Tilburg University

Sequencing Situations and Games with Non-Linear Cost Functions

Schouten, Jop; Saavedra-Nieves, Alejandro; Fiestras-Janeiro, G.

Publication date:
2020

Document Version
Early version, also known as pre-print

Link to publication in Tilburg University Research Portal

Citation for published version (APA):
Schouten, J., Saavedra-Nieves, A., & Fiestras-Janeiro, G. (2020). *Sequencing Situations and Games with Non-Linear Cost Functions*. (CentER Discussion Paper; Vol. 2020-006). CentER, Center for Economic Research.

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

Take down policy
If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Download date: 22. Sep. 2021

CentER

Discussion Paper

No. 2020-006

SEQUENCING SITUATIONS AND GAMES WITH NON-LINEAR COST FUNCTIONS

By

J. Schouten, A. Saavedra-Nieves,
M.G. Fiestras-Janeiro

4 March 2020

ISSN 0924-7815
ISSN 2213-9532

TILBURG ◆ UNIVERSITY

Sequencing situations and games with non-linear cost functions

J. Schouten*‡ A. Saavedra-Nieves§ M. G. Fiestras-Janeiro§

Abstract

This paper studies sequencing situations with non-linear cost functions. We show that the neighbor switching gains are now time-dependent, in contrast to the standard sequencing situations with linear cost functions, which complicate finding an optimal order and stable allocations. We derive conditions on the time-dependent neighbor switching gains in a (general) sequencing situation to guarantee convexity of the associated sequencing game. Moreover, we provide two procedures that uniquely specify a path from the initial order to an optimal order and we define two corresponding allocation rules that divide the neighbor switching gains equally in every step of the path. We show that the same conditions on the gains also guarantee stability for the allocations prescribed by these rules.

Keywords: sequencing games, non-linear cost functions, time-dependent neighbor switching gains, convexity, stable allocations
JEL classification: C44, C71

1 Introduction

In this paper, we deal with one-machine sequencing situations with, in addition, an initial order specified. Such a sequencing situation can be described by a set of players, each associated to a job that need to be processed on a single machine, an initial order that provides the initial processing rights on the machine, a processing time for each of the jobs and a cost function in terms of the completion time. The total costs are the sum of the costs of the players and the first goal is to find an optimal processing order that minimizes this total costs. By rearranging, the players can obtain cost savings and more specifically, the optimal order can be reached by interchanging consecutive players that are arranged different in the initial order than in the optimal order. The second goal is to find a suitable allocation that divides these total cost savings among the players.

Traditionally, the cost function is assumed to be linear, specified by linear cost coefficients. For these *standard sequencing situations*, the first goal is achieved by Smith (1956), who showed that in an optimal order, the players are arranged according to a (weakly) decreasing urgency index, which is the ratio of the linear cost coefficient and the processing time. Moreover, Curiel, Pederzoli, and Tijs (1989) defined the Equal Gain Splitting rule (EGS-rule), which is an allocation for a standard sequencing situation that divides the cost savings of every neighbor misplacement with regard to the initial order and an optimal order equally among the two players involved. Hereby, the second goal is also achieved for standard sequencing situations.

*CentER, Department of Econometrics & Operations Research, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands
‡E-mail: j.schouten@tilburguniversity.edu
§Departamento de Estatística e Investigación Operativa, Universidade de Vigo

Interestingly, the allocation specified by the EGS-rule for a standard sequencing situation (with linear cost functions) turned out to be a core-element of the associated sequencing game. A sequencing game (cf. Curiel et al., 1989) is a transferable utility cooperative game that is associated to a sequencing situation for which the worth of a coalition is defined as the maximal cost savings the coalition can obtain by admissible rearrangements with respect to the initial order. Curiel et al. (1989) showed that a standard sequencing game is a convex game.

Sequencing problems appear in different frameworks. The assumption of adding effects on the unavailability of the machine was considered by Liu, Lu, and Qi (2018). Yang, Sun, Hou, and Xu (2019) analyzed the influence of external players in the queue on the worth of the coalition. Chen, Huang, Wang, and Yang (2019) applied the ideas of sequencing in cloud manufacturing settings.

In this paper, we deal with sequencing situations with non-linear cost functions with as variable the number of time units spent in the system. We follow the above-mentioned lines (cf. Curiel et al., 1989) in the sense that we focus on conditions that guarantee convexity of the associated sequencing game and on allocations that provide core-elements of the associated sequencing game. In particular, we deal with three specific examples of non-linear cost functions, namely exponential, discounting and logarithmic cost functions. *Exponential sequencing situations* were recently introduced and analyzed by Saavedra-Nieves, Schouten, and Borm (2020), while sequencing situations with discounting cost functions, i.e. *discounting sequencing situations*, were introduced by Rothkopf (1966). In the same spirit, we consider sequencing situations with logarithmic cost functions. For *logarithmic sequencing situations*, we explicitly derive the expression for the cost savings obtained by two players if they interchange their positions. Moreover, we show that in an optimal order, players are arranged according to (weakly) increasing processing times.

This line of approach follows the direction of research in which assumptions of the standard sequencing model are relaxed or modified. For example, Slikker (2006) relaxed the assumption of cooperation between players, Lohmann, Borm, and Slikker (2014) modified the definition of the time a job spends in the system, and Musegaas, Borm, and Quant (2015) relaxed the set of admissible rearrangements. On the other hand, a lot of research is done in the direction of adding components to the model. For example, Hamers, Borm, and Tijs (1995) added ready times, Borm, Fiestras-Janeiro, Hamers, Sánchez, and Voorneveld (2002) studied due dates, Hamers, Klijn, and Van Velzen (2005) added precedence relations, Biskup (2008) reviewed the literature on scheduling with learning effects, and Estévez-Fernández, Borm, Calleja, and Hamers (2008) incorporated repeated players.

First, we focus on convexity of the associated sequencing games. Saavedra-Nieves et al. (2020) showed that, by imposing a set of conditions on the neighbor switching gains, the associated sequencing game of a (general) sequencing situation is convex. These conditions require the neighbor switching gains to be non-negative and non-decreasing for misplacements and non-positive for non-misplacements. We impose a second set of conditions in which the neighbor switching gains are required to be non-negative and non-increasing for misplacements and non-positive for non-misplacements. We add a proof that also for this second set of conditions, the associated sequencing game of a sequencing situation with an arbitrary non-linear cost function is convex. As a consequence, we have that discounting and logarithmic sequencing games are always convex.

Secondly, we focus on allocation rules for sequencing situations with non-linear cost functions.

For standard sequencing situations with linear cost functions, the cost savings obtained by two players who interchange their positions as they are ordered different in the optimal order than in the initial order are independent of the position of these two players. In other words, these neighbor switching gains are not dependent on the moment in time the players interchange their positions. In contrast, for sequencing situations with non-linear cost functions, the neighbor switching gains may be time-dependent. Hence, the neighbor switching gains depend on the path from the initial order to an optimal order.

This implies that it is not possible to directly apply the EGS-rule to sequencing situations with non-linear cost functions in the sense that the properties satisfied by the EGS-rule in the standard case are not maintained in general. In a standard sequencing situation, every path from the initial order to an optimal order leads to the same neighbor switching gains. For sequencing situations with non-linear cost functions, we have to specify which path to choose. We provide two different procedures that specify a path from the initial order to an optimal order:

- The *Growing Head* procedure. This procedure starts with the player that occupies the first position in the optimal order and consecutively moves this player to that position. Secondly, the player that is in the second position of the optimal order moves to that position and so on, successively until all players are in their positions in the optimal order.
- The *Growing Tail* procedure. This procedure reverses the idea of the Growing Head procedure and starts with the player that is in the last position of the optimal order.

To obtain an allocation, we adopt the idea of the EGS-rule and divide the neighbor switching gains in every step of a path from the initial order to an optimal order equally among the two players involved. This results in two distinct allocation rules, depending on the procedure that is used: the *Equal Gain Splitting Head rule (EGSH-rule)* and the *Equal Gain Splitting Tail rule (EGST-rule)*. We show that, by reusing the two different sets of conditions on the neighbor switching gains, both allocation rules prescribe a core-element of the associated sequencing game. In particular, we show that for discounting and logarithmic sequencing situations, the EGST-rule leads to a core-element, while for the three subclasses as defined in Saavedra-Nieves et al. (2020), the EGSH-rule results in a core-element.

This paper is structured as follows. Section 2 contains all preliminaries. Section 3 provides an analysis of exponential, discounting and logarithmic sequencing situations. Section 4 provides a result on convexity and Section 5 introduces two allocation rules for sequencing situations with an arbitrary non-linear cost function. Finally, Section 6 contains the concluding remarks.

2 Preliminaries

In a *(general) sequencing situation*, there is a finite, non-empty set of players N that each have a job that needs to be processed on a single machine. A *(processing) order* of the players is described by a bijective function $\sigma : N \to \{1, 2, \ldots, |N|\}$ in which $\sigma(i) = k$ means that the job of player i is in position k of the order σ. The set of all orders of N is denoted by $\Pi(N)$. Moreover, let $\sigma_0 \in \Pi(N)$ denote the initial processing order of the players, providing the initial processing rights on the machine. For every player $i \in N$, let $p_i \in \mathbb{R}_{++}$ denote the *processing time* of the job of player i and let the *cost function* of player i be given by $c_i : [0, \infty) \to \mathbb{R}$, where

$t \in [0, \infty)$ is the number of time units player i has spent in the system. Here, it is assumed that the machine starts processing at time $t = 0$ and that all jobs are present at $t = 0$.

Following the lines as described in Saavedra-Nieves et al. (2020), a *(general) sequencing situation* is represented by a tuple (N, σ_0, p, c), where $p = (p_i)_{i \in N}$ and $c = (c_i)_{i \in N}$ summarize the processing times and cost functions, respectively. The set of all sequencing situations with player set N is denoted by SEQ^N and a sequencing situation (N, σ_0, p, c) is also denoted by $(\sigma_0, p, c) \in SEQ^N$.

Let $(\sigma_0, p, c) \in SEQ^N$ be a sequencing situation and let $\sigma \in \Pi(N)$ be an order. The *set of predecessors* of player $i \in N$ with respect to σ is denoted by $P(\sigma, i)$ and given by $P(\sigma, i) = \{k \in N \mid \sigma(k) < \sigma(i)\}$, while the *set of followers* is denoted by $F(\sigma, i)$ and given by $F(\sigma, i) = \{k \in N \mid \sigma(k) > \sigma(i)\}$. The *starting time* of player $i \in N$ with respect to σ is denoted by $t_i(\sigma)$ and given by $t_i(\sigma) = \sum_{k \in P(\sigma,i)} p_k$. Similarly, the starting time of a group of players $I \subseteq N$ is denoted by $t_I(\sigma)$ and given by

$$t_I(\sigma) = \min_{i \in I} \{t_i(\sigma)\}.$$

Furthermore, the time player $i \in N$ spends in the system when the players follow the order σ is called the *completion time*, denoted by $C_i(\sigma)$ and given by $C_i(\sigma) = t_i(\sigma) + p_i$. The *total costs* of the order σ are denoted by $TC(\sigma)$ and given by

$$TC(\sigma) = \sum_{i \in N} c_i(C_i(\sigma)).$$

An order for which the total costs are minimized is called an *optimal order* and denoted by $\hat{\sigma}$, that is, $TC(\hat{\sigma}) \leq TC(\sigma)$ for all $\sigma \in \Pi(N)$. Given an optimal order $\hat{\sigma} \in \Pi(N)$, the *set of misplacements* contains all pairs of players that need to be interchanged in order to reach the optimal order from the initial order:

$$MP(\sigma_0, \hat{\sigma}) = \{(i, j) \in N \times N \mid \sigma_0(i) < \sigma_0(j) \text{ and } \hat{\sigma}(i) > \hat{\sigma}(j)\}.$$

In order to formally define the notion of a path from the initial order to an optimal order, as introduced by Saavedra-Nieves et al. (2020), we first define a *neighbor switch* associated to two orders $\sigma, \sigma' \in \Pi(N)$ as a pair of players $(i, j) \in N \times N$ for which it holds that $\sigma(j) = \sigma(i) + 1$ and $\sigma(i) = \sigma'(j), \sigma(j) = \sigma'(i)$ and $\sigma(k) = \sigma'(k)$ for all $k \in N \setminus \{i, j\}$. Now, we adopt the notion of a *path* from the initial order σ_0 to an optimal order $\hat{\sigma}$ as a sequence of orders $(\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_m)$ with $\sigma_m = \hat{\sigma}$ corresponding to neighbor switches (i_k, j_k), for every $k \in \{1, 2, \ldots, m\}$, associated to orders σ_{k-1} and σ_k such that there does not exist $k, \ell \in \{1, 2, \ldots, m\}$, $k \neq \ell$, such that $i_k = j_\ell$ and $j_k = i_\ell$. In other words, a path from the initial order to an optimal order consecutively interchanges misplacements and interchanges a particular misplacement exactly once. Hence, $m = |MP(\sigma_0, \hat{\sigma})|$.

