Optimal dynamic investment policy for different tax depreciation rates and economic depreciation rates

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Optimal Dynamic Investment Policy under Different Rates for Tax Depreciation and Economic Depreciation

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Abstract

This paper analyzes the consequences of incorporating a different rate for tax depreciation than for economic depreciation. Firms most often choose their tax depreciation rate in a strategic way. It would therefore be a coincidence if this optimization process leads to a tax depreciation rate that equals the economic depreciation rate. The implications of a difference between tax depreciation and economic depreciation are investigated in an optimal control model for the determination of the firm's optimal investment policy over time.

Key Words. Optimal control, investment policy, tax depreciation, economic depreciation.

1 Introduction

Since the choice of a depreciation method affects future taxable income, it is often chosen in a strategic way (see e.g. Ref. 1, for a thorough overview of the different incentives that influence this choice). Of course, this strategic behavior implies that it would be a coincidence if the resulting tax depreciation rate equals the rate of economic depreciation.

This motivates the set up of our paper where the aim is to determine optimal investment behavior, while there are different values for tax depreciation and economic depreciation. Consequently a model must be designed where the productive capital stock, which is affected by economic depreciation, differs from the tax base of assets. The tax base increases through investments and is reduced by tax depreciation. Moreover the development of the tax base over time is influenced by the fact that the firm is allowed to carry forward losses in order to reduce future tax payments. The firm's aim is to invest such that it maximizes shareholder value, which consists of the discounted dividend stream over time plus the discounted value of the firm at the planning horizon.

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The paper is organized as follows. In Section 2, the optimization problem of the firm is specified as an optimal control problem. Section 3 contains the mathematical analysis, whereas the two resulting optimal trajectories are extensively analyzed in Section 4. Section 5 contains some concluding remarks.

2 The Model

Let $K_1 = K_1(t)$ denote the stock of productive capital by time t, and $I = I(t)$ denote the gross investment rate at time t. Then introduce the standard formula for net investment,

$$\dot{K}_1 = I - \beta K_1, \quad K_1(0) > 0, \tag{1}$$

where $\beta > 0$ is the constant rate of technical depreciation.

With K_1 the firm produces goods which are sold on the market leading to a revenue denoted by $C(K_1)$. The revenue function is twice differentiable and satisfies $C(0) = 0$, and for all $K_1 > 0$

$$C'(K_1) > 0,$$
$$C''(K_1) < 0.$$

The new feature in this paper is the distinction between technical depreciation and tax depreciation. Consequently the capital good stock that represents the tax base of the assets will differ from the productive capital good stock K_1. Therefore we introduce $K_2 = K_2(t)$ being the capital good stock that represents the tax base of the assets. Furthermore the firm is allowed to carry over eventual losses to future periods. Denoting the tax depreciation rate by γ ($\gamma > 0$ and constant) the firm makes a loss in terms of taxable income if it holds that

$$C(K_1) - \gamma K_2 < 0.$$

The above implies that the evolution over time of the tax base is given by

$$\dot{K}_2 = I - \gamma K_2 - (C(K_1) - \gamma K_2)^-, \tag{2}$$

where $x^- = x$ if $x < 0$, and 0 otherwise. Hence, when losses are made they are added to K_2 so that in future periods they can be subtracted from before tax profits.

Investments are assumed to be irreversible (convincing arguments for this assumption can be found in, e.g., Ref. 2):

$$I \geq 0. \tag{3}$$

Concerning tax payments there is a fixed tax rate $T > 0$, so that the firm's tax payments equal

$$T\left(C\left(K_1\right) - \gamma K_2\right)^+ .$$

The firm behaves so that it maximizes its value for the shareholders. This value equals the discounted value of dividend payments over the planning period plus the value of the firm at the end of the planning period. This firm can choose to spend its net revenue (which is revenue minus taxes) on investments or dividends (denoted by $D = D(t)$), which leads to the following expression for the dividend payments which are restricted to be nonnegative:

$$D = C\left(K_1\right) - T\left(C\left(K_1\right) - \gamma K_2\right)^+ - I \geq 0. \tag{4}$$

In general, the final value of the firm depends on the level of the state variables at the planning horizon, and is thus of the form $f\left(K_1(z), K_2(z)\right)$, where z denotes the horizon date. This value represents the discounted dividend stream the firm generates from z onwards. To specify the final value we must find an expression for this discounted dividend stream. Most of the contributions in this area (see, e.g., Ref. 3) take the discounted value of equity as a proxy for this stream, but Ref. 4 has shown that the discounted value of equity underestimates the discounted dividend stream. Therefore we follow a different approach here. We start out by assuming that the planning period is long enough for the firm to reach its optimal long run value of productive capital stock denoted by K_1^* (we specify this value later). Since it is the optimal long run size, the firm has no incentive to deviate from this value of the productive capital stock, implying that from z onwards it will hold that

$$K_1 = K_1^*, \tag{5}$$

$$I = \beta K_1^*. \tag{6}$$

Concerning the development of K_2 we can obtain from (2) and (6) that

$$K_2(t) = K_2(z)\, e^{-\gamma(t-z)} + \left(1 - e^{-\gamma(t-z)}\right) \frac{\beta}{\gamma} K_1^*, \tag{7}$$

where it is implicitly assumed that the firm does not make any losses from time z onwards. (Later we will show that once the firm starts to make profits, i.e. $C(K_1) - \gamma K_2 > 0$, at time τ, it will not make losses for any $t > \tau$. This means that our assumption only fails if the firm did not make any profits before time z. We exclude this case in what follows.) Introduce a constant discount rate $r > 0$. Now we obtain from (4)-(7) that the final value of the firm is given by

$$\begin{aligned} f(K_1(z), K_2(z)) &= \int_z^{\infty} D(t)\, e^{-r(t-z)} dt \\ &= \frac{1-T}{r} C(K_1(z)) + \left(\frac{\gamma \beta T}{r(r+\gamma)} - \frac{\beta}{r} \right) K_1(z) + \frac{\gamma}{r+\gamma} T K_2(z). \end{aligned} \tag{8}$$

From (4) and (8) we finally derive that the objective of the firm can be mathematically expressed by

$$\begin{aligned} &\max_I \int_0^z \left(C(K_1) - T\left(C(K_1) - \gamma K_2\right)^+ - I \right) e^{-rt} dt \\ &+ \frac{1-T}{r} C(K_1(z)) + \left(\frac{\gamma \beta T}{r(r+\gamma)} - \frac{\beta}{r} \right) K_1(z) + \frac{\gamma}{r+\gamma} T K_2(z). \end{aligned} \tag{9}$$

The resulting optimal control problem is formed by optimizing expression (9), subject to the state equations (1), (2), and the constraints (3) and (4).

In order to limit the number of possible scenarios we add some additional assumptions. First, tax depreciation rates are only considered in the range

$$\beta \leq \gamma \leq r + \beta, \tag{10}$$

so that tax depreciation at least covers technical deterioration, but does not exceed the deterioration costs plus the discount rate (it will be difficult to get legal support for higher depreciation rates).

