A perfectness concept for multicriteria games

Borm, P.E.M.; van Megen, F.J.C.; Tijs, S.H.

Published in:
Mathematical Methods of Operations Research

Publication date:
1999

Math Meth Oper Res (1999) 49:401–412

Mathematical Methods
of Operations Research

A perfectness concept for multicriteria games

Peter Borm, Freek van Megen, Stef Tijs

Department of Econometrics and CentER, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands (e-mail: pemborm@kub.nl)

Abstract. This paper considers a refinement of equilibria for multicriteria games based on the perfectness concept of Selten (1975). Existence of perfect equilibrium points is shown and several characterizations are provided. Furthermore, contrary to the result for equilibria for multicriteria games, an example shows that there is no exact correspondence between perfect equilibrium points and the perfect Nash equilibria of the related trade-off games.

Key words: Multicriteria games, perfect equilibria, trade-off games

1 Introduction

Interactive decision situations with more than one decision maker and in which multiple objectives play a role, can be modelled by means of multicriteria games, an alternative name for games with vector payoffs.

A pioneering paper which deals with multicriteria games is Blackwell (1956). Here an analog of the minimax theorem is provided for repeated zero-sum games with vector payoffs based on the concepts of approachability/excludability of subsets of payoff vectors. Shapley (1959) defined the notion of equilibrium points for (one-shot) two-person games with vector payoffs and showed the correspondence between equilibria and Nash equilibria of so-called trade-off games.

For zero sum games this approach is elaborated in Zeleny (1976), Nieuwenhuis (1983) and Corley (1985), for non-zero sum games in Borm, van den Aarssen and Tijs (1988) and Ghose and Prasad (1989). Applications of zero-sum bicriterion games to combat games can be found in Prasad and Ghose (1988).

Manuscript received: March 1997

Existence theorems for equilibria in games with vector payoffs are provided in Wang (1991). Zhao (1991) describes Multiple Objective Mathematical Programming problems and defines equilibria for multiple objective games in strategic form by means of vector-maximization problems. He introduces properly efficient solutions for such games.

Up to now multicriteria analysis has attracted relatively little attention in the literature on games although the decision theoretic counterpart w.r.t. multiobjective programming is rather well-developed. We refer to Cochrane and Zeleny (1973), Cohon (1978) and more recently French et al. (1983), Chankong and Haimes (1983), Steuer (1986) and Vincke (1992). In our opinion multicriteria games can be of use in modelling various real-life situations where several objectives have to be taken into account such as in politics and management decisions, especially in situations in which the agents do not have an a priori opinion on the relative importance of the components of their payoff vectors.

This paper aims for a refinement theory for (weak) equilibria for non zero-sum multi-criteria games based on ideas and insights of the extensive literature on refinements for Nash equilibria in unicriterium games (cf. van Damme (1991)). In particular we introduce perfect equilibrium points in the sense of Selten (1975). Throughout the paper an example of a multicriteria production-inspection game illustrates the notions of equilibrium points, perfect equilibrium points and related trade-off games.

Section 2 contains the necessary definitions on multicriteria games and equilibrium points and recalls the correspondence between equilibria and Nash equilibria of related trade-off games.

Section 3 extends the characterization of a Nash equilibrium, the carriers being subsets of the best reply sets, to equilibrium points for multicriteria games in the sense that the carriers have to be subsets of a so-called efficient pure best reply sets. This characterization also reveals the possibility to order the strategies by means of levels of best reply sets, thus providing a first indication towards a properness concept à la Myerson (1978). The definition of perfect equilibria by means of perturbed games is generalized towards multicriteria games in section 4. Existence of perfect equilibrium points is shown using the characterization of equilibrium points given in section 3.

Section 5 describes two alternative characterizations of perfect equilibria á la van Damme (1991), one of them using the concept of ε-perfectness. Here it is also seen that contrary to the result for equilibrium points, there is no exact correspondence between perfect equilibria and perfect Nash equilibria of the corresponding trade-off games.

Section 6 concludes with some remarks on the case when one would apply a weaker concept of domination, providing stronger equilibria.

2 Equilibrium points for multicriteria games

We consider mixed extensions of n-person finite strategic multicriteria games. These are games with a player set $N = \{1, \ldots, n\}$ in which each player i has a finite set of pure strategies $S_i = \{s_{i1}, s_{i2}, \ldots, s_{im(i)}\}$.

Pure strategy combinations $s \in \Pi_{j=1}^n S_j$ provide to each player i "payoffs" given by an $r(i)$-vector valued function $K_i : \Pi_{j=1}^n S_j \to \mathbb{R}^{r(i)}$, i.e. player i takes $r(i)$ criteria into account. Considering mixed strategies we let $\Delta(S_i)$ represent the set of all probability measures on S_i for each player $i \in N$. The payoff functions K_i are extended to the set $\Pi_{j=1}^n \Delta(S_j)$ of all mixed strategy combinations in the obvious way.

A player's payoff is given by a vector instead off a scalar. Although usually a scalar-valued utility function is used, this definition covers cases where such functions are not explicitly available at first hand. Players may have no fixed opinion about the relative values of the vector coordinates or it may be practically impossible to weight the different objectives (see also Shapley (1959), Zeleny (1976)). Still, in such games, without weights on the objectives, a domination concept can be defined, which leads to a natural generalization of the notion of Nash equilibria in unicriterium games.

