On the open-loop Nash equilibrium in LQ-games

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by

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Abstract
In this paper we consider open-loop Nash equilibria of the linear-quadratic differential game. We present both necessary and sufficient conditions for existence of a unique solution for the finite-planning horizon case, and show that there exist situations where the set of associated Riccati differential equations has no solution, whereas the problem does have an equilibrium. The pursued analyses allows a simple study of convergence of the equilibrium strategy if the planning horizon expands. Conditions are given under which this strategy converges. A detailed study of the infinite planning horizon case is given. In particular we show that, in general, this problem has not a unique equilibrium. Furthermore, we show that the limit of the above mentioned converged strategy may be not an equilibrium for the infinite planning horizon problem. Particular attention is paid to computational aspects and the scalar case.

Keywords: Linear quadratic games, open-loop Nash equilibrium, solvability conditions, Riccati equations

I. Introduction

During the last decade there has been an increasing interest to study several problems in economics using a dynamic game theoretical setting. In particular in the area of environmental economics and macro-economic policy coordination this is a very natural framework to model problems (see e.g. de Zeeuw et al. (1991), Mäler (1992), Kaitala et al. (1992) and Dockner et al. (1985), Tabellini (1986), Fershtman et al. (1987), Petit (1989), Levine et al. (1994), van Aarle et al. (1995), Neck et al. (1995), Douven et al (1996)). In, e.g., policy coordination problems usually two basic questions arise i.e., first, are policies coordinated and, second, which information do the participating parties have. Usually both these points are rather unclear and, therefore, strategies for different possible scenarios are calculated and compared with eachother. One of these scenarios is the so-called open-loop strategy. This scenario can be interpreted as that the parties simultaneously determine their strategy, next submit their strategies to some authority who then enforces these plans as binding commitments. So, this strategy is based on the assumption that the parties act non-cooperatively and that the only information they have on the model is its present state and the model structure. In other words, open-loop decisions concern control functions of time only. Obviously, since according this scenario the participating parties can not react to eachother's policies, its economic relevance is mostly rather limited. However, as a benchmark to see how much parties can gain by playing other strategies, it plays a fundamental role. Due to its analytic tractability the open-loop Nash equilibrium strategy is in particular very popular for problems where the underlying model can be described by a (set of) linear differential equation(s) and the individual objectives, the parties are striving for, can be approximated by functions which quadratically penalize deviations from some (equilibrium) targets. Under the assumption that the parties only have a finite-planning horizon, this problem was first modeled and solved in a mathematically rigorous way by Starr and Ho in (1969). However, due to some inaccurate formulations it is, even in nowadays literature, an often encountered misunderstanding that this problem always has a unique Nash equilibrium strategy which can be obtained in terms of the solutions of a set of coupled matrix differential equations resembling (but more complicated than) the matrix Riccati equations which arise in optimal regulator theory. Eisele, who extended the Hilbert space approach of this problem taken by Lukes et al (1971), in (1982) already noted that there are some misleading formulations in the literature. But, probably due to the rather abstract approach he took, this point was not noted in the mainstream literature. So, in other words, there exist situations where the set of coupled matrix differential equations has no solution, whereas the problem does have an equilibrium. We will present such an example here and use the more direct simple Hamiltonian