Second, it is assumed that the revenue function satisfies the following requirements:

$$C'(K_1) > \beta, \tag{11}$$

$$C'(0) \geq \frac{r}{1-T} + \beta, \tag{12}$$

$$C'(\infty) \leq r + \beta, \tag{13}$$

$$C(K_1) > \frac{\beta}{1-T} K_1, \quad \text{for all } K_1 \leq K_1^*, \tag{14}$$

Expression (12) denotes that it is always optimal for this firm to start production since, for the stock of productive capital being equal to zero, marginal revenue exceeds the user cost of capital. The next inequality indicates that it cannot be optimal for the firm to keep on growing, since revenue net from depreciation costs falls below the time value of money if productive capital stock is sufficiently large. Finally, expression (14) implies that investments needed to replace deteriorated capital goods can always be financed out of the revenue after paying taxes, as long as productive capital stock is less than its optimal long run size.

3 Analysis of the Control Problem

3.1 Optimality conditions

Define the current value Hamiltonian

$$
\begin{aligned}
H(K_1, K_2, I, \lambda_1, \lambda_2) &= C(K_1) - T(C(K_1) - \gamma K_2)^+ - I \\
&\quad + \lambda_1 (I - \beta K_1) + \lambda_2 \left(I - \gamma K_2 + (\gamma K_2 - C(K_1))^+ \right),
\end{aligned} \tag{15}
$$

where $\lambda_i = \lambda_i(t), i = 1, 2$ are the current value adjoint variables. Define the Lagrangian

$$
\begin{aligned}
L(K_1, K_2, I, \lambda_1, \lambda_2, \eta_1, \eta_2) &= H(K_1, K_2, I, \lambda_1, \lambda_2) \\
&\quad + \eta_1 \left(C(K_1) - T(C(K_1) - \gamma K_2)^+ - I \right) + \eta_2 I,
\end{aligned} \tag{16}
$$

where $\eta_i = \eta_i(t), i = 1, 2$ are Lagrange multiplier functions.

The necessary conditions for optimality are:

$$L_I = -1 + \lambda_1 + \lambda_2 - \eta_1 + \eta_2 = 0, \tag{17}$$

$$
\begin{aligned}
\dot{\lambda}_1 = r\lambda_1 - L_{K_1} &= (r + \beta)\lambda_1 - (1 + \eta_1)\left(1 - T1_{(C - \gamma K_2 > 0)}\right) C'(K_1) \\
&\quad + \lambda_2 C'(K_1) 1_{(C - \gamma K_2 \leq 0)},
\end{aligned} \tag{18}
$$

$$\dot{\lambda}_2 = r\lambda_2 - L_{K_2} = \left(r + \gamma 1_{(C - \gamma K_2 > 0)}\right)\lambda_2 - (1 + \eta_1) T\gamma 1_{(C - \gamma K_2 > 0)}, \tag{19}$$

$$\eta_1 \left(C(K_1) - T\left(C\left(K_1\right) - \gamma K_2\right)^+ - I \right) = 0, \quad \eta_1 \geq 0, \tag{20}$$

$$\eta_2 I = 0, \quad \eta_2 \geq 0, \tag{21}$$

$$\lambda_1\left(z\right) = \frac{1-T}{r} C'\left(K_1\left(z\right)\right) + \left(\frac{\gamma\beta T}{r\left(r+\gamma\right)} - \frac{\beta}{r} \right), \tag{22}$$

$$\lambda_2\left(z\right) = \frac{\gamma}{r+\gamma} T. \tag{23}$$

Strictly speaking, the adjoint equations (18) and (19) only hold for $C\left(K_1\right) \neq \gamma K_2$, where the Lagrangian is differentiable. At the points where the Lagrangian is not differentiable the differential equations (18) and (19) must be replaced by the differential inclusions (see, e.g., Ref. 5):

$$\dot{\lambda}_1 \in \left[\, (r+\beta)\, \lambda_1 - (1+\eta_1)\,(1-T)\, C'\left(K_1\right) \;,\; (r+\beta)\, \lambda_1 - (1+\eta_1-\lambda_2)\, C'\left(K_1\right) \, \right], \tag{24}$$

$$\dot{\lambda}_2 \in \left[\, (r+\gamma)\, \lambda_2 - (1+\eta_1)\, T\gamma \;,\; r\lambda_2 \, \right]. \tag{25}$$

3.2 The Paths

From the complementary slackness conditions (20) and (21) it can be obtained that there are four possible paths. On each path we make a distinction between a situation where the firm's taxable income is positive (meaning that $C\left(K_1\right) - \gamma K_2 > 0$), and the complementary case where taxable income is non-positive. The first situation is denoted by Path $i_A (i = 1, ..., 4)$, and for the other situation we write Path i_B. Next, we provide a characterization of the paths.

Path 1: $\eta_1 = \eta_2 = 0$.

On this path it holds that

$$C(K_1) - I - T\left(C\left(K_1\right) - \gamma K_2\right)^+ \geq 0, \quad I \geq 0, \tag{26}$$

so that dividends as well as investments can be positive.

From (2) we obtain that on Path 1_B the evolution of the tax base K_2 is given by

$$\dot{K}_2 = I - C\left(K_1\right), \tag{27}$$

while it can be obtained from (26) that

$$C\left(K_1\right) - I \geq 0.$$

Hence, in any case the tax base K_2 is non-increasing on Path 1_B.

Furthermore, exploiting the optimality conditions (17)-(19) it is possible to prove that $\dot{K}_1 < 0$ on Path 1_B. The proof goes as follows. From equation (17) it is obtained that on this path it holds that

$$\lambda_1 + \lambda_2 = 1 \Rightarrow \dot{\lambda}_1 + \dot{\lambda}_2 = 0,$$

which implies, together with (18) and (19), that

$$r + \lambda_1 (\beta - C'(K_1)) = 0.$$

After differentiating this expression with respect to time it is obtained that

$$C''(K_1)\dot{K}_1 = -\frac{r}{\lambda_1^2}\dot{\lambda}_1 > 0,$$

where the greater-than-sign follows from the fact that (19) implies that $\dot{\lambda}_2 > 0$[1], so that from $\dot{\lambda}_1 + \dot{\lambda}_2 = 0$ it is obtained that $\dot{\lambda}_1 < 0$. Now, $\dot{K}_1 < 0$ is derived from $C''(K_1)\dot{K}_1 > 0$ and $C''(K_1) < 0$.

Path 2: $\eta_1 > 0, \eta_2 = 0$.

Here, no dividend is distributed so that all financial means are used for investment:

$$I = C(K_1) - T\left(C\left(K_1\right) - \gamma K_2\right)^+$$

which implies that investment is always positive since the tax rate T is less than one. Moreover, from (14) it is obtained that $\dot{K}_1 > 0$ on both Path 2_A and Path 2_B, if $K_1 < K_1^*$.

Concerning the development of K_2, on Path 2_A it holds that

$$\dot{K}_2 = (1 - T)\left(C\left(K_1\right) - \gamma K_2\right) > 0,$$

where the inequality sign follows from the fact that on an A-path it holds per definition that $C\left(K_1\right) - \gamma K_2 > 0$.

[1]It follows from (19) and (23), combined with the coupling procedure, that $\lambda_2 > 0$. Consequently, also $\dot{\lambda}_2 > 0$.

On Path 2_B we have

$$I = C(K_1),$$

and also (27) holds on Path 2_B. This leads to the conclusion that $\dot{K}_2 = 0$ on Path 2_B. Hence, we now know that

$$\frac{\partial (C(K_1) - \gamma K_2)}{\partial t} = C'(K_1)\dot{K}_1 - \gamma \dot{K}_2 > 0,$$

which implies that, given the fact that $C(K_1) - \gamma K_2$ needs to be less than zero on a B-path, Path 2_B is not sustainable on a sufficiently long time interval.