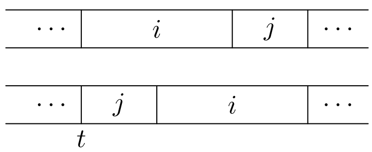

Figure 1 – *Interchanging players i and j, leading to the neighbor switching gain $g_{ij}(t)$.*

The players can jointly obtain cost savings by following a path from the initial order to an optimal order. More specifically, the *neighbor switching gain* of a neighbor switch $(i, j) \in N \times N$

at time $t \in [0, \infty)$ (see Figure 1) is defined by

$$g_{ij}(t) = c_i(t + p_i) + c_j(t + p_i + p_j) - c_i(t + p_i + p_j) - c_j(t + p_j). \quad (1)$$

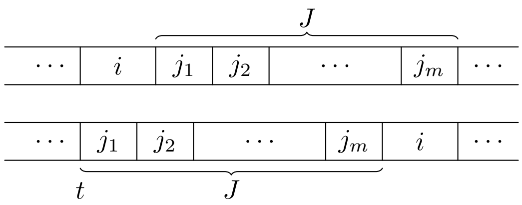

Figure 2 – *Interchanging player i with a group of players $\mathcal{J}$, leading to the gain $g_{iJ}(t)$.*

For notational convenience, we also define the consecutive neighbor switching gains of player $i \in N$ and a group $J \subseteq N$ at time $t \in [0, \infty)$ with player i directly in front of the group $J = \{j_1, \ldots, j_m\}$, which are ordered consecutively (see also Figure 2), denoted by $g_{iJ}(t)$ and defined by

$$g_{iJ}(t) = g_{ij_1}(t) + g_{ij_2}(t + p_{j_1}) + \ldots + g_{ij_m}(t + p_{j_1} + \ldots + p_{j_{m-1}}). \quad (2)$$

Similarly, we define the consecutive neighbor switching gains of player $j \in N$ and a group $I \subseteq N$ at time $t \in [0, \infty)$ with player j directly behind the group $I = \{i_1, \ldots, i_m\}$, which are ordered consecutively (see also Figure 3), denoted by $g_{Ij}(t)$ and defined by

$$g_{Ij}(t) = g_{i_m j}(t + p_{i_1} + \ldots + p_{i_{m-1}}) + g_{i_{m-1} j}(t + p_{i_1} + \ldots + p_{i_{m-2}}) + \ldots + g_{i_2 j}(t + p_{i_1}) + g_{i_1 j}(t). \quad (3)$$

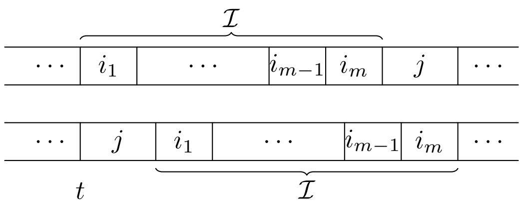

Figure 3 – *Interchanging player j with a group of players $\mathcal{I}$, leading to the gain $g_{Ij}(t)$.*

Obviously, for a path $(\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_m)$ with $\sigma_m = \hat{\sigma}$ from the initial order to an optimal order corresponding to neighbor switches $(i_1, j_1), (i_2, j_2), \ldots, (i_m, j_m)$, the total cost savings can be expressed in terms of the neighbor switching gains:

$$TC(\sigma_0) - TC(\hat{\sigma}) = \sum_{k=1}^{m} g_{i_k j_k}(t_{i_k}(\sigma_{k-1})).$$

How to allocate these joint total cost savings among the players is the central question in cooperative game theory. A *cooperative game* with transferable utility is represented by a tuple (N, v), where $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$ is the *characteristic function* which assigns to every *coalition* $S \in 2^N$ the *worth* of the coalition. Here, 2^N denotes the set of all subsets of N. The set of all cooperative games on N is denoted by TU^N and a cooperative game (N, v) is also denoted by $v \in TU^N$.

In this paper, we focus on two distinguished concepts within cooperative game theory: the core and convexity. Let $v \in TU^N$ be a cooperative game. Then, the *core* is denoted by $C(v)$ and given by

$$C(v) = \left\{ x \in \mathbb{R}^N \,\middle|\, v(N) = \sum_{i \in N} x_i \text{ and } v(S) \leq \sum_{i \in S} x_i \text{ for all } S \in 2^N \setminus \{\emptyset\} \right\}.$$

Here, the first condition is called *efficiency*, while the latter condition is called *stability*. Moreover, v is *convex* (cf. Shapley, 1971) if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for all $S, T \in 2^N \setminus \{\emptyset\}$.

Let $(\sigma_0, p, c) \in SEQ^N$ be a sequencing situation, $S \in 2^N \setminus \{\emptyset\}$ a coalition and $\sigma \in \Pi(N)$ an order. Then, σ is *admissible* for S with respect to σ_0 if $P(\sigma, i) = P(\sigma_0, i)$ for all $i \in N \setminus S$. In other words, all players outside S are in the same position in both σ and σ_0 and players in S can only interchange with other players in S if all players in between, according to the initial order, are also in S. The set of all admissible orders for S with respect to σ_0 is denoted by $\mathcal{A}(\sigma_0, S)$. We define the associated *sequencing game* $v \in TU^N$ by

$$v(S) = \max_{\sigma \in \mathcal{A}(\sigma_0, S)} \left\{ \sum_{i \in S} c_i(C_i(\sigma_0)) - \sum_{i \in S} c_i(C_i(\sigma)) \right\},$$

for all $S \in 2^N \setminus \{\emptyset\}$, that is, the worth of a coalition is equal to the maximal cost savings the coalition can achieve by admissible rearrangements with respect to σ_0.

An order σ *induces* an order σ_S if $\sigma_S \in \mathcal{A}(\sigma_0, S)$ and $P(\sigma, i) \cap S = P(\sigma_S, i) \cap S$ for all $i \in S$. In other words, all players outside S are in the same position in σ_S compared to σ_0 and all players in S are ordered in σ_S in the same order compared to σ. An *optimal order for* S, denoted by $\hat{\sigma}_S \in \Pi(N)$, is an admissible order for S with respect to σ_0 that minimizes the total costs for S, i.e. $TC(\hat{\sigma}_S) \leq TC(\sigma)$ for all $\sigma \in \mathcal{A}(\sigma_0, S)$. Lemma 2.1 (cf. Saavedra-Nieves et al., 2020) combines these two concepts and provides two conditions on the neighbor switching gains such that it is guaranteed that we can obtain optimal orders for coalitions from an optimal order for the grand coalition.

Lemma 2.1 [Saavedra-Nieves et al. (2020)] *Let $(\sigma_0, p, c) \in SEQ^N$ be a sequencing situation and let $\hat{\sigma} \in \Pi(N)$ be an optimal order. If*

1) *for all $t \in [0, \infty)$, $g_{ij}(t) \geq 0$ for all $(i,j) \in MP(\sigma_0, \hat{\sigma})$;*

2) *for all $t \in [0, \infty)$, $g_{ij}(t) \leq 0$ for all $(i,j) \notin MP(\sigma_0, \hat{\sigma})$,*

then, for every $S \in 2^N \setminus \{\emptyset\}$, the induced order $\hat{\sigma}_S$ is optimal for S.

Finally, S is *connected* with respect to σ if for all $i, j \in S$ and $k \in N$ for which $\sigma(i) < \sigma(k) < \sigma(j)$, it holds that $k \in S$. Following Borm et al. (2002), we use the following notation for some special connected coalitions:

$$\begin{aligned}
(i,j)_\sigma &= \{k \in N \mid \sigma(i) < \sigma(k) < \sigma(j)\}\,; \\
(i,j]_\sigma &= \{k \in N \mid \sigma(i) < \sigma(k) \leq \sigma(j)\}\,; \\
[i,j)_\sigma &= \{k \in N \mid \sigma(i) \leq \sigma(k) < \sigma(j)\}\,; \\
[i,j]_\sigma &= \{k \in N \mid \sigma(i) \leq \sigma(k) \leq \sigma(j)\}\,,
\end{aligned}$$

where $i, j \in N$ are two players such that $\sigma(i) < \sigma(j)$. Also, we will benefit from their following result:

Proposition 2.1 [Borm et al. (2002)] *Let $(\sigma_0, p, c) \in SEQ^N$ be a sequencing situation. Then, v is convex if and only if for all $i, j \in N$ such that $\sigma_0(i) < \sigma_0(j)$,*

$$v([i,j]_{\sigma_0}) - v([i,j)_{\sigma_0}) - v((i,j]_{\sigma_0}) + v((i,j)_{\sigma_0}) \geq 0.$$

Proposition 2.1 shows that in order to prove convexity, one only has to consider the special type of connected coalitions.

3 Sequencing situations with non-linear cost functions

This section is devoted to sequencing situations with non-linear cost functions. Traditionally, the focus is on *standard sequencing situations*, in which the cost functions are linear: $c_i(t) = \alpha_i t$ for all $t \in [0, \infty)$, where $\alpha_i \in \mathbb{R}_{++}$ is the *linear cost coefficient* of player $i \in N$. The set of all standard sequencing situations is denoted by $SSEQ^N$. For these standard sequencing situations, Smith (1956) showed that an optimal order can be reached by arranging the players according to weakly decreasing urgency, defined as the ratio of the linear cost coefficient and the processing time. The neighbor switching gains can be expressed in terms of the linear cost coefficients and processing times only:

$$g_{ij}(t) = \alpha_j p_i - \alpha_i p_j, \tag{4}$$

where $i, j \in N$ are two neighbors at time $t \in [0, \infty)$. Note that the neighbor switching gains are not time-dependent.

Recently, Saavedra-Nieves et al. (2020) studied *exponential sequencing situations*, in which the cost functions are exponential: $c_i(t) = e^{\alpha_i t}$ for all $t \in [o, \infty)$, where $\alpha_i \in \mathbb{R}_{++}$ is called the *exponential cost coefficient* of player $i \in N$. The set of all exponential sequencing situations is denoted by $ESEQ^N$. For an exponential sequencing situation $(\sigma_0, p, c) \in ESEQ^N$, the neighbor switching gain of two consecutive players $i, j \in N$ at time $t \in [0, \infty)$ is given by (cf. Saavedra-Nieves et al., 2020)

$$g_{ij}(t) = e^{\alpha_i(t+p_i)} + e^{\alpha_j(t+p_i+p_j)} - e^{\alpha_i(t+p_j+p_i)} - e^{\alpha_j(t+p_j)}. \tag{5}$$

For sequencing situations with an exponential cost function, Saavedra-Nieves et al. (2020) defined three subclasses that allow for a comparison index for determining an optimal order, like the urgency index, which is only based on the processing times and exponential cost coefficients. The following proposition summarizes the main results with regard to the neighbor switching gains for each of these three subclasses. Note that both statement 1) and 2) of Proposition 3.1 also hold for a standard sequencing situation.

Proposition 3.1 [Saavedra-Nieves et al. (2020)] *Let $(\sigma_0, p, c) \in ESEQ^N$ be an exponential sequencing situation such that one of the following three cases holds:*

i) there is an $\alpha \in \mathbb{R}_{++}$ such that, for all $i \in N$ and all $t \in [0, \infty)$, $c_i(t) = e^{\alpha t}$;

ii) there is a $p \in \mathbb{R}_{++}$ such that, for all $i \in N$, $p_i = p$;

iii) there are $\alpha_L, \alpha_H, p_L, p_H \in \mathbb{R}_{++}$ with $\alpha_L < \alpha_H$ and $p_L < p_H$ such that, for all $i \in N$, $\alpha_i \in \{\alpha_L, \alpha_H\}, p_i \in \{p_L, p_H\}$ and

$$e^{\alpha_H p_H} - e^{\alpha_L p_L} \leq e^{\alpha_H(p_L+p_H)} - e^{\alpha_L(p_L+p_H)}.$$

Let $\hat{\sigma} \in \Pi(N)$ be an optimal order. Then,

1) *$g_{ij}(t) \geq 0$ for all $t \in [0, \infty)$ and $g_{ij}(s) \leq g_{ij}(t)$ for all $s, t \in [0, \infty)$ with $s \leq t$, if $(i, j) \in MP(\sigma_0, \hat{\sigma})$;*

2) *$g_{ij}(t) \leq 0$ for all $t \in [0, \infty)$ and $g_{ij}(s) \geq g_{ij}(t)$ for all $s, t \in [0, \infty)$ with $s \leq t$, if $(i, j) \notin MP(\sigma_0, \hat{\sigma})$.*

Furthermore, Rothkopf (1966) studied *discounting sequencing situations*, in which the cost function of player $i \in N$ is given by $c_i(t) = \alpha_i(1 - e^{-rt})$. Here, $r \in \mathbb{R}_{++}$ denotes the *discount rate* and $\alpha_i \in \mathbb{R}_{++}$ the *discounting cost coefficient* of player $i \in N$. The set of all discounting sequencing situations with player set N is denoted by $DSEQ^N$.