approach to analyze the problem. In addition to its simplicity this approach has the advantage that it also permits an elementary geometric study of convergence of the equilibrium strategy if the planning horizon expands. Like in the optimal regulator theory it turns out that under some conditions it can be shown that this strategy converges. One nice property of this converged solution is, as we will see, that it is rather easy to calculate and much easier to implement than any finite planning horizon equilibrium solution. One would expect that this (converged) solution also solves the problem if the parties consider an infinite-planning horizon. Remarkably, however, the author was not able to trace a reference in literature dealing with this subject in a rigorous mathematical way. Particularly in the economic literature one usually sticks to considering the limiting behaviour of the discounted version of the above mentioned finite-planning horizon solution, and imposes some additional constraints (e.g. the no-Ponzi game condition (see e.g. van Aarle et al. (1995)) or transversality condition (see e.g. Neck et al. (1995))) on the solution of this problem.
The reason for studying an infinite-planning horizon can be motivated from at least two reasons. First, in economic growth theory it is usually difficult to justify the assumption that a firm (or government) has a finite-planning horizon t_f; for, why should it ignore profits earned after t_f (or utility of generations alive beyond t_f). Second, we will see that from a computational point of view the equilibrium strategies are much easier to implement and analyze than those for a finite-planning horizon.
We will solve here the infinite-planning horizon problem in a rigorous way. We like to point out already two remarkable points that we will see. First, we will see that although the problem may have a unique equilibrium strategy for an arbitrary finite-planning horizon, there may exist more than one equilibrium solution for the infinite-planning horizon case and, second, the limit of this unique finite-planning horizon equilibrium solution may be not a solution for the infinite-planning horizon problem. On the other hand we will see that it can be easily verified whether the limiting solution of the finite-planning horizon problem solves also the infinite-planning horizon case.
The outline of the paper is as follows. In section two we start by stating the problem analysed in this paper and show how both a necessary and sufficient condition, in terms of a rank condition on a matrix, can be derived for the existence of a unique open-loop Nash equilibrium using the Hamiltonian approach. Moreover, we present the relationship that exists between solvability of a set of Riccati differential equations and solvability of the problem. Before we present the convergence results of the finite-planning horizon equilibrium solution in section 4, we first consider the algebraic equations associated with the set of Riccati differential equations, and their solutions. In section 3 we show how all solutions of these equations can be determined from the eigenstructure of a certain matrix M. It turns out that the eigenval-

ues of the associated closed-loop system, obtained by applying the limiting equilibrium strategy, are completely determined by the eigenvalues of this matrix M. A number of the results presented in sections 3 an 4 are also reported by Abou-Kandil et al. (1993). Their presentation is, however, of a more analytic nature. Since the easiest way to present our results on the infinite-planning horizon is in geometric terms, it is more convenient to have a geometric formulation of their results too. Therefore we choose to give here a selfcontained geometric exposition including their results. The results on the infinite-planning horizon case are discussed in section 5. In particular we show that if the participating parties discount their future objectives, then the finite-planning horizon equilibrium solution converges to a limit. Generically, this limit is the unique solution to the infinite-plannning horizon case if the discount factor is large enough. Finally, in section 6 we study the scalar case which is of particular interest for many economic applications. We show that in the scalar case, under a mild regularity condition, everything works out fine. This, in the sense that the finite-planning equilibrium solution can be obtained by solving the set of Riccati differential equations and that the equilibrium solution converges to a stationary policy which stabilizes the closed-loop dynamics of the system. Furthermore this strategy is the unique equilibrium strategy of the infinite-planning horizon problem. By a stationary policy we mean in this context a policy which can be rewritten as a linear time-invariant function of the state of the system.
The paper ends with some concluding remarks. Parts of this paper were also reported in Engwerda et al. (1994,1995), Weeren (1995) and Engwerda (1996).

II. The finite-planning horizon case

In this paper we consider the problem where two parties (henceforth called players) try to minimize their individual quadratic performance criterion. Each player controls a different set of inputs to a single system. The system is described by a differential equation of arbitrary order. As already mentioned in the introduction we assume that both players have to formulate their strategy already at the moment the system starts to evolve, and this strategy can not be changed once the system runs. So, the players have to minimize their performance criterion based on the information that they only know the differential equation and its initial state. We are looking now for combinations of pairs of strategies of both players which are secure against any attempt by one player to unilaterally alter his strategy. That is, for those pairs of strategies which are such that if one player deviates from his strategy he will only lose. In the literature on dynamic games this problem is well-known as the open-loop Nash non-zero-sum linear quadratic differential game (see e.g. Starr and Ho (1969), Simaan and Cruz (1973), Başar and Olsder (1982) or Abou-Kandil and Bertrand (1986)). Formally the system we consider is as follows:

$$\dot{x} = Ax + B_1u_1 + B_2u_2,\ x(0) = x_0. \tag{1}$$

Here x is the n-dimensional state of the system, u_i is an m_i-dimensional (control) vector player i can manipulate, x_0 is the arbitrarily chosen initial state of the system, A, B_1, and B_2 are constant matrices of appropriate dimensions, and $\dot{x}$ denotes the time derivative of x.
The performance criterion player $i = 1, 2$ aims to minimize is:

$$J_1(u_1, u_2) := \frac{1}{2}x(t_f)^T K_{1f} x(t_f) + \frac{1}{2}\int_0^{t_f} \{x(t)^T Q_1 x(t) + u_1(t)^T R_{11} u_1(t) + u_2(t)^T R_{12} u_2(t)\} dt,$$

and

$$J_2(u_1, u_2) := \frac{1}{2}x(t_f)^T K_{2f} x(t_f) + \frac{1}{2}\int_0^{t_f} \{x(t)^T Q_2 x(t) + u_1(t)^T R_{21} u_1(t) + u_2(t)^T R_{22} u_2(t)\} dt.$$

All matrices that occur in the performance criteria are symmetric. Moreover, both Q_i and K_{if} are semi-positive definite and R_{ii} are positive definite.
In this section we consider in detail the existence of a unique open-loop Nash equilibrium of this differential game. Due to the stated assumptions both cost functionals $J_i, i = 1, 2$, are strictly convex functions of u_i for all admissible control functions $u_j, j \neq i$ and for all x_0. This implies that the conditions following from the minimum principle are both necessary and sufficient (see e.g. Başar and Olsder (1982, section 6.5)).
Minimization of the Hamiltonian

$$H_i = (x^T Q_i x + u_1^T R_{i1} u_1 + u_2^T R_{i2} u_2) + \psi_i^T (Ax + B_1u_1 + B_2u_2),\ i = 1, 2$$

with respect to u_i yields the optimality conditions (see e.g. Başar and Olsder (1982) or Abou-Kandil and Bertrand (1986)):

$$u_1^*(t) = -R_{11}^{-1} B_1^T \psi_1(t) \tag{2}$$

$$u_2^*(t) = -R_{22}^{-1} B_2^T \psi_2(t), \tag{3}$$

where the n-dimensional vectors $\psi_1(t)$ and $\psi_2(t)$ satisfy

$$\dot{\psi}_1(t) = -Q_1 x(t) - A^T \psi_1(t),\ \text{with } \psi_1(t_f) = K_{1f} x(t_f)$$

$$\dot{\psi}_2(t) = -Q_2 x(t) - A^T \psi_2(t),\ \text{with } \psi_2(t_f) = K_{2f} x(t_f)$$

and

$$\dot{x}(t) = Ax(t) - S_1\psi_1(t) - S_2\psi_2(t);\ x(0) = x_0.$$

Here $S_i = B_i R_{ii}^{-1} B_i^T, i = 1, 2$.
In other words, the problem has a unique open-loop Nash equilibrium if and only if the differential equation

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ \psi_1(t) \\ \psi_2(t) \end{pmatrix} = -\begin{pmatrix} -A \; S_1 \; S_2 \\ Q_1 \; A^T \; 0 \\ Q_2 \; 0 \; A^T \end{pmatrix}\begin{pmatrix} x(t) \\ \psi_1(t) \\ \psi_2(t) \end{pmatrix}$$

with boundary conditions $x(0) = x_0,\ \psi_1(t_f) - K_{1f}x(t_f) = 0$ and $\psi_2(t_f) - K_{2f}x(t_f) = 0$, has a unique solution. Denoting the state variable $(x^T(t)\ \psi_1^T(t)\ \psi_2^T(t))^T$ by $y(t)$, we can rewrite this two-point boundary value problem in the standard form

$$\dot{y}(t) = -My(t), \text{ with } Py(0) + Qy(t_f) = (x_0^T\ 0\ 0)^T, \tag{4}$$

