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Cost sharing methods for capacity restricted cooperative purchasing situations

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Abstract

This paper analyzes cooperative purchasing situations with two suppliers with limited supply capacity. In these *capacity restricted cooperative purchasing (CRCP) situations* we consider a group of cooperating purchasers for whom it turns out to be optimal as a group to order as much as possible at one supplier and the possible remainder at the other. To find suitable cost allocations of the total purchasing costs, a CRCP-situation is modeled as a cost sharing problem, in which the corresponding cost function is piecewise concave and the intervals of concavity are determined by the restricted capacity of the suppliers. It is seen that in the setting with piecewise concave cost functions, standard cost sharing mechanisms lose their axiomatic appeal, e.g. the serial cost sharing mechanism will neither satisfy *unit cost monotonicity (UCM)* nor *monotonic vulnerability for the absence of the smallest player (MOVASP)*. In fact, it is shown that these two properties are incompatible now. We introduce a new context specific class of *piecewise serial rules*, in which the vector of order quantities is divided into separate vectors for the different intervals of concavity, using a bankruptcy rule and subsequently apply the serial rule to each interval. We show that the *proportional rule* is the only bankruptcy rule for which the corresponding piecewise serial rule satisfies UCM. Moreover, we show that the piecewise serial rule corresponding to the *constrained equal losses rule* satisfies MOVASP.

Keywords: cooperative purchasing, cost sharing problems, piecewise concavity, bankruptcy rules, piecewise serial rules.
JEL classification: D71 (analysis of collective decision making: social choice), D23 (production and organizations: behavior, costs).

1 Introduction

Recently, the study of different types of cooperative purchasing (CP) situations — organizations that collaborate in their purchasing process — has focused on the bundling of purchasing volumes in order to obtain cost savings, see e.g. Schotanus (2007), Nagarajan, Sošić, and Zhang (2010) and Hezarkhani and Sošić (2019). Reasons for organizations to join a purchasing cooperative are numerous, but according to Tella and Virolainen (2005) the main motive is the

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cost savings due to the offered quantity discounts by the supplier. One of the main underlying assumptions in the cooperative purchasing literature in general is that the capacity of the supplier is sufficient to fulfill the total order of the group of purchasers. Although commonly assumed, one should realize that in practice the capacity of a supplier is limited. In particular, while a purchasing cooperative gets larger, the supplier's capacity might be exceeded and the cooperative has to use a second supplier. Capacity restrictions on the part of the suppliers in cooperative purchasing situations will be the main topic of this paper.

Not much literature can be found on capacity restrictions within cooperative purchasing. In general, however, cooperative purchasing has been studied from different perspectives. One stream of research focuses on several factors of risk: Berger, Gerstenfeld, and Zeng (2004) argue that maintaining a relationship with multiple suppliers can be a good strategy to decrease supply chain risks. Chen and Roma (2011) provide conditions under which cooperative purchasing is beneficial (or not). Hu, Duenyas, and Beil (2013) provide insights in situations in which cooperative purchasing is (dis)advantageous. Another stream of research focuses on supplier selection: Ghodsypour and O'Brien (1998) incorporated an analytical hierarchy process for allocating orders among suppliers based on both quantitative and qualitative criteria. Jayaraman and Srivastava (1999) developed a mixed integer programming model for selecting suppliers and for allocating the total order quantity among the selected suppliers. Ghodsypour and O'Brien (2001) provide an algorithm for quantity allocation where the possible suppliers have limited capacities. Alternatively, the group of purchasers can collaborate more drastically and can merge, as studied by Cho (2014).

Suppliers' optimal pricing strategies in a multiple supplier environment have been discussed by, e.g., Marvel and Yang (2008). More precisely, Marvel and Yang (2008) discuss pricing strategies when two suppliers face a purchasing cooperative. From a buyers' perspective, however, a purchasing cooperative with capacity restricted suppliers has not yet been studied.

In this paper, we consider a purchasing cooperative in which each participant has an individual order quantity with respect to a certain commodity. Think of a group of departments or ministries, a group of municipalities with a joined purchasing program or online group-buying markets (the latter is studied by Anand and Aron, 2003). Typically, the sum of the order quantities determines the unit price negotiated at a supplier. Instead of facing one supplier with sufficient supplies, as in the classical CP-situations, the cooperative faces two suppliers with (possibly) insufficient individual supplies, although the combined capacity of the two suppliers is assumed to be sufficient to cover the demands of the cooperative. The unit price of a supplier weakly decreases with the size of the total order, that is, however, up to his capacity bound. These unit prices or quantity discount schemes are not necessarily the same for both suppliers.

Within these *capacity restricted cooperative purchasing (CRCP) situations*, we are interested in finding the answers to two questions. Firstly, how to split the total order over the two suppliers such that the total purchasing costs are minimized? Secondly, how to adequately divide the total joint purchasing costs over the group of purchasers?

For the first question, we show that there is a straightforward solution. We show that it is optimal to order as much as possible at one supplier and the possible remainder at the other.

The second problem is more involved. Schotanus (2007), e.g., argued that finding a fair cost allocation method is one of the critical success factors for cooperative purchasing. Especially in the presence of differences in order quantity size, organizations with a large order quantity could get the feeling that organizations with a small order quantity profit from their size, without making any further contributions.

As Moulin (2002) points out, when there are no quantity discounts, the fair distribution of purchasing costs should simply follow Aristotle's proportionality: *Equals should be treated equally, and unequals, unequally in proportion to relevant similarities and differences.* However, since both suppliers have a weakly decreasing unit price function, quantity discounts will be present and we need to look for a more sophisticated allocation method.

For cost allocation methods of the purchasing costs in a CRCP-situation, there are two desirable properties. Firstly, the quantity discounts should be incorporated in the cost allocation, i.e, organizations with large order quantities do not pay a higher unit price than organizations with smaller order quantities. A second desirable effect of an allocation method is that organizations with large order quantities do profit, in terms of cost allocations, from the presence of players with smaller order quantities. Loosely formulated: the smaller players should not be free riders or profiteers.

To find suitable cost allocations, we model the CRCP-situation as a cost sharing problem. Generally, a cost sharing problem involves a set of users of a certain 'technology' and each of the users has an individual level of demanded output. To produce the total demanded output a certain level of input or costs is needed. The relationship between input and output, is represented by a cost function, where the function describes for each level of output the needed input in terms of associated costs. How to fairly distribute these joint costs is the central theme in the cost sharing literature. Moulin (2002) provided an overview of different types of cost sharing problems and allocation mechanisms.

In the CRCP-setting the output is the sum of the individual order quantities. The cost function of the cost sharing problem corresponding to a CRCP-situation provides for each level of order quantities, the minimal purchasing costs. These minimal purchasing costs follow from dividing the order quantities optimally over the two suppliers. The resulting cost sharing problem corresponding to a CRCP-situation then falls within the class of so-called *one-input-one-output-technologies*, such as airport problems (cf. Littlechild and Thompson, 1977) or single-product inventory problems. The common feature within this class is that the output is a single homogeneous divisible good.

We show that the cost function of a cost sharing problem corresponding to a CRCP-situation is non-decreasing and piecewise concave, and that the finite number of (maximal) intervals of concavity are determined by the restricted capacity of the suppliers. While a concave cost function implies unlimited increasing returns to scale, a piecewise concave cost function implies limited increasing returns to scale: after a certain output level, new investments are needed. According to Swoveland (1975), piecewise concave cost functions are a realistic representation of returns to scale in a production environment. Cost sharing problems with (almost) concave functions are studied in Karsten, Slikker, and Borm (2017), from the perspective of coalitional rationality and benefit ordering.

In the process of finding a suitable cost allocation method for CRCP-situations we start by considering the three main cost sharing rules: the Shapley-Shubik rule (Shubik, 1962), Aumann-Shapley pricing (Aumann and Shapley, 1974) and the serial cost sharing rule (Moulin and Shenker, 1992). There are two main arguments for Friedman and Moulin (1999) to conclude that from the three main cost sharing methods, the serial cost sharing rule is most appropriate for one-input-one-output-technologies such as the cost sharing problems we consider. First, since the cost sharing problem corresponding to a CRCP-situation concerns only homogeneous inputs and outputs, the Aumann-Shapley pricing boils down to average cost pricing, i.e. dividing the total purchasing costs proportionally (based on order quantities) over the buyers. As argued before, this method neglects the quantity discounts that are present in cooperative purchasing. Second, one of the main properties of the Shapley-Shubik rule is, that it is invariant to the scale in which the output is measured, which is not of any relevance in a situation in which order quantities are for a single good.

Furthermore, we will provide more direct arguments in favor of the serial cost sharing rule. In the setting of concave cost functions, the serial cost sharing rule satisfies properties that are attractive from the perspective of CRCP-situations. Firstly, the serial rule satisfies *unit cost monotonicity (UCM)*: when organization II has a higher order quantity than organization I, organization II does not pay a higher cost per unit than organization I. Secondly, the serial rule satisfies *monotonic vulnerability for the absence of the smallest player (MOVASP)*. When the organization with the smallest order quantity is not present in the cooperation, the group of remaining organizations can be split in a group of organizations for which the allocated costs decrease and a group of organizations for which the allocated costs increase. MOVASP requires that there is a monotonicity relation throughout: the larger the order quantity, the higher the increase (or the smaller the decrease) in cost allocation. In other words, the organization with the largest order quantity has either the least decrease or the highest increase in cost allocation when the smallest player is absent. Hence, it creates a group cohesiveness in which the organization with the smallest order quantity can contribute to lower cost allocations of organizations with larger order quantities.

However, we will also show that these rather compelling properties are lost in the setting of piecewise concave cost functions. We explicitly show that for cost sharing problems with piecewise concave cost functions, the serial rule in general does not satisfy UCM and MOVASP.

For this reason, we introduce a new and context specific class of cost sharing rules for cost sharing problems with piecewise concave cost functions. Using a bankruptcy rule[1], we first divide the vector of order quantities into separate vectors for the different intervals of concavity. Subsequently, for each interval and its corresponding vector we use the serial rule to allocate the costs of that specific interval over the organizations. Finally, by summing these allocated costs over all intervals we obtain the allocation prescribed by the piecewise serial rule.

In particular, we consider the piecewise serial rules where we divide the vector of order quantities into separate vectors, using the *proportional rule* and the *constrained equal losses rule*. It will be shown that the proportional rule is the only bankruptcy rule for which the corresponding piecewise serial rule satisfies UCM on the class of cost sharing problems with piecewise concave cost functions. With regard to the constrained equal losses rule, we will show that the corresponding piecewise serial rule satisfies MOVASP. Together, this shows incompatibility of UCM and MOVASP on the class of cost sharing problems with piecewise concave cost functions.

In the concluding section of the paper, we mainly focus on the incompatibility of UCM and MOVASP on the class of cost sharing problems corresponding to CRCP-situations, a subclass of cost sharing problems with piecewise concave cost functions. For illustrative purposes we conclude the paper with an example highlighting the differences between several rules for

[1] A *bankruptcy rule* prescribes, for each bankruptcy problem — an available amount of money and a vector of rightful claims — how to divide the available amount of money over the claims. For a detailed overview on bankruptcy rules see e.g. Thomson (2003).

CRCP-situations. This example also shows that piecewise serial rules seem fairly appropriate cost allocation mechanisms for CRCP-situations.

The structure of the paper is as follows. Section 2 formally describes a CRCP-situation. In Section 3, we model CRCP-situations as cost sharing problems in which the cost functions are piecewise concave. As an alternative to the serial cost sharing rule, we introduce piecewise serial rules in Section 4 and focus on the properties of UCM and MOVASP for the proportional and constrained equal losses-variants of piecewise serial rules. Section 5 provides a conclusion and discussion.

2 Capacity restrictions in cooperative purchasing

In a *capacity restricted cooperative purchasing (CRCP) situation*, there is a finite player set $N = \{1, \dots, n\}$, with $n \geq 2$ and a vector of individual order quantities $q \in \mathbb{R}^N_{++}$. There are two suppliers providing a particular commodity: A and B. The suppliers have capacities $Q_A, Q_B \in \mathbb{R}_{++}$ such that $\sum_{i \in N} q_i \leq Q_A + Q_B$. The suppliers have weakly decreasing unit price functions p_A and p_B which are smooth (e.g. twice differentiable with a continuous second derivative) on their respective domains. For A, $p_A : [0, Q_A] \to \mathbb{R}_+$ and for B, $p_B : [0, Q_B] \to \mathbb{R}_+$, are assumed to be such that

$$p'_A(t) \leq 0 \quad \text{for all } t \in [0, Q_A]$$

and

$$p'_B(t) \leq 0 \quad \text{for all } t \in [0, Q_B].$$

Moreover, it is natural to assume that the revenue of a supplier does not decrease if t increases and that the quantity discounts are such that the higher the total order quantity the lower the increase in revenue. For supplier A this is the case if the revenue function c_A of A, given by $c_A(t) = p_A(t)t$ for all $t \in [0, Q_A]$, is non-decreasing and concave. Hence, for all $t \in [0, Q_A]$, we assume

$$c'_A(t) \geq 0 \text{ and } c''_A(t) \leq 0.$$

Similarly for supplier B with revenue function c_B, given by $c_B(t) = p_B(t)t$ for all $t \in [0, Q_B]$, we assume

$$c'_B(t) \geq 0 \text{ and } c''_B(t) \leq 0,$$

for all $t \in [0, Q_B]$. For the remainder of this paper, we assume, without loss of generality, that

$$Q_A \leq Q_B,$$

and that the players are numbered in such a way that

$$q_1 \leq q_2 \leq \dots \leq q_n.$$

When we refer to a smaller player, we refer to a player with smaller order quantity and thus smaller index. When we refer to a larger or bigger player, we refer to a player with a larger order quantity or larger index.

