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Stochastic Cooperative Games: Superadditivity, Convexity, and Certainty Equivalents

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This paper extends the notions of superadditivity and convexity to stochastic cooperative games. It is shown that convex games are superadditive and have nonempty cores, and that these results also hold in the context of NTU games. Furthermore, a subclass of stochastic cooperative games to which one can associate a deterministic cooperative game is considered. It is shown that such a stochastic cooperative game satisfies properties like nonemptiness of the core, superadditivity, and convexity if and only if the corresponding deterministic game satisfies these properties. *Journal of Economic Literature* Classification Number: C71. © 1999 Academic Press

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1. INTRODUCTION

In general, the payoff of a coalition in a cooperative game is assumed to be known with certainty. In many cases, however, payoffs to coalitions are uncertain. This would not raise a problem, if the agents can await the realizations of the payoffs before deciding which coalitions to form and which allocations to settle on. But if the formation of coalitions and allocations has to take place before the payoffs are realized, standard cooperative game theory can not longer be applied.

Charnes and Granot (1973) considered cooperative games in stochastic characteristic function form. For these games the value $V(S)$ of a coalition S is allowed to be a stochastic variable. It was suggested to allocate the stochastic payoff of the grand coalition in two stages. In the first stage, so called prior payoffs are promised to the agents. These prior payoffs are determined in such a way that there is a fair chance that this promise will be realized. In the second stage the realization of the stochastic payoff is

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awaited and, subsequently, a possibly nonfeasible prior payoff vector has to be adjusted to this realization in some way. This approach was elaborated in Charnes and Granot (1976), Charnes and Granot (1977), and Granot (1977). Most of the adjustment processes use a specific choice of objections as an adjustment base.

Suijs, *et al.* (1995) also considered cooperative games with stochastic payoffs, but in a slightly different way than the authors above. The most significant differences between the model introduced by Charnes *et al.* (1973) and the model introduced by Suijs *et al.* (1995) is that the latter explicitly incorporates preferences on stochastic payoffs for each agent and allows each coalition to choose from several actions. In Suijs *et al.* (1995) the reader is provided with some applications of the model and suggestions for possible choices of preferences. Moreover, it is shown that for a special class of cooperative games with stochastic payoffs, the core of the game is nonempty if and only if the game is balanced.

In this paper we continue on the model introduced by Suijs *et al.* (1995). We extend the definitions of superadditivity and convexity for TU games to stochastic cooperative games. Furthermore, we show that a convex stochastic cooperative game is superadditive and has a nonempty core. We also consider a specific subclass of games in which the preferences of the agents are such that a stochastic payoff can be represented by its certainty equivalent, i.e., the amount of money for which an agent is indifferent between receiving the stochastic payoff and this amount. Using these certainty equivalents we can associate to each stochastic cooperative game within this class a deterministic cooperative game. Subsequently, we show that the core of the stochastic cooperative game is nonempty if and only if the core of the associated deterministic game is nonempty. Moreover, we show the equivalence between the notions of superadditivity and convexity for the two related games, respectively.

The paper is organized as follows. In Section 2 we recall the definitions concerning stochastic cooperative games and introduce the notions of superadditivity and convexity. Furthermore, we show that convexity implies a nonempty core. In Section 3 we present a subclass of games for which certainty equivalents are well defined and introduce for each stochastic cooperative game the corresponding deterministic game. Moreover, a relation is stated between these two types of games regarding the nonemptiness of the core, superadditivity and convexity.

2. STOCHASTIC COOPERATIVE GAMES

Let us first recall some of the definitions concerning stochastic cooperative games as introduced by Suijs *et al.* (1995). A stochastic cooperative game is described by a tuple $\Gamma = (N, \{A_S\}_{S\subset N}, \{X_S\}_{S\subset N}, \{\succsim_i\}_{i\in N})$, where N is the set of agents, A_S the nonempty and finite set of actions a coalition S can take, $X_S : A_S \to L^1(\mathbb{R})$ the payoff function of coalition S, assigning to each action $a \in A_S$ a stochastic payoff $X_S(a) \in L^1(\mathbb{R})$ with finite expectation, and $\succsim_i$ the preference relation of agent i over the set $L^1(\mathbb{R})$ of stochastic payoffs with finite expectation. We assume that for each player the preferences are complete, transitive, and continuous.[1] Furthermore, we assume that $\mathbb{P}(X_\varnothing(a) = 0) = 1$ for all $a \in A_\varnothing$. The class of all cooperative games with stochastic payoffs with agent set N is denoted by $SG(N)$. To simplify notation, however, we restrict our attention to the case that each coalition only has one action to take, that is, $|A_S| = 1$ for all $S \subset N$. So we can denote a stochastic cooperative game by $\Gamma = (N, \{X_S\}_{S\subset N}, \{\succsim_i\}_{i\in N})$. For a more extensive discussion of this model and some examples, we refer to Suijs *et al.* (1995) and Suijs *et al.* (1998).

An allocation of a stochastic payoff X_S to coalition S is described by a pair $(d, r) \in \mathbb{R}^S \times \mathbb{R}^S$ such that $\sum_{i\in S} d_i \leq 0$ and $\sum_{i\in S} r_i = 1$ and $r_i \geq 0$ for all $i \in S$. The payoff to agent $i \in S$ according to the allocation (d, r) equals $d_i + r_i X_S$. This payoff will henceforth be denoted by $(d, r)_i$. Note that the payoff $(d, r)_i$ is a stochastic variable. The set of all allocations for coalition S is denoted by $Z(S)$, and the set of all individually rational allocations by $IR(S)$. So, $IR(S) = \{(d, r) \in Z(S) | \forall_{i\in S} : d_i + r_i X_S \succsim_i X_{\{i\}}\}$.