where $M = \begin{pmatrix} -A & S_1 & S_2 \\ Q_1 & A^T & 0 \\ Q_2 & 0 & A^T \end{pmatrix}$, $P = \begin{pmatrix} I\,0\,0 \\ 0\,0\,0 \\ 0\,0\,0 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & 0 & 0 \\ -K_{1f} & I & 0 \\ -K_{2f} & 0 & I \end{pmatrix}$.
From (4) we have immediately that problem (1) has a unique open-loop Nash equilibrium if and only if

$$(P + Qe^{-Mt_f})y(0) = (x_0^T\ 0\ 0)^T,$$

has a unique solution for an arbitrary choice of x_0. Or, equivalently,

$$(Pe^{Mt_f} + Q)e^{-Mt_f}y(0) = (x_0^T\ 0\ 0)^T, \tag{5}$$

is uniquely solvable, for an arbitrary choice of x_0. It is easily verified that this is equivalent with the requirement that $Pe^{Mt_f} + Q$ is invertible.
Using the following notation:

$$H(t_f) := W_{11}(t_f) + W_{12}(t_f)K_{1f} + W_{13}(t_f)K_{2f},$$

with $W(t_f) = (W_{ij}(t_f))\ \{i, j = 1, 2, 3;\ W_{ij} \in R^{n\times n}\} := \exp(Mt_f)$,
elementary matrix analysis then shows that (see also Engwerda et al. (1994))

Theorem 1:
The two-player linear quadratic differential game (1) has a unique open-loop Nash equilibrium for every initial state if and only if matrix $H(t_f)$ is invertible. Moreover, the open-loop Nash equilibrium solution as well as the associated state trajectory can be calculated from the linear two-point boundary value problem (4). □

Next, consider the following set of coupled asymmetric Riccati-type differential equations:

$$\dot{K}_1 = -A^T K_1 - K_1 A - Q_1 + K_1 S_1 K_1 + K_1 S_2 K_2;\ K_1(t_f) = K_{1f} \tag{6}$$

$$\dot{K}_2 = -A^T K_2 - K_2 A - Q_2 + K_2 S_2 K_2 + K_2 S_1 K_1;\; K_2(t_f) = K_{2f} \quad (7)$$

<u>Remark 2:</u>

1) In the sequel we will denote solutions satisfying these equations (6,7) by $K_i(t, t_f)$. Also the notation $K_i(t)$ (or even K_i) will be used if it is clear which endpoint t_f is meant.

2) Note that the solutions $K_i(t, t_f)$ are, in general, not symmetric since both equations (6,7) contain just a term $K_i S_j K_j$ and no factor $K_j S_j K_i$.

3) From the equations (6,7) it is clear that whenever $K_i(t, t_f)$ exists, then for any $\delta > 0$, also $K_i(t + \delta, t_f + \delta)$ exists and equals $K_i(t, t_f)$. That is, the differential equations are time-invariant.

Let $K_i(t)$ satisfy this set of Riccati equations and assume that player i uses the strategy

$$u_1(t) = -R_{11}^{-1} B_1^T K_1(t) \Phi(t, 0) x_0 \quad (8)$$

$$u_2(t) = -R_{22}^{-1} B_2^T K_2(t) \Phi(t, 0) x_0, \quad (9)$$

where $\Phi(t, 0)$ is the solution of the transition equation $\dot{\Phi}(t, 0) = (A - S_1 K_1(t) - S_2 K_2(t))\Phi(t, 0);\ \Phi(0, 0) = I$.
Now, define $\psi_i(t) := K_i(t)\Phi(t, 0)x_0$. Then, obviously $\dot{\psi}_i(t) = \dot{K}_i(t)\Phi(t, 0)x_0 + K_i(t)\dot{\Phi}(t, 0)x_0$.
Substitution of $\dot{K}_i$ from (6,7) and $\dot{\Phi}(t, 0)$ yields

$$\dot{\psi}_i = (-A^T K_i - Q_i)\Phi(t, 0)x_0 = -A^T \psi_i - Q_i \Phi(t, 0)x_0.$$