A game $\Gamma = \langle \Delta(S_1), \dots, \Delta(S_n), K_1, \dots, K_n \rangle$ as above is called (a mixed extension of) an *n-person* $(r(1), \dots, r(n))$ *multicriteria game*. We denote the class of all such games by $MG(n, (r(1), \dots, r(n)))$.

A strategy combination $\sigma = (\sigma_1, \dots, \sigma_n) \in \Pi_{j=1}^n \Delta(S_j)$ gives to each player a payoff vector as an outcome. We compare these payoff vectors by means of a vector domination concept (cf. Shapley (1959), Borm et al. (1988)).

Let $P \subseteq \mathbb{R}^t$. Then $x \in P$ is undominated (in P) if $\{y \in \mathbb{R}^t \mid y > x\} \cap P = \emptyset$. Here $y > x$ if and only if $y_i > x_i$ for all $i \in \{1, \dots, t\}$. For a strategy $\sigma_{-i} \in \Pi_{j \neq i} \Delta(S_j)$ the polytope $P_i(\Gamma, \sigma_{-i})$ of all *attainable* vector payoffs for player i is defined by

$$P_i(\Gamma, \sigma_{-i}) = \text{conv}\{K_i(s_{i\ell}, \sigma_{-i}) \mid \ell \in \{1, \dots, m(i)\}\},$$

where 'conv' denotes the convex hull operator. Strategies $\sigma_i \in \Delta(S_i)$ which lead to undominated payoff vectors in $P_i(\Gamma, \sigma_{-i})$ are called *best replies* to σ_{-i}. The set of best replies to σ_{-i} in Γ is denoted by $B_i(\Gamma, \sigma_{-i})$.

A strategy combination $\sigma = (\sigma_1, \dots, \sigma_n) \in \Pi_{j=1}^n \Delta(S_j)$ of (mutual) best replies is called an *equilibrium point*. The set of equilibrium points for a game Γ is denoted by $E(\Gamma)$. In case $r(i) = 1$ for all $i \in N$ we deal with a mixed extension of a finite unicriterium game and equilibrium points correspond to Nash equilibria.

Notice that we used the notion of strong dominance and so in fact consider weak equilibria (cf. Shapley (1959)). This guarantees closedness of the equilibrium set, what we will use later on in the existence proof of perfect equilibrium points. In a sense this concept of domination is the strongest option available.

As mentioned in the introduction, Zhao (1991) uses proper efficiency to define equilibria. In example 2.3 below, Zhao's set of equilibria will be equal to the set of perfect equilibria. However, in general the solutions turn out to be different. Examples of games in which the set of properly efficient Nash equilibria is a strict subset of the set of perfect equilibrium points, and vice versa, are not hard to find.

There is a correspondence between the equilibrium points of the game Γ and the Nash equilibria of the corresponding trade-off games in which the various objectives of the players are weighted.

Definition 2.1. *Let* $\Gamma = \langle \Delta(S_1), \dots, \Delta(S_n), K_1, \dots, K_n \rangle \in MG(n, (r(1), \dots, r(n)))$ *be a multicriteria game. Given trade-off vectors* $\lambda(i) \in \Delta_{r(i)}$ *the trade-off unicriterium game* $\Gamma(\lambda)$ *is defined as the n-person game with (mixed) strategy spaces* $\Delta(S_i)$ *and payoff functions* $\bar{K}_i : \Pi_{j=1}^n \Delta(S_j) \to \mathbb{R}$ *given by* $\bar{K}_i(\sigma) =$

$\sum_{t=1}^{r(i)} \lambda(i)_t (K_i(\sigma))_t$ *for all* $i \in N$ *and* $\sigma \in \Pi_{j=1}^n \Delta(S_j)$. *Denote by* $N(\Gamma(\lambda))$ *the set of Nash equilibria for the strategic game* $\Gamma(\lambda)$.

Theorem 2.2. *(Shapley (1959)). For any game* $\Gamma \in MG(n, (r(1), \ldots, r(n)))$:

$$E(\Gamma) = \{\sigma \mid \sigma \in N(\Gamma(\lambda)), \lambda = (\lambda(1), \ldots, \lambda(n)) \in \Pi_{i=1}^n \Delta_{r(i)}\}$$

As a consequence of this theorem and the result that the set of Nash equilibria of a strategic form game is non empty (Nash (1951)), the set of equilibrium points is non empty.

As an illustration we consider a 2-person (2,2) multicriteria game which will also be used later on.

Example 2.3. [A production-inspection game] Consider the 2-person (2,2) multicriteria game $\Gamma = \langle \Delta(S_1), \Delta(S_2), K_1, K_2 \rangle$ where $S_1 = \{I, NI\}$, $S_2 = \{H, NH\}$ and the vector-valued payoff functions K_1 and K_2 are determined by the following two diagrams:

	H	NH	
I	$(-1,1)$	$(c-1,\frac{1}{2})$	$: K_1$.
NI	$(0,1)$	$(0,0)$	

	H	NH	
I	$(-1,1)$	$(-c-1,1)$	$: K_2$.
NI	$(-1,1)$	$(0,0)$	

Here c denotes a real number larger than 1.

One could consider this example as corresponding to a situation of Hygienical labour and Inspection, in which a factory and a bureau of inspection are the players. The factory has two objectives, to achieve some level of hygienical production and to minimize the production costs. The inspection bureau also has two objectives, to minimize inspection costs and to provide an acceptable level of hygiene in production.