The core of a stochastic cooperative game is defined as follows. Let $\Gamma \in SG(N)$ and $(d, r) \in Z(N)$. Then the allocation (d, r) is a core allocation for the game Γ if for each coalition S there is no allocation $(\tilde{d}, \tilde{r}) \in Z(S)$ such that $\tilde{d}_i + \tilde{r}_i X_S \succ_i d_i + r_i X_S$ for all $i \in S$. The set of all core allocations for Γ is denoted by $Core(\Gamma)$.

Now, let us introduce superadditivity for stochastic cooperative games. For this, recall that for both TU and NTU games the underlying idea of superadditivity is that two disjoint coalitions can do (weakly) better by forming one coalition. Therefore, we propose the following definition of superadditivity, which is not only applicable to stochastic cooperative games, but also to TU and NTU-games. Let $\Gamma \in SG(N)$. Then Γ is called superadditive if for any disjoint $S, T \subset N$ it holds that for every $(d^S, r^S) \in Z(S)$ and every $(d^T, r^T) \in Z(T)$ there exists an allocation $(d^{S\cup T}, r^{S\cup T}) \in Z(S \cup T)$ such that

$$\begin{aligned} d_i^{S\cup T} + r_i^{S\cup T} X_{S\cup T} &\succsim_i d_i^S + r_i^S X_S \quad \text{for all } i \in S, \\ d_i^{S\cup T} + r_i^{S\cup T} X_{S\cup T} &\succsim_i d_i^T + r_i^T X_T \quad \text{for all } i \in T. \end{aligned} \tag{1}$$

So whatever allocation the coalitions S and T agree on separately, they can always (weakly) improve their payoffs by forming one large coalition. For-

[1]The preferences $\succsim$ are continuous if for all $X \in L^1(\mathbb{R})$ the sets $\{Y \in L^1(\mathbb{R}) | Y \succsim X\}$ and $\{Y \in L^1(\mathbb{R}) | Y \precsim X\}$ are closed.

mulated in the context of NTU games, this definition reads as follows. For all disjoint $S, T \subset N$ it holds that for each allocation $x^S \in V(S)$ and each allocation $x^T \in V(T)$ there exists an allocation $x^{S \cup T} \in V(S \cup T)$ such that $x_i^{S \cup T} \geq x_i^S$ for all $i \in S$ and $x_i^{S \cup T} \geq x_i^T$ for all $i \in T$. Then it is not difficult to check that this definition is equivalent to the traditional definition of superadditivity for NTU games, i.e., for all disjoint $S, T \subset N$ it holds that $V(S) \times V(T) \subset V(S \cup T)$.

For our definition of convexity, we use the following definition from TU games as a basis. A TU game (N, v) is called convex if for each $U \subset N$ and each $S \subset T \subset N \backslash U$ it holds that

$$v(S \cup U) - v(S) \leq v(T \cup U) - v(T). \tag{2}$$

This means that for a coalition it is more profitable to join a larger coalition. Now, we apply this idea to stochastic cooperative games. Let $\Gamma \in SG(N)$. Then Γ is called convex if for each $U \subset N$ and each $S \subset T \subset N \backslash U$ the following statement is true. For all $(d^S, r^S) \in IR(S)$, all $(d^T, r^T) \in IR(T)$ and all $(d^{S \cup U}, r^{S \cup U}) \in Z(S \cup U)$ satisfying

$$d_i^{S \cup U} + r_i^{S \cup U} X_{S \cup U} \succsim_i d_i^S + r_i^S X_S,$$

for all $i \in S$, there exists an allocation $(d^{T \cup U}, r^{T \cup U}) \in Z(T \cup U)$ such that

$$\begin{aligned} d_i^{T \cup U} + r_i^{T \cup U} X_{T \cup U} &\succsim_i d_i^T + r_i^T X_T && \text{for all } i \in T, \text{ and} \\ d_i^{T \cup U} + r_i^{T \cup U} X_{T \cup U} &\succsim_i d_i^{S \cup U} + r_i^{S \cup U} X_{S \cup U} && \text{for all } i \in U. \end{aligned} \tag{3}$$

So whatever individually rational allocation the coalitions S and T agree on separately, then given an allocation for coalition $S \cup U$ such that coalition S is willing to let coalition U join, the members of coalition U can obtain (weakly) better payoffs by joining the larger coalition T.

Note that for a meaningful definition of convexity we need to consider individually rational allocations $(d^S, r^S) \in IR(S)$ and $(d^T, r^T) \in IR(T)$ instead of arbitrary allocations $(d^S, r^S) \in Z(S)$ and $(d^T, r^T) \in Z(T)$. To see this, take $U = \{i\}$. Since $(d^S, r^S) \in IR(S)$, the payoffs to the agents in coalition S are bounded from below, which implies that the payoff $(d^{S \cup \{i\}}, r^{S \cup \{i\}})_i$ for agent i is bounded from above. If one would allow $(d^S, r^S) \in Z(S)$, we can choose very bad payoffs (read not individually rational) for the agents $i \in S$. As a consequence, one can choose the allocation $(d^{S \cup \{i\}}, r^{S \cup \{i\}}) \in Z(S \cup \{i\})$ such that the payoff $(d^{S \cup \{i\}}, r^{S \cup \{i\}})_i$ is rated very good by agent i. But this would make it very hard if not impossible to find the allocation $(d^{T \cup \{i\}}, r^{T \cup \{i\}})$ that improves agent i's payoff. In that case, a stochastic cooperative game can fail the convexity condition due to an allocation $(d^S, r^S) \in Z(S) \backslash IR(S)$ that coalition S will never agree upon in the first place. Therefore, it is more appropriate to consider individually rational allocations only.

It is a straightforward exercise to show that in the context of TU games the definitions of convexity provided in (2) and the TU formulation of (3) are equivalent. As will become clear later on, the NTU formulation of convexity as given in (3) is not equivalent to either ordinal convexity (Vilkov, 1977) or cardinal convexity (Sharkey, 1981) for NTU games.

By taking $S = \varnothing$ in the definition of convexity it follows immediately that convex games are superadditive. The next theorem shows that the core of a convex stochastic cooperative game is nonempty.