It is now easily verified that $x(t) := \Phi(t, 0)x_0, \psi_1(t), \psi_2(t)$ satisfy the two-point boundary value problem (4), for an arbitrarily choice of x_0. Or, stated differently, (5) has a solution for an arbitrary choice of x_0. Elementary linear algebra shows that consequently $H(t_f)$ must be invertible. So, according theorem 1 the game has a unique equilibrium. Concluding, this gives a robust proof of the next theorem stated by Starr and Ho in (1969):

<u>Theorem 3:</u>
If the set of Riccati equations (6,7) has a solution then the two-player linear quadratic differential game (1) has a unique open-loop Nash equilibrium for every initial state.
Moreover, the equilibrium strategies are then given by (8,9). □

The following example shows that there exist situations where the set of Riccati differential equations (6,7) does not have a solution, whereas there exists an open-loop Nash equilibrium for the game.

Example 4:

Let $A = \begin{pmatrix} -1 & 0 \\ 0 & -5/22 \end{pmatrix}$, $B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $Q_2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $R_{11} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$, and $R_{22} = 1$.
Now, choose $t_f = 0.1$. Then, numerical calculation shows

$$H(0.1) = \begin{pmatrix} 1.1155 & 0.0051 \\ 0.0051 & 1.0230 \end{pmatrix} + \begin{pmatrix} 0.1007 & 0.1047 \\ 0.0964 & 0.2002 \end{pmatrix} K_{1f} +$$

$$\begin{pmatrix} 0.1005 & 0 \\ 0.0002 & 0 \end{pmatrix} K_{2f} =: V * (I \ K_{1f} \ K_{2f})^T.$$

Now, choose $K_{1f} = \begin{pmatrix} 1 & h_1 \\ h_1 & h_1^2+1 \end{pmatrix}$, with $h_1 = \frac{-V(1,2)-V(2,3)-V(2,5)*10}{V(2,4)}$, and $K_{2f} = \begin{pmatrix} 10 & h_2 \\ h_2 & \frac{h_2^2+1}{10} \end{pmatrix}$, with $h_2 = \frac{-V(2,2)-V(2,3)*K_{1f}(1,2)-V(2,4)*K_{1f}(2,2)}{V(2,5)}$. Then, clearly, both K_{1f} and K_{2f} are positive definite whereas the last row of $H(0.1)$ contains, by construction, only zeros. That is, $H(0.1) = \begin{pmatrix} 2.1673 & -752.6945 \\ 0 & 0 \end{pmatrix}$ is not invertible.
So, according to theorem 1 the game has not a unique open-loop Nash equilibrium for every initial state. Using the converse statement of theorem 3, this implies that the corresponding set of Riccati differential equations has no solution.
Next consider $H(0.11)$. Numerical calculation shows that with the system parameters as chosen above, $H(0.11)$ is invertible. So, according to theorem 1 again, the game does have an open-loop Nash equilibrium for $t_f = 0.11$. Since in this example $K_i(t, 0.1)$ does not exist for all $t \in [0, 0.1]$, we conclude (see remark 2.3) that $K_i(t, 0.11)$ neither exists for all $t \in [0.01, 0.11]$. So, the game does have a solution, whereas the set of Riccati differential equations has no solution. $\square$

Note that the above theorems are in fact local results. That is, they make just statements concerning existence of an equilibrium strategy for a fixed endpoint t_f in time. Below we show that existence of an equilibrium strategy for the game defined on the interval $[0, t_f]$ for all $t_f \in [0, t_1]$ is equivalent to the existence of a solution to the set of Riccati differential equations (6,7) on the interval $[0, t_1]$.
One part of this conjecture is rather immediate. Assume that we know that the set of Riccati differential equations has a solution on $[0, t_1]$. Then, see remark 2.3, also a solution $K_i(t, t_f)$ exists to this set of equations on the

interval $[0, t_f]$, for every point $t_f \in [0, t_1]$. So, according theorem 3 there will exist an open-loop equilibrium strategy for the game defined on the interval $[0, t_f]$, for every choice of $t_f \in [0, t_1]$.
Next, consider the other part of this conjecture. That is, assume that the open-loop game has a solution on the interval $[0, t_f]$ for all $t_f \in [0, t_1]$. Then, in particular, it follows from theorem 1 and (5) that

$$y_0 = e^{Mt_1} \begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix} H^{-1}(t_1)x_0.$$