The strategies the bureau can take are Inspection and No Inspection (I and NI in the diagrams). Those for the factory are Hygienical production and Non Hygienical production (H and NH). So player 1 corresponds to the bureau and player 2 to the factory. The first coordinate in the payoff vectors for player 1 depicts the negative costs (benefits) of inspection, the second coordinate depicts a satisfaction with the hygienical situation. The first coordinate for the factory depicts extra production costs, the second represents a hygienical satisfaction level. In this context the number c might be interpreted as a penalty which can be imposed if the production fails to be hygienical.

Let $p \in [0,1]$ represent the strategy of player 1 in which the probability upon I is p and $1-p$ is the probability upon NI. For player 2 we assume $q \in [0,1]$ to represent the strategy in which H is played with probability q and NH with $1-q$.

$K_1(1,q) = \left(-qc + c - 1, \frac{1}{2} + \frac{1}{2}q\right)$ and $K_1(0,q) = (0,q)$ for all $q \in [0,1]$, and therefore

$$B_1(\Gamma, q) = \begin{cases} \{1\} & \text{if } 0 \le q < 1 - \frac{1}{c} \\ [0,1] & \text{if } 1 - \frac{1}{c} \le q \le 1 \end{cases}$$

Further, $K_2(p,1) = (-1,1)$ and $K_2(p,0) = (p(-c-1), p)$ for all $p \in [0,1]$, so

$$B_2(\Gamma, p) = \begin{cases} \{1\} & \text{if } \frac{1}{1+c} < p < 1 \\ [0,1] & \text{if } 0 \le p \le \frac{1}{1+c} \text{ or } p = 1 \end{cases}$$

This implies that

$$E(\Gamma) = \left(\left[0, \frac{1}{1+c}\right] \times \left[1 - \frac{1}{c}, 1\right]\right) \cup \left(\left(\frac{1}{1+c}, 1\right) \times \{1\}\right) \cup (\{1\} \times [0,1])$$

The analysis above shows that equilibrium points in this simple game are those in which there is full inspection, those in which the factory produces in a hygienical way with probability 1 and those in which the chance upon inspection is small but the production is hygienical with a fair chance.

3 A characterization of equilibrium points in terms of carriers and efficient best reply sets

For a unicriterium $m \times n$ bimatrix game (A, B) Nash equilibria can be characterized in terms of carriers and best replies: the strategy pair $(p, q) \in \Delta_m \times \Delta_n$ is a Nash equilibrium if and only if

$$C(p) \subseteq PB_1(q) \quad \text{and} \quad C(q) \subseteq PB_2(p),$$

with $C(p) := \{i \in \{1, \ldots, m\} \mid p_i > 0\}$ denoting the carrier of p and $PB_1(q) = \{i \in \{1, \ldots, m\} \mid e_i A q = \max_k e_k A q\}$ denoting the set of pure best replies of player 1 to q. $C(q)$ and $PB_2(q)$ are defined analogously.

A similar characterization for equilibria of multicriteria games is not straightforward since the fact that two pure strategies lead to undominated payoff vectors w.r.t. some fixed strategy combination of the opponent does not imply that all probability measures on these two strategies lead to undominated payoff vectors. However, for multicriteria games we give a characterization of equilibrium points in terms of carriers and so called efficient pure best reply sets.

For a game $\Gamma = \langle \Delta(S_1), \ldots, \Delta(S_n), K_1, \ldots, K_n \rangle \in MG(n, (r(1), \ldots, r(n)))$ and strategies $\sigma = (\sigma_1, \ldots, \sigma_n)$ we introduce the following concepts: $C(\Gamma, \sigma_i) = \{t \in \{1, \ldots, m(i)\} \mid \sigma_i(s_{it}) > 0\}$ denotes the *carrier* of σ_i w.r.t. Γ. $I \subseteq \{1, \ldots, m(i)\}$ is called *efficient* for player i w.r.t. $\sigma_{-i} \in \Pi_{j \neq i} \Delta(S_j)$ in Γ if for all strategies $\sigma_i \in \Delta(S_i)$ with $C(\Gamma, \sigma_i) \subseteq I$ it holds that $K_i(\sigma)$ is undominated in $P_i(\Gamma, \sigma_{-i})$. Notice that for any efficient $I \subseteq \{1, \ldots, m(i)\}$ a subset $K \subseteq I$ is efficient too. $I \subseteq \{1, \ldots, m(i)\}$ is an *efficient pure best reply set* for player i

w.r.t. σ_{-i} in Γ if I is efficient w.r.t. σ_{-i} in Γ and there exists no efficient $K \subseteq \{1,\ldots,m(i)\}$ with $I \subseteq K$ and $I \neq K$.

Clearly, for any efficient set I an efficient pure best reply set $K \subseteq \{1,\ldots,m(i)\}$ exists such that $I \subseteq K$. Let $\mathscr{E}_i(\Gamma,\sigma_{-i})$ be the set of efficient pure best reply sets for player i w.r.t. σ_{-i} in Γ.

Example 3.1. Consider the 2-person (1,2) multicriteria game where the payoff functions for player 1 and player 2 are determined by

	s_{21}	s_{22}	s_{23}
s_{11}	−1	2	0
s_{12}	0	0	0

and

	s_{21}	s_{22}	s_{23}
s_{11}	(−1, 1)	(−3, 1)	(0, 0)
s_{12}	(−1, 1)	(0, 0)	(0, 0)

respectively.