THEOREM 2.1. *Let* Γ *be a stochastic cooperative game. If* Γ *is convex, then* $Core(\Gamma) \neq \varnothing$.

Proof. To improve the readability of this proof let us abbreviate the notation (d, r) of an allocation to a vector x. We construct an allocation in $Z(N)$ and show that it belongs to the core of the stochastic cooperative game.

Take $i \in N$, $S \subset N \backslash \{i\}$ and $x \in Z(S)$. Define

$$B(S, x, i) = \{y \in Z(S \cup \{i\}) | \forall_{j \in S} : y_j \succsim_j x_j\}$$

as the set of all allocations for coalition $S \cup \{i\}$ which all members of S weakly prefer to the allocation $x \in Z(S)$. Note that $B(S, x, i)$ is nonempty. Furthermore, let $b(S, x, i) \in B(S, x, i)$ be the most preferred allocation for player i in this set, that is, $b_i(S, x, i) \succsim_i y_i$ for all $y \in B(S, x, i)$. Since the preferences are assumed to be complete, transitive and continuous such a best allocation exists. The allocation x^n is now constructed as follows. Let $x^1 \in Z(\{1\})$ be individually rational. Thus $x^1 = X_{\{1\}}$. Take $x^2 \in Z(\{1, 2\})$ such that $x^2 = b(\{1\}, x^1, 2)$. Next, take $x^3 \in Z(\{1, 2, 3\})$ such that $x^3 = b(\{1, 2\}, x^2, 3)$. Continuing this procedure yields the allocation $x^n \in Z(N)$.

In order to prove that x^n is a core allocation, we prove for $k = 1, 2, \ldots, n$ that x^k is a core allocation for the corresponding subgame Γ^k with player set $\{1, 2, \ldots, k\}$. The proof of the latter statement goes by induction on k.

For $k = 1$ it is obvious that x^1 belongs to the core of Γ^1. Now suppose that $x^k \in Core(\Gamma^k)$ for $k = 1, 2, \ldots, m-1$. Let $x^m = b(\{1, 2, \ldots, m-1\}, x^{m-1}, m)$. To prove that $x^m \in Core(\Gamma^m)$, consider a coalition $S \subset \{1, 2, \ldots, m-1\}$.

Since $x^{m-1} \in Core(\Gamma^{m-1})$ and $x_j^m \succsim_j x_j^{m-1}$ for all $j \in S$, it follows that coalition S has no incentive to leave the coalition $\{1, 2, \ldots, m\}$.

Next, we also show that the coalition $S \cup \{m\}$ has no incentive to leave the coalition $\{1, 2, \ldots, m\}$ if x^m is allocated. For this, let $y^S \in IR(S)$ be such that $y_j^S \sim_j X_{\{j\}}$ for all $j \in S$ and let $y^{S \cup \{m\}} = b(S, y^S, m)$. Note that y^S exists by the continuity of $\{\succsim_i\}_{i \in S}$. So $y_m^{S \cup \{m\}}$ is the "best" payoff player m can obtain when cooperating with coalition S. Next, let $T =$

$\{1, 2, \ldots, m-1\}$ and $U = \{m\}$. Since $x^{m-1} \in IR(\{1, 2, \ldots, m-1\})$, it follows from convexity that there exists an allocation $z \in Z(\{1, 2, \ldots, m\})$ such that $z_j \succsim_j x_j^{m-1}$ for all $j \in \{1, 2, \ldots, m-1\}$ and $z_m \succsim_m y_m^{S \cup \{m\}}$. Since $z \in B(\{1, 2, \ldots, m-1\}, x^{m-1}, m)$ and $x^m = b(\{1, 2, \ldots, m-1\}, x^{m-1}, m)$ we have that $x_m^m \succsim_m z_m \succsim_m y_m^{S \cup \{m\}}$. From the fact that $y_m^{S \cup \{m\}}$ is the "best" payoff player m can obtain when cooperating with coalition S, there exists no individually rational allocation for coalition S that yields player m a strictly better payoff than x_m^m. Hence, coalition $S \cup \{m\}$ has no incentive to part company with the coalition $\{1, 2, \ldots, m\}$ when x^m is allocated. Consequently, we have that $x^m \in Core(\Gamma^m)$. Taking $m = n$ then gives that $x^n \in Core(\Gamma^n) = Core(\Gamma)$. Thus, $Core(\Gamma) \neq \varnothing$. ■

Note that we have actually shown that every subgame of a convex stochastic cooperative game has a nonempty core. Thus convexity implies that the game is totally balanced. Furthermore, the core allocation constructed in the proof can be interpreted as a "marginal vector." Since the order $1, 2, \ldots, n$ used in the proof is taken arbitrarily, it follows that for stochastic cooperative games satisfying the convexity condition formulated in (3), all marginal vectors belong to the core. The reverse of this statement, however, is not true. The following example shows that if all marginal vectors belong to the core, then the stochastic cooperative game need not be convex.

EXAMPLE 2.2. Let Γ be a 3-person stochastic cooperative game with $X_{\{i\}} = 0$ for all $i \in N$, $X_S = 7/4$ for $S = \{1, 2\}, \{1, 3\}$, $X_N = 3\,1/2$ and $X_{\{2,3\}} = Y$, where Y is a random variable attaining the value 0 or 4 with equal probability. Furthermore, each player maximizes his expected utility according to the following utility function:

$$u_i(t) = \begin{cases} 2t, & \text{if } t \leq 2, \\ t + 3, & \text{if } t > 2 \end{cases}$$