A CRCP-situation on N as described above, is summarized by $Z = (S, q)$, in which $S = (p_A, Q_A, p_B, Q_B)$ summarizes the suppliers' information. We denote the set of all such CRCP-situations on N by $\mathcal{Z}^N$.

The main assumption of this paper is that the players are purchasing cooperatively. The next example shows that in CRCP-situations, one can not readily determine (coalitional) stand-alone ordering costs to serve as potential reference values.

Example 2.1 Let $N = \{1, 2, 3\}$ and let $Z = (S, q) \in \mathcal{Z}^N$ with $q = (8, 9, 15)$ and $S = (p_A, Q_A, p_B, Q_B)$ be given by $Q_A = 16$, $Q_B = 20$ and

$$\begin{cases} p_A(t) & = 18 - \frac{1}{3}t, \quad t \in [0, 16]; \\ p_B(t) & = 25 - \frac{1}{2}t, \quad t \in [0, 20]. \end{cases}$$

One readily verifies that all assumptions on a CRCP-situation are satisfied. If player 1 would be on his own, he would like to order 8 at A with purchasing costs $c_A(8) = 122\frac{2}{3}$. Similarly, player 2 would like to order 9 at A with purchasing costs $c_A(9) = 135$. Also, player 3 prefers ordering 15 at supplier A with purchasing costs $c_A(15) = 195$. However, this is not a feasible combination of individual orders, since $8 + 9 + 15 > 16 = Q_A$. For this reason it is far from straightforward to adequately define appropriate individual stand-alone costs (let alone coalitional ones). $\triangle$

If the suppliers' capacities would have been unlimited or both $Q_A \geq \sum_{i \in N} q_i$ and $Q_B \geq \sum_{i \in N} q_i$, then the problem boils down to a regular cooperative purchasing situation, simply by taking the minimum of the two functions as unit price function p.

3 Cost sharing problems corresponding to CRCP-situations

As Example 2.1 shows, one can not readily use a cooperative game to analyze CRCP-situations. Instead, we will model and analyze CRCP-situations as cost sharing problems.

A *cost sharing problem on* $N = \{1, \ldots, n\}$ is represented by a pair (C, q), with demand vector $q \in \mathbb{R}^N_{++}$ such that $q_1 \leq \ldots \leq q_n$, and cost function $C : [0, \sum_{i \in N} q_i] \to \mathbb{R}_+$ for which it holds that $C(t)$ is non-decreasing and continuous in t and $C(0) = 0$. Here, the argument t in $C(t)$ represents the total demanded output. Let $\mathcal{CS}^N$ denote the corresponding class of cost sharing problems on N. When allowing for a variable finite player set, we will use the notation $\mathcal{CS}$ for the class of all such cost sharing problems. A *cost sharing rule* f *on* $\mathcal{CS}^N$ is a mapping $f : \mathcal{CS}^N \to \mathbb{R}^N$, such that $\sum_{i \in N} f_i(C, q) = C(\sum_{i \in N} q_i)$ and $f(C, q) \geq 0$ for all $(C, q) \in \mathcal{CS}^N$. A *cost sharing rule* f *on* $\mathcal{CS}$ is a mapping that assigns to each cost sharing problem $(C, q) \in \mathcal{CS}^N$, with an arbitrary finite player set N, such a vector $f(C, q)$. Also for specific subclasses of $\mathcal{CS}^N$, dropping the index N from the notation will mean that we allow for a variable player set.

A CRCP-situation corresponds to a special type of cost sharing problem. Let $Z = (S, q) \in \mathcal{Z}^N$ be a capacity restricted cooperative purchasing situation. Using the suppliers' information S, we determine the *minimal ordering costs* $D^S(t)$ for all $t \in [0, Q_A + Q_B]$ as follows:

$$D^S(t) = \min\{c_A(t_A) + c_B(t_B) \mid t_A + t_B = t, \;\; 0 \leq t_A \leq Q_A, \;\; 0 \leq t_B \leq Q_B\}.$$

Then, we can determine the corresponding cost sharing problem (C^Z, q), in which $C^Z(t) = D^S(t)$ represents the minimal ordering costs for each $t \in [0, \sum_{i \in N} q_i]$. It can be readily checked that $(C^Z, q) \in \mathcal{CS}^N$. Note that the value $C^Z(t)$ does not depend on N and q. Only the domain of the function C^Z is determined by $\sum_{i \in N} q_i$. The following theorem specifies this minimal ordering costs for three separate cases.

Theorem 3.1 *Let $Z = (S, q) \in \mathcal{Z}^N$ with $S = (p_A, Q_A, p_B, Q_B)$ be a capacity restricted cooperative purchasing situation. Then, for all $t \in [0, Q_A + Q_B]$,*

$$D^S(t) = \begin{cases} \min\{c_B(t), c_A(t)\} & \text{if } t \in [0, Q_A]; \\ \min\{c_B(t), c_A(Q_A) + c_B(t - Q_A)\} & \text{if } t \in (Q_A, Q_B]; \\ \min\{c_A(t - Q_B) + c_B(Q_B), \\ \qquad c_A(Q_A) + c_B(t - Q_A)\} & \text{if } t \in (Q_B, Q_A + Q_B]. \end{cases}$$

All proofs can be found in the Appendix. Theorem 3.1 directly implies the following.

Corollary 3.1 *Let $Z = (S, q) \in \mathcal{Z}^N$ with $S = (p_A, Q_A, p_B, Q_B)$ be a capacity restricted cooperative purchasing situation and let $(C^Z, q) \in \mathcal{CS}^N$ be the corresponding cost sharing problem. Then, for all $t \in [0, \sum_{i \in N} q_i]$,*

$$C^Z(t) = \min\{\ c_A(\min\{Q_A, t\}) + c_B(\max\{t - Q_A, 0\}), \\ c_A(\max\{t - Q_B, 0\}) + c_B(\min\{Q_B, t\})\ \}.$$

So, to minimize ordering costs, one has to compare two extreme policies: order as much as possible at one of the two suppliers first and the remaining part at the other one. Depending on the unit price functions and the total order quantity t one might prefer A 'first' or B 'first'.

The following two examples show how one can use Theorem 3.1 and Corollary 3.1 in finding the cost function of the cost sharing problem corresponding to a CRCP-situation.

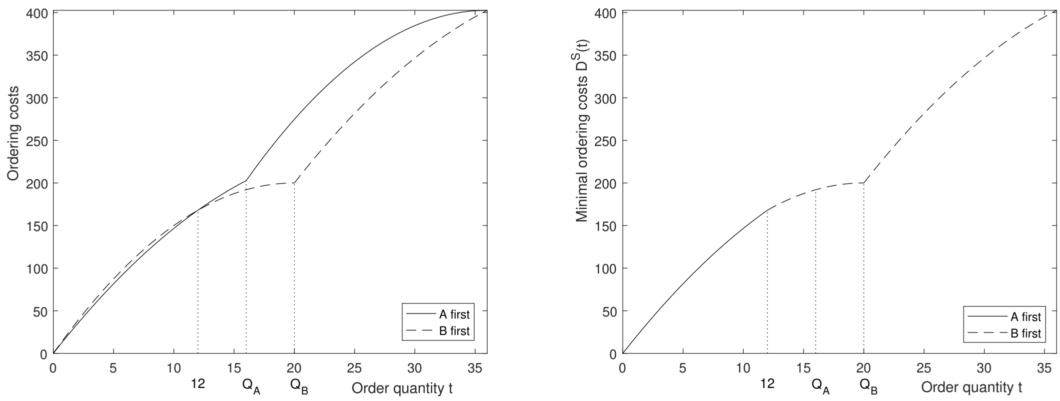

Figure 1 – *The two extreme policies and the cost function D^S in Example 3.1*

Example 3.1 Let $Z = (S, q) \in \mathcal{Z}^N$ with $S = (p_A, Q_A, p_B, Q_B)$ be given by $Q_A = 16$, $Q_B = 20$ and

$$\begin{cases} p_A(t) & = 18 - \frac{1}{3}t,\ t \in [0, 16]; \\ p_B(t) & = 20 - \frac{1}{2}t,\ t \in [0, 20]. \end{cases}$$

Note that we do not need exact specifications of N and q in order to compute the value $D^S(t)$ for all $t \in [0, 36]$. According to Theorem 3.1, one has to consider three cases: $t \in [0, 16], t \in (16, 20]$ and $t \in (20, 36]$. Since $c_A(t) \leq c_B(t)$, for all $t \in [0, 12]$ and $c_B(t) < c_A(t)$, for all $t \in (12, 16]$, we have that, for $t \in [0, 16]$,

$$D^S(t) = \begin{cases} c_A(t) & \text{if } t \in [0, 12]; \\ c_B(t) & \text{if } t \in (12, 16]. \end{cases}$$

Next, since $c_B(t) < c_A(16) + c_B(t-16)$, for all $t \in (16, 20]$, $D^S(t) = c_B(t)$, for all $t \in (16, 20]$.
Finally, since $c_B(20) + c_A(t-20) \leq c_A(16) + c_B(t-16)$, for all $t \in (20, 36]$, we have that $D^S(t) = c_B(20) + c_A(t-20)$, for all $t \in (20, 36]$.
Summarizing, we find that for all $t \in [0, Q_A + Q_B]$,

$$D^S(t) = \begin{cases} 18t - \frac{1}{3}t^2 & \text{if } t \in [0, 12]; \\ 20t - \frac{1}{2}t^2 & \text{if } t \in (12, 20]; \\ 200 + 18(t-20) - \frac{1}{3}(t-20)^2 & \text{if } t \in (20, 36], \end{cases} \tag{1}$$

as drawn in Figure 1 (right). Note that D^S is piecewise concave with two maximal intervals of concavity: $[0, 20]$ and $[20, 36]$. As stated in Corollary 3.1, the cost function C^Z, which is the restriction of D^S to the domain $[0, \sum_{i \in N} q_i]$, is the minimum of the following two policies: order as much as possible at A first and then go to B or order as much as possible at B first and then go to A. The cost functions of these two policies are shown in Figure 1 (left). In this case, there is only one switch between the two extreme policies, at $t = 12$. △

The following example shows that there can be several switches between the two extreme policies on the path between $t = 0$ and $t = Q_A + Q_B$.

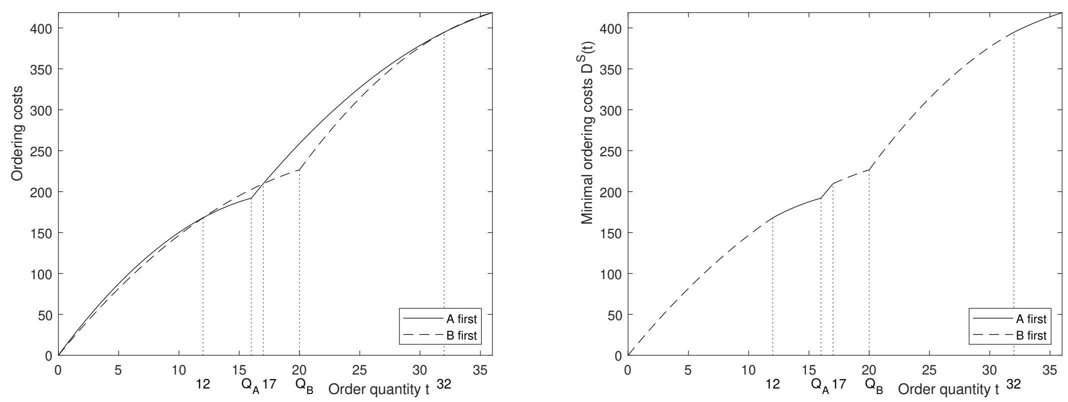

Figure 2 – *The two extreme policies and the cost function D^S in Example 3.2*

Example 3.2 Let $Z = (S, q) \in \mathcal{Z}^N$ with $S = (p_A, Q_A, p_B, Q_B)$ be given by $Q_A = 16$, $Q_B = 20$ and

$$\begin{cases} p_A(t) & = 20 - \frac{1}{2}t, \; t \in [0, 16]; \\ p_B(t) & = 18 - \frac{1}{3}t, \; t \in [0, 20]. \end{cases}$$

Figure 2 (left) depicts the cost functions corresponding to the two extreme policies (A first or B first). In this situation we see three switches: at $t = 12, t = 17$ and $t = 32$. This can also be seen from the following explicit expression of $D^S(t)$, as depicted in Figure 2 (right), for all $t \in [0, Q_A + Q_B]$:

$$D^S(t) = \begin{cases} c_B(t) & \text{if } t \in [0, 12]; \\ c_A(t) & \text{if } t \in (12, 16]; \\ c_A(16) + c_B(t-16) & \text{if } t \in (16, 17]; \\ c_B(t) & \text{if } t \in (17, 20]; \\ c_B(20) + c_A(t-20) & \text{if } t \in (20, 32]; \\ c_A(16) + c_B(t-16) & \text{if } t \in (32, 36]. \end{cases} \tag{2}$$

Note that D^S is piecewise concave and that there are three maximal intervals of concavity: $[0, 16]$, $[16, 20]$ and $[20, 36]$. $\triangle$

One can readily generalize the observations made in Example 3.1 and 3.2: the cost function D^S will be non-decreasing and piecewise concave with finitely many (maximal) intervals of concavity and hence, so will the cost function C^Z corresponding to a CRCP-situation $Z = (S, q)$.