For a discounting sequencing situation $(\sigma_0, p, c) \in DSEQ^N$, the neighbor switching gain of two consecutive players $i, j \in N$ at time $t \in [0, \infty)$ is, using Equation (1), given by (cf. Rothkopf, 1966)

$$\begin{aligned} g_{ij}(t) &= c_i(t + p_i) + c_j(t + p_i + p_j) - c_i(t + p_i + p_j) - c_j(t + p_j) \\ &= \alpha_i - \alpha_i e^{-r(t+p_i)} + \alpha_j - \alpha_j e^{-r(t+p_i+p_j)} - \alpha_i + \alpha_i e^{-r(t+p_i+p_j)} - \alpha_j + \alpha_j e^{-r(t+p_j)} \\ &= \alpha_i e^{-r(t+p_i+p_j)} + \alpha_j e^{-r(t+p_j)} - \alpha_i e^{-r(t+p_i)} - \alpha_j e^{-r(t+p_i+p_j)}. \end{aligned} \tag{6}$$

Rothkopf (1966) showed that (by using Equation (6), this can be readily verified), for a discounting sequencing situation $(\sigma_0, p, c) \in DSEQ^N$ and an order $\hat{\sigma} \in \Pi(N)$ it holds that $\hat{\sigma}$ is optimal if and only if

$$\text{for all } i, j \in N: \qquad \frac{\alpha_j e^{-rp_j}}{1 - e^{-rp_j}} < \frac{\alpha_i e^{-rp_i}}{1 - e^{-rp_i}} \Rightarrow \hat{\sigma}(i) < \hat{\sigma}(j). \tag{7}$$

Equation (7) can be used to define the pairs of players that should be interchanged and hence, are in the set of misplacements. For a discounting sequencing situation, all neighbor switching gains corresponding to misplacements are non-*negative* and non-*in*creasing, while the neighbor switching gains corresponding to non-misplacements are non-*positive* and non-*de*creasing. This partly contrasts Proposition 3.1, in which the neighbor switching gains corresponding to misplacements are non-*de*creasing and the gains corresponding to non-misplacements non-*in*creasing.

Proposition 3.2 *Let $(\sigma_0, p, c) \in DSEQ^N$ be a discounting sequencing situation. Let $i, j \in N$ be such that $\sigma_0(i) < \sigma_0(j)$ and let $\hat{\sigma} \in \Pi(N)$ be an optimal order. Then,*

1) *$g_{ij}(t) \geq 0$ for all $t \in [0, \infty)$ and $g_{ij}(s) \geq g_{ij}(t)$ for all $s, t \in [0, \infty)$ with $s \leq t$, if $(i, j) \in MP(\sigma_0, \hat{\sigma})$;*

2) *$g_{ij}(t) \leq 0$ for all $t \in [0, \infty)$ and $g_{ij}(s) \leq g_{ij}(t)$ for all $s, t \in [0, \infty)$ with $s \leq t$, if $(i, j) \notin MP(\sigma_0, \hat{\sigma})$.*

Proof: First, from Equation (6), it readily follows that $g'_{ij}(t) = -rg_{ij}(t)$ for all $t \in [0, \infty)$.

1) If $(i, j) \in MP(\sigma_0, \hat{\sigma})$, then,

$$\frac{\alpha_i e^{-rp_i}}{1 - e^{-rp_i}} \leq \frac{\alpha_j e^{-rp_j}}{1 - e^{-rp_j}}.$$

Hence,

$$\alpha_i e^{-rp_i} - \alpha_i e^{-r(p_i+p_j)} \leq \alpha_j e^{-rp_j} - \alpha_j e^{-r(p_i+p_j)},$$

and consequently, $g_{ij}(t) \geq 0$ for all $t \in [0, \infty)$. Moreover, since $r > 0$, $g'_{ij}(t) \leq 0$ for all $t \in [0, \infty)$, which implies that $g_{ij}(s) \geq g_{ij}(t)$ for all $s, t \in [0, \infty)$ with $s \leq t$.

2) If $(i, j) \notin MP(\sigma_0, \hat{\sigma})$, then,

$$\frac{\alpha_i e^{-rp_i}}{1 - e^{-rp_i}} \geq \frac{\alpha_j e^{-rp_j}}{1 - e^{-rp_j}}.$$

Hence,

$$\alpha_i e^{-rp_i} - \alpha_i e^{-r(p_i + p_j} \geq \alpha_j e^{-rp_j} - \alpha_j e^{-r(p_i + p_j)},$$

and consequently, $g_{ij}(t) \leq 0$ for all $t \in [0, \infty)$. Moreover, since $r > 0$, $g'_{ij}(t) \geq 0$ for all $t \in [0, \infty)$, which implies that $g_{ij}(s) \leq g_{ij}(t)$ for all $s, t \in [0, \infty)$ with $s \leq t$. □

Finally, in this paper we introduce the notion of a *logarithmic sequencing situation* which deals with logarithmic cost functions: $c_i(t) = \ln(\alpha_i t)$ for all $t \in [o, \infty)$, where α_i is called the *logarithmic cost coefficient* of player $i \in N$ and is such that $\ln(\alpha_i t) > 0$ for all $t > p_i$. The set of all logarithmic sequencing situations with player set N is denoted by $LSEQ^N$.

For a logarithmic sequencing situation $(\sigma_0, p, c) \in LSEQ^N$, the neighbor switching gain of two consecutive players $i, j \in N$ at time $t \in [0, \infty)$ is, using Equation (1), given by

$$\begin{aligned} g_{ij}(t) &= c_i(t + p_i) + c_j(t + p_i + p_j) - c_i(t + p_i + p_j) - c_j(t + p_j) \\ &= \ln(\alpha_i(t + p_i)) + \ln(\alpha_j(t + p_i + p_j)) - \ln(\alpha_i(t + p_j + p_i)) - \ln(\alpha_j(t + p_j)) \\ &= \ln(t + p_i) + \ln(t + p_i + p_j) - \ln(t + p_j + p_i) - \ln(t + p_j) \\ &= \ln(t + p_i) - \ln(t + p_j) = \ln\left(\frac{t + p_i}{t + p_j}\right). \end{aligned} \tag{8}$$

Equation (8) shows that the neighbor switching gains for a logarithmic sequencing situation are time-dependent. Moreover, the logarithmic cost coefficients turn out to be irrelevant from an optimization perspective. Even more, the total costs consist of a fixed part of the sum of the logarithmic cost coefficients and another part that is dependent on the order. Formally, for a logarithmic sequencing situation $(\sigma_0, p, c) \in LSEQ^N$ and an order $\sigma \in \Pi(N)$, we have that

$$TC(\sigma) = \sum_{i \in N} c_i(C_i(\sigma)) = \sum_{i \in N} \ln(\alpha_i C_i(\sigma)) = \sum_{i \in N} \ln(\alpha_i) + \sum_{i \in N} \ln(C_i(\sigma)). \tag{9}$$

Consequently, an optimal order can be determined by considering the processing times only. The following theorem shows that in an optimal order, the players are arranged according to weakly increasing processing times. This is also known as the so-called Shortest Processing Time first (SPT) rule. The structure of the proof is due to Saavedra-Nieves et al. (2020).

Theorem 3.1 *Let $(\sigma_0, p, c) \in LSEQ^N$ be a logarithmic sequencing situation and let $\hat{\sigma} \in \Pi(N)$ be an order. Then it holds that $\hat{\sigma}$ is optimal if and only if*

$$\textit{for all } i, j \in N: \qquad p_i < p_j \Rightarrow \hat{\sigma}(i) < \hat{\sigma}(j). \tag{10}$$

Proof: First, we prove that, if $\hat{\sigma}$ is an optimal order, then it satisfies Equation (10). For this, assume that $\hat{\sigma}$ is an optimal order and suppose for the sake of contradiction that $\hat{\sigma}$ does not satisfy Equation (10). Then there are $i, j \in N$ such that $p_i < p_j$, while $\hat{\sigma}(i) > \hat{\sigma}(j)$. W.l.o.g. we can assume that $\hat{\sigma}(i) = \hat{\sigma}(j) + 1$, i.e., that players i and j are neighbors in $\hat{\sigma}$. We show that

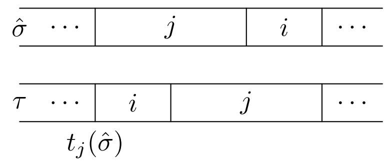

Figure 4 – *Interchanging players i and j from $\hat{\sigma}$ to τ.*

interchanging players i and j lead to an order for which the total costs are less than the total costs of $\hat{\sigma}$.

Define this order $\tau \in \Pi(N)$ as follows: $\tau(k) = \hat{\sigma}(k)$ for all $k \in N \setminus \{i, j\}$, $\tau(i) = \hat{\sigma}(j)$ and $\tau(j) = \hat{\sigma}(i)$ (see also Figure 4). Then it holds that,

$$\begin{aligned} TC(\hat{\sigma}) - TC(\tau) &= \sum_{k \in N} \ln(\alpha_k C_k(\hat{\sigma})) - \sum_{k \in N} \ln(\alpha_k C_k(\tau)) \\ &= \ln(\alpha_j C_j(\hat{\sigma})) + \ln(\alpha_i C_i(\hat{\sigma})) - \ln(\alpha_i C_i(\tau)) - \ln(\alpha_j C_j(\tau)) \\ &= \ln(\alpha_j(t_j(\hat{\sigma}) + p_j)) + \ln(\alpha_i(t_j(\hat{\sigma}) + p_i + p_j)) \\ &\qquad - \ln(\alpha_i(t_j(\hat{\sigma}) + p_i)) - \ln(\alpha_j(t_j(\hat{\sigma}) + p_i + p_j)) \\ &= \ln(t_j(\hat{\sigma}) + p_j) - \ln(t_j(\hat{\sigma}) + p_i) > 0. \end{aligned}$$

Here, the second equality follows from the fact that $C_k(\hat{\sigma}) = C_k(\tau)$ for all $k \in N \setminus \{i, j\}$ and the strict inequality from the fact that $\ln(t + p_j) - \ln(t + p_i) > 0$ for all $t \in [0, \infty)$, since $p_i < p_j$. Hence, $TC(\hat{\sigma}) > TC(\tau)$.

Secondly, we prove that if two orders both satisfy Equation (10), then both orders yield the same total costs. For this, consider two different orders $\sigma, \sigma' \in \Pi(N), \sigma \neq \sigma'$ that both satisfy Equation (10). It is readily seen that the only differences between the two orders can be within a block of players with identical processing times. Hence, using Equation (9), it follows that $TC(\sigma) = TC(\sigma')$. □

Using Theorem 3.1, it is easily seen that, in order to reach an optimal order from the initial order, at least the pairs of players $i, j \in N$ for which it holds that $\sigma_0(i) < \sigma_0(j)$ and $p_i > p_j$ should be interchanged. Obviously, other optimal orders can be obtained by interchanging, besides the necessarily ones, even more pairs of players with identical processing times. Similar to Proposition 3.2, we show that all neighbor switching gains corresponding to misplacements are non-*negative* and non-*in*creasing, while the neighbor switching gains are non-*positive* and non-*de*creasing for non-misplacements.

Proposition 3.3 *Let $(\sigma_0, p, c) \in LSEQ^N$ be a logarithmic sequencing situation. Let $i, j \in N$ be such that $\sigma_0(i) < \sigma_0(j)$ and let $\hat{\sigma} \in \Pi(N)$ be an optimal order. Then,*

1) $g_{ij}(t) \geq 0$ for all $t \in [0, \infty)$ and $g_{ij}(s) \geq g_{ij}(t)$ for all $s, t \in [0, \infty)$ with $s \leq t$, if $(i, j) \in MP(\sigma_0, \hat{\sigma})$;

2) $g_{ij}(t) \leq 0$ for all $t \in [0, \infty)$ and $g_{ij}(s) \leq g_{ij}(t)$ for all $s, t \in [0, \infty)$ with $s \leq t$, if $(i, j) \notin MP(\sigma_0, \hat{\sigma})$.

Proof: First, we have that $g_{ij}(t) = \ln(t + p_i) - \ln(t + p_j)$ and $g'_{ij}(t) = \frac{1}{t+p_i} - \frac{1}{t+p_j}$ for all $t \in [0, \infty)$.

1) If $(i,j) \in MP(\sigma_0, \hat{\sigma})$, then, $p_i \geq p_j$. Hence, $\ln(t+p_i) \geq \ln(t+p_j)$ and $\frac{1}{t+p_i} \leq \frac{1}{t+p_j}$. Consequently, $g_{ij}(t) \geq 0$ and $g'_{ij}(t) \leq 0$ for all $t \in [0, \infty)$. The latter implies that $g_{ij}(s) \geq g_{ij}(t)$ for all $s, t \in [0, \infty)$ with $s \leq t$.

2) If $(i,j) \notin MP(\sigma_0, \hat{\sigma})$, then, $p_i \leq p_j$. Hence, $\ln(t+p_i) \leq \ln(t+p_j)$ and $\frac{1}{t+p_i} \geq \frac{1}{t+p_j}$. Consequently, $g_{ij}(t) \leq 0$ and $g'_{ij}(t) \geq 0$ for all $t \in [0, \infty)$. The latter implies that $g_{ij}(s) \leq g_{ij}(t)$ for all $s, t \in [0, \infty)$ with $s \leq t$. □

4 Convexity

Convexity is an appealing property for a cooperative game, because for a convex cooperative game, the marginal costs of joining a coalition are lower if the coalition is larger. This provides a definite drive to cooperate. Moreover, the core of a convex game is equal to the convex hull of all marginal vectors (cf. Shapley, 1971 and Ichiishi, 1981) and hence, the Shapley value (cf. Shapley, 1953) is the barycentre of the core.