The expected utilities the players can obtain in the various coalitions are depicted in Fig. 1. Note that this game is superadditive. The marginal vectors of this game yield expected utilities of $(3\,1/2, 3\,1/2, 0)$, $(3\,1/2, 0, 3\,1/2)$ and $(0, 3\,1/2, 3\,1/2)$, respectively. Moreover, the marginal vectors belong to the core of the game. This game, however, is not convex. To see this, let $S = \{3\}$, $T = \{2, 3\}$ and $U = \{1\}$. Take $(d^S, r^S) \in Z(S)$ arbitrary and take $(d^{S \cup U}, r^{S \cup U}) \in Z(S \cup U)$ such that $d = (0, 0)$ and $r = (1, 0)$, i.e., player 1 receives 7/4 and player 2 receives 0. The expected utilities of this allocation equal $3\,1/2$ for player 1 and 0 for player 2 (see also the point x in Fig. 1). Next, let $(d^T, r^T) \in Z(T)$ be such that $d = (0, 0)$ and $r = (1/2, 1/2)$. So, both players 2 and 3 receive $1/2Y$. The expected utilities then equal 2 for both players (see also the point y in Fig. 1). Now if the game is convex

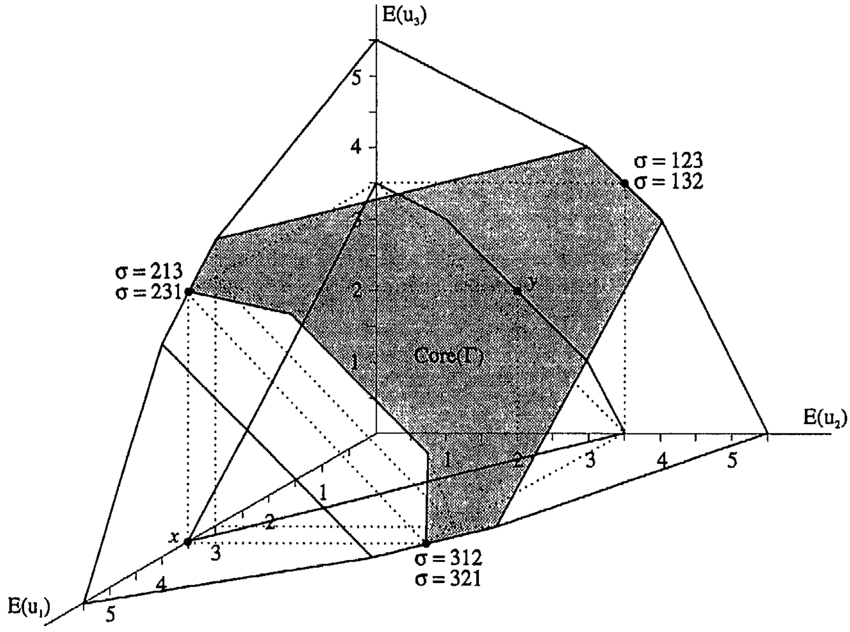

FIG. 1. A non-convex stochastic cooperative game.

there must exist an allocation $(d^{T\cup U}, r^{T\cup U}) \in Z(T \cup U)$ such that the expected utilities are at least 2 for players 2 and 3 and at least 3 1/2 for player 1. Since $T \cup U = N$, this means that we have to allocate $X_N = 3\,1/2$. If players 2 and 3 must receive an expected utility of at least 2, this implies that each player receives at least 1. Since only 3 1/2 can be allocated, this implies that player 1 can receive at most 1 1/2, yielding a utility of at most 3. Hence, the game cannot be convex.

The previous results also hold for NTU games if we formulate convexity based on (3) in the following way. An NTU game (N, V) satisfies convexity if for every $U \subset N$ and every $S \subset T \subset N\backslash U$ the following statement holds. For every individually rational $x^S \in V(S)$, every individually rational $x^T \in V(T)$ and every $x^{S\cup U} \in V(S \cup U)$ such that $x_i^{S\cup U} \geq x_i^S$ for all $i \in S$, there exists $x^{T\cup U} \in V(T \cup U)$ satisfying

$$x_i^{T\cup U} \geq x_i^T \quad \text{for all } i \in T, \text{ and}$$
$$x_i^{T\cup U} \geq x_i^{S\cup U} \quad \text{for all } i \in U.$$

In this way, convex NTU games are totally balanced and every marginal vector belongs to the core. This also implies that our definition of convexity is not equivalent to ordinal or cardinal convexity, since for both notions of convexity not all marginal vectors belong to the core.

3. CERTAINTY EQUIVALENTS

In this section we focus on a special class of stochastic cooperative games to which one can associate a deterministic cooperative game. For the games in this subclass the preferences $\{\succsim_i\}_{i\in N}$ are such that for each $i \in N$ there exists a function $m_i : L^1(\mathbb{R}) \to \mathbb{R}$ satisfying

(M1) for all $X, Y \in L^1(\mathbb{R}) : X \succsim_i Y$ if and only if $m_i(X) \geq m_i(Y)$,

(M2) for all $d \in \mathbb{R} : m_i(d) = d$,

(M3) for all $X \in L^1(\mathbb{R}) : m_i(X - m_i(X)) = 0$,

(M4) for all $X \in L^1(\mathbb{R})$ and all $d, d' \in \mathbb{R}$ with $d < d' : m_i(d + X) < m_i(d' + X)$.

The interpretation is that agent i is indifferent between receiving the amount of money $m_i(X)$ with certainty and receiving the stochastic payoff X. The amount $m_i(X)$ is called the certainty equivalent of X. Condition (M1) states that agent i weakly prefers one stochastic payoff to another one if and only if the certainty equivalent of the first is greater than or equal to the certainty equivalent of the latter. Condition (M2) states that the certainty equivalent of a deterministic payoff d equals d itself. From the conditions (M1) and (M2) it then follows that $X \sim_i m_i(X)$ for all $X \in L^1(\mathbb{R})$. Condition (M3) states that an agent is indifferent between receiving the stochastic payoff $X - m_i(X)$ and receiving the payoff zero. Finally, condition (M4) states that the preferences over stochastic payoffs of the form $d + X$ are monotonically increasing in d. Note that condition (M3) is not implied by the other three conditions as the following example shows.