4 Cost sharing rules for piecewise concave cost functions

In this section, we develop a new class of cost sharing rules, all based on the serial cost sharing rule (Moulin and Shenker, 1992), that are suitable for cost sharing problems with piecewise concave cost functions. In particular, these rules will provide adequate allocation mechanisms for CRCP-situations.

4.1 The serial cost sharing rule

In the introduction we argued that, of the traditional cost sharing rules, the serial cost sharing rule is the most suitable rule for the class of cost sharing problems under consideration, i.e., one-input-one-output-technologies.

The serial cost sharing rule is based on the requirement that a player's costs should not depend on the size of the order quantity of larger players. For a concave cost function, this requirement implies that smaller players profit less from the economies of scale than the larger players. If we think of CRCP-situations in which large players generally account for more quantity discounts, this seems a suitable solution method for dividing costs that follow from purchasing cooperatively. The serial (cost sharing) rule, Ser, allocates the costs in the following way.

Definition [Moulin and Shenker, 1992]: The *serial rule*, Ser, on the class $\mathcal{CS}$ of cost sharing problems is such that for all $i \in N$,

$$Ser_i(C, q) = \sum_{j=1}^{i} \frac{C(s_j) - C(s_{j-1})}{n - j + 1},$$

where $s_0 = 0$ and, for all $i \in N$,

$$s_i = \sum_{j=1}^{i-1} q_j + (n - i + 1)q_i.$$

Clearly, for $(C, q) \in \mathcal{CS}^N$ and $i \in N \backslash \{n\}$, it holds that $Ser_{i+1}(C, q) = Ser_i(C, q) + \frac{C(s_{i+1}) - C(s_i)}{n-i}$. Alternatively, the serial rule can be reformulated as in the following lemma.

Lemma 4.1 [Friedman and Moulin, 1999] *Let $(C, q) \in \mathcal{CS}^N$. Then, for all $i \in N$,*

$$Ser_i(C, q) = \frac{C(s_i)}{n - i + 1} - \sum_{j=1}^{i-1} \frac{C(s_j)}{(n - j + 1)(n - j)}.$$

The main property that characterizes the serial rule is *independence of the size of larger demands (ISLAD)*. ISLAD (cf. Moulin and Shenker, 1994) requires for all $(C, q) \in \mathcal{CS}^N$, all $i, j \in N$ with $q_i \leq q_j$, and for all $(C, \bar{q}) \in \mathcal{CS}^N$ with $\bar{q} = ((q_k)_{k \in N \backslash \{j\}}, r)$, with $r \geq q_j$, that $Ser_i(C, q) =$

$Ser_i(C, \bar{q})$. The serial rule also satisfies basic properties as *demand monotonicity*, i.e., for all $(C, q) \in \mathcal{CS}^N$ and all $i \in N \backslash \{n\}$, we have that $Ser_i(C, q) \leq Ser_{i+1}(C, q)$, and *symmetry*, i.e. for all $(C, q) \in \mathcal{CS}^N$ and all $i, j \in N$ with $q_i = q_j$, we have that $Ser_i(C, q) = Ser_j(C, q)$.

Let $\mathcal{CCS}^N$ denote the subclass of all $(C, q) \in \mathcal{CS}^N$ in which the cost function C is concave. On $\mathcal{CCS}$, the serial rule satisfies two attractive properties: UCM and MOVASP.

Definition: A cost sharing rule f satisfies *unit cost monotonicity (UCM)* on $\mathcal{C} \subseteq \mathcal{CS}^N$ if for all $(C, q) \in \mathcal{C}$ and for all $i \in N \backslash \{n\}$,

$$\frac{f_i(C, q)}{q_i} \geq \frac{f_{i+1}(C, q)}{q_{i+1}}.$$

Proposition 4.1 *The serial rule satisfies UCM on $\mathcal{CCS}^N$.*

This implies that the serial rule assigns a (weakly) lower cost per unit to organizations with a (weakly) higher demanded output. Moreover, the serial rule satisfies MOVASP, which ensures that also the organization with the smallest demanded output contributes to lower cost allocations of organizations with larger demanded output.

Definition: A cost sharing rule f satisfies *monotonic vulnerability for the absence of the smallest player (MOVASP)* on $\mathcal{C} \subseteq \mathcal{CS}$ if for all $(C, q) \in \mathcal{C}$ with $(C, q) \in \mathcal{CS}^N$ and $(C, q_{|N \backslash \{1\}}) \in \mathcal{C}$ and for all $i \in N \setminus \{1, 2\}$,[2]

$$f_i(C, q) - f_i(C, q_{|N\backslash\{1\}}) \leq f_{i-1}(C, q) - f_{i-1}(C, q_{|N\backslash\{1\}}).$$

Proposition 4.2 *The serial rule satisfies MOVASP on $\mathcal{CCS}$.*

Unfortunately, on the larger class of cost sharing problems with piecewise concave cost functions, the serial rule loses both UCM and MOVASP as is seen in the following two examples.

Example 4.1 Let $(C, q) \in \mathcal{CS}^N$ be the cost sharing problem with $N = \{1, 2, 3\}$, $q = (8, 9, 15)$ and piecewise concave cost function given by

$$C(t) = \begin{cases} 18t - \frac{1}{3}t^2 & \text{if } t \in [0, 12]; \\ 20t - \frac{1}{2}t^2 & \text{if } t \in (12, 20]; \\ 200 + 18(t - 20) - \frac{1}{3}(t - 20)^2 & \text{if } t \in (20, 32]. \end{cases} \tag{3}$$

One readily checks that

$$\begin{aligned} Ser_1(C, q) =& \frac{C(24)}{3} = 88\frac{8}{9}, \\ Ser_2(C, q) =& \frac{C(24)}{3} + \frac{C(26) - C(24)}{2} = 103\frac{5}{9}, \\ Ser_3(C, q) =& \frac{C(24)}{3} + \frac{C(26) - C(24)}{2} + (C(32) - C(26)) = 175\frac{5}{9}. \end{aligned}$$

Hence,

$$\frac{Ser_1(C, q)}{q_1} = 11\frac{9}{81} < \frac{Ser_2(C, q)}{q_2} = 11\frac{41}{81} < \frac{Ser_3(C, q)}{q_3} = 11\frac{57}{81},$$

which is, in fact, opposite to the ordering prescribed by UCM. △

[2] With minor abuse of notation, in $(C, q_{|N\backslash\{1\}})$, the players are labeled $\{2, 3, \ldots, n\}$ with $q_2 \leq q_3 \leq \ldots \leq q_n$.

Example 4.2 Let $(C,q) \in \mathcal{CS}^N$ with $N = \{1,2,3,4\}$, $q = (2,4,9,15)$ and piecewise concave cost function given by

$$C(t) = \begin{cases} 18t - \frac{1}{3}t^2 & \text{if } t \in [0,12]; \\ 20t - \frac{1}{2}t^2 & \text{if } t \in (12,16]; \\ 192 + 18(t-16) - \frac{1}{3}(t-16)^2 & \text{if } t \in (16,17]; \\ 18t - \frac{1}{3}t^2 & \text{if } t \in (17,20]; \\ 226\frac{2}{3} + 20(t-20) - \frac{1}{2}(t-20)^2 & \text{if } t \in (20,32]. \end{cases} \tag{4}$$

One readily checks that

$$Ser(C,q) = \big(30\frac{2}{3}, 50\frac{4}{9}, 108\frac{7}{9}, 186\frac{7}{9}\big),$$

while

$$Ser(C,q_{|N\setminus\{1\}}) = (56, 104\frac{1}{3}, 194\frac{1}{3}).$$

Hence, e.g.,

$$Ser_3(C,q) - Ser_3(C,q_{|N\setminus\{1\}}) = 4\frac{4}{9} > -5\frac{5}{9} = Ser_2(C,q) - Ser_2(C,q_{|N\setminus\{1\}}),$$

contradicting MOVASP. △

4.2 Piecewise serial rules

In this subsection, we modify the serial rule into piecewise serial rules that are suitable for cost sharing problems with piecewise concave cost functions. We will pinpoint a specific piecewise serial rule that satisfies UCM and a specific piecewise serial rule that satisfies MOVASP. Moreover, it is seen that these two properties are incompatible on the class of all cost sharing problems with piecewise concave cost functions.

Let $\mathcal{CCS}^{N,m} \subseteq \mathcal{CS}^N$ with $m \in \{1,2,\dots\}$ denote the class of cost sharing problems where the cost function is piecewise concave with exactly m maximal intervals of concavity.

First, we explain the idea for a piecewise serial rule by means of an example and then present the formal definition. For this we need to introduce the notion of bankruptcy problems and bankruptcy rules. A *bankruptcy problem* involves a monetary estate E that has to be divided among a finite group of claimants N, all having a non-negative justifiable monetary claim d_i, $i \in N$, on the estate. We summarize these claims into a vector $d = (d_i)_{i\in N}$. The set of all bankruptcy problems (E,d) on N is denoted by $\mathcal{B}^N$. A *bankruptcy rule* $\varphi : \mathcal{B}^N \to \mathbb{R}^N$ assigns to every bankruptcy problem $(E,d) \in \mathcal{B}^N$ a vector $\varphi(E,d) \in \mathbb{R}^N$, such that

$$\sum_{i\in N} \varphi_i(E,d) = \min\{E, \sum_{j\in N} d_j\},$$

and $0 \leq \varphi(E,d) \leq d$. Furthermore, we assume from the onset that a bankruptcy rule satisfies both *estate monotonicity* (for all $(E,d) \in \mathcal{B}^N$ and all $(E',d) \in \mathcal{B}^N$ with $E' \geq E$, we have $\varphi(E,d) \leq \varphi(E',d)$) and *symmetry* (for all $(E,d) \in \mathcal{B}^N$ with $d_i = d_j$ for some $i,j \in N$ with $i \neq j$, $\varphi_i(E,d) = \varphi_j(E,d)$).

The *proportional rule*, $PROP$, divides the estate proportionally over the claimants, i.e., for $(E, d) \in \mathcal{B}^N$ and $i \in N$,

$$PROP_i(E, d) = \min\left\{d_i \frac{E}{\sum_{j\in N} d_j}, d_i\right\}.$$

The *constrained equal losses-rule*, CEL, is such that for $(E, d) \in \mathcal{B}^N$ and $i \in N$,

$$CEL_i(E, d) = \begin{cases} \max\{d_i - \lambda, 0\} & \text{if } \sum_{j\in N} d_j > E; \\ d_i & \text{if } \sum_{j\in N} d_j \leq E, \end{cases}$$

where $\lambda \in \mathbb{R}_{++}$ is such that $\sum_{j\in N} \max\{d_j - \lambda, 0\} = E$.

Example 4.3 Reconsider the cost sharing problem $(C, q) \in \mathcal{CCS}^{N,2}$ with $N = \{1, 2, 3\}$ and $q = (8, 9, 15)$ as in Example 4.1 in which the cost function C is given by

$$C(t) = \begin{cases} 18t - \frac{1}{3}t^2 & \text{if } t \in [0, 12]; \\ 20t - \frac{1}{2}t^2 & \text{if } t \in (12, 20]; \\ 200 + 18(t - 20) - \frac{1}{3}(t - 20)^2 & \text{if } t \in (20, 32]. \end{cases}$$

Note that this cost function is the restriction of the function as specified by Equation (1) and hence, visualised in Figure 1. As can be seen in Figure 1 (left), C has two maximal intervals of concavity: $[0, 20]$ and $[20, 32]$. If we could divide the vector q over these two intervals, i.e., find a suitable vector $x^1 \in \mathbb{R}^N$ with $\sum_{j\in N} x^1_j = 20$ for the first interval and a suitable vector $x^2 = q - x^1$ for the second interval, we could apply the serial rule on each of these two cost sharing problems.