In this section, we study the sequencing games that are associated to sequencing situations with non-linear cost functions. In particular, following the lines of Saavedra-Nieves et al. (2020), the associated sequencing game of a standard sequencing situation (with a linear cost function) is called a *standard sequencing game*, the associated game of an exponential sequencing situation (with an exponential cost function) is called an *exponential sequencing game*, the one associated with a discounting sequencing situation is called a *discounting sequencing game* and finally, the associated game of a logarithmic sequencing situation is called a *logarithmic sequencing game.*

For standard sequencing situations, Curiel et al. (1989) showed that the associated standard sequencing games are convex. For exponential sequencing situations, Saavedra-Nieves et al. (2020) provided three subclasses (see also Proposition 3.1) that yield a convex exponential sequencing game. The proof of this result is based on a general result for a sequencing situation with an arbitrary non-linear cost function, which provides three conditions on the neighbor switching gains to guarantee convexity for the associated exponential sequencing games:

Theorem 4.1 [Saavedra-Nieves et al. (2020)] *Let $(\sigma_0, p, c) \in SEQ^N$ be a sequencing situation, $\hat{\sigma} \in \Pi(N)$ an optimal order and $v \in TU^N$ the associated sequencing game. If*

i) for all $t \in [0, \infty)$, $g_{ij}(t) \geq 0$ for all $(i,j) \in MP(\sigma_0, \hat{\sigma})$;

ii) for all $t \in [0, \infty)$, $g_{ij}(t) \leq 0$ for all $(i,j) \notin MP(\sigma_0, \hat{\sigma})$;

iii) for all $s, t \in [0, \infty)$ with $s \leq t$, $g_{ij}(s) \leq g_{ij}(t)$ for all $(i,j) \in MP(\sigma_0, \hat{\sigma})$,

then, v is convex.

Combining Theorem 4.1 and Proposition 3.1, the following result immediately follows.

Corollary 4.1 [Saavedra-Nieves et al. (2020)] *Let $(\sigma_0, p, c) \in ESEQ^N$ be an exponential sequencing situation such that one of the following three cases holds:*

i) there is an $\alpha \in \mathbb{R}_{++}$ such that, for all $i \in N$ and all $t \in [0, \infty)$, $c_i(t) = e^{\alpha t}$;

ii) there is a $p \in \mathbb{R}_{++}$ such that, for all $i \in N$, $p_i = p$;

iii) there are $\alpha_L, \alpha_H, p_L, p_H \in \mathbb{R}_{++}$ with $\alpha_L < \alpha_H$ and $p_L < p_H$ such that, for all $i \in N$, $\alpha_i \in \{\alpha_L, \alpha_H\}, p_i \in \{p_L, p_H\}$ and

$$e^{\alpha_H p_H} - e^{\alpha_L p_L} \leq e^{\alpha_H (p_L + p_H)} - e^{\alpha_L (p_L + p_H)}.$$

Let $v \in TU^N$ be the associated exponential sequencing game. Then, v is convex.

In Theorem 4.1, convexity for the associated game is guaranteed if the neighbor switching gains of the sequencing situation are non-*negative* and non-*de*creasing for misplaced pairs of players and non-positive for non-misplaced pairs of players. Next, we show that it also suffices to require that the neighbor switching gains are non-*in*creasing for misplacements, together with non-*negativity* for misplacements and non-*positivity* for non-misplacements. The proof of this result follows the same structure as the proof of Theorem 4.1 from Saavedra-Nieves et al. (2020).

Theorem 4.2 *Let $(\sigma_0, p, c) \in SEQ^N$ be a sequencing situation, $\hat{\sigma} \in \Pi(N)$ an optimal order and $v \in TU^N$ the associated sequencing game. If*

i) for all $t \in [0,\infty)$, $g_{ij}(t) \geq 0$ for all $(i,j) \in MP(\sigma_0, \hat{\sigma})$;

ii) for all $t \in [0,\infty)$, $g_{ij}(t) \leq 0$ for all $(i,j) \notin MP(\sigma_0, \hat{\sigma})$;

iii) for all $s,t \in [0,\infty)$ with $s \leq t$, $g_{ij}(s) \geq g_{ij}(t)$ for all $(i,j) \in MP(\sigma_0, \hat{\sigma})$,

then, v is convex.

Proof: To prove convexity, we show that

$$v([i,j]_{\sigma_0}) - v([i,j)_{\sigma_0}) \geq v((i,j]_{\sigma_0}) - v((i,j)_{\sigma_0}),$$

for all $i,j \in N$ with $\sigma_0(i) < \sigma_0(j)$, which is sufficient according to Proposition 2.1. Let $i,j \in N$ with $\sigma_0(i) < \sigma_0(j)$ (see Figure 5).

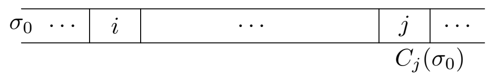

Figure 5 – *Players $i,j \in N$ in the initial order σ_0.*

According to Lemma 2.1, the induced orders $\hat{\sigma}_{[i,j]_{\sigma_0}}, \hat{\sigma}_{[i,j)_{\sigma_0}}, \hat{\sigma}_{(i,j]_{\sigma_0}}$ and $\hat{\sigma}_{(i,j)_{\sigma_0}}$ are optimal for $[i,j]_{\sigma_0}, [i,j)_{\sigma_0}, (i,j]_{\sigma_0}$ and $(i,j)_{\sigma_0}$, respectively. First, consider the case where $\hat{\sigma}(i) < \hat{\sigma}(j)$.

Figure 6 – *Schematic overview of the first case.*

In Figure 6, the order of the relevant players is shown for the different induced orders. Here, $\mathcal{J} \subseteq N$ is defined as the set of players that have to switch with player j:

$$\mathcal{J} = \{k \in (i,j)_{\sigma_0} \mid (k,j) \in MP(\sigma_0, \hat{\sigma})\}.$$

Hence,

$$v([i,j]_{\sigma_0}) - v([i,j)_{\sigma_0}) = g_{Jj}(C_j(\sigma_0) - p_j - \sum_{k \in \mathcal{J}} p_k)$$
$$= v((i,j]_{\sigma_0}) - v((i,j)_{\sigma_0}),$$

where $g_{Jj}(C_j(\sigma_0) - p_j - \sum_{k \in \mathcal{J}} p_k)$ is the consecutive neighbor switching gain as defined in Equation (2).

Secondly, consider the case where $\hat{\sigma}(i) > \hat{\sigma}(j)$. Figure 7 provides the order of the relevant players for the different induced orders. Here, $\mathcal{I}, \mathcal{J} \subseteq N$ are defined as follows:

$$\mathcal{I} = \{\ell \in (i,j)_{\sigma_0} \mid \hat{\sigma}(j) < \hat{\sigma}(\ell) < \hat{\sigma}(i)\};$$
$$\mathcal{J} = \{k \in (i,j)_{\sigma_0} \mid \hat{\sigma}(i) < \hat{\sigma}(k)\}.$$

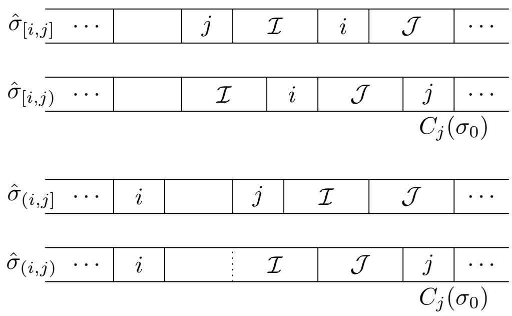

Figure 7 – *Schematic overview of the first case.*

Hence,

$$v([i,j]_{\sigma_0}) - v([i,j)_{\sigma_0}) = g_{Jj}(C_j(\sigma_0) - p_j - \sum_{k \in \mathcal{J}} p_k)$$
$$+ g_{ij}(C_j(\sigma_0) - p_i - p_j - \sum_{k \in \mathcal{J}} p_k)$$
$$+ g_{Ij}(C_j(\sigma_0) - p_i - p_j - \sum_{\ell \in \mathcal{I}} p_\ell - \sum_{k \in \mathcal{J}} p_k),$$

and

$$v((i,j]_{\sigma_0}) - v((i,j)_{\sigma_0}) = g_{Jj}(C_j(\sigma_0) - p_j - \sum_{k \in \mathcal{J}} p_k)$$
$$+ g_{Ij}(C_j(\sigma_0) - p_j - \sum_{\ell \in \mathcal{I}} p_\ell - \sum_{k \in \mathcal{J}} p_k).$$

First, note that $g_{ij}(C_j(\sigma_0) - p_i - p_j - \sum_{k \in \mathcal{J}} p_k) \geq 0$, since $(i,j) \in MP(\sigma_0, \hat{\sigma})$. Secondly,

$$C_j(\sigma_0) - p_i - p_j - \sum_{\ell \in \mathcal{I}} p_\ell - \sum_{k \in \mathcal{J}} p_k < C_j(\sigma_0) - p_j - \sum_{\ell \in \mathcal{I}} p_\ell - \sum_{k \in \mathcal{J}} p_k,$$

since $p_i > 0$. Using condition *iii)*, we then have that

$$g_{Ij}(C_j(\sigma_0) - p_i - p_j - \sum_{\ell \in \mathcal{I}} p_\ell - \sum_{k \in \mathcal{J}} p_k) \geq g_{Ij}(C_j(\sigma_0) - p_j - \sum_{\ell \in \mathcal{I}} p_\ell - \sum_{k \in \mathcal{J}} p_k),$$

and hence,

$$v([i,j]_{\sigma_0}) - v([i,j)_{\sigma_0}) \geq v((i,j]_{\sigma_0}) - v((i,j)_{\sigma_0}). \qquad \square$$

As a direct consequence of Theorem 4.2, we have that, using Propositions 3.2 and 3.3, any discounting sequencing game and any logarithmic sequencing game is convex.

Corollary 4.2 *The two following statements hold:*

1) Let $(\sigma_0, p, c) \in DSEQ^N$ be a discounting sequencing situation and let $v \in TU^N$ be the associated discounting sequencing game. Then, v is convex.

2) Let $(\sigma_0, p, c) \in LSEQ^N$ be a logarithmic sequencing situation and let $v \in TU^N$ be the associated logarithmic sequencing game. Then, v is convex.

It is worth recalling that the Shapley value of a convex cooperative game is the barycentre of the core. Although this result is very positive, the computation of the Shapley value may become a difficult task. For this reason, we are aiming to find simpler alternative procedures for sharing the cost savings.

5 Cost savings allocation rules

In this section, we introduce two specific allocation rules that can be directly computed from the sequencing situation itself. Both rules use the ideas of the Equal Gain Splitting Rule for standard sequencing problems, now applied in the more general setting. We are also interested in the game-theoretical properties that they satisfy. In particular, we study if the allocations we obtain are stable, in the sense that they belong to the core of the associated cooperative game, for the sequencing situations analyzed in this paper.

For standard sequencing situations, the *Equal Gain Splitting Rule (EGS-rule)* (cf. Curiel et al., 1989) provides such an allocation. This means that, for $(\sigma_0, p, c) \in SSEQ^N$, we have that $EGS(\sigma_0, p, c) \in C(v)$, where v denotes the associated standard sequencing game, i.e.,

$$v(S) \leq \sum_{i \in S} EGS_i(\sigma_0, p, c),$$

for all $S \in 2^N \setminus \{\emptyset\}$ and

$$v(N) = \sum_{i \in N} EGS_i(\sigma_0, p, c).$$

Formally, the allocation prescribed by the EGS-rule, which is independent of the choice of an optimal order $\hat{\sigma} \in \Pi(N)$, is given by

$$EGS(\sigma_0, p, c) = \sum_{(i,j) \in MP(\sigma_0, \hat{\sigma})} \frac{1}{2}(\alpha_j p_i - \alpha_j p_i) \mathbb{1}_{\{i,j\}},$$

where $\mathbb{1}_{\{i,j\}} \in \mathbb{R}^N$ is such that, for all $k \in N$,

$$(\mathbb{1}_{\{i,j\}})_k = \begin{cases} 1, & \text{if } k = i \text{ or } k = j; \\ 0, & \text{otherwise.} \end{cases}$$

We have the following expression for the EGS-rule, according to Curiel, Potters, Prasad, Tijs, and Veltman (1993) and Curiel, Potters, Prasad, Tijs, and Veltman (1994):

$$EGS_i(\sigma_0, p, c) = \frac{1}{2}\Big(v(P(\sigma_0, i) \cup \{i\}) - v(P(\sigma_0, i))\Big) + \frac{1}{2}\Big(v(F(\sigma_0, i) \cup \{i\}) - v(F(\sigma_0, i))\Big).$$

Hence, the EGS-rule can be expressed by using the marginal contributions of a player when the players are joining the coalition according to the initial order and by using the marginal contributions of a player when the players are joining the coalition according the reversed initial order.

Furthermore, since for a standard sequencing situation it holds that $g_{ij}(t) = \alpha_j p_i - \alpha_i p_j$ for every misplacement $(i, j) \in MP(\sigma_0, \hat{\sigma})$ at time $t \in [0, \infty)$, the EGS-rule thus divides the neighbor switching gain for every misplaced pair of players equally between the two players involved. Since the neighbor switching gains are not dependent on the moment in time both players interchange their positions, the EGS-rule does not dependent on the path from the initial order to an optimal order. In other words, every path from the initial order to an optimal order lead to the same neighbor switching gains and hence, to the same allocation.

However, in sequencing situations with non-linear cost functions, the neighbor switching gains may be time-dependent. For example, for sequencing situations with exponential, discounting or logarithmic cost functions, the exact value of the neighbor switching gain of a misplaced pair of players depends on the moment the players interchange their positions. Hence, the neighbor switching gains depend on the path from the initial order to an optimal order. The following example shows this for a logarithmic sequencing situation.