EXAMPLE 3.1. Take the utility function equal to $u_i(x) = \sqrt{x+1} - 1$ and let $\succsim_i$ be such that $X \succsim_i Y$ if and only if $E(u_i(X)) \geq E(u_i(Y))$. Take X such that $\mathbb{P}(X = 0) = \mathbb{P}(X = 1) = 1/2$. Then $E(u_i(X)) = 1/2(\sqrt{2} - 1)$ and $m_i(X) = u_i^{-1}(E(u_i(X))) = 1/2(\sqrt{2} - 1/2)$. Note that m_i satisfies (M1), (M2) and (M4). However, we have that

$$m_i(X - m_i(X)) = \tfrac{1}{2}\left(\sqrt{\tfrac{5}{4} - \tfrac{1}{2}\sqrt{2}} - 1\right) + \tfrac{1}{2}\left(\sqrt{\tfrac{7}{4} - \tfrac{1}{2}\sqrt{2}} - 1\right) \neq 0.$$

Conditions (M3) and (M4) are equivalent with condition (M5) below:

(M5) for all $X \in L^1(\mathbb{R})$ and all $d \in \mathbb{R} : m_i(d + X) = d + m_i(X)$.

Obviously, condition (M5) implies conditions (M3) and (M4). For the converse, suppose that $m_i(d+X) > d + m_i(X)$ for some $d \in \mathbb{R}$ and some $X \in L^1(\mathbb{R})$. Then we get the following contradiction:

$$\begin{aligned} 0 &= m_i(d + X - (m_i(d+X))) < m_i(d + X - (d + m_i(X))) \\ &= m_i(X - m_i(X)) = 0. \end{aligned}$$

Here the first and the last equality follow from condition (M3) and the inequality follows from condition (M4). Of course, a similar argument holds if one would suppose that $m_i(d+X) < d + m_i(X)$.

The set of all stochastic cooperative games with agent set N for which the preference relations $(\succsim_i)_{i\in N}$ satisfy the conditions (M1)–(M4) is denoted by $MG(N)$. Furthermore, the subclass of $MG(N)$ where all payoffs are deterministic is denoted by $DG(N)$.

Next, we give three examples of preferences for which the certainty equivalent satisfies conditions (M1)–(M4).

EXAMPLE 3.2. Consider expected utility maximizers based on the utility function $u(t) = a_1 + a_2 \cdot e^{bt}$, $(t \in \mathbb{R})$, where $a_1, \in \mathbb{R}$ and $a_2, b \neq 0$. Then the conditions (M1)–(M4) are satisfied. The certainty equivalent of $X \in L^1(\mathbb{R})$ can be defined by $m(X) = u^{-1}(E(u(X)))$. It is easy to check that m satisfies conditions (M1), (M2) and (M4). For condition (M3), let $X \in L^1(\mathbb{R})$. Then $u^{-1}(\tau) = (1/b)\log((\tau - a_1)/a_2)$ and

$$\begin{aligned} m(X - m(X)) &= u^{-1}(E(u(X - m(X)))) \\ &= \frac{1}{b}\log\left(\frac{1}{a_2}\left(\int a_1 + a_2 \cdot e^{b(t-m(X))}\, dF_X(t) - a_1\right)\right) \\ &= -m(X) + \frac{1}{b}\log\left(\frac{1}{a_2}\left(\int a_1 + a_2 \cdot e^{bt} dF_X(t) - a_1\right)\right) \\ &= -m(X) + m(X) = 0. \end{aligned}$$

Finally, note that u is a monotonically increasing and concave function if $a_2 < 0$ and $b < 0$. Consequently, an agent with such a utility function is risk averse[2].

EXAMPLE 3.3. Let the preferences $\succsim_\alpha$ be such that for $X, Y \in L^1(\mathbb{R})$ it holds that $X \succsim_\alpha Y$ if $u_\alpha^X \geq u_\alpha^Y$, where $\alpha \in (0,1)$ and $u_\alpha^X = \sup\{t \mid F_X(t) < \alpha\}$ denotes the α-quantile of the distribution function F_X of X. This type of preference appears, for example, in insurance. They are used by insurance

[2]A utility function u induces risk averse, risk neutral or risk loving behavior if for all $X \in L^1(\mathbb{R})$ it holds that $u(E(X)) \geq E(u(X))$, $u(E(X)) = E(u(X))$, or $u(E(X)) \leq E(u(X))$, respectively.

companies if the premium is determined on the basis of the percentile principle (also called chance constrained premium). With the certainty equivalent of $X \in L^1(\mathbb{R})$ given by $m(X) = u_\alpha^X$, then conditions (M1)–(M4) are satisfied.

EXAMPLE 3.4. Let the preferences $\succsim^b$ be such that for $X, Y \in L^1(\mathbb{R})$ it holds that $X \succsim^b Y$ if $E(X) + b\sqrt{V(X)} \geq E(Y) + b\sqrt{V(Y)}$, where $V(X)$ denotes the variance of X. This type of preference can be found, for example, in portfolio decision theory, where an agent's evaluation of a portfolio depends on the expected revenue of the portfolio and the standard deviation of the revenue. With the certainty equivalent of $X \in L^1(\mathbb{R})$ given by $m(X) = E(X) + b\sqrt{V(X)}$, then conditions (M1)–(M4) are satisfied.