Path 3: $\eta_1 = 0, \eta_2 > 0$.

The complementary slackness condition (21) implies that on this path no investments will be carried out, so that

$$\dot{K}_1 = -\beta K_1 < 0,$$
$$\dot{K}_2 = -\gamma K_2 < 0.$$

Path 4: $\eta_1 > 0, \eta_2 > 0$.

The complementary slackness conditions (20) and (21) imply that

$$I = 0 = C(K_1) - T(C(K_1) - \gamma K_2)^+.$$

The implication is that K_1 must be zero on this path. However, from (1) and (3) it can be obtained that

$$K_1 > 0.$$

Hence, Path 4 is an infeasible path and does not need to be considered any further.

3.3 Synthesizing the paths

In the previous subsection we presented the paths that the optimal solution can consist of. Next, we have to determine in what sequence these paths occur. This can be done by applying a formal path coupling procedure (see, e.g., Ref. 3). This procedure exploits the continuity properties of state and co-state variables in order to check whether one path can precede another path. The following proposition provides useful information for this path coupling procedure, since it implies that a B-path can never be preceded by an A-path.

Proposition 3.1 *If at time τ it holds that $C(K_1) - \gamma K_2 \leq 0$, then the implication is that $C(K_1) - \gamma K_2 < 0$ at any time $t < \tau$.*

Proof. See Appendix A. ■

Also from an economic point of view this is an interesting result, because it says that whenever there exists a time interval where the firm pays no taxes, this must be an initial interval.

The path coupling procedure starts at the end of the planning period and then works backwards in time. So, first it is determined which path can be a final path. For a sufficiently long time period the firm typically ends in a steady state path where dividend is paid to the shareholders. This is shown in various other dynamic models of the firm (see, e.g., Ref. 3). In this paper we only consider cases where the firm ends up in a steady path. Concerning the final path the following proposition can be established:

Proposition 3.2 *If the final path is a steady state path, then Path 1_A is the final path. On this path the productive capital stock is implicitly given by*

$$C'(K_1^*) = \left(\frac{r}{1-T} + \gamma\right)\left(\frac{r+\beta}{r+\gamma}\right). \tag{28}$$

K_1^* is the unique steady state level of capital stock at which marginal revenue of investment equals marginal cost and the firm has a positive after tax income.

Proof. See Appendix B. ■

The coupling procedure goes on by checking which path(s) can precede the final path, followed by checking which path(s) can precede the path(s) before the final path, and so on and so forth. Once no feasible predecessor can be found the procedure terminates. A sequence of paths starting with an initial path, before which no other path can occur, and ending with a final path, is called a master trajectory. Carrying out this exercise for our model we arrive at the following proposition.

Proposition 3.3 *Application of the path coupling procedure leads to the following three master trajectories:*[2]

MT1: Path $2_B \rightarrow$ *Path* $2_A \rightarrow$ *Path* 1_A,

MT2: Path $2_B \rightarrow$ *Path* $1_B \rightarrow$ *Path* $3_B \rightarrow$ *Path* $3_A \rightarrow$ *Path* 1_A,

MT3: Path $3_B \rightarrow$ *Path* $1_B \rightarrow$ *Path* $3_B \rightarrow$ *Path* $3_A \rightarrow$ *Path* 1_A.

Proof. See Appendix C. ■

It follows from theorem 2 on p. 285 in Ref. 6 that an optimal control exists. Since for every initial state of the firm there is only one solution that satisfies the necessary conditions, this solution is optimal. Depending on the initial state of the firm, its optimal policy will start at some point during one of the master trajectories. If for example the firm's taxable income is positive at the initial point of time (i.e. $C(K_1) - \gamma K_2 > 0$), the optimal policy begins at the A-part of a master trajectory.

On Path 3 the firm does not invest so that both capital stocks are decreasing. In Subsection 3.2 it was found that on Path 1_B productive capital stock is decreasing while K_2 is non-increasing. Therefore, it can be concluded that MT3 is a master trajectory on which it is optimal to decrease the firm's size until the optimal steady state is reached. Hence, this trajectory will only be applied when the firm is "large" in the beginning, i.e. when marginal cost initially exceeds marginal revenue. In the next section optimal dynamic investment behavior of initially "small" firms, i.e. when marginal revenue initially exceeds marginal cost is analyzed.

On the two master trajectories MT1 and MT2 the firm starts out on Path 2_B, which is a B-path so that $C(K_1) - \gamma K_2 \leq 0$. Furthermore, on this path it holds that $\lambda_1 + \lambda_2 > 1$. Now it depends on which of these two inequalities changes first its sign, which master trajectory will be applied: if it happens first that $C(K_1) - \gamma K_2$ becomes positive, then Path 2_B passes into Path 2_A and MT1 will hold. On the other hand, if it happens first that $\lambda_1 + \lambda_2$ becomes equal to one, then Path 2_B will be followed by Path 1_B so that we are now on MT2. Apparently, the initial value of $\gamma K_2 - C(K_1)$ will be relatively large on MT2 compared to MT1. Therefore, in the next section, the scenario

[2]Strictly speaking, in MT2 and MT3, the sequence $1_B \rightarrow 3_B$ can also be $1_B \rightarrow 3_B \rightarrow 1_B \rightarrow 3_B \rightarrow \cdots \rightarrow 1_B \rightarrow 3_B$. This however can be excluded for functions $C(\cdot)$ with $C'''(\cdot) \leq 0$, e.g. functions in the class $C(K_1) = aK_1 - bK_1^2$.

where MT1 applies is denoted by "moderate initial losses", while the scenario for MT2 is called "large initial losses".

4 Economic Analysis

In this section the economic implications of the two master trajectories MT1 and MT2 are discussed. Both start out with Path 2_B on which $\lambda_1 + \lambda_2 > 1$. The economic interpretation of a co-state variable is that it equals the discounted contribution to the objective from a marginal increase of the corresponding state variable. Consider now a situation where the firm carries out an additional unit of investment. On the one hand this increases the firm's expenditures by one unit, but on the other hand it generates an additional unit of K_1 and of K_2. The contribution to the objective of this extra unit of K_1 and K_2 equals $\lambda_1 + \lambda_2$. Hence, starting out by Path 2_B is only optimal if at the initial point of time it holds that marginal contribution to the objective of investment exceeds marginal cost. Therefore it is optimal to invest maximally (Path 2!). As already mentioned at the end of the previous section, here we only analyze firms that are "small" in the beginning, i.e. for which initially marginal revenue exceeds marginal cost.

4.1 A firm with moderate initial losses

The solution that holds in this scenario is depicted in Figure 1. At the initial point of time it holds that the firm's taxable income is negative, i.e. $C(K_1) - \gamma K_2 < 0$, so that the firm does not need to pay any profit tax. This implies that a B-path is applied first. Furthermore, the marginal contribution to the objective of investment exceeds marginal cost, which means that it is optimal for the firm to start out investing maximally. This happens on Path 2. Therefore, the firm starts out on Path 2_B.