Example 5.1 Let $(\sigma_0, p, c) \in LSEQ^N$ be a logarithmic sequencing situation, where $N = \{1, 2, 3\}$, $\sigma_0 = (1, 2, 3)$, $\alpha_i = 1$ for all $i \in N$, and $p_1 = 4, p_2 = 3$ and $p_3 = 2$. The total costs for all possible orders are given below.

σ	$TC(\sigma)$
$(1, 2, 3)$	5.5294
$(1, 3, 2)$	5.3753
$(2, 1, 3)$	5.2417
$(2, 3, 1)$	4.9053
$(3, 1, 2)$	4.6821
$(3, 2, 1)$	4.4998

Obviously, $\hat{\sigma} = (3, 2, 1)$ is the unique optimal order and $MP(\sigma_0, \hat{\sigma}) = \{(1, 2), (1, 3), (2, 3)\}$. Hence, there are two paths from the initial order to the optimal order that repairs all neighbor misplacements:

$$\sigma_0 = (1, 2, 3) \xrightarrow{g_{23}(p_1)} (1, 3, 2) \xrightarrow{g_{13}(0)} (3, 1, 2) \xrightarrow{g_{12}(p_3)} (3, 2, 1) = \hat{\sigma},$$

corresponding to neighbor switches $(2, 3), (1, 3)$ and $(1, 2)$ respectively, and

$$\sigma_0 = (1, 2, 3) \xrightarrow{g_{12}(0)} (2, 1, 3) \xrightarrow{g_{13}(p_2)} (2, 3, 1) \xrightarrow{g_{23}(0)} (3, 2, 1) = \hat{\sigma},$$

corresponding to neighbor switches $(1, 2), (1, 3)$ and $(2, 3)$ respectively. Below, the values of the neighbor switching gains are given.

$g_{12}(0)$	0.2877
$g_{12}(p_3)$	0.1823
$g_{13}(0)$	0.6932
$g_{13}(p_2)$	0.3365
$g_{23}(0)$	0.4055
$g_{23}(p_1)$	0.1542

It can be clearly seen that the neighbor switching gains depend on the choice of the path from the initial order to the optimal order. △

In order to obtain a reasonable allocation in sequencing situations with non-linear cost functions, we adopt the idea behind the EGS-rule of splitting the neighbor switching gains equally between the two neighbors involved. In addition, we need to specify which path to choose to reach an optimal order from the initial order. Together, this yields an allocation rule of prescribing allocations to sequencing situations with non-linear cost functions.

5.1 Specifying a path

We start out by focusing on the choice of the path from the initial order to an optimal order that repairs all neighbor misplacements. Below, we prescribe two procedures that specify such a path. For the first procedure, named the *Growing Head* procedure, we start with the player that occupies the first position in an optimal order. In the initial order, this player may not be in the first position, but in a different position. The Growing Head procedure starts by consecutively moving this player to the first position, that is, this player consecutively switches with the players in front (according to the initial order) of him, until he reaches the first position. Secondly, we consider the player that is in second position in the optimal order. Again, we consecutively move this player to the second position. We continue this process until all players are positioned in the position according to the optimal order. Formally, the Growing Head procedure is defined as follows:

Procedure 5.1 [Growing Head] *Let $(\sigma_0, p, c) \in SEQ^N$ be a sequencing situation and let $\hat{\sigma} \in \Pi(N)$ be an optimal order.*

Step 1: *For the first step, set $j = \hat{\sigma}^{-1}(1)$, i.e., j is the player that is in the first position according to $\hat{\sigma}$. Consider the path $(\sigma_0, \ldots, \sigma_1)$ corresponding to neighbor switches (i, j) for every $i \in I$, where $I = P(\sigma_0, j)$. Here, $\sigma_1 \in \Pi(N)$ is the order in which $\sigma_1(j) = 1$, $\sigma_1(k) = \sigma_0(k) + 1$ for all $k \in P(\sigma_0, j)$ and $\sigma_1(k) = \sigma_0(k)$ for all $k \in F(\sigma_0, j)$.*

σ_0	I	j	$F(\sigma_0, j)$
σ_1	j	I	$F(\sigma_0, j)$

Figure 8 – *Step 1 of the Growing Head procedure.*

For $\ell > 1$ until $\ell = |N| - 1$, perform the following step:

Step ℓ: *Set $j = \hat{\sigma}^{-1}(\ell)$ and $k = \hat{\sigma}^{-1}(\ell - 1)$, i.e., j is the player that is in the ℓth position according to $\hat{\sigma}$ and k the player that is $\ell - 1$th position. Note that $\sigma_{\ell-1}(k) = \sigma_\ell(k) = \ell - 1$. Consider the path $(\sigma_{\ell-1}, \ldots, \sigma_\ell)$ corresponding to neighbor switches (i, j) for every $i \in I$, where $I = (k, j)_{\sigma_{\ell-1}}$. Here, $\sigma_\ell \in \Pi(N)$ is the order in which $\sigma_\ell(j) = \ell$, $\sigma_\ell(i) = \sigma_{\ell-1}(i) + 1$ for all $i \in I$, $\sigma_\ell(r) = \sigma_{\ell-1}(r)$ for all $r \in F(\sigma_{\ell-1}, j)$ and $\sigma_\ell(r) = \sigma_{\ell-1}(r)$ for all $r \in P(\sigma_{\ell-1}, k) \cup \{k\}$.*

$\sigma_{\ell-1}$			$\cdots$	k	I	j	$F(\sigma_{\ell-1}, j)$
σ_ℓ			$\cdots$	k	j	I	$F(\sigma_{\ell-1}, j)$

Figure 9 – *Step ℓ of the Growing Head procedure.*

◁

The second procedure, named the *Growing Tail* procedure, reverses the idea of the first procedure: instead of starting with the player that is in the first position in the optimal order, we now start with the player that is in the last position. We consecutively move this player to the back, in a similar way as in the Growing Head procedure. Formally, the Growing Tail procedure is defined as follows:

Procedure 5.2 [Growing Tail] *Let $(\sigma_0, p, c) \in SEQ^N$ be a sequencing situation and let $\hat{\sigma} \in \Pi(N)$ be an optimal order.*

Step 1: *For the first step, set $i = \hat{\sigma}^{-1}(|N|)$, i.e., i is the player that is in the last position according to $\hat{\sigma}$. Consider the path $(\sigma_0, \ldots, \sigma_1)$ corresponding to neighbor switches (i, j) for every $j \in J$, where $J = F(\sigma_0, i)$. Here, $\sigma_1 \in \Pi(N)$ is the order in which $\sigma_1(i) = |N|$, $\sigma_1(k) = \sigma_0(k)$ for all $k \in P(\sigma_0, i)$ and $\sigma_1(k) = \sigma_0(k) - 1$ for all $k \in F(\sigma_0, i)$.*

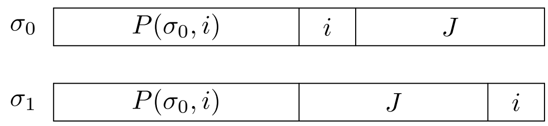

Figure 10 – *Step 1 of the Growing Tail procedure.*

For $\ell > 1$ until $\ell = |N| - 1$, perform the following step:

Step ℓ: *Set $i = \hat{\sigma}^{-1}(|N| - \ell + 1)$ and $k = \hat{\sigma}^{-1}(|N| - \ell + 2)$, i.e., i is the player that is in the $|N| - \ell + 1$th position according to $\hat{\sigma}$ and k the player that is $|N| - \ell + 2$th position. Note that $\sigma_{\ell-1}(k) = \sigma_\ell(k) = |N| - \ell + 2$. Consider the path $(\sigma_{\ell-1}, \ldots, \sigma_\ell)$ corresponding to neighbor switches (i, j) for every $j \in J$, where $J = (i, k)_{\sigma_{\ell-1}}$. Here, $\sigma_\ell \in \Pi(N)$ is the order in which $\sigma_\ell(i) = |N| - \ell + 1$, $\sigma_\ell(j) = \sigma_{\ell-1}(j) - 1$ for all $j \in J$, $\sigma_\ell(r) = \sigma_{\ell-1}(r)$ for all $r \in P(\sigma_{\ell-1}, i)$ and $\sigma_\ell(r) = \sigma_{\ell-1}(r)$ for all $r \in F(\sigma_{\ell-1}, k) \cup \{k\}$.*

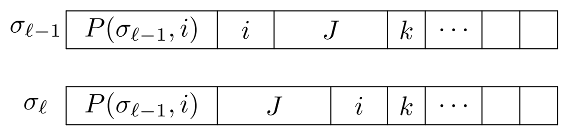

Figure 11 – *Step ℓ of the Growing Tail procedure .*

◁

In the following example, which is a continuation of Example 5.1, it is shown that both procedures specify a path from the initial order to the optimal order.

Example 5.2 Reconsider the logarithmic sequencing situation $(\sigma_0, p, c) \in LSEQ^N$, as described in Example 5.1, with $N = \{1, 2, 3\}$, $\sigma_0 = (1, 2, 3)$, $\alpha_i = 1$ for all $i \in N$, and $p_1 = 4, p_2 = 3$ and $p_3 = 2$. Recall that $\hat{\sigma} = (3, 2, 1)$ is the unique optimal order and $MP(\sigma_0, \hat{\sigma}) = \{(1, 2), (1, 3), (2, 3)\}$. We first perform the Growing Head procedure, according to Procedure 5.1. Note that $\hat{\sigma}(3) = 1$, so we start by consecutively moving player 3 to the front:

$$\sigma_0 = (1, 2, 3) \to (1, 3, 2) \to (3, 1, 2) = \sigma_1.$$

In the second step, we move player 2 to the second position, since $\hat{\sigma}(2) = 2$:

$$\sigma_1 = (3, 1, 2) \to (3, 2, 1) = \sigma_2 = \hat{\sigma}.$$

Note that this path corresponds to the first path as described in Example 5.1.

Next, we perform the Growing Tail procedure, following Procedure 5.2. Note that $\hat{\sigma}(1) = 3$, so we start by consecutively moving player 1 to the back:

$$\sigma_0 = (1, 2, 3) \to (2, 1, 3) \to (2, 3, 1) = \sigma_1.$$

In the second step, we move player 2 to the second position, since $\hat{\sigma}(2) = 2$:

$$\sigma_1 = (2, 3, 1) \to (3, 2, 1) = \sigma_2 = \hat{\sigma}.$$

Note that this path corresponds to the second path as described in Example 5.1. △

Starting with an initial order, one can follow either one of the two procedures in order to reach an optimal order. For example, by consecutively moving players to the front, forming an optimal order by letting the head grow larger and larger, the Growing Head procedure specifies a path from the initial order to an optimal order. To check that such a path indeed corresponds to the Growing Head procedure, one has to trace the steps of the procedure. An alternative way of checking this makes use of the characteristics of the Growing Head (and the Growing Tail) procedure. In a path specified by the Growing Head procedure, if players i and j interchange positions, then it should hold that all players before player j in the optimal order are already in their position according to the optimal order. This exactly reflects the idea of letting the head of an optimal order grow larger. Similarly, in a path specified by the Growing Tail procedure, if players i and j interchange positions, then all players after player i in the optimal order are already in their position according to the optimal order. Together, we obtain the following corollary as a consequence of both Procedures 5.1 and 5.2.

Corollary 5.1 *Let $(\sigma_0, p, c) \in SEQ^N$ be a sequencing situation and let $\hat{\sigma} \in \Pi(N)$ be an optimal order. Then,*

1) the Growing Head procedure specifies the unique path from σ_0 to $\hat{\sigma}$ such that, for all neighbor switches (i, j) associated to two consecutive orders σ and τ, it holds that $\tau(k) = \hat{\sigma}(k)$ for all $k \in P(\hat{\sigma}, j)$;

2) the Growing Tail procedure specifies the unique path from σ_0 to $\hat{\sigma}$ such that, for all neighbor switches (i, j) associated to two consecutive orders σ and τ, it holds that $\tau(k) = \hat{\sigma}(k)$ for all $k \in F(\hat{\sigma}, i)$.

5.2 Extending the EGS-rule

In addition to the choice of a path from the initial order to an optimal order, we have to specify how to divide the corresponding neighbor switching gains in every step in such a path. Analogous to the EGS-rule, we divide the neighbor switching gains equally between the two neighbors involved. Hence, we obtain two cost savings allocation rules based on the two procedures.