For each game in $MG(N)$ we define an associated game in $DG(N)$ in the following way. Let Γ be an element of $MG(N)$. Consider a coalition S and an allocation (d, r) for this coalition. Since the stochastic payoff equals $(d, r)_i = d_i + r_i X_S$ for each agent $i \in S$, the certainty equivalent for agent i equals $m_i((d, r)_i)$. Consequently, the certainty equivalent of (d, r) for S equals $\sum_{i \in S} m_i((d, r)_i)$. Moreover, the following property shows that an allocation $(d, r) \in Z(S)$ is Pareto optimal for coalition S if and only if

$$\sum_{i \in S} m_i((d, r)_i) = \max_{(\hat{d}, \hat{r}) \in Z(S)} \sum_{i \in S} m_i((\hat{d}, \hat{r})_i),$$

PROPOSITION 3.5. *Let $S \subset N$. An allocation $(d, r) \in Z(S)$ is Pareto optimal for S if and only if*

$$\sum_{i \in S} m_i((d, r)_i) = \max_{(\hat{d}, \hat{r}) \in Z(S)} \sum_{i \in S} m_i((\hat{d}, \hat{r})_i). \tag{4}$$

Proof. We start with proving the "if" part of the statement. Let $(d, r) \in Z(S)$ satisfy expression (4). Suppose that (d, r) is not a Pareto optimal allocation. Then there exists an allocation $(\tilde{d}, \tilde{r}) \in Z(S)$ such that $(\tilde{d}, \tilde{r})_i \succ_i (d, r)_i$, for all $i \in S$. From condition (M1) it then follows that

$$\max_{(\hat{d}, \hat{r}) \in Z(S)} \sum_{i \in S} m_i((\hat{d}, \hat{r})_i) = \sum_{i \in S} m_i((d, r)_i) < \sum_{i \in S} m_i((\tilde{d}, \tilde{r})_i).$$

This is a contradiction. Consequently, we must have that (d, r) is a Pareto optimal allocation.

For the "only if" part, let $(d, r) \in Z(S)$ be Pareto optimal and suppose that (d, r) does not satisfy expression (4). Then there exists an allocation $(\hat{d}, \hat{r}) \in Z(S)$ such that $\sum_{i \in S} m_i((d, r)_i) < \sum_{i \in S} m_i((\hat{d}, \hat{r})_i)$. Next, define $(\tilde{d}, \hat{r}) \in Z(S)$ by

$$\tilde{d}_i = \hat{d}_i - m_i((\hat{d}, \hat{r})_i) + m_i((d, r)_i) + \frac{1}{|N|}\left(\sum_{j \in S} m_j((\hat{d}, \hat{r})_j) - \sum_{j \in S} m_j((d, r)_i)\right),$$

for all $i \in S$. Then for all $i \in S$ it holds that

$$\begin{aligned} m_i((\tilde{d},\hat{r})_i) &= m_i(\tilde{d}_i + \hat{r}_i X_S) \\ &= m_i(\hat{d}_i + \hat{r}_i X_S) - m_i((\hat{d},\hat{r})_i)) \\ &\quad + m_i((d,r)_i) + \frac{1}{|N|}\Big(\sum_{j\in S} m_j((\hat{d},\hat{r})_j) - \sum_{j\in S} m_j((d,r)_i)\Big) \\ &= m_i((d,r)_i) + \frac{1}{|N|}\Big(\sum_{j\in S} m_j((\hat{d},\hat{r})_j) - \sum_{j\in S} m_j((d,r)_i)\Big) \\ &> m_i((d,r)_i), \end{aligned}$$

where the second equality follows from linearity condition (M5) and the third equality follows from condition (M3). Then condition (M1) implies that $(\tilde{d},\hat{r})_i \succ_i (d,r)_i$ for all $i \in S$. This contradicts the Pareto optimality of (d,r). Hence, expression (4) is satisfied. ■

Consequently, it is optimal for a coalition S to maximize the expression $\sum_{i\in S} m_i((d,r)_i)$ over all possible allocations. Therefore we define the deterministic payoff of a coalition S as the maximum of $\sum_{i\in S} m_i((d,r)_i)$ over all allocations $(d,r) \in Z(S)$. Note that if $\sum_{i\in S} m_i((d,r)_i)$ is continuous in r this maximum always exists. Indeed, from the linearity condition (M5) it follows that

$$\sum_{i\in S} m_i((d,r)_i) = \sum_{i\in S}(d_i + m_i(r_i X_S)) = \sum_{i\in S} m_i(r_i X_S).$$

Consequently we have that $\max_{(d,r)\in Z(S)} \sum_{i\in S} m_i((d,r)_i)$ is independent of the choice of d. Hence, the maximum is actually taken over the compact set $\{r \in [0,1]^S \mid \sum_{i\in S} r_i = 1\}$.

We define for each stochastic cooperative game Γ with stochastic payoffs given by $\Gamma = (N, \{X_S\}_{S\subset N}, \{\succsim_i\}_{i\in N}) \in MG(N)$ the associated cooperative game Δ_Γ with deterministic payoffs by $\Delta_\Gamma(N) = (N, \{x_S\}_{S\subset N}, \{\succsim_i\}_{i\in N}) \in DG(N)$ with

$$x_S = \max_{(d,r)\in Z(S)} \sum_{i\in S} m_i((d,r)_i), \tag{5}$$

for all $S \subset N$.

Next, we show that a stochastic cooperative game satisfies properties like nonemptiness of the core, superadditivity and convexity if and only if the corresponding deterministic game also satisfies these properties. The proofs of these propositions are given in the Appendix.

PROPOSITION 3.6. *Let $\Gamma \in MG(N)$. Then*

$$Core(\Gamma) \neq \emptyset \text{ if and only if } Core(\Delta_\Gamma) \neq \emptyset. \tag{6}$$

Note that expression (6) in Proposition 3.6 can be replaced by a similar statement in terms of allocations, i.e., if $(d, r) \in Z(N)$ and $y \in \mathbb{R}^N$ are such that $m_i((d, r)_i) = y_i$ for all $i \in N$, then

$$(d, r) \in Core(\Gamma) \text{ if and only if } y \in Core(\Delta_\Gamma).$$

Moreover, it is not difficult to show that $y \in Core(\Delta_\Gamma)$ if and only if $\sum_{i \in S} y_i \geq x_S$ for all $S \subset N$ and $\sum_{i \in N} y_i = x_N$. So, for the class of games $MG(N)$ the problem of finding a core allocation of a stochastic cooperative game is reduced to the problem of finding a core allocation of the corresponding deterministic game, which is a similar problem as finding a core allocation of a transferable utility game.

