, t_4, λ^c and λ^f, meaning that the optimal E^c and E^f are known at each instant of time if these variables are known. The six variables have to satisfy a number of inequalities such as

$$0 < t_1 < t_2 < t_3 < t_4; \quad \lambda^c > 0; \quad \lambda^f > 0 \tag{17}$$

Moreover, it should be the case that Eq. (12) is satisfied, implying that

$$\int_0^{t_4} E^f(t)\mathrm{d}t = \int_{t_2}^{t_3} [\bar{p} - P^c(t)]\mathrm{d}t = S_0^f \tag{18}$$

and

$$\int_0^{t_4} E^c(t)\mathrm{d}t = \int_0^{t_1} [\bar{p} - P^c(t)]\mathrm{d}t + \int_{t_1}^{t_2} [\bar{p} - P^m(t)]\mathrm{d}t + \int_{t_3}^{t_4} [\bar{p} - P^m(t)]\mathrm{d}t = S_0^c \tag{19}$$

We should also have

$$P^c(t_1) = P^m(t_1) \tag{20}$$

$$P^m(t_4) = \bar{p} \tag{21}$$

Eq. (20) has to hold to guarantee the continuity of instantaneous profits when the cartel switches from supplying at the competitive price to supplying at the monopoly price. Eq. (21) must hold because there is no supply after t_4. Finally, there should exist a constant υ such that

$$\left[e^{-rt_j}\left(P^m(t_j) - k^c\right) - \lambda^c\right]\left[\bar{p} - P^m(t_j\right] = \upsilon\left(\bar{p} - P^c(t_j)\right) \quad \text{for } j = 2,3 \qquad (22)$$

These equations were already discussed in the previous section.

After elimination of υ from the two equations in (22) we have five equations in the six variables, which will be denoted by $g_j(x) := g_j(t_1, t_2, t_3, t_4, \lambda^c, \lambda^f) = 0$ ($j = 1, 2, \ldots, 5$). These functions are given in Appendix A, and, in a tedious but straightforward way, total cartel profits can be written as a function π of x. This π is also given in Appendix A. Proceeding along these lines, we reduce the original optimal control problem to a standard Lagrange problem of the form:

$$\underset{x}{\text{maximize}}\, \pi(x) \ \text{ subject to } g_j\ (x) = 0 \quad (j = 1,2,\ldots,5)$$

This problem can be solved using standard techniques. It is not necessarily true that the solution satisfies Eq. (17). If it does not it should be assumed that $t_1 = t_2$ or even $t_3 = t_4$ and $t_1 = t_2$. In the latter case there is no monopoly phase and we cannot determine λ^c. However, the value to the cartel of an initial marginal increase of its stock then follows from the value function, which gives the maximal profits for the cartel given S_0^c and S_0^f. In the end a solution will be found, because by Filippovs existence theorem the original optimal control problem has a solution.

By way of illustration we consider an example where $\bar{p} = 50$, $k^f = 23$, $k^c = 12$, $S_0^f = 30$, $S_0^c = 300$ and $r = 0.08$. If the open-loop equilibrium is calculated along the lines of Ulph/Newbery (with a continuous price trajectory) it is found that

$$C^c = [0,\ 11.81), \quad F = [11.81,\ 14.13), \quad C^m = [14.13,\ 25.91]$$

Cartel profits amount to 3265.80. The 'correct' open-loop equilibrium is

$$C^c = [0,\ 11.00), \quad C^m = [11.00,\ 13.45] \cup (16.90,\ 27.52),$$
$$F = [13.45,\ 16.90)$$

Cartel profits now amount to 3294.93.

5. Conclusions

In this paper we have derived the open-loop von Stackelberg equilibrium in the cartel-vs.-fringe model. Mathematically speaking, the main difficulty has been that the model specifying the problem for the cartel is an optimal control model where the constraint qualification fails to hold. As far as the qualitative characterization of the equilibrium was concerned this difficulty was circumvented by deriving ad hoc necessary conditions rather than by invoking sophisticated control theory (although this theory showed the route to these conditions). When dealing with the way to actually calculate the equilibrium, we were able to derive the value function, parametrized by six variables. As far as the economics of the exercise is concerned, we are fully aware of the fact that the open-loop von Stackelberg equilibrium concept is not appropriate when it leads to time-inconsistent outcomes, as is the case if the stock of the cartel is sufficiently large. However, it has been shown that our method at least yields the correct solution to the open-loop von Stackelberg game.

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Appendix A

Eqs. (17)–(21) can be written as

$$(\bar{p} - k^f)(rt_3 - rt_2) - \lambda^f(e^{rt_3} - e^{rt_2}) = rS_0^f \tag{A1}$$

$$(\bar{p} - k^f)rt_1 - \lambda^f(e^{rt_1} - 1) + \tfrac{1}{2}(\bar{p} - k^c)(rt_2 - rt_1) - \tfrac{1}{2}\lambda^c(e^{rt_2} - e^{rt_1}) + \tfrac{1}{2}(\bar{p} - k^c)(rt_4 - rt_3) - \tfrac{1}{2}\lambda^c(e^{rt_4} - e^{rt_3}) = rS_0^c \tag{A2}$$

$$k^f + \lambda^f e^{rt_1} = \tfrac{1}{2}(\bar{p} + k^c) + \tfrac{1}{2}\lambda^c e^{rt_1} \tag{A3}$$

$$\tfrac{1}{2}(\bar{p} + k^c) + \tfrac{1}{2}\lambda^c e^{rt_4} = \bar{p} \tag{A4}$$

$$e^{-rt_2}\left[\tfrac{1}{2}(\bar{p} - k^c) + \tfrac{1}{2}\lambda^c e^{rt_2}\right]\left[\tfrac{1}{2}(\bar{p} - k^c) - \tfrac{1}{2}\lambda^c e^{rt_2}\right]/(\bar{p} - k^f - \lambda^f e^{rt_2}) = e^{-rt_3}\left[\tfrac{1}{2}(\bar{p} - k^c) + \tfrac{1}{2}\lambda^c e^{rt_3}\right]\left[\tfrac{1}{2}(\bar{p} - k^c) - \tfrac{1}{2}\lambda^c e^{rt_3}\right]/(\bar{p} - k^f - \lambda^f e^{rt_3}) \tag{A5}$$

The profits of the cartel are given by

$$r\pi := -(k^f - k^c)(\bar{p} - k^f)(e^{-rt_1} - 1) - (k^f - k^c)\lambda^f rt_1 + (\bar{p} - k^f)\lambda^f rt_1 - (\lambda^f)^2(e^{rt_1} - 1) - \tfrac{1}{4}(\bar{p} - k^c)^2(e^{-rt_2} - e^{-rt_1}) - \tfrac{1}{4}(\lambda^c)^2(e^{rt_2} - e^{rt_1}) - \tfrac{1}{4}(\bar{p} - k^c)^2(e^{-rt_4} - e^{-rt_3}) - \tfrac{1}{4}(\lambda^c)^2(e^{rt_4} - e^{rt_3}) \tag{A6}$$

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