Definition 5.1 The *Equal Gain Splitting Head Rule (EGSH-rule)* specifies for every sequencing situation $(\sigma_0, p, c) \in SEQ^N$ and every optimal order $\hat{\sigma} \in \Pi(N)$ the following allocation:

$$EGSH(\sigma_0, p, c, \hat{\sigma}) = \sum_{\ell=1}^{|N|-1} \left(\frac{1}{2} g_{Ij}(t_I(\sigma_{\ell-1})) \mathbb{1}_{\{j\}} + \sum_{i \in I} \frac{1}{2} g_{ij}(t_i(\sigma_{\ell-1})) \mathbb{1}_{\{i\}} \right),$$

where $j = \hat{\sigma}^{-1}(\ell)$ for every $\ell \in \{1, 2, \ldots, |N| - 1\}$, σ_ℓ for every $\ell \in \{1, 2, \ldots, |N| - 1\}$ according to the Growing Head procedure, and

$$I = \begin{cases} P(\sigma_0, j), & \text{if } \ell = 1; \\ (k, j)_{\sigma_{\ell-1}}, & \text{if } \ell \in \{2, 3, \ldots, |N| - 1\}. \end{cases}$$

Similarly, the *Equal Gain Splitting Tail Rule (EGST-rule)* specifies for every sequencing situation $(\sigma_0, p, c) \in SEQ^N$ and every optimal order $\hat{\sigma} \in \Pi(N)$ the following allocation:

$$EGST(\sigma_0, p, c, \hat{\sigma}) = \sum_{\ell=1}^{|N|-1} \left(\frac{1}{2} g_{iJ}(t_i(\sigma_{\ell-1}))\mathbb{1}_{\{i\}} + \sum_{j \in J} \frac{1}{2} g_{ij}(t_j(\sigma_{\ell-1}) - p_i)\mathbb{1}_{\{j\}} \right),$$

where $i = \hat{\sigma}^{-1}(|N| - \ell + 1)$ for every $\ell \in \{1, 2, \ldots, |N| - 1\}$, σ_ℓ for every $\ell \in \{1, 2, \ldots, |N| - 1\}$ according to the Growing Tail procedure, and

$$J = \begin{cases} F(\sigma_0, i), & \text{if } \ell = 1; \\ (i, k)_{\sigma_{\ell-1}}, & \text{if } \ell \in \{2, 3, \ldots, |N| - 1\}. \end{cases} \qquad \triangleleft$$

By definition, both the EGSH-rule and the EGST-rule lead to efficient allocations, that is,

$$\sum_{i \in N} EGSH_i(\sigma_0, p, c, \hat{\sigma}) = v(N) = \sum_{i \in N} EGST_i(\sigma_0, p, c, \hat{\sigma}),$$

where $v(N)$ is the worth of the grand coalition of the sequencing game associated to a sequencing situation $(\sigma_0, p, c) \in SEQ^N$ and $\hat{\sigma} \in \Pi(N)$ an optimal order. Additionally, we can guarantee stability for the EGSH-rule, that is,

$$\sum_{i \in S} EGSH_i(\sigma_0, p, c, \hat{\sigma}) \geq v(S),$$

for all $S \in 2^N \setminus \{\emptyset\}$, if the neighbor switching gains corresponding to misplacements are non-*de*creasing. Similarly,

$$\sum_{i \in S} EGST_i(\sigma_0, p, c, \hat{\sigma}) \geq v(S),$$

for all $S \in 2^N \setminus \{\emptyset\}$, if the neighbor switching gains corresponding to misplacements are non-*in*creasing. This is formulated by the following theorem.

Theorem 5.1 *Let $(\sigma_0, p, c) \in SEQ^N$ be a sequencing situation, $\hat{\sigma} \in \Pi(N)$ an optimal order and $v \in TU^N$ the associated sequencing game. Assume that, for all $t \in [0, \infty)$,*

i) $g_{ij}(t) \geq 0$ for all $(i, j) \in MP(\sigma_0, \hat{\sigma})$;

ii) $g_{ij}(t) \leq 0$ for all $(i, j) \notin MP(\sigma_0, \hat{\sigma})$.

Then the following two statements hold:

1) If, for all $s, t \in [0, \infty)$ with $s \leq t$, $g_{ij}(s) \leq g_{ij}(t)$ for all $(i, j) \in MP(\sigma_0, \hat{\sigma})$, then $EGSH(\sigma_0, p, c, \hat{\sigma}) \in C(v)$;

2) If, for all $s, t \in [0, \infty)$ with $s \leq t$, $g_{ij}(s) \geq g_{ij}(t)$ for all $(i, j) \in MP(\sigma_0, \hat{\sigma})$, then $EGST(\sigma_0, p, c, \hat{\sigma}) \in C(v)$.

Proof: By definition, both allocations prescribed by the EGSH-rule and the EGST-rule are efficient. For stability, it suffices to restrict to connected coalitions. Hence, let $S \in 2^N \setminus \{\emptyset\}$ be a connected coalition. Then,

$$v(S) = \sum_{(i,j)\in MP(\sigma_0,\hat{\sigma}),i,j\in S} g_{ij}(t_{ij}^S),$$

where t_{ij}^S is the starting time of player i for which players i and j switch positions if the group of players S is rearranging to its optimal position. Let $(i,j) \in MP(\sigma_0, \hat{\sigma})$ with $i, j \in S$. For both statements, we can use Lemma 2.1 and either the Growing Head procedure or the Growing Tail procedure, respectively, to obtain a direct expression for t_{ij}^S:

1) Using the Growing Head procedure (according to Procedure 5.1), we see that

$$t_{ij}^S = \sum_{k\in P(\sigma_0,i)} p_k + \sum_{k\in F(\sigma_0,i)\cap P(\hat{\sigma},j)\cap S} p_k; \tag{11}$$

2) Using the Growing Tail procedure (according to Procedure 5.2), we see that

$$t_{ij}^S = \sum_{k\in P(\sigma_0,S)} p_k + \sum_{k\in P(\hat{\sigma},i)\cap P(\sigma_0,j)\cap S} p_k. \tag{12}$$

On the other hand, we denote by t_{ij} the starting time of player i for which the players i and j switch positions if all players (the grand coalition N) are rearranging to its optimal position. Again, we can derive a direct expression for t_{ij}, depending on the procedure we use:

1) Using the Growing Head procedure, we see that

$$t_{ij} = \sum_{k\in P(\sigma_0,i)} p_k + \sum_{k\in F(\sigma_0,i)\cap P(\hat{\sigma},j)} p_k; \tag{13}$$

2) Using the Growing Tail procedure, we see that

$$t_{ij} = \sum_{k\in P(\hat{\sigma},i)\cap P(\sigma_0,j)} p_k. \tag{14}$$

Next, we show that $g_{ij}(t_{ij}^S) \leq g_{ij}(t_{ij})$ in both cases.

1) From Equation (11) and Equation (13) and by using the fact that $F(\sigma_0,i)\cap P(\hat{\sigma},j)\cap S \subseteq F(\sigma_0,i)\cap P(\hat{\sigma},j)$, it follows that $t_{ij}^S \leq t_{ij}$. Consequently, we have that $g_{ij}(t_{ij}^S) \leq g_{ij}(t_{ij})$;
2) From Equation (12) and Equation (14) and by using the fact that for every $k \in P(\hat{\sigma},i)\cap P(\sigma_0,j)$, we have that either $k \in P(\sigma_0,S)$, if $k \notin S$, or $k \in P(\hat{\sigma},i)\cap P(\sigma_0,j)\cap S$, if $k \in S$, it follows that $t_{ij} \leq t_{ij}^S$. Consequently, we have that $g_{ij}(t_{ij}^S) \leq g_{ij}(t_{ij})$.

Hence, in both cases,

$$\sum_{(i,j)\in MP(\sigma_0,\hat{\sigma}),i,j\in S} g_{ij}(t_{ij}) \geq \sum_{(i,j)\in MP(\sigma_0,\hat{\sigma}),i,j\in S} g_{ij}(t_{ij}^S) = v(S). \tag{15}$$

Finally, note that, for every $(i,j) \in MP(\sigma_0,\hat{\sigma})$ with $i,j \in S$, we have that the corresponding neighbor switching gain $g_{ij}(t_{ij})$ is divided equally between players i and j. This means that, when adding all allocations of the players for either the EGSH-rule or the EGST-rule, we have that

$$\sum_{k\in S} EGSH_k(\sigma_0, p, c, \hat{\sigma}) \geq \sum_{(i,j)\in MP(\sigma_0,\hat{\sigma}),i,j\in S} g_{ij}(t_{ij}),$$

and

$$\sum_{k\in S} EGST_k(\sigma_0, p, c, \hat{\sigma}) \geq \sum_{(i,j)\in MP(\sigma_0,\hat{\sigma}),i,j\in S} g_{ij}(t_{ij}).$$

Consequently, by combining this with Equation (15), we have that

$$\sum_{k\in S} EGSH_k(\sigma_0, p, c, \hat{\sigma}) \geq v(S),$$

and

$$\sum_{k\in S} EGST_k(\sigma_0, p, c, \hat{\sigma}) \geq v(S). \qquad \square$$

For exponential sequencing situations, we see that for the subclasses as stated in Proposition 3.1, we can combine Theorem 5.1 with Proposition 3.1 to see that the EGSH-rule leads to core-elements. Moreover, for both discounting sequencing situations and logarithmic sequencing situations, we see that, by combining Theorem 5.1 with Propositions 3.2 and 3.3, the EGST-rule leads to allocations that are core-elements of the associated (discounting or logarithmic) sequencing games. Together, this yields the following corollary.

Corollary 5.2 *The following three statements hold:*

1) *Let $(\sigma_0, p, c) \in ESEQ^N$ be an exponential sequencing situation such that one of the following three cases holds:*

   i) *there is an $\alpha \in \mathbb{R}_{++}$ such that, for all $i \in N$ and all $t \in [0, \infty)$, $c_i(t) = e^{\alpha t}$;*

   ii) *there is a $p \in \mathbb{R}_{++}$ such that, for all $i \in N$, $p_i = p$;*

   iii) *there are $\alpha_L, \alpha_H, p_L, p_H \in \mathbb{R}_{++}$ with $\alpha_L < \alpha_H$, $p_L < p_H$ such that, for all $i \in N$, $\alpha_i \in \{\alpha_L, \alpha_H\}, p_i \in \{p_L, p_H\}$ and*

   $$e^{\alpha_H p_H} - e^{\alpha_L p_L} \leq e^{\alpha_H(p_L+p_H)} - e^{\alpha_L(p_L+p_H)}.$$

   Let $\hat{\sigma} \in \Pi(N)$ be an optimal order and $v \in TU^N$ the associated exponential sequencing game. Then,

   $$EGSH(\sigma_0, p, c, \hat{\sigma}) \in C(v).$$

2) *Let $(\sigma_0, p, c) \in DSEQ^N$ be a discounting sequencing situation, $\hat{\sigma} \in \Pi(N)$ an optimal order and $v \in TU^N$ the associated discounting sequencing game. Then,*

   $$EGST(\sigma_0, p, c, \hat{\sigma}) \in C(v).$$

3) *Let $(\sigma_0, p, c) \in LSEQ^N$ be a logarithmic sequencing situation, $\hat{\sigma} \in \Pi(N)$ an optimal order and $v \in TU^N$ the associated logarithmic sequencing game. Then,*

   $$EGST(\sigma_0, p, c, \hat{\sigma}) \in C(v).$$

Note that, for standard sequencing situations, the allocations specified by both the EGSH-rule and the EGST-rule boil down to the allocation prescribed by the EGS-rule, since every path from the initial order to an optimal order lead to the same allocation. However, for, e.g., logarithmic, discounting, and exponential sequencing situations, the EGSH-rule and the EGST-rule can prescribe different allocations. We illustrate this for the logarithmic case in Example 5.3 (similar examples can be found for each of the other two cases). Besides, it can be also seen that the allocation prescribed by the EGST-rule for such logarithmic sequencing situation is also in the core of the associated sequencing game. The example is a continuation of Examples 5.1 and 5.2.

Example 5.3 Reconsider the logarithmic sequencing situation $(\sigma_0, p, c) \in LSEQ^N$, as described in Examples 5.1 and 5.2, with $N = \{1,2,3\}$, $\sigma_0 = (1,2,3)$, $\alpha_i = 1$ for all $i \in N$, and $p_1 = 4, p_2 = 3$ and $p_3 = 2$. Recall that $\hat{\sigma} = (3,2,1)$, $MP(\sigma_0, \hat{\sigma}) = \{(1,2),(1,3),(2,3)\}$ and the corresponding neighbor switching gains as given below.