The following two propositions state a similar result as Proposition 3.6 with respect to superadditivity and convexity.

PROPOSITION 3.7. *Let $\Gamma \in MG(N)$. Then Γ is superadditive if and only if Δ_Γ is superadditive.*

PROPOSITION 3.8. *Let $\Gamma \in MG(N)$. Then Γ is convex if and only if Δ_Γ is convex.*

EXAMPLE 3.9. Let Γ be a 3-person stochastic cooperative game with $X_{\{i\}} = 0$ for all $i \in N$, $X_N = 2$ and $X_S = Y$ for all 2-person coalitions with Y the uniform distribution on $[0, 2]$. Furthermore, let the players be expected utility maximizers with utility function $u_i(t) = -e^{-t/\alpha_i}$ where $\alpha_1 = 1$, $\alpha_2 = 2$ and $\alpha_3 = 4$. Now we can calculate the deterministic game Δ_Γ by applying Proposition 3.5 and the following result from Wilson (1968), which says that an allocation $(d, r) \in Z(S)$ is Pareto optimal if and only if $\sum_{i \in S} d_i = 0$ and $r_i = \alpha_i / (\sum_{j \in S} \alpha_j)$ for all $i \in S$. This yields that $x_{\{i\}} = 0$ for all $i \in N$, $x_{\{1,2\}} = 0.9446$, $x_{\{1,3\}} = 0.9667$, $x_{\{2,3\}} = 0.9722$ and $x_{\{1,2,3\}} = 2$. Since the deterministic game Δ_G is a convex game , Proposition 3.8 implies that the game Γ is convex.

Finally, it is not difficult to show that the results stated in this section still hold when the definition of an allocation is adjusted in the following way. Instead of (d, r) one could define an allocation for coalition S as a tuple $(d, Y) \in \mathbb{R}^S \times L^1(\mathbb{R})^S$ where $d \in \mathbb{R}^S$ is such that $\sum_{i \in S} d_i \leq 0$ and $Y \in L^1(\mathbb{R})^S$ is such that $\sum_{i \in S} Y_i = X_S$. The stochastic payoff of agent $i \in S$ then equals $d_i + Y_i$.

APPENDIX

Proof of Proposition 3.6. Let $\Gamma \in MG(N)$ such that $Core(\Delta_\Gamma) = \emptyset$. Suppose $(d, r) \in Core(\Gamma)$. Since $Core(\Delta_\Gamma) = \emptyset$, there exists a coalition $S \subset N$ such that $\sum_{i \in S} m_i((d, r)_i) < x_S$. From Proposition 3.5 it then follows that coalition S can improve the allocation (d, r). Hence, $(d, r) \notin Core(\Gamma)$. Consequently, we must have $Core(\Gamma) = \emptyset$.

Next, let $Core(\Gamma) = \emptyset$ and suppose that $y \in Core(\Delta_\Gamma)$. Let $(d, r) \in Z(N)$ be such that $\sum_{i \in N} m_i((d, r)_i) = x_N$. Define $(\tilde{d}, \tilde{r}) \in Z(N)$ by $\tilde{d}_i = y_i - m_i((d, r)_i) + d_i$, for all $i \in N$ and $\tilde{r}_i = r_i$ for all $i \in N$. Then we have

$$\begin{aligned} m_i((\tilde{d}, \tilde{r})_i) &= \tilde{d}_i + m_i(\tilde{r}_i X_N) \\ &= y_i - m_i((d, r)_i) + m_i(d_i + r_i X_N) \\ &= y_i - m_i((d, r)_i) + m_i((d, r)_i) = y_i, \end{aligned}$$

for all $i \in N$. Since $Core(\Gamma) = \emptyset$, there exists a coalition $S \subset N$ with an allocation $(\hat{d}, \hat{r}) \in Z(S)$ such that $(\hat{d}, \hat{r})_i \succ_i (\tilde{d}, \tilde{r})_i$ or, equivalently, $m_i((\hat{d}, \hat{r})_i) > m_i((\tilde{d}, \tilde{r})_i)$ holds for all $i \in S$. But this leads to the following contradiction:

$$x_S \leq \sum_{i \in S} y_i = \sum_{i \in S} m_i((\tilde{d}, \tilde{r})_i) < \sum_{i \in N} m_i((\hat{d}, \hat{r})_i) \leq x_S.$$

Hence, $y \notin Core(\Delta_\Gamma)$, thus $Core(\Delta_\Gamma) = \emptyset$. ■

Proof of Proposition 3.7. Let Γ be superadditive. Take $S, T \subset N$ disjoint and let $(d^S, r^S) \in Z(S)$ and $(d^T, r^T) \in Z(T)$ be Pareto optimal, that is, $\sum_{i \in S} m_i((d^S, r^S)_i) = x_S$ and $\sum_{i \in T} m_i((d^T, r^T)_i) = x_T$. Then superadditivity implies that there exists $(d^{S \cup T}, r^{S \cup T}) \in Z(S \cup T)$ such that $(d^{S \cup T}, r^{S \cup T})_i \succsim_i (d^S, r^S)_i$ for all $i \in S$ and $(d^{S \cup T}, r^{S \cup T})_i \succsim_i (d^T, r^T)_i$ for all $i \in T$. Hence, it holds that

$$\begin{aligned} x_{S \cup T} &\geq \sum_{i \in S \cup T} m_i((d^{S \cup T}, r^{S \cup T})_i) \\ &\geq \sum_{i \in S} m_i((d^S, r^S)_i) + \sum_{i \in T} m_i((d^T, r^T)_i) = x_S + x_T. \end{aligned}$$

Hence, Δ_Γ is superadditive.