The set $P_2(\Gamma,s_{11})$ of attainable payoff vectors to s_{11} for player 2 equals $\operatorname{conv}\{(-1,1),(-3,1)(0,0)\}$. Clearly $J \in \mathscr{E}_2(\Gamma,s_{11})$ if and only if $J = \{1,2\}$ or $J = \{1,3\}$.

The previous observations are summarized in the next theorem.

Theorem 3.2. *Let $\Gamma \in MG(n,(r(1),\ldots,r(n)))$ and $\sigma \in \Pi_{j=1}^{n}\Delta(S_j)$. Then for all $i \in N$ $\sigma_i \in B_i(\Gamma,\sigma_{-i}) \Longleftrightarrow C(\Gamma,\sigma_i) \subseteq I$ for some $I \in \mathscr{E}_i(\Gamma,\sigma_{-i})$*

This approach also provides an opportunity to order the set of pure strategies into levels of best replies. A description of the procedure is given in section 6.

4 A perfectness concept for multicriteria games

The ideas in this section are related to the work of Selten (1975). We define a perfect equilibrium point as a limit point of a sequence of equilibria of perturbed multicriteria games.

Perturbed games are derived from the original game by demanding that every pure strategy has to be chosen with a positive probability. We therefore call a vector $\varepsilon = (\varepsilon^1,\varepsilon^2,\ldots,\varepsilon^n) \in \Pi_{i=1}^{n}\mathbb{R}^{m(i)}$ a *mistake vector* if $\sum_{t=1}^{m(i)}\varepsilon_t^i < 1$ for all $i \in N$ and $\varepsilon > 0$. Now we define the ε-perturbed game $\Gamma(\varepsilon)$ for a game $\Gamma = \langle \Delta(S_1),\ldots,\Delta(S_n),K_1,\ldots,K_n\rangle$ and a mistake vector $\varepsilon \in \Pi_{i=1}^{n}\mathbb{R}^{m(i)}$. As pure strategy set for player i we take

$$S_i(\varepsilon) = \{s_{i1}(\varepsilon),\ldots,s_{im(i)}(\varepsilon)\} \tag{4.1}$$

where $s_{it}(\varepsilon)$ denotes the mixed strategy in $\Delta(S_i)$ which gives probability ε^i_k to s_{ik} if $k \neq t$ and probability $(1 - \sum_{k \neq t} \varepsilon^i_k)$ to s_{it}.

The payoff functions for the game $\Gamma(\varepsilon)$ are just the functions K_i restricted to the new domain.

The ε-perturbed game $\Gamma(\varepsilon) = \langle \Delta(S_1(\varepsilon)), \ldots, \Delta(S_n(\varepsilon)), K_1, \ldots, K_n \rangle$ itself is an n-person $(r(1), \ldots, r(n))$ multicriteria game, so carriers, payoff polytopes and efficient pure best reply sets are properly defined.

Notice that each mixed strategy in the perturbed game can be identified with a mixed strategy in the original game. Therefore, with minor abuse of notation, we obtain $\Delta(S_i(\varepsilon)) \subset \Delta(S_i)$.

In theorem 4.1 we will show that the efficient pure best reply sets of player i w.r.t. a strategy $\sigma_{-i} \in \Pi_{j \neq i} \Delta(S_j(\varepsilon))$ in both Γ and $\Gamma(\varepsilon)$ coincide. In the proof we will use the following mappings between the original and the perturbed strategy spaces for all $i \in N$. Let $f_i : \Delta(S_i) \rightarrow \Delta(S_i(\varepsilon))$ be defined by

$$f_i(\sigma_i)(s_{it}(\varepsilon)) := \sigma_i(s_{it}) \text{ for all } \sigma_i \in \Delta(S_i) \text{ and } t \in \{1, \ldots, m(i)\}$$

Alternatively, because $\Delta(S_i(\varepsilon)) \subset \Delta(S_i)$, $f_i(\sigma_i)$ can be expressed in the following way

$$f_i(\sigma_i)(s_{it}) = \varepsilon^i_t + \left(1 - \sum_{k=1}^{m(i)} \varepsilon^i_k\right) \sigma_i(s_{it}) \text{ for all } \sigma_i \in \Delta(S_i) \text{ and } t \in \{1, \ldots, m(i)\}$$

Clearly, f_i is *continuous, dominance preserving* and *bijective* where $f_i^{-1} : \Delta(S_i(\varepsilon)) \rightarrow \Delta(S_i)$ is given by

$$f_i^{-1}(\tilde{\sigma}_i)(s_{it}) = \frac{\tilde{\sigma}_i(s_{it}) - \varepsilon^i_t}{\left(1 - \sum_{k=1}^{m(i)} \varepsilon^i_k\right)} \text{ for all } \tilde{\sigma}_i \in \Delta(S_i(\varepsilon)) \text{ and } t \in \{1, \ldots, m(i)\}.$$

Furthermore $C(\Gamma, \sigma_i) = C(\Gamma(\varepsilon), f_i(\sigma_i))$ for all $\sigma_i \in \Delta(S_i)$.

Theorem 4.1. *Let $\Gamma \in MG(n, (r(1), \ldots, r(n)))$, ε a mistake vector in $\Pi_{i=1}^n \mathbb{R}^{m(i)}$ and $\sigma \in \Pi_{j=1}^n \Delta(S_j(\varepsilon))$. Then $\mathscr{E}_i(\Gamma, \sigma_{-i}) = \mathscr{E}_i(\Gamma(\varepsilon), \sigma_{-i})$ for all $i \in N$.*

Proof: Let $i \in N$. It suffices to show that any efficient set I w.r.t. σ_{-i} in Γ is also efficient w.r.t. σ_{-i} in $\Gamma(\varepsilon)$ and conversely.