$g_{12}(0)$	0.2877
$g_{12}(p_3)$	0.1823
$g_{13}(0)$	0.6932
$g_{13}(p_2)$	0.3365
$g_{23}(0)$	0.4055
$g_{23}(p_1)$	0.1542

Recall from Example 5.2 that the Growing Head procedure specifies the following path from the initial order to the optimal order:

$$\sigma_0 = (1,2,3) \xrightarrow{g_{23}(p_1)} (1,3,2) \xrightarrow{g_{13}(0)} (3,1,2) = \sigma_1 \xrightarrow{g_{12}(p_3)} (3,2,1) = \sigma_2 = \hat{\sigma}.$$

Hence, by using Definition 5.1, the allocation prescribed by the EGSH-rule is

$$\begin{aligned} EGSH(\sigma_0, p, c, \hat{\sigma}) &= \frac{1}{2} g_{\{1,2\}3}(0) \mathbb{1}_{\{3\}} + \frac{1}{2} g_{13}(0) \mathbb{1}_{\{1\}} + \frac{1}{2} g_{23}(p_1) \mathbb{1}_{\{2\}} \\ &\qquad + \frac{1}{2} g_{\{1\}2}(p_3) \mathbb{1}_{\{2\}} + \frac{1}{2} g_{12}(p_3) \mathbb{1}_{\{1\}} \\ &= \frac{1}{2} g_{23}(p_1) \mathbb{1}_{\{2,3\}} + \frac{1}{2} g_{13}(0) \mathbb{1}_{\{1,3\}} + \frac{1}{2} g_{12}(p_3) \mathbb{1}_{\{1,2\}} \\ &= \frac{1}{2} \left(g_{13}(0) + g_{12}(p_3), g_{23}(p_1) + g_{12}(p_3), g_{23}(p_1) + g_{13}(0) \right) \\ &= \frac{1}{2} \left(0.6932 + 0.1823, 0.1542 + 0.1823, 0.1542 + 0.6932 \right) \\ &= (0.4377, 0.1682, 0.4236). \end{aligned}$$

Example 5.2 also shows that the Growing Tail procedure specifies the following path from the initial order to the optimal order:

$$\sigma_0 = (1,2,3) \xrightarrow{g_{12}(0)} (2,1,3) \xrightarrow{g_{13}(p_2)} (2,3,1) \xrightarrow{g_{23}(0)} (3,2,1) = \hat{\sigma}.$$

Hence, by using Definition 5.1, the allocation prescribed by the EGST-rule is

$$\begin{aligned} EGST(\sigma_0, p, c, \hat{\sigma}) &= \frac{1}{2} g_{1\{2,3\}}(0) \mathbb{1}_{\{1\}} + \frac{1}{2} g_{12}(0) \mathbb{1}_{\{2\}} + \frac{1}{2} g_{13}(p_2) \mathbb{1}_{\{4\}} \\ &\qquad + \frac{1}{2} g_{2\{3\}}(0) \mathbb{1}_{\{2\}} + \frac{1}{2} g_{23}(0) \mathbb{1}_{\{3\}} \\ &= \frac{1}{2} g_{12}(0) \mathbb{1}_{\{1,2\}} + \frac{1}{2} g_{13}(p_2) \mathbb{1}_{\{1,3\}} + \frac{1}{2} g_{23}(0) \mathbb{1}_{\{2,3\}} \\ &= \frac{1}{2} \left(g_{12}(0) + g_{13}(p_2), g_{12}(0) + g_{23}(0), g_{13}(p_2) + g_{23}(0) \right) \\ &= \frac{1}{2} \left(0.2877 + 0.3365, 0.2877 + 0.4055, 0.3365 + 0.4055 \right) \\ &= (0.3121, 0.3466, 0.3710). \end{aligned}$$

Consequently,

$$EGSH(\sigma_0, p, c, \hat{\sigma}) = (0.4377, 0.1682, 0.4236) \neq (0.3121, 0.3466, 0.3710) = EGST(\sigma_0, p, c, \hat{\sigma}).$$

Furthermore, with regard to the associated logarithmic sequencing game $v \in TU^N$, note that $v(\{1,2\}) = g_{12}(0)$ and $v(\{2,3\}) = g_{23}(p_1)$. The logarithmic sequencing game is shown below.

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$v(S)$	0	0	0	0	0.2877	0	0.1542	1.0296

We see that indeed, in line with Corollary 5.2, $EGST(\sigma_0, p, c, \hat{\sigma}) \in C(v)$. Moreover, we see that also $EGSH(\sigma_0, p, c, \hat{\sigma}) \in C(v)$. △

The last observation of Example 5.3, the fact that also $EGSH(\sigma_0, p, c, \hat{\sigma}) \in C(v)$, can be proven for any logarithmic sequencing situation with three players, as is shown below.

Lemma 5.1 *Let $(\sigma_0, p, c) \in LSEQ^N$ with $|N| = 3$ be a logarithmic sequencing situation, $\hat{\sigma} \in \Pi(N)$ an optimal order and $v \in TU^N$ the associated logarithmic sequencing game. Then,*

$$EGSH(\sigma_0, p, c, \hat{\sigma}) \in C(v).$$

Proof: Set $N = \{1,2,3\}$ and w.l.o.g. assume that $\sigma_0 = (1,2,3)$. If

$$\hat{\sigma} \in \{(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2)\},$$

then there is only one path from the initial order to the optimal order that repairs all misplacements. Hence, $EGSH(\sigma_0, p, c, \hat{\sigma}) = EGST(\sigma_0, p, c, \hat{\sigma})$. Using Corollary 5.2, we see that $EGSH(\sigma_0, p, c, \hat{\sigma}) \in C(v)$.

If $\hat{\sigma} = (3,2,1)$, then $p_3 < p_2 < p_1$, according to Theorem 3.1. Moreover, $MP(\sigma_0, \hat{\sigma}) = \{(1,2), (1,3), (2,3)\}$ and the path as prescribed by the Growing Head procedure is given by

$$\sigma_0 = (1,2,3) \xrightarrow{g_{23}(p_1)} (1,3,2) \xrightarrow{g_{13}(0)} (3,1,2) \xrightarrow{g_{12}(p_3)} (3,2,1) = \hat{\sigma}.$$

The associated logarithmic sequencing game v can be expressed in terms of the neighbor switching gains:

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$v(S)$	0	0	0	0	$g_{12}(0)$	0	$g_{23}(p_1)$	$g_{23}(p_1) + g_{13}(0) + g_{12}(p_3)$

Furthermore, we can also express the allocation prescribed by the EGSH-rule in terms of the neighbor switching gains:

$$EGSH(\sigma_0, p, c, \hat{\sigma}) = \frac{1}{2}(g_{12}(p_3) + g_{13}(0), g_{12}(p_3) + g_{23}(p_1), g_{13}(0) + g_{23}(p_1)).$$

Note that $v(S) \leq \sum_{i \in S} EGSH_i(\sigma_0, p, c, \hat{\sigma})$ for all $S \in \{\{1\}, \{2\}, \{3\}, \{1,3\}\}$, since $v(S) = 0$ for these coalitions, while $EGSH(\sigma_0, p, c, \hat{\sigma}) \in \mathbb{R}^N_+$. With regard to coalition $\{1,2\} \in 2^N \setminus \{\emptyset\}$, we notice the following, using the expression for the neighbor switching gains as in Equation (8):

$$\begin{aligned}
&EGSH_1(\sigma_0, p, c, \hat{\sigma}) + EGSH_2(\sigma_0, p, c, \hat{\sigma}) - v(\{1,2\}) \\
&\qquad = \frac{1}{2}\left(g_{12}(p_3) + g_{13}(0) + g_{12}(p_3) + g_{23}(p_1)\right) - g_{12}(0) \\
&\qquad = \frac{1}{2}\ln\left(\frac{(p_1+p_3)\cdot(p_1+p_3)\cdot p_1\cdot(p_1+p_2)}{(p_2+p_3)\cdot(p_2+p_3)\cdot p_3\cdot(p_1+p_3)}\right) - \ln\left(\frac{p_1}{p_2}\right) \\
&\qquad = \frac{1}{2}\ln\left(\frac{(p_1+p_3)\cdot p_1\cdot(p_1+p_2)}{(p_2+p_3)\cdot p_3\cdot(p_2+p_3)}\cdot\frac{p_2\cdot p_2}{p_1\cdot p_1}\right) \\
&\qquad > \frac{1}{2}\ln\left(\frac{(p_1+p_3)\cdot p_1\cdot(p_1+p_2)}{(p_2+p_3)\cdot p_2\cdot(p_2+p_3)}\cdot\frac{p_2\cdot p_2}{p_1\cdot p_1}\right) \\
&\qquad = \frac{1}{2}\ln\left(\frac{(p_1+p_3)\cdot(p_1+p_2)}{(p_2+p_3)\cdot(p_2+p_3)}\cdot\frac{p_2}{p_1}\right) \\
&\qquad \geq \frac{1}{2}\ln\left(\frac{(p_1+p_3)\cdot(p_1+p_1)}{(p_2+p_3)\cdot(p_1+p_3)}\cdot\frac{p_2}{p_1}\right) \\
&\qquad = \frac{1}{2}\ln\left(\frac{2p_1}{p_2+p_3}\cdot\frac{p_2}{p_1}\right) \\
&\qquad > \frac{1}{2}\ln\left(\frac{2p_1}{p_2+p_2}\cdot\frac{p_2}{p_1}\right) \\
&\qquad = \frac{1}{2}\ln\left(\frac{p_1}{p_2}\cdot\frac{p_2}{p_1}\right) \\
&\qquad = \frac{1}{2}\ln(1) = 0,
\end{aligned}$$

where we used that $p_3 < p_2$ for the first and third inequality and $p_3 < p_1$ together with the fact that $g_{13}(t)$ is non-increasing, such that $\frac{p_1+p_2}{p_2+p_3} \geq \frac{p_1+p_1}{p_1+p_3}$ for the second inequality. Hence,

$$EGSH_1(\sigma_0, p, c, \hat{\sigma}) + EGSH_2(\sigma_0, p, c, \hat{\sigma}) \geq v(\{1,2\}).$$

Moreover, we have that

$$\begin{aligned}
EGSH_2(\sigma_0, p, c, \hat{\sigma}) + EGSH_3(\sigma_0, p, c, \hat{\sigma}) &= g_{23}(p_1) + \frac{1}{2}\left(g_{12}(p_3) + g_{13}(0)\right) \\
&\geq g_{23}(p_1) = v(\{2,3\}),
\end{aligned}$$

where the inequality follows from the fact that all neighbor switching gains corresponding to misplacements are non-negative. Finally, we have that

$$\begin{aligned}
&EGSH_1(\sigma_0, p, c, \hat{\sigma}) + EGSH_2(\sigma_0, p, c, \hat{\sigma}) + EGSH_3(\sigma_0, p, c, \hat{\sigma}) \\
&\qquad = g_{23}(p_1) + g_{13}(0) + g_{12}(p_3) = v(N).
\end{aligned}$$

Consequently, $EGSH(\sigma_0, p, c, \hat{\sigma}) \in C(v)$. □

For logarithmic sequencing situations with more than three players, the allocation prescribed by the EGSH-rule is not necessarily a core-element, as the following example shows.

Example 5.4 Let $(\sigma_0, p, c) \in LSEQ^N$ be a logarithmic sequencing situation, where $N = \{1, 2, 3, 4\}$, $\sigma_0 = (1, 2, 3, 4)$, $\alpha_i = 1$ for all $i \in N$ and the processing times as specified in the table below.

	player 1	player 2	player 3	player 4
p_i	2.96	1.8	1.78	1.75

The total costs for all 24 orders are given below.

σ	$TC(\sigma)$	σ	$TC(\sigma)$	σ	$TC(\sigma)$
$(1, 2, 3, 4)$	6.6384	$(2, 3, 1, 4)$	5.8561	$(3, 4, 1, 2)$	5.8232
$(1, 2, 4, 3)$	6.6338	$(2, 3, 4, 1)$	5.6516	$(3, 4, 2, 1)$	25.6263
$(1, 3, 2, 4)$	6.6342	$(2, 4, 1, 3)$	5.8431	$(4, 1, 2, 3)$	6.0977
$(1, 3, 4, 2)$	6.6265	$(2, 4, 3, 1)$	5.6431	$(4, 1, 3, 2)$	6.0946
$(1, 4, 2, 3)$	6.6233	$(3, 1, 2, 4)$	6.1256	$(4, 2, 1, 3)$	5.815
$(1, 4, 3, 2)$	6.6202	$(3, 1, 4, 2)$	6.118	$(4, 2, 3, 1)$	5.615
$(2, 1, 3, 4)$	6.141	$(3, 2, 1, 4)$	5.845	$(4, 3, 1, 2)$	5.8062
$(2, 1, 4, 3)$	6.1364	$(3, 2, 4, 1)$	5.6404	$(4, 3, 2, 1)$	5.6093

Obviously, $\hat{\sigma} = (4, 3, 2, 1)$ is the unique optimal order and

$$MP(\sigma_0, \hat{\sigma}) = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}.$$

The Growing Head procedure specifies the following path from the initial order to the optimal order:

$$\begin{aligned} \sigma_0 = (1, 2, 3, 4) &\to (1, 2, 4, 3) \to (1, 4, 2, 3) \to (4, 1, 2, 3) = \sigma_1 \\ &\to (4, 1, 3, 2) \to (4, 3, 1, 2) = \sigma_2 \\ &\to (4, 3, 2, 1) = \hat{\sigma}. \end{aligned}$$

The corresponding neighbor switching gains are shown in the table below.

$g_{34}(p_1 + p_2)$	0.0046
$g_{24}(p_1)$	0.0106
$g_{14}(0)$	0.5256
$g_{23}(p_1 + p_4)$	0.0031
$g_{13}(p_4)$	0.2884
$g_{12}(p_3 + p_4)$	0.1969

Hence, the EGSH-rule specifies the following allocation:

$$EGSH(\sigma_0, p, c, \hat{\sigma}) = (0.5054, 0.1053, 0.1480, 0.2704).$$

Then, we see that for coalition $\{1,2,3\} \in 2^N \setminus \{\emptyset\}$, we have that

$$\begin{aligned} v(\{1,2,3\}) &= TC(\sigma_0) - TC((3,2,1,4)) = 6.6384 - 5.845 \\ &= 0.7935 > 0.7587 \\ &= EGSH_1(\sigma_0,p,c,\hat{\sigma}) + EGSH_2(\sigma_0,p,c,\hat{\sigma}) + EGSH_3(\sigma_0,p,c,\hat{\sigma}), \end{aligned}$$

where v denotes the associated logarithmic sequencing game. Hence, $EGSH(\sigma_0,p,c,\hat{\sigma}) \notin C(v)$. △

However, Lemma 5.1 is a specific result for logarithmic sequencing situations. Counterexamples can be obtained for other sequencing problems under the same conditions of Theorem 5.1 that logarithmic sequencing problems. Example 5.5 shows that the allocation specified by the EGSH-rule is not a core-element in discounting sequencing situations with three players.