For this, the demands, given by q, are considered as claims on their preferred interval $[0, 20]$ (since in this interval, the returns to scale are larger than in the other interval) and we use a bankruptcy rule φ to determine $x^1 = \varphi(20, q)$. Arguing that large players should obtain a lower cost per unit than small players and therefore should be allocated a relatively higher part of the preferred interval $[0, 20]$, we can opt for the constrained equal losses rule CEL. In that case it gives $x^1 = CEL(20, (8, 9, 15)) = (4, 5, 11)$ and $x^2 = (8, 9, 15) - (4, 5, 11) = (4, 4, 4)$. Subsequently, on the interval $[0, 20]$ we face the cost sharing problem (C^1, x^1) with $C^1(t) = C(t)$ for $t \in [0, 20]$ and on the second interval we face the cost sharing problem (C^2, x^2) with $C^2(t) = C(t + 20) - C(20) = C(t + 20) - 200$ for all $t \in [0, 12]$. Using the serial rule as the leading principle for concave cost functions, it is readily verified that $Ser(C^1, x^1) = (56, 63, 81)$ and $Ser(C^2, x^2) = (56, 56, 56)$. Consequently, for the cost allocation specified by the CEL-piecewise serial rule, denoted by $\Psi^{CEL}(C, q)$, we have

$$\Psi^{CEL}(C, q) = (56, 63, 81) + (56, 56, 56) = (112, 119, 137).$$

On the other hand, one could also argue that the players should have 'relatively equal' rights to all of the intervals, which can be realized by dividing q proportionally over the intervals. Then, $x^1 = PROP(20, (8, 9, 15)) = (5, 5\frac{5}{8}, 9\frac{3}{8})$ and $x^2 = (3, 3\frac{3}{8}, 5\frac{5}{8})$. Subsequently, since

$$Ser(C^1, x^1) \approx (62.5, 65.2, 72.3)$$

and

$$Ser(C^2, x^2) \approx (45, 49.4, 73.6),$$

we have that (using a similar notation)

$$\Psi^{PROP}(C, q) \approx (107.5, 144.6, 145.9).$$

△

Definition: Let $m \in \{1, 2, \ldots\}$ and let φ be a bankruptcy rule on $\mathcal{B}^N$. The *φ-piecewise serial rule* $\Psi^\varphi : \mathcal{CCS}^{N,m} \to \mathbb{R}^N$ is defined in the following way. Let $(C, q) \in \mathcal{CCS}^{N,m}$ be a cost sharing problem and let $[t_0, t_1], [t_1, t_2], \ldots, [t_{m-1}, t_m]$ with $0 = t_0 < t_1 < t_2 < \ldots < t_{m-1} < t_m = \sum_{i \in N} q_i$ be the maximal intervals of concavity of C. With $q^1(\varphi) = q$, recursively compute the vectors $q^r(\varphi) \in \mathbb{R}^N$ and $x^r(\varphi) \in \mathbb{R}^N$ for $r = 1, \ldots, m$, in the following way:

$$\begin{cases} x^r(\varphi) & = \varphi(t_r - t_{r-1}, q^r(\varphi)); \\ q^{r+1}(\varphi) & = q^r(\varphi) - x^r(\varphi). \end{cases}$$

Let $C^r : [0, t_r - t_{r-1}] \to \mathbb{R}_+$ denote the modified cost function on the interval $[t_{r-1}, t_r]$ given by

$$C^r(t) = C(t + t_{r-1}) - C(t_{r-1}),$$

for all $t \in [0, t_r - t_{r-1}]$, such that $(C^r, x^r(\varphi))$ is a cost sharing problem with a concave cost function[3]. Finally, we set

$$\Psi^\varphi(C, q) = \sum_{r=1}^{m} Ser(C^r, x^r(\varphi)).$$

It is readily seen that for a bankruptcy rule φ, the corresponding φ-piecewise serial rule Ψ^φ is efficient and satisfied demand monotonicity. For a symmetric bankruptcy rule φ, the corresponding φ-piecewise serial rule is symmetric as well. We will specifically focus on Ψ^{PROP} and on Ψ^{CEL} as allocation methods for cost sharing problems with piecewise concave cost functions.

Importantly, both $PROP$ and CEL satisfy *order preservation.* A bankruptcy rule φ satisfies order preservation if for all $(E, d) \in \mathcal{B}^N$ and all $i, j \in N$, with $d_i \leq d_j$ both $\varphi_i(E, d) \leq \varphi_j(E, d)$ and $d_i - \varphi_i(E, d) \leq d_j - \varphi_j(E, d)$. It is readily checked that for a bankruptcy rule φ that satisfies order preservation, both claims and allocations underlying the recursive procedure for the corresponding φ-piecewise serial rule are non-decreasing over the players in each maximal interval of concavity, that is,

$$x_1^r(\varphi) \leq x_2^r(\varphi) \leq \ldots \leq x_n^r(\varphi)$$

and

$$q_1^r(\varphi) \leq q_2^r(\varphi) \leq \ldots \leq q_n^r(\varphi),$$

for all $r \in \{1, \ldots, m\}$.

The following theorem shows that the $PROP$-piecewise serial rule satisfies unit cost monotonicity. Recall that all proofs can be found in the Appendix.

Theorem 4.1 *Ψ^{PROP} satisfies UCM on $\mathcal{CCS}^{N,m}$ for all $m \in \{1, 2, \ldots\}$.*

Next, we show that Ψ^{PROP} is the unique piecewise serial rule that satisfies unit cost monotonicity.

Theorem 4.2 *Let φ be a bankruptcy rule such that Ψ^φ satisfies UCM on $\mathcal{CCS}^{N,m}$ for all $m \in \{1, 2 \ldots\}$. Then, $\varphi = PROP$.*

[3]To draw this conclusion, we realize that we need that $x^r(\varphi)$ is non-decreasing over the players. For arbitrary φ, this may not be the case and therefore, the players should be reordered. However, to avoid a notational overburden, this reordering is left implicit in the definition. If needed in the proofs, the reordering of the players will be made explicit.

The next example shows that Ψ^{PROP} does not satisfy MOVASP on $\mathcal{CCS}$.

Example 4.4 Let $N = \{1, 2, 3, 4\}$ and let $(C, q) \in \mathcal{CCS}^{N,3}$ be a cost sharing problem given by $q = (2, 5, 6, 9)$ and

$$C(t) = \begin{cases} 18t - \frac{1}{3}t^2 & \text{if } t \in [0, 12]; \\ 20t - \frac{1}{2}t^2 & \text{if } t \in (12, 16]; \\ 192 + 18(t-16) - \frac{1}{3}(t-16)^2 & \text{if } t \in (16, 17]; \\ 18t - \frac{1}{3}t^2 & \text{if } t \in (17, 18]; \\ 216 + 20(t-18) - \frac{1}{2}(t-18)^2 & \text{if } t \in (18, 22]. \end{cases} \tag{5}$$

The function C has three maximal intervals of concavity: $[0, 16]$, $[16, 18]$ and $[18, 22]$. It can be calculated that

$$\Psi^{PROP}(C, q) = (33.6, 71.8, 80.5, 102.1),$$

while

$$\Psi^{PROP}(C, q_{|N\setminus\{1\}}) \approx (72.6, 80.8, 100.6).$$

Thus,

$$\begin{aligned} \Psi_2^{PROP}(C, q) - \Psi_2^{PROP}(C, q_{|N\setminus\{1\}}) \approx -0.8 & \\ < -0.3 \approx \Psi_3^{PROP}(C, q) - \Psi_3^{PROP}(C, q_{|N\setminus\{1\}}) & \\ < 1.5 \approx \Psi_4^{PROP}(C, q) - \Psi_4^{PROP}(C, q_{|N\setminus\{1\}}), & \end{aligned}$$

exactly reverse to the order prescribed by MOVASP. △

Interestingly, the CEL-piecewise serial rule does satisfy MOVASP on $\mathcal{CCS}^m$ for all $m \in \{1, 2, \ldots\}$.

Theorem 4.3 *Ψ^{CEL} satisfies MOVASP on $\mathcal{CCS}^m$ for all $m \in \{1, 2, \ldots\}$.*

5 Conclusion and discussion

In this paper, we introduced capacity restricted cooperative purchasing situations and modeled them as cost sharing problems. This section mainly focuses on the class of cost sharing problems that indeed stem from capacity restricted cooperative purchasing situations. This especially in relation to the incompatibility result of UCM and MOVASP on the larger class $\mathcal{CCS}^m$ of cost sharing problems with piecewise concave cost functions with exactly m maximal intervals of concavity. We denoted $(C^Z, q) \in \mathcal{CS}^N$ as the cost sharing problem corresponding to the CRCP-situation $Z = (S, q) \in \mathcal{Z}^N$. Let $\mathcal{ZCCS}^N$ denote the class of all such cost sharing problems on fixed N, and $\mathcal{ZCCS}$ the class of all such cost sharing problems on variable but finite N.

The cost function C^Z of a cost sharing problem $(C^Z, q) \in \mathcal{ZCCS}^N$ turned out to be piecewise concave with finitely many (maximal) intervals of concavity. Hence,

$$\mathcal{ZCCS}^N \subseteq \bigcup_{m=1}^{\infty} \mathcal{CCS}^{N,m}.$$

With regard to the serial rule and the piecewise serial rules, we focused on the class of cost sharing problems with concave and piecewise concave cost functions, respectively. For example, in Example 4.1 we showed that the serial rule does not satisfy UCM on the class of piecewise concave cost functions. In fact, the cost sharing problem as described in Example 4.1 originates from the CRCP-situation as described in Example 3.1, as can be seen by comparing the cost function D^S of Equation (1) with the cost function C of Equation (3). Hence, the serial rule does not satisfy UCM on $\mathcal{ZCCS}^N$.

Similarly, the cost function C of Equation (4) in Example 4.2 is the restriction of the cost function D^S of Equation (2) that corresponds to the CRCP-situation as described in Example 3.2. Hence, the serial rule does not satisfy MOVASP on $\mathcal{ZCCS}$.

Also the cost sharing problem of Example 4.4, in which it is shown that Ψ^{PROP} does not satisfy MOVASP on $\mathcal{CCS}$, corresponds to a CRCP-situation. It can be readily checked that the cost function C of Equation (5) is the restriction of the cost function D^S of the CRCP-situation $Z = (S, q) \in \mathcal{Z}^N$ with $S = (p_A, Q_A, p_B, Q_B)$ be given by $Q_A = 16$, $Q_B = 18$ and

$$\begin{cases} p_A(t) = 20 - \frac{1}{2}t, & t \in [0, 16]; \\ p_B(t) = 18 - \frac{1}{3}t, & t \in [0, 18]. \end{cases}$$

Hence, the $PROP$-piecewise serial rule does not satisfy MOVASP on $\mathcal{ZCCS}$.

In Theorem 4.2, we show that the properties UCM and MOVASP are incompatible on $\mathcal{CCS}$ in the sense that the only φ-piecewise serial rule that satisfies UCM is the $PROP$-piecewise serial rule. The proof of this theorem however does not immediately guarantee that UCM and MOVASP are also incompatible on $\mathcal{ZCCS}$.

To investigate, among other things, whether indeed UCM and MOVASP are incompatible on the class $\mathcal{ZCCS}$, we illustrate differences and similarities between the two piecewise serial rules Ψ^{PROP} and Ψ^{CEL} and the classical serial rule Ser for specific numerical cost sharing problems arising from CRCP-situations using simulation techniques.

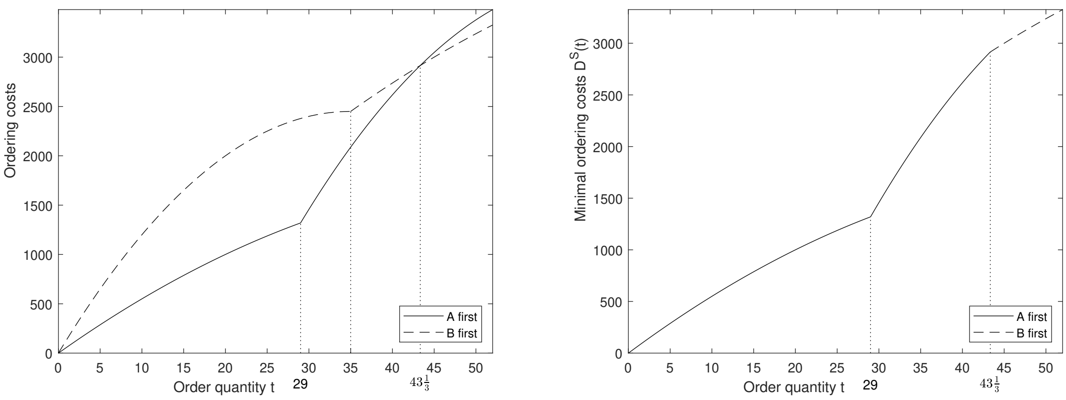

Figure 3 – *The two extreme policies and the cost function D^S for the simulation*

As input, we take CRCP-situations $Z = (S, q) \in \mathcal{Z}^N$ with $S = (p_A, Q_A, p_B, Q_B)$ be given by $Q_A = 29$, $Q_B = 35$ and

$$\begin{cases} p_A(t) = 60 - \frac{1}{2}t, & t \in [0, 29], \\ p_B(t) = 140 - 2t, & t \in [0, 35]. \end{cases}$$

We also keep $\sum_{i \in N} q_i = 52$, such that the cost function C^Z of the associated cost sharing problem $(C^Z, q) \in \mathcal{ZCCS}^N$ remains the same for all CRCP-situations. This cost function is depicted in Figure 3 and has 2 maximal intervals of concavity: $[0, 29]$ and $[29, 52]$.