Take $I \in \mathscr{E}_i(\Gamma, \sigma_{-i})$ and suppose that I is not efficient w.r.t. σ_{-i} in $\Gamma(\varepsilon)$. We can choose $\tilde{\sigma}_i \in \Delta(S_i(\varepsilon))$ with $C(\Gamma(\varepsilon), \tilde{\sigma}_i) \subseteq I$ such that $K_i(\tilde{\sigma}_i, \sigma_{-i})$ is dominated in $P_i(\Gamma(\varepsilon), \sigma_{-i})$. Hence there exists a strategy $\hat{\sigma}_i \in \Delta(S_i(\varepsilon))$ such that $K_i(\hat{\sigma}_i, \sigma_{-i}) - K_i(\tilde{\sigma}_i, \sigma_{-i}) > 0$. Consequently also

$$K_i(f_i^{-1}(\hat{\sigma}_i), \sigma_{-i}) - K_i(f_i^{-1}(\tilde{\sigma}_i), \sigma_{-i})$$
$$= \frac{1}{1 - \sum_{k=1}^{m(1)} \varepsilon^i_k} (K_i(\hat{\sigma}_i, \sigma_{-i}) - K_i(\tilde{\sigma}_i, \sigma_{-i})) > 0.$$

This contradicts the fact that I is efficient w.r.t. σ_{-i} in Γ since

$$C(\Gamma, f_i^{-1}(\tilde{\sigma})) = C(\Gamma(\varepsilon), \tilde{\sigma}_i) \subseteq I.$$

The proof of the converse statement is similar and is left to the reader. □

Definition 4.2. *Let* $\Gamma = \langle \Delta(S_1), \ldots, \Delta(S_n), K_1, \ldots, K_n \rangle \in MG(n, (r(1), \ldots, r(n)))$. *A strategy combination* $\sigma = (\sigma_1, \ldots, \sigma_n) \in \Pi_{j=1}^n \Delta(S_j)$ *is called* **perfect** *for* Γ *if there exist a sequence* $\{\varepsilon(k)\}_{k=1}^\infty$ *of mistake vectors converging to 0 and a sequence* $\{\sigma(k)\}_{k=1}^\infty$ *such that* $\sigma(k) \in E(\Gamma(\varepsilon(k)))$ *for each* k *and* $\lim_{k \to \infty} \sigma(k) = \sigma$.

The following observations can be made.

Theorem 4.3. *Let* $\Gamma \in MG(n, (r(1), \ldots, r(n)))$. *Then*
(1) *there exists at least one perfect strategy combination,*
(2) *if* σ *is perfect for* Γ, *then* σ *is an equilibrium point of* Γ, *and*
(3) *if* $r(i) = 1$ *for all* $i \in N$, *then perfect equilibrium points correspond to perfect Nash equilibria.*

Proof: (3) is obvious and (1) can be proved using the compactness of the strategy space $\Pi_{j=1}^n \Delta(S_j)$. For the proof of (2) we will use theorem 3.2. Let $\sigma \in \Pi_{j=1}^n \Delta(S_j)$ be perfect in Γ. Take a sequence $\{\varepsilon(k)\}_{k=1}^\infty$ of mistake vectors and $\{\sigma(k)\}_{k=1}^\infty$ of strategy combinations such that $\lim_{k \to \infty} \varepsilon(k) = 0$, $\sigma(k) \in E(\Gamma(\varepsilon(k)))$ for all k and $\lim_{k \to \infty} \sigma(k) = \sigma$. By theorem 3.2, we only need to show that $C(\Gamma, \sigma_i) \subseteq I$ for some $I \in \mathscr{E}_i(\Gamma, \sigma_{-i})$. For every $t \in C(\Gamma, \sigma_i)$ and sufficiently large k it holds that $\sigma_i(s_{it}) > \varepsilon_i^t(k)$ and hence $\sigma(k)_i(s_{it}) > \varepsilon_i^t(k)$.

This implies $C(\Gamma, \sigma_i) \subseteq C(\Gamma(\varepsilon(k)), \sigma(k)_i)$ for large k. Since $\sigma(k)$ is an equilibrium point of the perturbed game $\Gamma(\varepsilon(k))$, theorem 3.2 implies the existence of $I^k \in \mathscr{E}_i(\Gamma(\mathscr{E}(k)), \sigma(k)_{-i})$, such that $C(\Gamma(\varepsilon(k)), \sigma(k)_i) \subseteq I^k$. By theorem 4.1 it holds that $\mathscr{E}_i(\Gamma(\varepsilon(k)), \sigma(k)_{-i}) = \mathscr{E}_i(\Gamma, \sigma(k)_{-i})$.

Therefore $C(\Gamma, \sigma_i) \subseteq C(\Gamma(\varepsilon(k)), \sigma(k)_i) \subseteq I^k$ for large k for some $I^k \in \mathscr{E}_i(\Gamma, \sigma(k)_{-i})$. Draw a subsequence $\{\sigma(\ell)\}_{\ell=1}^\infty$ such that $I^\ell = \tilde{I}$ for all ℓ. Since $\lim_{\ell \to \infty} \sigma(\ell)_{-i} = \sigma_{-i}$ and $\tilde{I}$ is efficient for all $\sigma(\ell)_{-i}$ in Γ, $\tilde{I}$ is efficient for σ_{-i} in Γ. So we can find a set $I \in \mathscr{E}_i(\Gamma, \sigma_{-i})$ with $\tilde{I} \subseteq I$. So we may conclude that there is an $I \in \mathscr{E}_i(\Gamma, \sigma_{-i})$ such that $C(\Gamma, \sigma_i) \subseteq \tilde{I} \subseteq I$. □

It is not difficult to show that the set $PE(\Gamma)$ of all perfect equilibria for Γ is closed in $\Pi_{j=1}^n \Delta(S_j)$.