In Subsection 3.2 it was proven that on Path 2_B taxable income $C(K_1) - \gamma K_2$ increases over time. At the moment that $C(K_1) - \gamma K_2$ becomes positive the firm has to pay taxes, and thus Path 2_B passes into Path 2_A. The firm invests maximally on Path 2_A which implies that

$$\dot{K}_1 = I - \beta K_1 = (1 - T)\, C(K_1) + T\gamma K_2 - \beta K_1. \tag{29}$$

[Place Figure 1 about here]

On this path all revenues are retained and used for investments. As can be obtained from expression (29), K_1 as well as K_2 contribute to investment. With K_1 the firm produces goods which are sold on the market leading to an after tax revenue $(1-T)\,C\,(K_1)$. Tax depreciation, which linearly depends on K_2, can be subtracted from the firm's income before paying taxes. This reduces the tax payments with $T\gamma K_2$, so that this amount can be interpreted as revenue generated by the tax base K_2.

From (14) it can be obtained that (29) implies that productive capital stock increases on Path 2_A, provided that $K_1 \leq K_1^*$ (it will turn out later that this inequality holds on Path 2_A). Since the revenue function exhibits decreasing returns to scale, $\dot{K}_1 > 0$ implies that marginal revenue decreases over time. At the moment that it becomes equal to marginal cost, it need not be optimal anymore to invest maximally. Therefore, as soon as this happens, Path 2_A passes into Path 1_A.

Path 1_A is the final path of this master trajectory. From (17) it is obtained that on this path it holds that

$$\lambda_1 + \lambda_2 = 1. \tag{30}$$

Substitution of the transversality conditions (22) and (23) into this expression gives

$$C'(K_1(z)) = \left(\frac{r}{1-T} + \gamma\right)\left(\frac{r+\beta}{r+\gamma}\right). \tag{31}$$

Comparing this expression with (28) learns that $K_1(z) = K_1^*$.

From (19) it is obtained that

$$\lambda_2 \binom{>}{<} \frac{\gamma}{r+\gamma} T \Longleftrightarrow \dot{\lambda}_2 \binom{>}{<} 0.$$

This implies that the transversality condition (23) can only be satisfied if $\dot{\lambda}_2 = 0$ on the whole final Path 1_A. Now, from (30) it is derived that also $\dot{\lambda}_1 = 0$, which, via (18), in turn implies that $\dot{K}_1 = 0$ on this path. Hence, it holds that $K_1(t) = K_1^*$ on Path 1_A throughout. The implication is that on the final Path 1_A only replacement investments are carried out, i.e. $I(t) = \beta K_1^*$. The remaining revenue is used to pay dividends to the shareholders. Substitution of this investment level into (2) gives the following evolution of the tax base on this path:

$$\dot{K}_2 = I - \gamma K_2 = \beta K_1^* - \gamma K_2.$$

This implies that

$$\dot{K}_2 \binom{>}{<} 0 \Longleftrightarrow K_2 \binom{<}{>} \frac{\beta K_1^*}{\gamma},$$

from which it can be concluded that K_2 converges to $\frac{\beta K_1^*}{\gamma}$ on Path 1_A.

The new feature in our paper is that we have different values for technical depreciation rate (β) and tax depreciation rate (γ). Therefore it is interesting to find out in what way this model feature affects the optimal long run level of productive capital stock K_1^*, which is implicitly given by (28) (or (31)). First, if the depreciation rates were equal, it is obtained from (28) that

$$(1-T)(C'(K_1^*) - \beta) = r. \tag{32}$$

Hence, the optimal long run capital stock level is determined such that marginal revenue, net from depreciation and tax, equals the shareholder time preference rate. This is the usual formula for the final steady state level of capital stock in dynamic models of the firm (see, e.g., Ref. 3).

From (28) it is further obtained that

$$\frac{\partial C'(K_1^*)}{\partial \gamma} = \frac{-rT\,(r+\beta)}{(r+\gamma)^2\,(1-T)} < 0 \Rightarrow \frac{\partial K_1^*}{\partial \gamma} > 0. \tag{33}$$

It can thus be concluded that the optimal long run level of productive capital stock increases with γ. Economically, this is easy to understand, since a higher value of γ implies that capital is depreciated faster so that in earlier periods more can be subtracted from revenue before paying tax. In this way, tax payments are deferred to later periods so that investments become more profitable due to the time preference rate r. The highest value of γ under consideration here equals $r + \beta$ (cf. (10)). Then the corresponding value of K_1^* is implicitly given by

$$(1-T)(C'(K_1^*) - \gamma) = r - \frac{(1-T)r(r+\beta)}{2r+\beta}.$$

Finally, let us return to Path 2_A, where it was concluded that $\dot{K}_1 > 0$ provided that $K_1 < K_1^*$. This inequality indeed holds, since on Path 2_A capital stock increases to the steady state level

K_1^* so that the stock of productive capital must be below this steady state level on Path 2_A (note that shrinking, while approaching K_1^*, is impossible on Path 2_A, because (14) implies that $(1-T)\,C\,(K_1^*)+T\gamma K_2-\beta K_1^*>0$ for any positive K_2, so that via (29) it is obtained that $\dot{K}_1>0$ when K_1 approaches K_1^* on Path 2_A).

4.2 A firm with large initial losses

The solution in this case is depicted in Figure 2. As in the previous subsection, at the initial point of time taxable income is negative and the firm is that small (i.e. low level of the stock of productive capital) that the marginal contribution to the objective of investment exceeds marginal cost. Therefore, also here the optimal policy is to apply Path 2_B, thus start out investing at the maximal level, implying that no dividends are paid out.

The difference with the case in the previous subsection is that, at the moment the marginal contribution to the objective of investment equals marginal cost, taxable income is still negative. Hence, a B-path must still be applied at this stage, which implies that Path 2_B will pass into Path 1_B. Since the marginal contribution of investment to the objective is given by $\lambda_1+\lambda_2$ and marginal cost of investment is always equal to the expenditure of one unit of money, we can derive that $\lambda_1+\lambda_2$ must decrease on Path 2_B to make the coupling with Path 1_B possible. Concerning the time derivative of λ_1 on Path 2_B we can derive from (17) and (18) that

$$\dot{\lambda}_1=(r+\beta-C'(K_1))\,\lambda_1.$$

Furthermore , from (19) it is easily obtained that $\dot{\lambda}_2=r\lambda_2>0$. Hence, in order to make it possible that $\lambda_1+\lambda_2$ decreases on Path 2_B, it must hold that

$$C'(K_1)>r+\beta. \tag{34}$$

[Place Figure 2 about here]

From the path coupling procedure (see Appendix C), it is obtained that, at the moment that Path 2_B passes into Path 1_B, the productive capital stock exceeds K_1^*. This means that on Path

2_B there exists a point of time where the firm kept on growing while productive capital stock had already reached its optimal long run value. The reason that it was not optimal to stop growth there, is that at that particular point of time taxable income was negative. Note that K_1^* was determined on Path 1_A as the steady state level on which marginal revenue equals marginal cost, but under the condition that taxable income is positive. However, when taxable income is negative, marginal revenue of investment equals $C'(K_1^*)$, while under positive tax income marginal revenue equals $(1-T)\ C'(K_1^*) + T\gamma$. Now, it can be concluded from (10) and (34) that $\gamma < C'(K_1^*)$, so that marginal revenue under negative taxable income exceeds marginal revenue under positive taxable income. Hence, given the fact that for positive taxable income marginal revenue equals marginal cost, it must hold that for negative taxable income marginal revenue exceeds marginal cost when K_1 equals K_1^* on Path 2_B. Therefore it is optimal for the firm to keep on growing at that stage.