To create a CRCP-situation, we set $N = \{1, 2, 3, 4\}$ and randomly generate an integer-valued vector q of order quantities such that the sum of the order quantities equals 52. We compare two separate scenarios: $q_i \in \{10, 11, 12, 13, 14, 15, 16\}$ for all $i \in N$ and $q_i \in \{1, 5, 9, 13, 17, 21, 25\}$ for all $i \in N$, that is, a first scenario with 'small' differences between the possible order quantities and a second scenario with 'big' differences. For each generated instance (C, q), $Ser(C, q)$, $\Psi^{PROP}(C, q)$ and $\Psi^{CEL}(C, q)$ are calculated. Per integer value of the order quantity (independent of the corresponding player), the corresponding allocation for each of the cost sharing rules is stored and, in the end, averaged over the number of times it has occurred in the simulated instances.

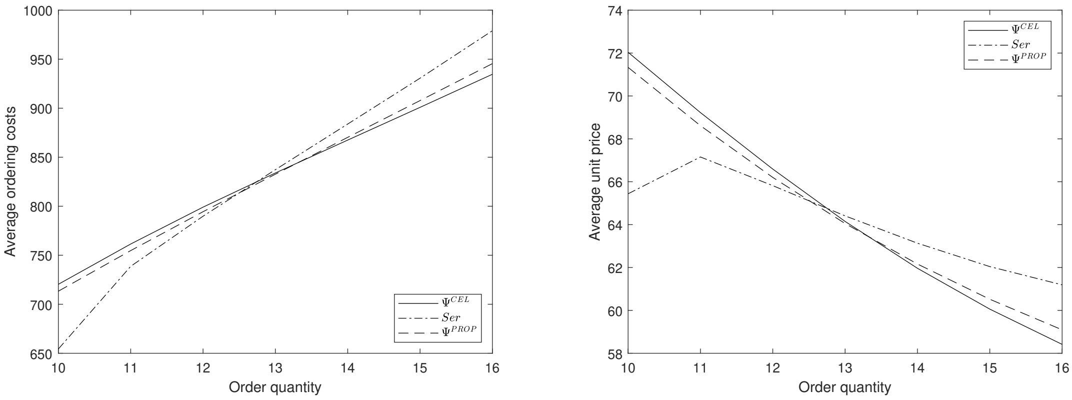

Figure 4 – *A comparison of the cost sharing rules: Total costs and unit costs for a scenario with 'small' differences*

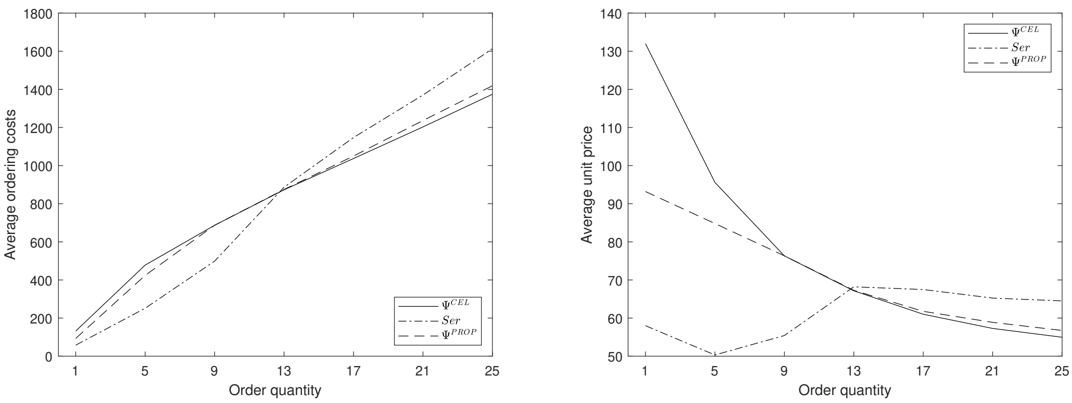

Figure 5 – *A comparison of the cost sharing rules: Total costs and unit costs for a scenario with 'big' differences*

In Figure 4 we plotted the (average) total costs and costs per time unit allocated by Ser, Ψ^{PROP} and Ψ^{CEL} for a player i with order quantity $q_i \in \{10, 11, 12, 13, 14, 15, 16\}$, i.e., in the scenario with 'small' differences. Figure 5 provides the same data for a player i with $q_i \in \{1, 5, 9, 13, 17, 21, 25\}$, in the scenario with 'big' differences.

We can make the following observations. On average, the serial cost sharing rule Ser allocates more costs to the larger players than the two piecewise serial rules. For middle players the cost allocations are almost the same, while Ser allocates less costs to smaller players. The differences between Ψ^{PROP} and Ψ^{CEL} are smaller, although larger players prefer Ψ^{CEL} over Ψ^{PROP}. The reason for this is that using CEL as the mechanism for allocating q over the intervals of concavity recursively, will result in larger players having relatively bigger shares in the first intervals than in later ones.

It is clearly visible that Ser does not satisfy UCM, while Ψ^{PROP} does satisfy UCM. For Ψ^{CEL}, the data seems to suggest that UCM is satisfied too for cost sharing problems arising from CRCP-situations.

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A Proofs

Theorem 3.1 *Let $Z = (S, q) \in \mathcal{Z}^N$ with $S = (p_A, Q_A, p_B, Q_B)$ be a capacity restricted cooperative purchasing situation. Then, for all $t \in [0, Q_A + Q_B]$,*

$$D^S(t) = \begin{cases} \min\{c_B(t), c_A(t)\} & \text{if } t \in [0, Q_A]; \\ \min\{c_B(t), c_A(Q_A) + c_B(t - Q_A)\} & \text{if } t \in (Q_A, Q_B]; \\ \min\{c_A(t - Q_B) + c_B(Q_B), & \\ \qquad c_A(Q_A) + c_B(t - Q_A)\} & \text{if } t \in (Q_B, Q_A + Q_B]. \end{cases}$$

Proof. Let $t \in [0, Q_A + Q_B]$. Then,

$$\begin{aligned} D^S(t) &= \min\{c_A(t_A) + c_B(t_B) \mid t_A + t_B = t, 0 \leq t_A \leq Q_A, 0 \leq t_B \leq Q_B\} \\ &= \min\{c_A(t_A) + c_B(t - t_A) \mid t_A \in [\max\{t - Q_B, 0\}, \min\{Q_A, t\}]\}. \end{aligned}$$

Note that the interval $[\max\{t - Q_B, 0\}, \min\{Q_A, t\}]$ is non-empty since $Q_A, Q_B \in \mathbb{R}_{++}$. Let $g : [\max\{t - Q_B, 0\}, \min\{Q_A, t\}] \to \mathbb{R}_+$ be defined by,

$$g(t_A) = c_A(t_A) + c_B(t - t_A),$$

for all $t_A \in [\max\{t - Q_B, 0\}, \min\{Q_A, t\}]$. Note that

$$g'(t_A) = c'_A(t_A) - c'_B(t - t_A)$$

and that

$$g''(t_A) = c''_A(t_A) + c''_B(t - t_A) \leq 0.$$

Hence, g is concave and thus the minimum of g can be found at one of the boundaries of the domain of g: $t_A = \max\{t - Q_B, 0\}$ or $t_A = \min\{Q_A, t\}$, in the minimum.
If $t \leq Q_A$, then in the minimum, $t_A = 0$ (and $t_B = t$) or $t_A = t$ (and $t_B = 0$) and consequently

$$D^S(t) = \min\{c_B(t), c_A(t)\}.$$

If $Q_A < t \leq Q_B$, then in the minimum, $t_A = 0$ (and $t_B = t$) or $t_A = Q_A$ (and $t_B = t - Q_A$). Consequently

$$D^S(t) = \min\{c_B(t), c_A(Q_A) + c_B(t - Q_A)\}.$$

If $t > Q_B$, then in the minimum, $t_A = t - Q_B$ (and $t_B = Q_B$) or $t_A = Q_A$ (and $t_B = t - Q_A$). Consequently

$$D^S(t) = \min\{c_A(t - Q_B) + c_B(Q_B), c_A(Q_A) + c_B(t - Q_A)\}. \qquad \square$$

The proof of Proposition 4.1 is based on some basic properties of concave functions. These properties are summarized in the following proposition.

Proposition A.1 *Let X be a convex subset of $\mathbb{R}$ and let $f : X \to \mathbb{R}$ with $f(0) = 0$ be a non-decreasing and concave function. Then,*

i) $\frac{f(x)}{x} \geq \frac{f(y)}{y}$ *for all $x, y \in X$ with $0 < x \leq y$;*

ii) $f(x+z) - f(x) \geq f(y+z) - f(y)$ *for all $x, y, z \in X$ with $0 < x \leq y$ and $z > 0$;*

iii) $\frac{f(y)-f(x)}{y-x} \geq \frac{f(z)-f(x)}{z-x} \geq \frac{f(z)-f(y)}{z-y}$ *for all $x, y, z \in X$ with $0 < x < y < z$.*

Proposition 4.1 *The serial rule satisfies UCM on $\mathcal{CCS}^N$.*

Proof. Let $(C, q) \in \mathcal{CCS}^N$. For all $i \in N\backslash\{n\}$, we have that

$$\begin{aligned}
\frac{Ser_i(C,q)}{q_i} - \frac{Ser_{i+1}(C,q)}{q_{i+1}} &= \frac{Ser_i(C,q)}{q_i} - \frac{Ser_i(C,q)}{q_{i+1}} - \frac{C(s_{i+1}) - C(s_i)}{(n-i)q_{i+1}} \\
&= \frac{(q_{i+1} - q_i)Ser_i(C,q)}{q_i q_{i+1}} - \frac{C(s_{i+1}) - C(s_i)}{(n-i)q_{i+1}} \\
&= \frac{(q_{i+1} - q_i)Ser_i(C,q)}{q_i q_{i+1}} - \frac{(q_{i+1} - q_i)(C(s_{i+1}) - C(s_i))}{(n-i)(q_{i+1} - q_i)q_{i+1}} \\
&= \frac{(q_{i+1} - q_i)Ser_i(C,q)}{q_i q_{i+1}} - \frac{(q_{i+1} - q_i)(C(s_{i+1}) - C(s_i))}{(s_{i+1} - s_i)q_{i+1}},
\end{aligned}$$

by noting that $(n-i)(q_{i+1} - q_i) = s_{i+1} - s_i$ for the last equality. Hence, to show that

$$\frac{Ser_i(C,q)}{q_i} \geq \frac{Ser_{i+1}(C,q)}{q_{i+1}},$$

for all $i \in N\backslash\{n\}$, it is sufficient to show that

$$\frac{Ser_i(C,q)}{q_i} \geq \frac{C(s_{i+1}) - C(s_i)}{s_{i+1} - s_i},$$

for all $i \in N\backslash\{n\}$.