Example 4.4. For the game Γ of example 2.3 we found $\{1\} \times [0,1] \subset E(\Gamma)$. However for any $q \in [0,1)$ the strategy combination $(1, q)$ is not perfect. It is seen that any probability distribution $\tilde{p}$ in $\Delta(S_1(\varepsilon))$ close to $p = 1$ has the property that $B_2(\Gamma(\varepsilon), \tilde{p}) = \{1\}$. (Use theorem 3.2 and 4.1.) This implies that for any sequence of mistake vectors $\{(\varepsilon(k))\}_{k=1}^\infty$ and any sequence $\{(p^k, q^k)\}_{k=1}^\infty$ such that $(p^k, q^k) \in E(\Gamma(\varepsilon(k)))$, $\lim_{k \to \infty} \varepsilon(k) = 0$ it holds that $q^k \to 1$. Notice that all other equilibrium points are perfect.

5 Characterizations of perfect equilibrium points

In this section we provide alternative characterizations of perfect equilibria inspired by the characterizations of perfect Nash equilibria given by van Damme (1991).

For this we first introduce the concept of ε-perfectness for completely mixed strategy combinations.

Definition 5.1. *Let $\Gamma \in MG(n,(r(1),\ldots,r(n)))$ and $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$.*

A strategy combination σ with $C(\Gamma,\sigma_i) = \{1,\ldots,m(i)\}$ for all $i \in N$ is called **ε-perfect** *if there exists for every $i \in N$ an $I_i \in \mathscr{E}_i(\Gamma,\sigma_{-i})$ such that $\sigma_i(s_{it}) \leq \varepsilon$ for all $t \notin I_i$.*

Theorem 5.2. *Let $\Gamma \in MG(n,(r(1),\ldots,r(n)))$ and $\hat{\sigma} \in \Pi_{j=1}^{n}\Delta(S_j)$.*

The following three assertions are equivalent.

(1) *$\hat{\sigma}$ is a perfect equilibrium point for Γ.*

(2) *There is a sequence $\{\varepsilon(k)\}_{k=1}^{\infty}$ of positive real numbers converging to 0 and a sequence of completely mixed strategy combinations $\{\sigma(k)\}_{k=1}^{\infty}$ in $\Pi_{j=1}^{n}\Delta(S_j)$ converging to $\hat{\sigma}$ such that $\sigma(k)$ is $\varepsilon(k)$-perfect for all k.*

(3) *There is a sequence $\{\sigma(k)\}_{k=1}^{\infty}$ of completely mixed strategies such that $\hat{\sigma}_i \in B_i(\Gamma,\sigma(k)_{-i})$ for all k and all $i \in N$ and $\lim_{k\to\infty}\sigma(k) = \hat{\sigma}$.*

Proof: We show that (1) implies (2), (2) implies (3) and (3) implies (1).

(1) $\Rightarrow$ (2): Assume $\hat{\sigma}$ is perfect. Take a sequence $\{\delta(k)\}_{k=1}^{\infty}$ of mistake vectors converging to 0 and a sequence $\{\sigma(k)\}_{k=1}^{\infty}$ of equilibria in the perturbed games $\Gamma(\delta(k))$ with $\lim_{k\to\infty}\sigma(k) = \hat{\sigma}$. Take $\varepsilon(k) = \max\{(\delta(k))_t^i \mid i \in N, t \in \{1,\ldots,m(i)\}\}$ for each k. Then $\lim_{k\to\infty}\varepsilon(k) = 0$ and $\sigma(k)$ is an $\varepsilon(k)$-perfect pair for each k.

(2) $\Rightarrow$ (3): Suppose (2) holds. Take a sequence $\{\varepsilon(k)\}_{k=1}^{\infty}$ of positive real numbers converging to 0 and a sequence $\{\sigma(k)\}_{k=1}^{\infty}$ of $\varepsilon(k)$-perfect pairs tending to $\hat{\sigma}$.

Let $i \in N$. For each $k \in \mathbb{N}$, there is an $I^k \in \mathscr{E}_i(\Gamma,\sigma(k)_{-i})$ with $\sigma(k)_i(s_{it}) \leq \varepsilon(k)$ for all $t \notin I^k$. If $t \in C(\Gamma,\hat{\sigma}_i)$ then there exists a number $N_t \in \mathbb{N}$, chosen large enough, with $\hat{\sigma}_i(s_{it}) > \varepsilon(k)$ and $\sigma(k)_i(s_{it}) > \varepsilon(k)$ for all $k \geq N_t$.

Take $N_{\max} = \max\{N_t \mid t \in C(\Gamma,\hat{\sigma}_i)\}$. For all $k \geq N_{\max}$ and all $t \in C(\Gamma,\hat{\sigma}_i)$ we have $\hat{\sigma}_i(s_{it}) > \varepsilon(k)$, $\sigma(k)_i(s_{it}) > \varepsilon(k)$ and so $C(\Gamma,\hat{\sigma}_i) \subseteq I^k$.