Let Δ_Γ be superadditive. Take $S, T \subset N$ disjoint and let $(d^S, r^S) \in Z(S)$ and $(d^T, r^T) \in Z(T)$. So, $\sum_{i \in S} m_i((d^S, r^S)_i) \leq x_S$ and $\sum_{i \in T} m_i((d^T, r^T)_i) \leq x_T$. Now, let $(d^{S \cup T}, r^{S \cup T}) \in Z(S \cup T)$ be Pareto optimal and define

$$\delta_i = \begin{cases} m_i(d^S, r^S)_i) + d_i^{S \cup T} - m_i(d^{S \cup T}, r^{S \cup T})_i) & \text{if } i \in S, \\ m_i(d^T, r^T)_i) + d_i^{S \cup T} - m_i(d^{S \cup T}, r^{S \cup T})_i) & \text{if } i \in T. \end{cases}$$

Then applying condition (M5) yields that $m_i((\delta, r^{S \cup T})_i) = m_i((d^S, r^S)_i)$ if $i \in S$ and $m_i((\delta, r^{S \cup T})_i) = m_i((d^T, r^T)_i)$ if $i \in T$. Since

$$\sum_{i \in S \cup T} m_i((\delta_i, r^{S \cup T})_i) \leq x_S + x_T \leq x_{S \cup T},$$

there exists a Pareto optimal allocation $(\hat{d}, \hat{r}) \in Z(S \cup T)$ (see Proposition 3.5) which everyone (weakly) prefers to $(\delta, r^{S \cup T})$. Hence, Γ is superadditive. ■

Proof of Proposition 3.8. Let Γ be a convex game. Take $U \in N$ and $S \subset T \subset N \backslash U$. Next, let $(d^S, r^S) \in IR(S)$, $(d^T, r^T) \in IR(T)$ be Pareto optimal for S and T, respectively. Now, take $(d^{S\cup U}, r^{S\cup U}) \in Z(S \cup U)$ such that it is Pareto optimal and $(d^S, r^S)_i \sim_i (d^{S\cup U}, r^{S\cup U})_i$ for all $i \in S$. By the convexity of Γ there exists a Pareto optimal allocation $(d^{T\cup U}, r^{T\cup U}) \in Z(T \cup U)$ which (weakly) improves everyone's payoff. Hence,

$$\begin{aligned} x_{S\cup U} - x_S &= \sum_{i \in S\cup U} m_i((d^{S\cup U}, r^{S\cup U})_i) - \sum_{i\in S} m_i((d^S, r^S)_i) \\ &= \sum_{i\in U} m_i((d^{S\cup U}, r^{S\cup U})_i) \\ &\leq \sum_{i\in U} m_i((d^{T\cup U}, r^{T\cup U})_i) \\ &= x_{T\cup U} - \sum_{i\in T} m_i((d^{T\cup U}, r^{T\cup U})_i) \\ &\leq x_{T\cup U} - x_T. \end{aligned}$$

Hence, Δ_Γ is convex.

Let Δ_Γ be convex, $U \in N$ and $S \subset T \subset N \backslash U$. Take $(d^S, r^S) \in IR(S)$, $(d^T, r^T) \in IR(T)$ and $(d^{S\cup U}, r^{S\cup U}) \in Z(S \cup U)$ such that each allocation is Pareto optimal and $(d^{S\cup U}, r^{S\cup U})_i \succsim_i (d^S, r^S)_i$ for all $i \in S$. Note that such an allocation exists by the superadditivity of Γ. Moreover,

$$\sum_{i\in U} m_i((d^{S\cup U}, r^{S\cup U})_i) \leq x_{S\cup U} - x_S. \tag{7}$$

Next, take $(d^{T\cup U}, r^{T\cup U}) \in Z(T \cup U)$ such that it is Pareto optimal and $(d^{T\cup U}, r^{T\cup U})_i \sim_i (d^T, r^T)_i$ for all $i \in S$. That such an allocation indeed exists follows from the superadditivity of Γ and the fact that the certainty equivalents satisfy (M5). Define $\delta_i = d_i^{T\cup U}$ for all $i \in T$ and

$$\begin{aligned} \delta_i = {} & m_i((d^{S\cup U}, r^{S\cup U})_i) + d_i^{T\cup U} - m_i((d^{T\cup U}, r^{T\cup U})_i) \\ & + \frac{1}{|U|}\left(x_{T\cup U} - x_T - \sum_{i\in U} m_i((d^{S\cup U}, r^{S\cup U})_i)\right), \end{aligned}$$

for all $i \in U$. From

$$\begin{aligned} \sum_{i\in T\cup U} \delta_i &= x_{T\cup U} - x_T + \sum_{i\in T\cup U} d_i^{T\cup U} - \sum_{i\in U} m_i((d^{T\cup U}, r^{T\cup U})_i) \\ &\leq x_{T\cup U} - x_T - \sum_{i\in U} m_i((d^{T\cup U}, r^{T\cup U})_i) \\ &= x_{T\cup U} - x_T - \sum_{i\in T\cup U} m_i((d^{T\cup U}, r^{T\cup U})_i) + \sum_{i\in T} m_i((d^{T\cup U}, r^{T\cup U})_i) \\ &= x_{T\cup U} - x_T - x_{T\cup U} + x_T = 0, \end{aligned}$$

it follows that $(\delta, r^{T\cup U}) \in Z(T \cup U)$. Then applying condition (M5) yields for all $i \in U$ that

$$m_i(\delta, r^{T\cup U})_i) = m_i((d^{S\cup U}, r^{S\cup U})_i) + \frac{1}{|U|}\left(x_{T\cup U} - x_T - \sum_{i\in U} m_i((d^{S\cup U}, r^{S\cup U})_i)\right).$$

Since $x_{T\cup U} - x_T - \sum_{i\in U} m_i((d^{S\cup U}, r^{S\cup U})_i) \geq 0$ by (7) and the convexity of Δ_Γ, it follows that $m_i(\delta, r^{T\cup U})_i) \geq m_i((d^{S\cup U}, r^{S\cup U})_i)$ and, consequently, that $(\delta, r^{T\cup U})_i \succsim_i (d^{S\cup U}, r^{S\cup U})_i$ for all $i \in U$. Hence, Γ is convex. ■

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