The previous paragraph learns that at the start of Path 1_B productive capital stock exceeds its long run level. Therefore, with application of Path 1_B the firm starts a contraction phase. At Path 1_B investment is still positive but below depreciation level. On this path investment is determined such that the marginal contribution to the objective of investment equals its marginal cost (i.e. $\lambda_1 + \lambda_2 = 1$). To accomplish this, productive capital stock decreases over time, so that, as time passes, higher marginal revenue counterbalances the fact that the moment of time that tax must be paid comes nearer. At a certain point of time investment must be negative in order to keep the marginal contribution to investment equal to marginal cost. Since investment is irreversible, this is not possible. The second best thing to do at this stage is to keep investment equal to zero, but the consequence is that the marginal contribution to the objective of investment falls below marginal cost of investment. This explains the occurrence of Path 3_B after Path 1_B.

It is easy to show that on the sequence we had so far, thus Path 2_B $\rightarrow$Path 1_B $\rightarrow$Path 3_B, taxable income increases over time. At the moment taxable income becomes positive, Path 3_B passes into Path 3_A. Then the firm starts to pay tax, while still refraining from investment. This phase stops at the moment that productive capital stock reaches its optimal long run level K_1^* where investment jumps to replacement level and the remaining revenue is paid out as dividends to the shareholders. This happens on Path 1_A, which is extensively analyzed in the previous subsection.

5 Concluding remarks

This paper has studied the standard dynamic investment model of a firm, extended with a different depreciation rate for tax depreciation than for economic depreciation. This model feature was motivated by the fact that firms can have several incentives to choose a tax depreciation method that differs from economic depreciation. Furthermore, it was possible for the firm to carry forward losses in order to reduce future tax payments. The results were derived by optimal control theory and application of a formal synthesizing procedure that goes back to Ref. 7.

So-called master trajectories were developed that ended up in a steady state equilibrium where dividend is paid out. For firms that are initially "small", it turned out that different master trajectories can occur: one in which the firm grows maximally until the steady state equilibrium is reached, and another one where initial maximal growth is followed by a contraction phase, which goes on until the same steady state equilibrium is reached. Using these master trajectories, the solution path of firms with different initial states can be determined as well.

Another important consequence of the difference between the economic depreciation rate and the tax depreciation rate is that the optimal long run capital stock level changes. Deferring tax payments to a later date by choosing a higher tax depreciation rate implies that the firm should grow to a higher level.

This paper determined the optimal investment behavior of a firm for a given rate of tax depreciation. A very interesting extension would be to introduce this tax depreciation rate as a control variable in the model. Then, the investment rate and the tax depreciation rate are simultaneously optimized over time.

Appendices

A Proof of proposition 3.1

Proof. Define the auxiliary function $g(I) = \frac{\partial}{\partial t}(C(K_1) - \gamma K_2)$. We find:

$$g(I) = C'(K_1)\dot{K}_1 - \gamma \dot{K}_2 = (C'(K_1) - \gamma)I + \gamma C(K_1) - C'(K_1)\beta K_1,$$

which is linear in the investments I. This implies that

$$\begin{aligned} & g(0) \leq g(I) \leq g(C(K_1)) \quad \text{if } C'(K_1) \geq \gamma, \\ \text{and} \quad & g(0) > g(I) > g(C(K_1)) \quad \text{if } C'(K_1) < \gamma. \end{aligned}$$

Given that $\gamma \geq \beta$, $C(.)$ is concave, and $C(K_1) \leq \gamma K_2$ (B-path), it holds that:

$$\begin{aligned} g(0) & = \gamma C(K_1) - \beta C'(K_1)K_1, \\ & > \gamma C(K_1) - \beta C(K_1), \\ & \geq 0, \end{aligned}$$

and

$$\begin{aligned} g(C(K_1)) & = C'(K_1)C(K_1) - C'(K_1)\beta K_1, \\ & = C'(K_1)\big[C(K_1) - \beta K_1\big], \\ & > 0. \end{aligned}$$

This implies that $g(I) > 0$ for all $I \in [0, C(K_1)]$, so that $C(K_1) - \gamma K_2$ is strictly increasing over time when the firm is on a B path. ■

B Proof of proposition 3.2

Proof. To show that path 1_A can be the final path, the necessary conditions (35) - (37),

$$\dot{\lambda}_1 = (r + \beta)\lambda_1 - C'(K_1)(1 - T), \tag{35}$$

$$\dot{\lambda}_2 = (r + \gamma)\lambda_2 - \gamma T, \tag{36}$$

$$\lambda_1 + \lambda_2 = 1. \tag{37}$$

together with the transversality conditions have to be solved. The transversality conditions imply that $\lambda_2(z) = \frac{T\gamma}{r+\gamma}$. Together with (36) one has:

$$\dot{\lambda}_2 = 0$$

(37) implies that $\dot{\lambda}_1 + \dot{\lambda}_2 = 0$. So it follows that $\dot{\lambda}_1 = 0$. Given (35) and (36) this implies that:

$$r + \beta\lambda_1 + \gamma(1 - \lambda_1) - C'(K_1)(1 - T) - T\gamma = 0,$$

Combined with $\dot{\lambda}_i = 0$, this yields the equilibrium:

$$\lambda_1 = \frac{r + (1-T)\gamma}{r+\gamma},$$
$$\lambda_2 = \frac{T\gamma}{r+\gamma},$$
$$C'(K_1) = \left(\frac{r}{1-T} + \gamma\right)\left(\frac{r+\beta}{r+\gamma}\right).$$

Hence, the transversality conditions imply that the co-state variables must be constant on this path, which implies that also K_1 is constant, so this is a steady state path. Remains the question if this steady state can be maintained until infinity. This can be done by investing $I(t) = \beta K_1^*$, where K_1^* satisfies (28). For these investments one has $C(K_1^*) > I$, so $D > 0$, since $C'(K_1) > \beta$. So one obtains a steady state with $\dot{K}_1 = \dot{\lambda}_1 = \dot{\lambda}_2 = 0$.
Moreover, suppose that on path 1_A, $\lambda_2(t) < T\gamma/(r+\gamma)$ (resp. $>$). Then (36) implies that λ_2 is decreasing (resp. increasing) so that $\lambda_2(u) < T\gamma/(r+\gamma)$ (resp. $>$) for $u \geq t$. A similar argument holds for λ_1, since $\lambda_1 + \lambda_2 = 1$. This implies that if the firm is on path 1_A, but not in a steady state (it does not satisfy the transversality conditions), then it cannot satisfy the transversality conditions without leaving path 1_A.

It now remains to show that the other paths cannot be the final paths.
For path 1_B, the only equilibrium is:

$$\lambda_1 = 1,$$

$$\lambda_2 = 0,$$

$$C'(K_1) = r + \beta,$$

so that the equilibrium conditions imply that $\dot{K}_1 = 0$. Consequently, one has

$$\dot{K}_2(t) = \beta K_1^* - C(K_1^*).$$

It is verified easily that $\dot{K}_2 < 0$, so that path 2_B has to be abandoned at a certain point in time. This argument holds for all paths i_B, independent of the values of λ_i. The transversality conditions can be satisfied, but when the time-horizon is long enough, it is not possible to end in a situation where no taxes are paid.

Finally, notice that on path 2, one has $\dot{K}_1 > 0$, and on path 3, one has $\dot{K}_1 < 0$. Therefore it is seen that an equilibrium is not possible. We can therefore conclude that the final path is 1_A, and that this is a steady state path.