Using theorem 3.2 we find $\hat{\sigma}_i \in B_i(\Gamma,\sigma(k)_{-i})$.

(3) $\Rightarrow$ (1): Let $\{\sigma(k)\}_{k=1}^{\infty}$ be a sequence of completely mixed strategies converging to $\hat{\sigma}$ such that $\hat{\sigma}_i \in B_i(\Gamma,\sigma(k)_{-i})$ for all k and all $i \in N$.

We define for all $i \in N$

$$(\varepsilon(k))_t^i = \begin{cases} \frac{1}{k} & \text{if } t \in C(\Gamma,\hat{\sigma}_i) \\ \sigma(k)_i(s_{it}) & \text{if } t \notin C(\Gamma,\hat{\sigma}_i) \end{cases}$$

Clearly, $\lim_{k\to\infty}(\varepsilon(k))_t^i = 0$ and $((\varepsilon(k))^1,\ldots,(\varepsilon(k))^n) \in \Pi_{i=1}^{n}\mathbb{R}^{m(i)}$ is a mistake vector if k is large enough. It suffices to show that $\sigma(k) \in E(\Gamma(\varepsilon(k)))$ for large k. Let $i \in N$.

For large k, $\sigma(k)_i(s_{it}) > \frac{1}{k} = (\varepsilon(k))_t^i$ if $t \in C(\Gamma,\hat{\sigma}_i)$ and $\sigma(k)_i(s_{it}) = (\varepsilon(k))_t^i$ if $t \notin C(\Gamma,\hat{\sigma}_i)$. This implies that $C(\Gamma(\varepsilon(k)),\sigma(k)_i) = C(\Gamma,\hat{\sigma}_i)$ for large k. Since $\hat{\sigma}_i \in B_i(\Gamma,\sigma(k)_{-i})$ we can find $I^k \in \mathscr{E}_i(\Gamma,\sigma(k)_{-i})$ with $C(\Gamma,\hat{\sigma}_i) \subseteq I^k$. Consequently, $C(\Gamma(\varepsilon(k)),\sigma(k)_i) \subseteq I^k$ for large k.

This implies that $\sigma(k) \in E(\Gamma(\varepsilon(k)))$ for large k. □

Recall that any equilibrium point for a finite multicriteria game Γ is a Nash equilibrium in a related trade-off game and, conversely, that every (trade-off) Nash equilibrium is an equilibrium. For perfect equilibrium points this type of result does not hold: a perfect equilibrium need not be a perfect Nash equilibrium of a trade-off game. Nevertheless the reverse statement can be made: any perfect Nash equilibrium of a trade-off game is a perfect equilibrium.

In theorem 5.3 $PN(\Gamma)$ denotes the set of perfect Nash equilibria for a unicriterium game Γ.

Theorem 5.3. *Let* $\Gamma \in MG(n, (r(1), \ldots, r(n)))$. *Then* $PE(\Gamma) \supseteq \{\sigma \mid \sigma \in PN(\Gamma(\lambda)), \lambda = (\lambda(1), \ldots, \lambda(n)) \in \Pi_{i=1}^{n} \Delta_{r(i)}\}$

Proof: Let $\lambda(i) \in \Delta_{r(i)}$ for all $i \in N$ and $\sigma \in PN(\Gamma(\lambda))$. Take a sequence of mistake vectors $\{\varepsilon(k)\}_{k=1}^{\infty}$ converging to 0 and a sequence of completely mixed strategy combinations $\{\sigma(k)\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} \sigma(k) = \sigma$ and $\sigma(k) \in N(\Gamma(\lambda)(\varepsilon(k)))$.

Then $\sigma(k) \in E(\Gamma(\varepsilon(k)))$ (see Theorem 2.2) and thus $\sigma \in PE(\Gamma)$. □

Example 5.4. For the game of example 2.3 and 4.4 we have that $\left(\frac{1}{1+c}, 1\right)$ constitutes a perfect equilibrium point. The trade-off games for which this is a Nash equilibrium are the games $\Gamma(\lambda)$, where $\lambda(1) = (0, 1)$ and $\lambda(2) = (0, 1)$ or $\lambda(2) = (x, 1-x)$ with $x = \left[\frac{1}{2}, 1\right]$. The pay-off diagram for player 1 is in these cases:

1	$\frac{1}{2}$
1	0

It is clear that for player 1 the strategy $p = \frac{1}{1+c}$ (put probability $\frac{1}{1+c}$ on the first row and $\frac{c}{1+c}$ on the second one) is dominated by the strategy $p = 1$. Therefore it can not be a perfect Nash equilibrium in these trade-off games.

6 Concluding remarks

A next step in refining equilibria for multicriteria games might be inspired on the notion of proper equilibria (Myerson (1978)), using explicitly the possibility to define levels of best reply sets. For $\Gamma = \langle \Delta(S_1), \ldots, \Delta(S_n), K_1, \ldots, K_n \rangle \in MG(n, (r(1), \ldots, (r(n)))$ and $\sigma \in \Pi_{j=1}^{n} \Delta(S_j)$ the first level of best replies of player i w.r.t. σ_{-i} in Γ is the set of all pure strategies contained in the efficient pure best reply sets w.r.t. σ_{-i}. The second level is constructed by considering the best replies if pure strategies in the first level are not taken into account. Formally it can be seen in this way:

$$\begin{cases} M^1(i) := \{1, \ldots, m(i)\} \\ \mathscr{E}_i^1(\Gamma, \sigma_{-i}) := \mathscr{E}_i(\Gamma, \sigma_{-i}) \end{cases}$$

and for every $k \in \mathbb{N}$: $M^k(i) := \{t \in M^{k-1}(i) \mid t \notin I \text{ for all } I \in \mathscr{E}_i^{k-1}(\Gamma, \sigma_{-i})\}$.