Let $i = 1$. Obviously, if $q_1 = q_2$, then $Ser_1(C,q) = Ser_2(C,q)$ and hence, $\frac{Ser_1(C,q)}{q_1} = \frac{Ser_2(C,q)}{q_2}$. If $q_1 \neq q_2$, then

$$\begin{aligned} \frac{Ser_1(C,q)}{q_1} - \frac{C(s_2) - C(s_1)}{s_2 - s_1} &= \frac{C(s_1)}{nq_1} - \frac{C(s_2) - C(s_1)}{s_2 - s_1} \\ &= \frac{C(s_1)}{s_1} - \frac{s_1 C(s_2) - s_1 C(s_1)}{s_1(s_2 - s_1)} \\ &= \frac{(s_2 - s_1)C(s_1) - s_1 C(s_2) + s_1 C(s_1)}{s_1(s_2 - s_1)} \\ &= \frac{s_2 C(s_1) - s_1 C(s_2)}{s_1(s_2 - s_1)} \geq 0, \end{aligned}$$

where the inequality follows from Proposition A.1 part *i)*, by using that $0 < s_1 \leq s_2$ and the fact that C is a concave function. Proceeding by induction, take $i \in N\backslash\{n\}$, $i > 1$ and assume that for all $j \in N\backslash\{n\}$, $j < i$,

$$\frac{Ser_j(C,q)}{q_j} \geq \frac{Ser_{j+1}(C,q)}{q_{j+1}},$$

which is equivalent to

$$Ser_j(C,q) \geq \frac{q_j}{q_{j+1}} Ser_{j+1}(C,q).$$

If $q_i = q_{i+1}$, then $Ser_i(C,q) = Ser_{i+1}(C,q)$ and hence,

$$\frac{Ser_i(C,q)}{q_i} = \frac{Ser_{i+1}(C,q)}{q_{i+1}}.$$

If $q_i < q_{i+1}$, we distinguish between two cases: first, if $q_1 = q_2 = \ldots = q_{i-1} = q_i$, then $s_1 = s_2 = \ldots = s_i = nq_i$ and hence,

$$\begin{aligned} \frac{Ser_i(C,q)}{q_i} - \frac{C(s_{i+1}) - C(s_i)}{s_{i+1} - s_i} &= \frac{C(s_1)}{nq_i} + \frac{C(s_2) - C(s_1)}{(n-1)q_i} + \ldots + \frac{C(s_i) - C(s_{i-1})}{(n-i)q_i} - \frac{C(s_{i+1}) - C(s_i)}{s_{i+1} - s_i} \\ &= \frac{C(s_1)}{nq_i} - \frac{C(s_{i+1}) - C(s_i)}{s_{i+1} - s_i} \\ &= \frac{C(s_1)}{s_1} - \frac{C(s_{i+1}) - C(s_1)}{s_{i+1} - s_1} \\ &= \frac{s_{i+1} C(s_1) - s_1 C(s_{i+1})}{s_1(s_{i+1} - s_1)} \geq 0, \end{aligned}$$

where the inequality follows from Proposition A.1 part *i)*, by using that $0 < s_1 \leq s_{i+1}$ and the fact that C is a concave function.

Secondly, if there exists a $k \in N$ such that $q_k < q_i$ and $q_\ell = q_i$ for all $\ell \in \{k+1, \dots, i-1\}$. Then, $s_k < s_i < s_{i+1}$. Hence,

$$\begin{aligned}\frac{C(s_{i+1}) - C(s_i)}{s_{i+1} - s_i} &\leq \frac{C(s_i) - C(s_k)}{s_i - s_k} \\ &= \frac{C(s_i) - C(s_k)}{(n-k)(q_i - q_k)} \\ &= \frac{Ser_i(C,q) - Ser_k(C,q)}{q_i - q_k} \\ &\leq \frac{Ser_i(C,q) - \frac{q_k}{q_i} Ser_i(C,q)}{q_i - q_k} \\ &= \frac{Ser_i(C,q)}{q_i},\end{aligned}$$

where the first inequality follows from Proposition A.1 part *iii)* and the second inequality from the induction hypothesis. □

Proposition 4.2 *The serial rule satisfies MOVASP on $\mathcal{CCS}$.*

Proof. Let $(C,q) \in \mathcal{CCS}^N$. Set $\Delta Ser_j = Ser_j(C,q) - Ser_j(C, q_{|N\setminus\{1\}})$ for all $j \in N\setminus\{1\}$. We have to show that

$$\Delta Ser_2 \geq \Delta Ser_3 \geq \ldots \geq \Delta Ser_n.$$

According to Lemma 4.1 we have, for all $i \in \{2, 3, \dots, n\}$,

$$Ser_i(C,q) = \frac{C(s_i)}{n-i+1} - \sum_{j=1}^{i-1} \frac{C(s_j)}{(n-j+1)(n-j)}$$

and, also (with some technical variation[4])

$$Ser_i(C, q_{|N\setminus\{1\}}) = \frac{C(s_i - q_1)}{n-i+1} - \sum_{j=2}^{i-1} \frac{C(s_j - q_1)}{(n-j+1)(n-j)}.$$

Hence, for all $i \in \{2, 3, \dots, n\}$,

$$\Delta Ser_i = \frac{C(s_i) - C(s_i - q_1)}{n-i+1} - \frac{C(s_1)}{n(n-1)} - \sum_{j=2}^{i-1} \frac{C(s_j) - C(s_j - q_1)}{(n-j+1)(n-j)}.$$

Consequently, for $i \in \{3, 4, \dots, n\}$,

$$\begin{aligned}\Delta Ser_i - \Delta Ser_{i-1} = {} & \frac{C(s_i) - C(s_i - q_1)}{n-i+1} - \frac{C(s_1)}{n(n-1)} - \sum_{j=2}^{i-1} \frac{C(s_j) - C(s_j - q_1)}{(n-j+1)(n-j)} \\ & - \frac{C(s_{i-1}) - C(s_{i-1} - q_1)}{n-i+2} + \frac{C(s_1)}{n(n-1)} + \sum_{j=2}^{i-2} \frac{C(s_j) - C(s_j - q_1)}{(n-j+1)(n-j)}\end{aligned}$$

[4]Let $0 = s_0, s_1, s_2, \dots, s_n$ be the intermediate points that are used to compute $Ser(C,q)$. Let $0 = s'_1, s'_2, s'_3, \dots, s'_n$ be the intermediate points used to compute $Ser(C, q_{|N\setminus\{1\}})$. Clearly, $s'_j = s_j - q_1$ for all $j \in N\setminus\{1\}$.

$$= \frac{C(s_i) - C(s_i - q_1)}{n-i+1} - \frac{C(s_{i-1}) - C(s_{i-1} - q_1)}{(n-i+2)(n-i+1)} - \frac{C(s_{i-1}) - C(s_{i-1} - q_1)}{n-i+2}$$

$$= \frac{C(s_i) - C(s_i - q_1)}{n-i+1} - \frac{C(s_{i-1}) - C(s_{i-1} - q_1)}{n-i+1}$$
$$\leq \frac{C(s_{i-1}) - C(s_{i-1} - q_1)}{n-i+1} - \frac{C(s_{i-1}) - C(s_{i-1} - q_1)}{n-i+1} = 0,$$

where the inequality follows from Proposition A.1 part *ii)*, by using that $0 < s_{i-1} - q_1 \leq s_i - q_1$ and $q_1 > 0$. □

Theorem 4.1 *Ψ^{PROP} satisfies UCM on $\mathcal{CCS}^{N,m}$ for all $m \in \{1, 2, \ldots\}$.*

Proof. The theorem clearly holds for $m = 1$, by using Proposition 4.1 and the fact that $\mathcal{CCS}^{N,1} = \mathcal{CCS}^N$. Let $m \in \{2, 3, \ldots\}$ and let $(C, q) \in \mathcal{CCS}^{N,m}$ and denote by $[t_0, t_1], [t_1, t_2], \ldots, [t_{m-1}, t_m]$ with $0 = t_0 < t_1 < t_2 < \ldots < t_{m-1} < t_m = \sum_{i \in N} q_i$ the maximal intervals of concavity of C. For the remainder of the proof, we abbreviate $x^r(PROP)$ and $q^r(PROP)$ to x^r and q^r for all $r \in \{1, \ldots, m\}$. Since $PROP$ satisfies order preservation, we have, for all $r \in \{1, \ldots, m\}$,

$$\begin{cases} x_1^r \leq x_2^r \leq \ldots \leq x_n^r; \\ q_1^r \leq q_2^r \leq \ldots \leq q_n^r. \end{cases}$$

Moreover, due to the nature of the proportional rule, $x_1^r > 0$ and $q_1^r > 0$, for all $r \in \{1, \ldots, m\}$. Hence, for all $r \in \{1, \ldots, m\}$, $(C^r, x^r) \in \mathcal{CCS}^N$, and consequently, Proposition 4.1 implies that for all $i \in N \backslash \{n\}$,

$$\frac{Ser_i(C^r, x^r)}{x_i^r} \geq \frac{Ser_{i+1}(C^r, x^r)}{x_{i+1}^r}. \tag{6}$$

For the rest of the proof, fix $i \in N \backslash \{n\}$. For UCM of Ψ^{PROP}, we have to prove that

$$\frac{\Psi_i^{PROP}(C, q)}{q_i} \geq \frac{\Psi_{i+1}^{PROP}(C, q)}{q_{i+1}}.$$

We first prove that $\frac{x_i^r}{q_i} = \frac{x_{i+1}^r}{q_{i+1}}$ for all $r \in \{1, 2, \ldots, m\}$. Clearly,

$$\frac{PROP_i(t_1, q)}{q_i} = \frac{PROP_{i+1}(t_1, q)}{q_{i+1}},$$

and thus,

$$\frac{x_i^1}{q_i} = \frac{x_{i+1}^1}{q_{i+1}}.$$

Proceeding by induction, take $r \in \{2, \ldots, m\}$ and assume that, for all $s \in \{1, \ldots, r-1\}$,

$$\frac{x_i^s}{q_i} = \frac{x_{i+1}^s}{q_{i+1}}.$$

Then,

$$\frac{q_i^r}{q_i} = \frac{q_i^{r-1} - x_i^{r-1}}{q_i} = \ldots = \frac{q_i - \sum_{s=1}^{r-1} x_i^s}{q_i} = 1 - \sum_{s=1}^{r-1} \frac{x_i^s}{q_i}$$
$$\stackrel{\text{(IH)}}{=} 1 - \sum_{s=1}^{r-1} \frac{x_{i+1}^s}{q_{i+1}} = \frac{q_{i+1} - \sum_{s=1}^{r-1} x_{i+1}^s}{q_{i+1}} = \ldots = \frac{q_{i+1}^{r-1} - x_{i+1}^{r-1}}{q_{i+1}} = \frac{q_{i+1}^r}{q_{i+1}},$$

where IH stands for the induction hypothesis. Moreover,

$$\frac{x_i^r}{q_i^r} = \frac{PROP_i(t_r - t_{r-1}, q^r)}{q_i^r} = \frac{PROP_{i+1}(t_r - t_{r-1}, q^r)}{q_{i+1}^r} = \frac{x_{i+1}^r}{q_{i+1}^r}.$$

Hence,

$$\frac{x_i^r}{q_i} = \frac{x_i^r}{q_i^r}\frac{q_i^r}{q_i} = \frac{x_{i+1}^r}{q_{i+1}^r}\frac{q_{i+1}^r}{q_{i+1}} = \frac{x_{i+1}^r}{q_{i+1}}.$$

We may conclude that, for all $r \in \{1, \ldots, m\}$,

$$\frac{x_i^r}{q_i} = \frac{x_{i+1}^r}{q_{i+1}}. \tag{7}$$

Using Equations (6) and (7) we find

$$\frac{\Psi_i^{PROP}(C,q)}{q_i} = \sum_{r=1}^{m} \frac{Ser_i(C^r, x^r)}{q_i} = \sum_{r=1}^{m} \frac{Ser_i(C^r, x^r)}{x_i^r}\frac{x_i^r}{q_i}$$
$$\stackrel{(7)}{=} \sum_{r=1}^{m} \frac{Ser_i(C^r, x^r)}{x_i^r}\frac{x_{i+1}^r}{q_{i+1}}$$
$$\stackrel{(6)}{\geq} \sum_{r=1}^{m} \frac{Ser_{i+1}(C^r, x^r)}{x_{i+1}^r}\frac{x_{i+1}^r}{q_{i+1}}$$
$$= \sum_{r=1}^{m} \frac{Ser_{i+1}(C^r, x^r)}{q_{i+1}} = \frac{\Psi_{i+1}^{PROP}(C,q)}{q_{i+1}}.$$ □

Theorem 4.2 *Let φ be a bankruptcy rule such that Ψ^φ satisfies UCM on $\mathcal{CCS}^{N,m}$ for all $m \in \{1, 2 \ldots\}$. Then, $\varphi = PROP$.*

Proof. Suppose for the sake of contradiction that $\varphi \neq PROP$. Then, there exists $(E,d) \in \mathcal{B}^N$ with $d_1 \leq \ldots \leq d_n$, $\sum_{j \in N} d_j > E$ and $i \in N$ with $d_i > 0, d_{i+1} > 0$ and $d_i \neq d_{i+1}$ such that

$$\frac{\varphi_i(E,d)}{d_i} > \frac{\varphi_{i+1}(E,d)}{d_{i+1}} \tag{8}$$

or

$$\frac{\varphi_i(E,d)}{d_i} < \frac{\varphi_{i+1}(E,d)}{d_{i+1}}. \tag{9}$$

For both cases (8) and (9), we show that there exists a cost sharing problem $(C,q) \in \mathcal{CCS}^{N,2}$ for which

$$\frac{\Psi_i^\varphi(C,q)}{q_i} < \frac{\Psi_{i+1}^\varphi(C,q)}{q_{i+1}},$$

contradicting UCM.