Example 5.5 Let $(\sigma_0,p,c) \in DSEQ^N$ be a discounting sequencing situation, where $N = \{1,2,3\}$, $r = 0.8838$, $\sigma_0 = (1,2,3)$ and the discounting cost coefficients and processing times as shown in the next table.

	player 1	player 2	player 3
α_i	0.1768	0.9070	0.5041
p_i	0.8371	0.9450	0.6142

That leaves only six orders, for which the total costs are given below.

σ	$TC(\sigma)$
$(1,2,3)$	1.2551
$(1,3,2)$	1.2546
$(2,1,3)$	1.0972
$(2,3,1)$	1.0461
$(3,1,2)$	1.1368
$(3,2,1)$	1.0451

Clearly, $\hat{\sigma} = (3,2,1)$ is the unique optimal order and $MP(\sigma_0,\hat{\sigma}) = \{(1,2),(1,3),(2,3)\}$. The Growing Head procedure specifies the following path from σ_0 to $\hat{\sigma}$:

$$\begin{aligned} \sigma_0 = (1,2,3) &\rightarrow (1,3,2) \rightarrow (3,1,2) = \sigma_1 \\ &\rightarrow (3,2,1) = \hat{\sigma}. \end{aligned}$$

By using the previous table, the corresponding neighbor switching gains are $g_{23}(p_1) = 0.0005$, $g_{13}(0) = 0.1178$ and $g_{12}(p_3) = 0.0918$, respectively. Thus, we obtain the following allocation specified by the EGSH-rule:

$$EGSH(\sigma_0,p,c,\hat{\sigma}) = (0.1048, 0.0461, 0.0592).$$

To conclude, if we take coalition $\{1,2\} \in 2^N \setminus \{\emptyset\}$, we have that

$$\begin{aligned} v(\{1,2\}) &= TC(\sigma_0) - TC((2,1,3)) = 0.1579 \\ &> 0.1509 = EGSH_1(\sigma_0,p,c,\hat{\sigma}) + EGSH_2(\sigma_0,p,c,\hat{\sigma}), \end{aligned}$$

where v denotes the associated discounting sequencing game. Thus, $EGSH(\sigma_0,p,c,\hat{\sigma}) \notin C(v)$. △

Moreover, no result can be established to prove that $EGST(\sigma_0, p, c, \hat{\sigma}) \in C(v)$ for an exponential sequencing situations $(\sigma_0, p, c) \in ESEQ^N$ with three players. The concluding example by Saavedra-Nieves et al. (2020) is a counterexample for this.

The following table provides an overview of the results whether the two cost savings allocation rules, EGSH-rule and EGST-rule, are core-elements of the associated sequencing games or not.

	$EGSH(\sigma_0, p, c, \hat{\sigma}) \in C(v)$	$EGST(\sigma_0, p, c, \hat{\sigma}) \in C(v)$
Exponential sequencing situation $(\sigma_0, p, c) \in ESEQ^N$ satisfying one of the three cases as described in Proposition 3.1	Yes (Corollary 5.2)	No (Counterexample by Saavedra-Nieves et al., 2020)
Discounting sequencing situation $(\sigma_0, p, c) \in DSEQ^N$	No (Example 5.5)	Yes (Corollary 5.2)
Logarithmic sequencing situation $(\sigma_0, p, c) \in LSEQ^N$	Yes for $\|N\| = 3$ (Lemma 5.1) No for $\|N\| \geq 4$ (Example 5.4)	Yes (Corollary 5.2)
Standard sequencing situation $(\sigma_0, p, c) \in SSEQ^N$	Yes (EGS-rule)	Yes (EGS-rule)

Table 1 – *Overview of the results on the two cost savings allocation rules*

The final two examples illustrate two issues regarding the EGSH-rule and the EGST-rule. First, the next example shows that, if the conditions of Theorem 5.1 are not satisfied for a particular exponential sequencing situation (and hence, we are not in one of the subclasses as defined above), the EGSH-rule and the EGST-rule may not lead to allocations that are core-elements. This example is also extracted from Saavedra-Nieves et al. (2020).

Example 5.6 Let $(\sigma_0, p, c) \in ESEQ^N$ be an exponential sequencing situation, where $N = \{1, 2, 3\}$, $\sigma_0 = (1, 2, 3)$ and the exponential cost coefficients and processing times as specified in the table below.

	player 1	player 2	player 3
α_i	1.880	1.904	1.902
p_i	1.205	1.940	1.976

The total costs for all orders are given below.

σ	$TC(\sigma)$
$(1, 2, 3)$	17394
$(1, 3, 2)$	17595
$(2, 1, 3)$	17396
$(2, 3, 1)$	16933
$(3, 1, 2)$	17599
$(3, 2, 1)$	16949

Obviously, $\hat{\sigma} = (2, 3, 1)$ is the unique optimal order and $MP(\sigma_0, \hat{\sigma}) = \{(1, 2), (1, 3)\}$. Moreover, there is only one path from the initial order to the optimal order:

$$\sigma_0 = (1, 2, 3) \rightarrow (2, 1, 3) \rightarrow (2, 3, 1) = \hat{\sigma}.$$

Hence, $EGSH(\sigma_0, p, c, \hat{\sigma}) = EGST(\sigma_0, p, c, \hat{\sigma})$. Note that $g_{12}(0) = 17394 - 17595 = -1.5783 < 0$, which shows that the conditions of Theorem 5.1 are not satisfied. Consequently,

$$EGSH_2(\sigma_0, p, c, \hat{\sigma}) = EGST_2(\sigma_0, p, c, \hat{\sigma}) = \frac{1}{2} g_{12}(0) < 0 = v(\{2\}),$$

where v denotes the associated exponential sequencing game. This shows that both allocations prescribed by either the EGSH-rule or the EGST-rule are not core-elements of the exponential sequencing game. △

The last example illustrates that for a logarithmic sequencing situation with two different optimal orders, the allocations prescribed by the EGSH-rule are different for each of the optimal orders. The example can be easily modified by rearranging the processing times among the players to obtain a logarithmic sequencing situation with two different optimal orders where the allocations prescribed by the EGST-rule are different for each of the optimal orders.

Example 5.7 Let $(\sigma_0, p, c) \in LSEQ^N$ be a logarithmic sequencing situation, where $N = \{1, 2, 3\}$, $\sigma_0 = (1, 2, 3)$, $\alpha_i = 1$ for all $i \in N$, and $p_1 = 4$ and $p_2 = p_3 = 2$. The total costs for all possible orders are given below.

σ	$TC(\sigma)$
$(1, 2, 3)$	5.2575
$(1, 3, 2)$	5.2575
$(2, 1, 3)$	4.5643
$(2, 3, 1)$	4.1589
$(3, 1, 2)$	4.5643
$(3, 2, 1)$	4.1589

Obviously, there are two optimal orders, $\hat{\sigma}_1 = (2, 3, 1)$ and $\hat{\sigma}_2 = (3, 2, 1)$. For the first optimal order $\hat{\sigma}_1$, we have that $MP(\sigma_0, \hat{\sigma}_1) = \{(1, 2), (1, 3)\}$. Hence, there is only one path from the initial order to this optimal order that repairs all neighbor misplacements:

$$\sigma_0 = (1, 2, 3) \xrightarrow{g_{12}(0)} (2, 1, 3) \xrightarrow{g_{13}(p_2)} (2, 3, 1) = \hat{\sigma}_1,$$

corresponding to neighbor switches $(1, 2)$ and $(1, 3)$ respectively. Using the fact that $g_{12}(0) = 0.69315$ and $g_{13}(p_2) = 0.40547$, we obtain the following allocation specified by the EGSH-rule:

$$EGSH(\sigma_0, p, c, \hat{\sigma}_1) = (0.5493, 0.3466, 0.2027).$$

Similarly, $MP(\sigma_0, \hat{\sigma}_2) = \{(1, 2), (1, 3), (2, 3)\}$ and the path from the initial order to this second optimal order $\hat{\sigma}_2$ is given by:

$$\sigma_0 = (1, 2, 3) \xrightarrow{g_{23}(p_1)} (1, 3, 2) \xrightarrow{g_{13}(0)} (3, 1, 2) \xrightarrow{g_{12}(p_3)} (3, 2, 1) = \hat{\sigma}_2,$$

corresponding to the neighbor switches $(2, 3), (1, 3)$ and $(1, 2)$ respectively. Using the fact that $g_{23}(p_1) = 0, g_{13}(0) = 0.69315$ and $g_{12}(p_3) = 0.40547$, we obtain the following allocation specified by the EGSH-rule:

$$EGSH(\sigma_0, p, c, \hat{\sigma}_2) = (0.5493, 0.2027, 0.3466).$$

Consequently,

$$EGSH(\sigma_0, p, c, \hat{\sigma}_1) \neq EGSH(\sigma_0, p, c, \hat{\sigma}_2).$$ △

Example 5.7 shows the importance of fixing a specific optimal order first, as different optimal orders may lead to different allocations. Analogous conclusions are obtained when different paths exist from the initial order to the optimal one. In both cases, a linear combination of the different allocations obtained is naturally bearable if there is no preference between one path or another.

6 Concluding remarks

In this paper, we studied some sequencing situations with non-linear cost functions. We focused on the convexity of the associated sequencing games as well as on allocation rules that prescribe core-allocations. For both aspects, we derive conditions on the time-dependent neighbor switching gains to guarantee the desired outcome.

In particular, we showed that the two extensions of the EGS-rule based on either the Growing Head procedure or the Growing Tail procedure result in core-allocations if certain conditions are satisfied. These results could be generalized in a similar way as the EGS-rule is generalized by Hamers, Suijs, Tijs, and Borm (1996). Instead of dividing the neighbor switching gains equally among the two neighbors involved, the *split core* (cf. Hamers et al., 1996) consists of allocations for which every neighbor switching gain is divided among the two neighbors in an arbitrary way (providing that each player that is involved in a neighbor switch obtains a non-negative part of the gain).

This idea of generalizing the EGS-rule can also be applied to both the EGSH-rule and the EGST-rule. In these allocations, the neighbor switching gains are also divided equally among the two neighbors. However, one could imagine that different divisions of the gains may be possible. Note that the results obtained in this paper are also valid for this generalized allocations. In particular, Theorem 5.1 does not depend on equal division, which ensures that all these generalized allocations are also (under particular conditions as prescribed in Theorem 5.1) core-elements of the associated sequencing game.

References

Biskup, D. (2008). A state-of-the-art review on scheduling with learning effects. *European Journal of Operational Research*, **188**, 315–329.

Borm, P., G. Fiestras-Janeiro, H. Hamers, E. Sánchez, and M. Voorneveld (2002). On the convexity of games corresponding to sequencing situations with due dates. *European Journal of Operational Research*, **136**, 616–634.

Chen, J., G. Q. Huang, J.-Q. Wang, and C. Yang (2019). A cooperative approach to service booking and scheduling in cloud manufacturing. *European Journal of Operational Research*, **273**, 861 – 873.

Curiel, I., G. Pederzoli, and S. Tijs (1989). Sequencing games. *European Journal of Operational Research*, **40**, 344–351.

Curiel, I., J. Potters, R. Prasad, S. Tijs, and B. Veltman (1993). Cooperation in one machine scheduling. *Zeitschrift für Operations Research*, **38**, 113–129.

Curiel, I., J. Potters, R. Prasad, S. Tijs, and B. Veltman (1994). Sequencing and cooperation. *Operations Research*, **42**, 566–568.

Estévez-Fernández, A., P. Borm, P. Calleja, and H. Hamers (2008). Sequencing games with repeated players. *Annals of Operations Research*, **158**, 189–203.

Hamers, H., P. Borm, and S. Tijs (1995). On games corresponding to sequencing situations with ready times. *Mathematical Programming*, **69**, 471–483.

Hamers, H., F. Klijn, and B. Van Velzen (2005). On the convexity of precedence sequencing games. *Annals of Operations Research*, **137**, 161–175.

Hamers, H., J. Suijs, S. Tijs, and P. Borm (1996). The split core for sequencing games. *Games and Economic Behavior*, **15**, 165–176.

Ichiishi, T. (1981). Super-modularity: applications to convex games and to the greedy algorithm for LP. *Journal of Economic Theory*, **25**, 283–286.

Liu, Z., L. Lu, and X. Qi (2018). Cost allocation in rescheduling with machine unavailable period. *European Journal of Operational Research*, **266**, 16 – 28.

Lohmann, E., P. Borm, and M. Slikker (2014). Sequencing situations with Just-in-Time arrival, and related games. *Mathematical Methods of Operations Research*, **80**, 285–305.

Musegaas, M., P. Borm, and M. Quant (2015). Step out–Step in sequencing games. *European Journal of Operational Research*, **246**, 894–906.

Rothkopf, M. (1966). Scheduling independent tasks on parallel processors. *Management Science*, **12**, 437–447.

Saavedra-Nieves, A., J. Schouten, and P. Borm (2020). On interactive sequencing situations with exponential cost functions. *European Journal of Operations Research*, **280**, 78–89.

Shapley, L. (1953). A value for n-person games. *Annals of Mathematics Studies*, **28**, 307–317.

Shapley, L. (1971). Cores of convex games. *International Journal of Game Theory*, **1**, 11–26.

Slikker, M. (2006). Relaxed sequencing games have a nonempty core. *Naval Research Logistics*, **53**, 235–242.

Smith, W. (1956). Various optimizers for single-stage production. *Naval Research Logistics*, **3**, 59–66.

Yang, G., H. Sun, D. Hou, and G. Xu (2019). Games in sequencing situations with externalities. *European Journal of Operational Research*, **278**, 699 – 708.