$I \subseteq M^k(i)$ is *k-th level efficient* if for all strategies $\sigma_i \in \Delta(S_i)$ with $C(\Gamma, \sigma_i) \subseteq I$ it holds that $K_i(\sigma)$ is not dominated by any $K_i(\hat{\sigma})$ with $C(\Gamma, \hat{\sigma}) \subseteq M^k(i)$. $I \subseteq M^k(i)$ is an *k-th level efficient pure best reply set* if I is k-th level efficient and there is no k-th level efficient set $K \subseteq M^k(i)$ with $I \subseteq K$ and $I \neq K$. $\mathscr{E}_i^k(\Gamma, \sigma_{-i})$ is the set of k-th level efficient pure best reply sets for player i w.r.t. σ_{-i} in Γ.

Alternatively, strictly perfect equilibria (Okada (1984)) or stability concepts as e.g. introduced by Kohlberg and Mertens (1986) might be studied.

A second remark is on the domination concept used in the definition of equilibria. We could have used a weaker domination concept: for $P \subseteq \mathbb{R}^t$ we call $x \in P$ undominated (in P) if $\{y \in \mathbb{R}^t \mid y \geq x\} \cap P = \{x\}$, where $y \geq x$ if and only if $y_i \geq x_i$ for all $i \in \{1, \ldots, t\}$.

For this specific choice we denote the set of (strong) equilibria of a multicriteria game Γ by $SE(\Gamma)$. Clearly $SE(\Gamma) \subseteq E(\Gamma)$. Moreover, it can be shown that

$$\{\sigma \mid \sigma \in N(\Gamma(\lambda)), \lambda = (\lambda(1), \ldots, \lambda(n)) \in \Pi_{i=1}^n \Delta_{r(i)}^0\} \subseteq SE(\Gamma),$$

where $\Delta_{r(i)}^0 = \{\lambda \in \mathbb{R}_+^{r(i)} \mid \sum_{t=1}^{r(i)} \lambda_t = 1, \lambda_t > 0 \text{ all } t\}$.

If we would define perfect strategy combinations similarly to section 4 using the weaker domination concept, one can not immediately conclude that every perfect combination is an equilibrium point, due to the fact that $SE(\Gamma)$ need not be closed. It is (example 5.4) clear that $\left(\frac{1}{1+c}, 1\right)$ is a perfect pair in the strong sense too since it is a limit of totally mixed equilibria. It is however not a (strong) equilibrium point itself, for $K_1(0, 1) = (0, 1)$ and $(-p, 1) = K_1(p, 1)$ for all $p > 0$.

References

Blackwell D (1956). An analog of the minimax theorem for vector payoffs. Pacific Journal of Mathematics 6:1–8

Borm P, Tijs S, van den Aarssen J (1988) Pareto equilibrium in multi-objective games. Methods of Operations Research 60:302–312

Chankong V, Haimes Y (1983) Multi-objective decision making. North-Holland, Amsterdam

Cochrane J, Zeleny M (1973) Multiple criteria decision making. University of South Carolina Press, Columbia

Cohon JL (1978) Multi-objective programming and planning. Academic Press, New York

Corley HN (1985) Games with vector payoffs. Journal of Optimization Theory and Applications 47:491–498

van Damme EEC (1991) Stability and perfection of Nash equilibria. Springer Verlag, Berlin

French S, Hartley R, Thomas LC, White DJ (1983) Multi-objective decision making. Academic Press, London

Ghose D, Prasad UR (1989) Solution concepts in two-person multicriteria games. Journal of Optimization Theory and Applications 63:167–189

Kohlberg E, Mertens JF (1986) On the strategic stability of equilibria. Econometrica 54:1003–1037

Myerson RB (1978) Refinements of the Nash equilibrium point concept. International Journal of game Theory 7:73–80

Nash JF (1951) Non-cooperative games. Annals of Mathematics 54:286–195

Nieuwenhuis JW (1983) Some minimax theorems in vector-valued functions. Journal of Optimization Theory and Applications 40:463–475

Okada A (1984) Strictly perfect equilibrium points of bimatrix games. International Journal of game Theory 13:145–154

Prasad UR, Ghose D (1988) Formulation and analysis of combat problems as zero-sum bicriterion differential games. Journal of Optimization Theory and Applications 59:1–24

Selten R (1975) Reexamination of the perfectness concept for equilibrium points in extensive games. International Journal of Game Theory 4:25–55

Shapley LS (1959) Equilibrium points in games with vector payoffs. Naval Research Logistics Quarterly 1:57–61.

Steuer R (1986) Multiple criteria optimization: theory, computation, and application. John Wiley and Sons, New York

Vincke P (1992) Multicriteria decision-aid. John Wiley and Sons, Chichester

Wang S (1991) An existence theorem of a Pareto equilibrium. Applied Mathematical Letters 3:61–63

Zeleny M (1976) Games with multiple payoffs. International Journal of Game Theory 4:179–191

Zhao J (1991) The equilibria of a multiple objective game. International Journal of Game Theory 20:171–182