Suppose (8) holds. Consider $(C, q) \in \mathcal{CCS}^{N,2}$ with $q = d$ and the cost function C given by

$$C(t) = \begin{cases} 2t & \text{if } t \in [0, E]; \\ 5t - 3E & \text{if } t \in (E, \sum_{j \in N} d_j]. \end{cases}$$

Note that C has two maximal intervals of concavity: $[0, E]$ and $[E, \sum_{j \in N} d_j]$. For the first interval, we have that $C^1(t) = 2t$ for all $t \in [0, E]$ and $q^1(\varphi) = q = d$. For $x^1(\varphi)$, the players should be reordered in such a way that $x^1(\varphi)$ is non-decreasing over the players. Let $\sigma : N \to \{1, \ldots, n\}$ be a bijection such that for all $k \in \{1, \ldots, n\}$, $\sigma^{-1}(k)$ is the player with the kth lowest value allocated by φ, that is,

$$\varphi_{\sigma^{-1}(1)}(E, d) \leq \varphi_{\sigma^{-1}(2)}(E, d) \leq \ldots \leq \varphi_{\sigma^{-1}(n)}(E, d).$$

Next, let $x^1_k(\varphi) = \varphi_{\sigma^{-1}(k)}(E, d)$ for all $k \in \{1, \ldots, n\}$, or equivalently, $x^1_{\sigma(j)}(\varphi) = \varphi_j(E, d)$ for all $j \in N$, which means that player $j \in N$ is on position $\sigma(j)$ of $x^1(\varphi)$. Then, $(C^1, x^1(\varphi))$ is a cost sharing problem on (positions) $\{1, 2, \ldots, n\}$ and[5]

$$Ser(C^1, x^1(\varphi)) = 2x^1(\varphi).$$

For the second interval, we have that $C^2(t) = 5t$ for all $t \in [0, \sum_{j \in N} d_j - E]$. For $q^2(\varphi)$, let $\bar{\sigma} : N \to \{1, \ldots, n\}$ be a bijection such that

$$d_{\bar{\sigma}^{-1}(1)} - \varphi_{\bar{\sigma}^{-1}(1)}(E, d) \leq \ldots \leq d_{\bar{\sigma}^{-1}(n)} - \varphi_{\bar{\sigma}^{-1}(n)}(E, d).$$

Next, let $x^2_k(\varphi) = d_{\bar{\sigma}^{-1}(k)} - \varphi_{\bar{\sigma}^{-1}(k)}(E, d)$ for all $k \in \{1, \ldots, n\}$, or equivalently, $q^2_{\bar{\sigma}(j)}(\varphi) = d_j - \varphi_j(E, d)$ for all $j \in N$. Moreover, $x^2(\varphi) = \varphi(\sum_{j \in N} d_j - E, q^2(\varphi)) = q^2(\varphi)$. Then, $(C^2, x^2(\varphi))$ is a cost sharing problem on (positions) $\{1, 2, \ldots, n\}$ and

$$Ser(C^2, x^2(\varphi)) = 5x^2(\varphi).$$

Consequently,

$$\begin{aligned} \Psi^{\varphi}_i(C, q) &= Ser_{\sigma(i)}(C^1, x^1(\varphi)) + Ser_{\bar{\sigma}(i)}(C^2, x^2(\varphi)) \\ &= 2x^1_{\sigma(i)}(\varphi) + 5x^2_{\bar{\sigma}(i)}(\varphi) \\ &= 2\varphi_i(E, d) + 5(d_i - \varphi_i(E, d)) \\ &= 5d_i - 3\varphi_i(E, d), \end{aligned}$$

and similarly,

$$\Psi^{\varphi}_{i+1}(C, q) = 5d_{i+1} - 3\varphi_{i+1}(E, d).$$

Hence, using that $q = d$,

$$\begin{aligned} \frac{\Psi^{\varphi}_i(C, q)}{q_i} - \frac{\Psi^{\varphi}_{i+1}(C, q)}{q_{i+1}} &= 5 - 3\frac{\varphi_i(E, d)}{d_i} - 5 + 3\frac{\varphi_{i+1}(E, d)}{d_{i+1}} \\ &= 3\left(\frac{\varphi_{i+1}(E, d)}{d_{i+1}} - \frac{\varphi_i(E, d)}{d_i}\right) < 0, \end{aligned}$$

[5]In general, for a cost sharing problem $(C, q) \in \mathcal{CS}^N$ with a linear cost function $C(t) = \alpha t + \beta$, with $\alpha, \beta \in \mathbb{R}$, it can be readily checked that $Ser(C, q) = \alpha q$.

where the inequality follows from Equation (8).

Next, suppose (9) holds. First, let $\bar{\sigma} : N \to \{1, \ldots, n\}$ be a bijection such that

$$d_{\bar{\sigma}^{-1}(1)} - \varphi_{\bar{\sigma}^{-1}(1)}(E, d) \leq \ldots \leq d_{\bar{\sigma}^{-1}(n)} - \varphi_{\bar{\sigma}^{-1}(n)}(E, d).$$

Secondly, let $\varepsilon > 0$ be such that

$$\varepsilon < \min \{r_k \mid r_k > 0, k \in \{1, \ldots, n\}\},$$

where, for every $k \in \{1, \ldots, n\}$,

$$r_k = \sum_{\ell=1}^{k-1} (d_{\bar{\sigma}^{-1}(\ell)} - \varphi_{\bar{\sigma}^{-1}(\ell)}(E, d)) + (n - k + 1)(d_{\bar{\sigma}^{-1}(k)} - \varphi_{\bar{\sigma}^{-1}(k)}(E, d)).$$

In particular,

$$\varepsilon < r_n = \sum_{\ell=1}^{n} d_{\bar{\sigma}^{-1}(\ell)} - \sum_{\ell=1}^{n} \varphi_{\bar{\sigma}^{-1}(\ell)}(E, d) = \sum_{j \in N} d_j - E.$$

Consider $(C^\varepsilon, q) \in \mathcal{CCS}^{N,2}$ with $q = d$ and the cost function C^ε given by

$$C^\varepsilon(t) = \begin{cases} 2t & \text{if } t \in [0, E]; \\ 5t - 3E & \text{if } t \in (E, E + \varepsilon]; \\ t + E + 4\varepsilon & \text{if } t \in (E + \varepsilon, \sum_{j \in N} d_j]. \end{cases}$$

Note that C has two maximal intervals of concavity: $[0, E]$ and $[E, \sum_{j \in N} d_j]$. For the first interval, we have that $C^1(t) = 2t$ for all $t \in [0, E]$ and $q^1(\varphi) = q = d$. For $x^1(\varphi)$, let $\sigma : N \to \{1, \ldots, n\}$ be a bijection such that

$$\varphi_{\sigma^{-1}(1)}(E, d) \leq \varphi_{\sigma^{-1}(2)}(E, d) \leq \ldots \leq \varphi_{\sigma^{-1}(n)}(E, d).$$

Let $x^1_k(\varphi) = \varphi_{\sigma^{-1}(k)}(E, d)$ for all $k \in \{1, \ldots, n\}$. Then, $(C^1, x^1(\varphi))$ is a cost sharing problem on (positions) $\{1, 2, \ldots, n\}$ and

$$Ser(C^1, x^1(\varphi)) = 2x^1(\varphi).$$

For the second interval, we have that

$$C^2(t) = \begin{cases} 5t, & \text{if } t \leq \varepsilon \\ t + 4\varepsilon, & \text{otherwise,} \end{cases}$$

for all $t \in [0, \sum_{j \in N} d_j - E]$. For $q^2(\varphi)$, we can use the bijection $\bar{\sigma}$ as defined above to reorder the players such that $q^2(\varphi)$ is non-decreasing over the players. That is, let $q^2_k(\varphi) = d_{\bar{\sigma}^{-1}(k)} - \varphi_{\bar{\sigma}^{-1}(k)}(E, d)$ for all $k \in \{1, \ldots, n\}$ and note that again, $x^2(\varphi) = q^2(\varphi)$. To determine which part of the function C^2 applies, we consider the intermediate points used to compute $Ser(C^2, x^2(\varphi))$: $s^2_0 = 0$,

$$s^2_1 = nx^2_1(\varphi) = n(d_{\bar{\sigma}^{-1}(1)} - \varphi_{\bar{\sigma}^{-1}(1)}(E, d)),$$

and, for all $k \in \{2, \ldots, n\}$,

$$\begin{aligned} s_k^2 &= \sum_{\ell=1}^{k-1} x_\ell^2(\varphi) + (n-k+1)x_k^2(\varphi) \\ &= \sum_{\ell=1}^{k-1} (d_{\bar{\sigma}^{-1}(\ell)} - \varphi_{\bar{\sigma}^{-1}(\ell)}(E,d)) + (n-k+1)(d_{\bar{\sigma}^{-1}(k)} - \varphi_{\bar{\sigma}^{-1}(k)}(E,d)). \end{aligned}$$

Note that $s_k^2 = r_k$ for all $k \in \{1, \ldots, n\}$. Therefore, for all $k \in \{1, \ldots, n\}$, if $s_k^2 = 0$, then $C^2(s_k^2) = 0$, and if $s_k^2 > 0$, then $s_k^2 > \varepsilon$ and hence, $C^2(s_k^2) = s_k^2 + 4\varepsilon$. This implies that,

$$Ser(C^2, x^2(\varphi)) = x^2(\varphi).$$

Consequently,

$$\begin{aligned} \Psi_i^\varphi(C,q) &= Ser_{\sigma(i)}(C^1, x^1(\varphi)) + Ser_{\bar{\sigma}(i)}(C^2, x^2(\varphi)) \\ &= 2x^1_{\sigma(i)}(\varphi) + x^2_{\bar{\sigma}(i)}(\varphi) \\ &= 2\varphi_i(E,d) + d_i - \varphi_i(E,d) \\ &= d_i + \varphi_i(E,d), \end{aligned}$$

and similarly,

$$\Psi_{i+1}^\varphi(C,q) = d_{i+1} + \varphi_{i+1}(E,d).$$

Hence, using that $q = d$,

$$\begin{aligned} \frac{\Psi_i^\varphi(C,q)}{q_i} - \frac{\Psi_{i+1}^\varphi(C,q)}{q_{i+1}} &= 1 + \frac{\varphi_i(E,d)}{d_i} - 1 - \frac{\varphi_{i+1}(E,d)}{d_{i+1}} \\ &= \frac{\varphi_i(E,d)}{d_i} - \frac{\varphi_{i+1}(E,d)}{d_{i+1}} < 0, \end{aligned}$$

where the inequality follows from Equation (9). □

The proof of Theorem 4.3 uses, among other things, an obvious property of CEL as a bankruptcy rule as provided below in Proposition A.2.

Proposition A.2 *Let $(E,d) \in \mathcal{B}^N$ be such that $N = \{1, 2, \ldots, n\}$ with $d_1 \leq d_2 \leq \ldots \leq d_n$ and $d_2 + d_3 + \ldots + d_n \geq E$. Then the following two statements hold:*

i) If $CEL_1(E,d) = 0$, then $CEL_j(E, d_{|N\backslash\{1\}}) = CEL_j(E,d)$ for all $j \in N\backslash\{1\}$;

ii) If $CEL_1(E,d) > 0$, then

$$CEL_j(E,d) = d_j - \frac{\sum_{k\in N} d_k - E}{n},$$

for all $j \in N$, and

$$CEL_j(E, d_{|N\backslash\{1\}}) = CEL_j(E,d) + \frac{CEL_1(E,d)}{n-1},$$

for all $j \in N\backslash\{1\}$.

Theorem 4.3 Ψ^{CEL} *satisfies MOVASP on* $\mathcal{CCS}^m$ *for all* $m \in \{1, 2, \ldots\}$.

Proof. The theorem clearly holds for $m = 1$, using Proposition 4.2 and the fact that $\mathcal{CCS}^1 = \mathcal{CCS}$. Let $m \in \{2, 3, \ldots\}$ and let $(C, q) \in \mathcal{CCS}^{N,m}$. Let as before, the maximal intervals of concavity of C be given by $[t_0, t_1], [t_1, t_2], \ldots, [t_{m-1}, t_m]$ with $0 = t_0 < t_1 < t_2 < \ldots < t_{m-1} < t_m = \sum_{i \in N} q_i$, with modified cost functions $C^r : [0, t_r - t_{r-1}] \to \mathbb{R}_+$, for $r \in \{1, \ldots, m\}$, defined by $C^r(t) = C(t + t_{r-1}) - C(t_{r-1})$, for all $t \in [0, t_r - t_{r-1}]$.

Abbreviate for all $r \in \{1, \ldots, m\}$, $q^r(CEL)$ to q^r and $x^r(CEL)$ to x^r and recall that these are the vectors in $\mathbb{R}^N$ which are defined recursively as in the definition of $\Psi^{CEL}(C, q)$. Similarly, denote by $\bar{q}^r$ and $\bar{x}^r$, $r \in \{1, \ldots, m\}$, the respective vectors in $\mathbb{R}^{N \setminus \{1\}}$ as provided in the definition of $\Psi^{CEL}(C, q_{|N \setminus \{1\}})$.