To prove that K_1^* is the unique steady state level at which marginal revenue equals marginal cost and taxable income is positive, consider the firm in a steady state at time t^*. This implies that investments equal $I(t) = \beta K_1(t^*)$ for all $t \geq t^*$. We now determine the marginal value of an additional investment at time t^*. Due to the extra investment x at time t^*, the evolution over time of the capital stock and the tax base after t^* is given by:

$$\tilde{K}_1(t,x) = K_1(t^*) + xe^{-\beta(t-t^*)},$$

$$\tilde{K}_2(t,x) = K_2(t) + xe^{-\gamma(t-t^*)}.$$

Since taxable income is positive, the marginal value generated by the additional investment is then given by:

$$\begin{aligned}
&\frac{\partial}{\partial x}_{|x=0} \int_{t^*}^{\infty} e^{-r(t-t^*)}\big[(1-T)C\big(K_1(t^*) + xe^{-\beta(t-t^*)}\big) + T\gamma\big(K_2(t) + xe^{-\gamma(t-t^*)}\big)\big]dt \\
&= \int_{t^*}^{\infty} e^{-r(t-t^*)}\big[(1-T)C'(K_1(t^*))e^{-\beta(t-t^*)} + T\gamma e^{-\gamma(t-t^*)}\big]dt \\
&= \frac{1}{\beta+r}(1-T)C'(K_1(t^*)) + \gamma T\frac{1}{\gamma+r}.
\end{aligned}$$

It now follows that the latter expression equals 1 (the marginal cost of the additional investment), iff

$$C'(K_1(t^*)) = \left(\frac{r}{1-T} + \gamma\right)\left(\frac{r+\beta}{r+\gamma}\right).$$

Since $C'(.)$ is strictly decreasing, this implies that $K_1(t^*) = K_1^*$. ■

C The path coupling procedure

To obtain the optimal sequence of paths, a formal synthesizing procedure (path coupling) is applied. It determines which path(s) can precede a given path, exploiting the continuity of state- and costate variables, and the necessary conditions for optimality.

The following definition will be helpful.

Definition C.1 *We define $\tilde{\gamma}$ to be the fixed point of the equation $C'(K_1^*(\gamma)) = \gamma$. Given* (28), *this implies that:*

$$\tilde{\gamma} = \frac{1}{2}\beta + \sqrt{\frac{1}{4}\beta^2 + \frac{(r+\beta)r}{1-T}}.$$

Straightforward calculations yield:

$$\tilde{\gamma} > \frac{1}{2}\beta + \sqrt{\frac{1}{4}(\beta + 2r)^2} = \beta + r,$$

so that $\tilde{\gamma} \notin [\beta, \beta + r]$.

We now proceed to determine the master trajectories. The following notation will be used. The instant at which path i is coupled before path j will be denoted $t_{i,j}$ (irrespective of whether they are A or B paths). The time instant just before this coupling will be denoted $t_{i,j}^-$.

Lemma C.1 *The following is a master trajectory:*

$$MT1 : 2_B \rightarrow 2_A \rightarrow 1_A.$$

Proof. In order to show that a sequence of paths is a master trajectory, one has to show that the paths can be coupled before each other without violating the optimality conditions, and that no other path can precede the first path in the trajectory. We will therefore subsequently show that

i) Path 2_A can be coupled before path 1_A.

ii) Path 2_B can be coupled before the sequence $2_A \rightarrow 1_A$.

iii) Nothing can be coupled before the sequence $2_B \rightarrow 2_A$.

[ad i)] The fact that λ_1 and λ_2 are continuous (see e.g. Ref. 8) implies that, in order to couple path 2_A before 1_A, one must have $\eta_1(t_{2,1}) = 0$.

[ad ii)] We first describe the dynamics on path 2_A, given that it is coupled before 1_A.

Path 2_A has $I = (1-T)C(K_1) + \gamma T K_2 \geq (1-T)C(K_1)$, and

$$\dot{\lambda}_1 = (r+\beta)\lambda_1 - (1+\eta_1)C'(K_1)(1-T), \tag{38}$$

$$\dot{\lambda}_2 = (r+\gamma)\lambda_2 - (1+\eta_1)\gamma T, \tag{39}$$

$$\lambda_1 + \lambda_2 = 1 + \eta_1. \tag{40}$$

Condition (14) therefore implies that $\dot{K_1} > 0$. Since $C(.)$ is concave, this implies that $C'(K_1)$ is decreasing over time on path 2_A.

$\dot{\lambda}_1, \dot{\lambda}_2$, and therefore $\dot{\eta}_1$ are continuous on $2_A \to 1_A$. Furthermore, one has $\eta_2(t_{2,1}) = 0$. Since $\dot{\lambda}_1(t_{2,1}) = 0$ and $\dot{\lambda}_1$ is continuous, and $C'(.)$ decreases over time, it follows that $\dot{\lambda}_1(t_{2,1}^-) \leq 0$.

Furthermore, since on path 2_A one has $\eta_1 \geq 0$, we know that $\eta_1(t_{2,1}^-) \geq 0$. This, together with (39), implies that $\dot{\lambda}_2(t_{2,1}^-) \leq 0$. From (40), it now follows that $\dot{\eta}_1(t_{2,1}^-) = \dot{\lambda}_1(t_{2,1}^-) + \dot{\lambda}_2(t_{2,1}^-) \leq 0$.

Furthermore, $\dot{\lambda}_i$ remain non-increasing, since if $\dot{\lambda}_i$ would be zero at time τ because of the change in λ_i, it follows that $\dot{\lambda}_i(\tau^-) \leq 0$, since $\dot{\eta}_1(\tau^-) = \dot{\lambda}_1(\tau^-) + \dot{\lambda}_2(\tau^-) \leq 0$, and $C'(K_1)$ decreases over time.

So we can conclude that on path 2_A before the final path, λ_1, λ_2 and η_1 are non-increasing.

The fact that $\dot{\eta}_1 < 0$ on path 2_A implies that only path 2_B can precede 2_A. Since at the coupling point $t_{2,2}$, one has $C(K_1) - \gamma K_2 = 0$, the costate variables do not have to be differentiable at $t_{2,2}$. Instead, (24) and (25) apply.

[ad iii)] In order to determine what can precede path 2_B, we look at the dynamics of the costates. On path 2_B, one has:

$$\dot{\eta}_1^B = [r + \beta - C'(K_1)]\,\lambda_1 + r\lambda_2. \tag{41}$$

This is equal to:

$$\dot{\eta}_1^B = [r + \beta - C'(K_1)]\,[1+\eta_1] + [C'(K_1) - \beta]\,\lambda_2.$$

Furthermore, the necessary conditions of this path are:

$$\dot{\lambda}_1 = (r+\beta)\lambda_1 - (1+\eta_1-\lambda_2)C'(K_1), \tag{42}$$

$$\dot{\lambda}_2 = r\lambda_2. \tag{43}$$

Now, checking the change in η_1 over time yields:

$$\frac{\partial \dot{\eta}_1^B}{\partial t} = -[1+\eta_1-\lambda_2]C''(K_1)\dot{K}_1 + \dot{\lambda}_2[C'(K_1)-\beta] + \dot{\eta}_1\left[(r+\beta)-C'(K_1)\right]. \tag{44}$$

Given that:

- (43) implies that λ_2 is increasing on path 2_B,
- $C'(K_1) > r+\beta$ (in the subsequent paths the firm will grow towards K_1^*, with $C'(K_1^*) > r+\beta$),
- (14) implies that $C'(K_1)$ is decreasing on path 2_B ($\dot{K}_1 > 0$),

it follows from (44) that, if $\dot{\eta}_1^B(\tau) < 0$ at some time τ, then $\dot{\eta}_1^B(t) < 0$ for all $t < \tau$.