Since $\Psi^{CEL}(C, q) = \sum_{r=1}^{m} Ser(C^r, x^r)$ and $\Psi^{CEL}(C, q_{|N \setminus \{1\}}) = \sum_{r=1}^{m} Ser(C^r, \bar{x}^r)$, by definition, in order to show MOVASP it suffices to show that, for all $i \in \{3, \ldots, n\}$ and for all $r \in \{1, \ldots, m\}$,

$$Ser_i(C^r, x^r) - Ser_i(C^r, \bar{x}^r) \leq Ser_{i-1}(C^r, x^r) - Ser_{i-1}(C^r, \bar{x}^r). \tag{10}$$

So, let $i \in \{3, \ldots, n\}$ and $r \in \{1, \ldots, m\}$.

We need to fix some notation regarding the intermediate points needed to calculate $Ser(C^r, x^r)$ and $Ser(C^r, \bar{x}^r)$. Regarding $Ser(C^r, x^r)$, these points are denoted by

$$0 = s_0^r, s_1^r, \ldots, s_{n-1}^r, s_n^r,$$

with, for $\ell \in \{1, \ldots, n\}$,

$$s_\ell^r = \sum_{k=1}^{\ell-1} x_k^r + (n - \ell + 1) x_\ell^r.$$

Similarly, the intermediate points regarding $Ser(C^r, \bar{x}^r)$ are denoted by

$$0 = \bar{s}_1^r, \bar{s}_2^r, \ldots, \bar{s}_{n-1}^r, \bar{s}_n^r,$$

with, for all $\ell \in \{1, \ldots, n\}$,

$$\bar{s}_\ell^r = \sum_{k=2}^{\ell-1} \bar{x}_k^r + (n - \ell + 1) \bar{x}_\ell^r.$$

First, assume that $q^r_{|N \setminus \{1\}} \neq \bar{q}^r$. Then, obviously, $r > 1$ and $q_k^r \neq \bar{q}_k^r$ for some $k \in \{2, \ldots, n\}$. Then, as a consequence of Proposition A.2, we have that $x_1^s > 0$ for some $s \in \{1, \ldots, r-1\}$. Assume w.l.o.g. that s is the first index for which this happen, i.e. $x_1^1 = \ldots = x_1^{s-1} = 0$. This implies that $q^s_{|N \setminus \{1\}} = \bar{q}^s$ and, by using Proposition A.2,

$$\begin{cases} x_j^s &= q_j^s - \frac{\sum_{h \in N} q_h^s - (t_s - t_{s-1})}{n}; \\ \bar{x}_j^s &= x_j^s + \frac{x_1^s}{n-1}, \end{cases}$$

for all $j \in N \setminus \{1\}$ and consequently,

$$\begin{cases} q_j^{s+1} &= q_j^s - x_j^s = \frac{\sum_{h \in N} q_h^s - (t_s - t_{s-1})}{n}; \\ \bar{q}_j^{s+1} &= \bar{q}_j^s - \bar{x}_j^s = q_j^s - x_j^s - \frac{x_1^s}{n-1} = \frac{\sum_{h \in N} q_h^s - (t_s - t_{s-1})}{n} - \frac{x_1^s}{n-1}, \end{cases}$$

are independent of $j \in N\backslash\{1\}$. Thus, in particular,

$$\begin{cases} q_i^{s+1} &= q_{i-1}^{s+1}; \\ \bar{q}_i^{s+1} &= \bar{q}_{i-1}^{s+1}, \end{cases}$$

and, by using the symmetry of CEL,

$$\begin{cases} x_i^{s+1} &= x_{i-1}^{s+1}; \\ \bar{x}_i^{s+1} &= \bar{x}_{i-1}^{s+1}. \end{cases}$$

Subsequently, we have that

$$\begin{cases} q_i^{\ell} &= q_{i-1}^{\ell}; \\ \bar{q}_i^{\ell} &= \bar{q}_{i-1}^{\ell}, \end{cases}$$

and

$$\begin{cases} x_i^{\ell} &= x_{i-1}^{\ell}; \\ \bar{x}_i^{\ell} &= \bar{x}_{i-1}^{\ell}, \end{cases}$$

for all $\ell \in \{s+1, \ldots, m\}$ and in particular, $x_i^r = x_{i-1}^r$ and $\bar{x}_i^r = \bar{x}_{i-1}^r$. The first equality implies that, by using the symmetry of the serial rule,

$$Ser_i(C^r, x^r) = Ser_{i-1}(C^r, x^r),$$

while the latter inequality implies that, by again using the symmetry of the serial rule,

$$Ser_i(C^r, \bar{x}^r) = Ser_{i-1}(C^r, \bar{x}^r).$$

Consequently, Equation (10) holds.

In the rest of the proof, we assume that $q^r_{|N\backslash\{1\}} = \bar{q}^r$. We distinguish between three cases:

(I) $\sum_{k \in N} q_k^r \leq t_r - t_{r-1}$;

(II) $\sum_{k \in N} q_k^r > t_r - t_{r-1} > \sum_{k \in N\backslash\{1\}} q_k^r$;

(III) $\sum_{k \in N\backslash\{1\}} q_k^r \geq t_r - t_{r-1}$.

Case I
In this case,

$$\sum_{k \in N\backslash\{1\}} \bar{q}_k^r = \sum_{k \in N\backslash\{1\}} q_k^r \leq \sum_{k \in N} q_k^r \leq t_r - t_{r-1}.$$

Hence,

$$\begin{cases} x^r = CEL(t_r - t_{r-1}, q^r) = q^r; \\ \bar{x}^r = CEL(t_r - t_{r-1}, \bar{q}^r) = \bar{q}^r = q^r_{|N\backslash\{1\}}. \end{cases}$$

Since $(C^r, q^r) \in \mathcal{CCS}^N$, Proposition 4.2 implies that

$$\begin{aligned} Ser_i(C^r, x^r) - Ser_i(C^r, \bar{x}^r) &= Ser_i(C^r, q^r) - Ser_i(C^r, q^r_{|N\backslash\{1\}}) \\ &\leq Ser_{i-1}(C^r, q^r) - Ser_{i-1}(C^r, q^r_{|N\backslash\{1\}}) \\ &= Ser_{i-1}(C^r, x^r) - Ser_{i-1}(C^r, \bar{x}^r). \end{aligned}$$

Case II
In this case,

$$\bar{x}^r_k = CEL_k(t_r - t_{r-1}, \bar{q}^r) = CEL_k(t_r - t_{r-1}, q^r_{|N\backslash\{1\}}) = q^r_k,$$

for all $k \in N\backslash\{1\}$. Moreover, with

$$\varepsilon = q^r_1 - \left(\sum_{h\in N} q^r_h - (t_r - t_{r-1}) \right),$$

we have that $\varepsilon > 0$. Furthermore, since

$$q^r_1 - \frac{\sum_{h\in N} q^r_h - (t_r - t_{r_1})}{n} > q^r_1 - \frac{q^r_1}{n} > 0,$$

we have, for all $k \in N$,

$$x^r_k = CEL_k(t_r - t_{r-1}, q^r) = q^r_k - \frac{\sum_{h\in N} q^r_h - (t_r - t_{r-1})}{n}.$$

Then, for all $\ell \in \{2, \dots, n\}$,

$$\begin{aligned} s^r_\ell =& \sum_{k=1}^{\ell-1} x^r_k + (n-\ell+1)x^r_\ell \\ =& \sum_{k=1}^{\ell-1} \left(q^r_k - \frac{\sum_{h\in N} q^r_h - (t_r - t_{r-1})}{n} \right) + (n-\ell+1) \left(q^r_\ell - \frac{\sum_{h\in N} q^r_h - (t_r - t_{r-1})}{n} \right) \\ =& (n-\ell+1)q^r_\ell + \sum_{k=1}^{\ell-1} q^r_k - [(n-\ell+1) + (\ell-1)] \left(\frac{\sum_{h\in N} q^r_h - (t_r - t_{r-1})}{n} \right) \\ =& (n-\ell+1)q^r_\ell + \sum_{k=1}^{\ell-1} q^r_k - \left(\sum_{h\in N} q^r_h - (t_r - t_{r-1}) \right) \\ =& (n-\ell+1)q^r_\ell + \sum_{k=2}^{\ell-1} q^r_k + q^r_1 - \left(\sum_{h\in N} q^r_h - (t_r - t_{r-1}) \right) \\ =& (n-\ell+1)\bar{x}^r_\ell + \sum_{k=2}^{\ell-1} \bar{x}^r_k + \varepsilon \\ =& \bar{s}^r_\ell + \varepsilon. \end{aligned}$$

Consequently,

$$\begin{aligned}
Ser_i(C^r, x^r) - Ser_{i-1}(C^r, x^r) - \Big(Ser_i(C^r, \bar{x}^r) - Ser_{i-1}(C^r, \bar{x}^r)\Big) \\
&= \frac{C^r(s_i^r) - C^r(s_{i-1}^r)}{n-i+1} - \frac{C^r(\bar{s}_i^r) - C^r(\bar{s}_{i-1}^r)}{n-i+1} \\
&= \frac{C^r(\bar{s}_i^r + \varepsilon) - C^r(\bar{s}_{i-1}^r + \varepsilon)}{n-i+1} - \frac{C^r(\bar{s}_i^r) - C^r(\bar{s}_{i-1}^r)}{n-i+1} \\
&= \frac{1}{n-i+1}\Big(C^r(\bar{s}_i^r + \varepsilon) - C^r(\bar{s}_i^r) - C^r(\bar{s}_{i-1}^r + \varepsilon) + C^r(\bar{s}_{i-1}^r)\Big) \\
&\leq 0,
\end{aligned}$$

where we used Proposition A.1 part *ii)* for the inequality, since $0 < \bar{s}_{i-1}^r \leq \bar{s}_i^r$ and $\varepsilon > 0$.

Case III
In this case,

$$\sum_{k \in N \setminus \{1\}} q_k^r \geq t_r - t_{r-1},$$

and,

$$\begin{aligned}
\bar{x}_\ell^r &= CEL_\ell(t_r - t_{r-1}, \bar{q}^r) \\
&= CEL_\ell(t_r - t_{r-1}, q^r_{|N\setminus\{1\}}) \\
&= CEL_\ell(t_r - t_{r-1}, q^r) + \frac{CEL_1(t_r - t_{r-1}, q^r)}{n-1} \\
&= x_\ell^r + \frac{x_1^r}{n-1},
\end{aligned}$$

for all $\ell \in \{2, \ldots, n\}$, where the third equality is a consequence of Proposition A.2. Hence,

$$\bar{s}_2^r = (n-1)\bar{x}_2^r = (n-1)(x_2^r + \frac{x_1^r}{n-1}) = (n-1)x_2^r + x_1^r = s_2^r,$$

and for all $\ell \in \{3, \ldots, n\}$,

$$\begin{aligned}
\bar{s}_\ell^r &= \sum_{k=2}^{\ell-1} \bar{x}_k^r + (n-\ell+1)\bar{x}_\ell^r \\
&= \sum_{k=2}^{\ell-1} (x_k^r + \frac{x_1^r}{n-1}) + (n-\ell+1)(x_\ell^r + \frac{x_1^r}{n-1}) \\
&= (n-\ell+1)x_\ell^r + \sum_{k=2}^{\ell-1} x_k^r + [(n-\ell+1) + (\ell-2)]\frac{x_1^r}{n-1} \\
&= (n-\ell+1)x_\ell^r + \sum_{k=2}^{\ell-1} x_k^r + x_1^r \\
&= (n-\ell+1)x_\ell^r + \sum_{k=1}^{\ell-1} x_k^r \\
&= s_\ell^r.
\end{aligned}$$

Thus, $\bar{s}^r_\ell = s^r_\ell$ for all $\ell \in \{2,\dots,n\}$. Consequently, for all $\ell \in \{2,3,\dots,n\}$,

$$\begin{aligned}
Ser_\ell(C^r,x^r) - Ser_\ell(C^r,\bar{x}^r) &= \sum_{k=1}^{\ell} \frac{C^r(s^r_k) - C^r(s^r_{k-1})}{n-k+1} - \sum_{k=2}^{\ell} \frac{C^r(\bar{s}^r_k) - C^r(\bar{s}^r_{k-1})}{n-k+1} \\
&= \sum_{k=1}^{\ell} \frac{C^r(s^r_k) - C^r(s^r_{k-1})}{n-k+1} - \left(\sum_{k=3}^{\ell} \frac{C^r(\bar{s}^r_k) - C^r(\bar{s}^r_{k-1})}{n-k+1} + \frac{C^r(\bar{s}^r_2) - C^r(\bar{s}^r_1)}{n-1} \right) \\
&= \frac{C^r(s^r_1) - C^r(s^r_0)}{n} + \frac{C^r(\bar{s}^r_1) - C^r(s^r_1)}{n-1} \\
&= \frac{C^r(s^r_1)}{n} - \frac{C^r(s^r_1)}{n-1},
\end{aligned}$$

is independent of ℓ. Hence,

$$Ser_i(C^r,x^r) - Ser_i(C^r,\bar{x}^r) = Ser_{i-1}(C^r,x^r) - Ser_{i-1}(C^r,\bar{x}^r),$$

which proves Equation (10) with an equality. □