Proposition 3.1 implies that only a B-path can precede path 2_B. Then, in order to couple another path before path 2_B, η_1^B has to be 0 at that coupling instant $t_{.,2}$. The above then implies that $\dot{\eta}_1^B(t_{2,2})$ has to be positive. Indeed, suppose that $\dot{\eta}_1^B(t_{2,2}) < 0$, then $\dot{\eta}_1^B(\tau) < 0$ for all $\tau \in [t_{.,2}, t_{2,2}]$. Then it is impossible to have $\eta_1^B(t_{.,2}) = 0$.

To see whether it is possible that $\dot{\eta}_1^B(t_{2,2}) \geq 0$, we consider the relation between the dynamics of η_1 on path 2_A and 2_B. On path 2_A one has:

$$\dot{\eta}_1^A = \dot{\lambda}_1 + \dot{\lambda}_2 = (\gamma-\beta)\lambda_2 - (1+\eta_1)\left[(1-T)C'(K_1) + T\gamma - (r+\beta)\right]. \tag{45}$$

Combined with (41), this implies:

$$\dot{\eta}_1^B = \dot{\eta}_1^A - [(1+\eta_1)T - \lambda_2](C'(K_1)-\gamma). \tag{46}$$

Now since $\dot{\eta}_1^A(t_{2,2}) \leq 0$ and $\lambda_2 \leq (1+\eta_1)\frac{T\gamma}{r+\gamma} < (1+\eta_1)T$, it follows that $\dot{\eta}_1^B(t_{2,2})$ can only be positive if $C'(K_1) < \gamma$.

At path 2_A and 2_B on this master trajectory, one has $C'(K_1) > C'(K_1^*)$. Therefore $C'(K_1) < \gamma$ implies $C'(K_1^*) < \gamma$. This implies that $\gamma > \tilde{\gamma}$. Path 2_B can therefore not be preceded by another path for $\gamma \in [\beta, r+\beta]$, since $\tilde{\gamma} > r+\beta$.

This concludes the proof. ■

Lemma C.2 *The following is a master trajectory:* [3]

$$MT2 : 2_B \to 1_B \to 3_B \to 3_A \to 1_A.$$

Proof. Similarly to the proof of lemma C.1, we will show that:

i) Path 3_A can be coupled before path 1_A.

ii) Path 3_B can be coupled before $3_A \to 1_A$.

iii) Path 1_B can be coupled before path $3_B \to 3_A \to 1_A$ if $C'(K_1) > r + \beta$.

iv) Path 2_B can be coupled before path $1_B \to 3_B \to 3_A \to 1_A$.

v) Nothing can be coupled before $2_B \to 1_B$.

[ad *i*)] When coupling 3_A before the terminating path, it holds that $\eta_2(t_{3,1}) = 0$.

[ad *ii*)] We first describe the dynamics on path 3_A before 1_A. On path 3_A, one has:

$$\dot{\lambda}_1 = (r + \beta)\lambda_1 - (1 - T)C'(K_1), \tag{47}$$

$$\dot{\lambda}_2 = (r + \gamma)\lambda_2 - \gamma T. \tag{48}$$

Since the dynamics of λ_2 are the same on paths 3_A and 1_A, one has $\dot{\lambda}_2 = 0$ during path 3_A (see (48)). Furthermore, since path 3_A is a shrink-path in capital stock, one has $\frac{\partial C'(K_1)}{\partial t} = C''(K_1)\dot{K}_1 > 0$, which, with (47), implies that $\dot{\lambda}_1 > 0$. So $\dot{\eta}_2 < 0$ and only path 3_B can precede path 3_A.

The coupling $3_B \to 3_A$ is quite trivial. At the coupling instant, one has $C(K_1) - \gamma K_2 = 0$, and $\eta_2 > 0$.

[ad *iii*) and *iv*)] We first show that the sequence $3_B \to 3_A$ can only be preceded by another path if $C'(K_1) > r + \beta$.

[3] Strictly speaking, the sequence $1_B \to 3_B$ can also be $1_B \to 3_B \to 1_B \to 3_B \to \cdots \to 1_B \to 3_B$. This however can be excluded for functions $C(\cdot)$ with $C'''(\cdot) \leq 0$, e.g. functions in the class $C(K_1) = aK_1 - bK_1^2$.

At the coupling point $t_{.,3}$, η_2 has to be zero, and at a certain point after the coupling instant η_2 has to be positive (in order to couple path 3_B before 3_A). Therefore, $\lambda_1(t_{.,3}) + \lambda_2(t_{.,3}) = 1$, and $\dot{\eta}_2(t_{.,3}) \geq 0$, which implies for path 3_B:

$$\begin{aligned} \dot{\eta}_2 &= -\dot{\lambda}_1 - \dot{\lambda}_2, \\ &= -(r+\beta)\lambda_1 + C'(K_1) - (r + C'(K_1))\lambda_2, \\ &< -(r+\beta) + C'(K_1). \end{aligned}$$

Therefore, $\dot{\eta}_2(t_{.,3}) \geq 0$ iff $C'(K_1) > r + \beta$ at time $t_{.,3}$, so that, when a path is coupled before path 3_B, one has $C'(K_1) > r + \beta$.

Proposition 3.1 implies that only a B-path can precede path 3_B.

It can be verified easily that the coupling of 1_B before 3_B is feasible iff $C'(K_1) < r + \beta$ at time $t_{1,3}$.

The coupling of 2_B before 1_B is quite trivial, since the continuity of co-states is clearly maintained, and only the investment strategy changes.

[ad v)] We show that when path 2_B is coupled before another B-path, it cannot be preceded by another path.

Preceding paths (1_B or 3_B) have $\dot{\lambda}_1 + \dot{\lambda}_2 \leq 0$. This implies that $C'(K_1) > r + \beta$. We then find from (42), (43) and (40), together with the fact that 2_B is a growth path, that just before the coupling instant, one has $\dot{\eta}_1(t^-_{2,.}) \leq 0$. This implies:

$$\begin{aligned} \dot{\eta}_1(t^-_{2,.}) &= (r+\beta)\lambda_1 - (1 + \eta_1 - \lambda_2)C'(K_1) + r\lambda_2, \\ &= r\lambda_2 - [C'(K_1) - (r+\beta)]\,\lambda_1, \\ &\leq 0. \end{aligned}$$

Furthermore,

$$\frac{\partial \dot{\eta}_1}{\partial t} = [(r+\beta) - C'(K_1)]\,\dot{\lambda}_1 - C''(K_1)\dot{K}_1\lambda_1 + r\dot{\lambda}_2. \tag{49}$$

Since $\lambda_1 > 0$ ($\lambda_1 + \lambda_2 \geq 1$, and $\lambda_2 \leq \frac{T\gamma}{r+\gamma}$), and $\dot{\lambda}_1 = [(r+\beta) - C'(K_1)]\lambda_1 < 0$, this is clearly positive, so $\dot{\eta}_1$ has increased to its negative value. Therefore, it is negative along the path. This implies that only a path with $\eta_1 > 0$ can precede path 2_B, but proposition 3.1 implies that path 2_A can not precede path 2_B.

This concludes the proof. ■

Lemma C.3 *The following is a master trajectory:* [4]

$$MT3 : 3_B \to 1_B \to 3_B \to 3_A \to 1_A.$$

Proof. Goes along the same lines as the proof of lemma C.2 and is therefore omitted. ■

In the following proposition, we prove that there are no other master trajectories, so that the optimal investment strategies are determined.

Proof of proposition 3.3 It suffices to show that $MT1$, $MT2$ and $MT3$ are the only master trajectories of problem (9).

In order to prove this, it is necessary to show that:

i) $MT1$, $MT2$ and $MT3$ are master trajectories.

ii) There is no part of $MT1$, $MT2$ or $MT2$ that can be preceded by paths that are not in these master trajectories, and

iii) There are no other possible couplings before the final path 1_A.

In the following these two points will be addressed.

[ad i)] See lemmas C.1, C.2 and C.3.

[ad ii)] In the proofs of lemmas C.1, C.2 and C.3, it is made clear that the couplings in MT1 are unique, and that the couplings for MT2 and MT3 are unique taking into account footnote (3). There were no other couplings possible than the ones that resulted in the master trajectories.

[ad iii)] In order to prove this part, all paths other than 2_A and 3_A must be proven to be not feasible before the final path 1_A.

- *Path 1_B before Path 1_A:* Path 1_B is a shrinkpath. When coupling path 1_B before 1_A at time $t_{1,1}$, $\dot{\lambda}_1$ and $\dot{\lambda}_2$ can be discontinuous (see (24) and (25)). K_1 however is continuous.

[4] Strictly speaking, the sequence $1_B \to 3_B$ can also be $1_B \to 3_B \to 1_B \to 3_B \to \cdots \to 1_B \to 3_B$. This however can be excluded for functions $C(\cdot)$ with $C'''(\cdot) \leq 0$, e.g. functions in the class $C(K_1) = aK_1 - bK_1^2$.

Therefore at time $t_{1,1}^-$ we have $C'(K_1) = C'(K_1^*)$. Since $\lambda_1 + \lambda_2 = 1$, this implies:

$$\begin{aligned}
& \dot{\lambda}_1 + \dot{\lambda}_2 = 0, \\
\Rightarrow \quad & \left\{(r+\beta) - \tfrac{r+\beta}{1-T}\tfrac{r+(1-T)\gamma}{r+\gamma}\right\}\tfrac{r+(1-T)\gamma}{r+\gamma} + r\tfrac{T\gamma}{r+\gamma} = 0, \\
\Rightarrow \quad & -T\tfrac{r+\beta}{1-T}\tfrac{r+(1-T)\gamma}{r+\gamma} = -T\gamma, \\
\Rightarrow \quad & \gamma = \bar{\gamma}.
\end{aligned}$$

Since $\bar{\gamma} > r + \beta$, the proof is complete.

- *Path 2_B before 1_A:* Path 2_B can precede path 1_A. Using the fact that $\dot{\lambda}_1 + \dot{\lambda}_2 = \dot{\eta}_1 < 0$, the proof is similar to the case $1_B \rightarrow 1_A$.

  Furthermore we know that nothing can precede $2_B \rightarrow 1_A$. This implies that $2_B \rightarrow 1_A$ could be a master trajectory. This master trajectory is a special case of master trajectory 1, that starts with path 2_B and has path 2_A for an infinitesimal small interval. However it is just one special case, dependent on all starting values and variables, such that full investments result in:

  $$\begin{aligned}
  & C(K_1) = \gamma K_2, \\
  & C'(K_1) = C'(K_1^*),
  \end{aligned}$$

  at the coupling instant $t_{2,\cdot}$. Therefore we neglect this possibility, since it is a boundary case and a special case of $MT1$.

- *Path 3_B before 1_A:* This coupling is not feasible. Using the fact that $\dot{\lambda}_1 + \dot{\lambda}_2 = -\dot{\eta}_2 > 0$, the proof is similar to the case $1_B \rightarrow 1_A$. This coupling is only feasible when $\gamma > \tilde{\gamma}$.

This completes the proof. □

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List of Figures

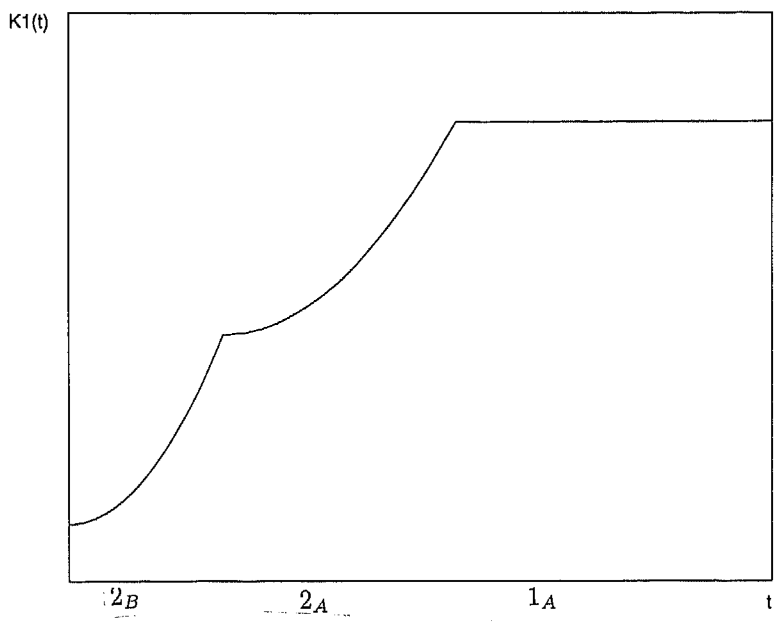

Figure 1: $K_1(t)$ for moderate initial losses.

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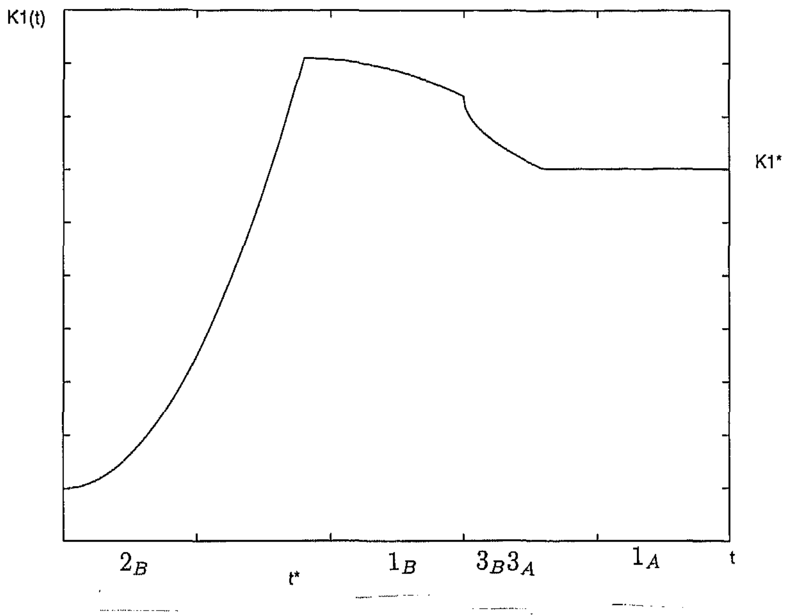

Figure 2: $K_1(t)$ for large initial losses.

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