Since $y(t) = e^{-Mt}y_0$, it follows that the entries of $y(t)$ can be rewritten as

$$x(t) = (I\ 0\ 0)e^{M(t_1-t)} \begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix} H^{-1}(t_1)x_0 \tag{10}$$

$$\psi_1(t) = (0\ I\ 0)e^{M(t_1-t)} \begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix} H^{-1}(t_1)x_0 \tag{11}$$

$$\psi_2(t) = (0\ 0\ I)e^{M(t_1-t)} \begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix} H^{-1}(t_1)x_0 \tag{12}$$

Using the previously introduced notation for $H(t)$, we see that (10) can be rewritten as $x(t) = H(t_1 - t)H^{-1}(t_1)x_0$.
Since by assumption $H(t_f)$ is invertible for all $t_f \in [0, t_1]$ (see theorem 1), in particular matrix $H(t_1 - t)$ is invertible. Therefore, it follows that $H^{-1}(t_1)x_0 = H^{-1}(t_1 - t)x(t)$.
Substitution of this expression into the equations for $\psi_i, i = 1, 2$, in (11,12) yields:

$$\psi_1(t) = G_1(t_1 - t)H^{-1}(t_1 - t)x(t) \text{ and} \tag{13}$$

$$\psi_2(t) = G_2(t_1 - t)H^{-1}(t_1 - t)x(t) \tag{14}$$

for some continuously differentiable matrix functions $G_i, i = 1, 2$, and $H^{-1}(.)$.
Now, denote $G_i(t_1 - t)H^{-1}(t_1 - t)$ by $K_i(t), i = 1, 2$. Then, from (13), (14) it follows that $\dot{\psi}_i = \dot{K}_i x + K_i\dot{x}, i = 1, 2$.
From (2,3) we have that $\psi_1(t)$ and $\psi_2(t)$ satisfy

$$\dot{\psi}_1(t) = -Q_1x(t) - A^T\psi_1(t), \text{ with } \psi_1(t_1) = K_{1f}x(t_1),$$

$$\dot{\psi}_2(t) = -Q_2x(t) - A^T\psi_2(t), \text{ with } \psi_2(t_1) = K_{2f}x(t_1)$$

and

$$\dot{x}(t) = Ax(t) - S_1\psi_1(t) - S_2\psi_2(t);\ x(0) = x_0.$$

Substitution of $\dot{\psi}_i$ and $\psi_i, i = 1,2$ into these formulas yields

$(\dot{K}_1+A^T K_1+K_1 A+Q_1-K_1 S_1 K_1-K_1 S_2 K_2)e^{Mt}x_0 = 0$ with $(K_1(t_1)-K_{1f})e^{Mt_1}x_0 = 0$, and

$(\dot{K}_2+A^T K_2+K_2 A+Q_2-K_2 S_2 K_2-K_2 S_1 K_1)e^{Mt}x_0 = 0$ with $(K_2(t_1)-K_{2f})e^{Mt_1}x_0 = 0$,

for arbitrarily chosen x_0.
From this it follows that $K_i(t), i = 1,2$ satisfy the set of Riccati differential equations (6,7). This proves the following result:

<u>Theorem 5:</u>
The following statements are equivalent:
1) For all $t_f \in [0, t_1]$ there exists a unique open-loop Nash equilibrium for the two-player linear quadratic differential game (1) defined on the interval $[0, t_f]$.
2) $H(t)$ is invertible for all $t_f \in [0, t_1]$.
3) The set of Riccati differential equations (6,7) has a solution on $[0, t_1]$. $\square$

The above theorem shows that for both computational purposes and for a better theoretical understanding of the open-loop problem it would be nice to have a global existence result for the set of Riccati differential equations (6,7). Up to now this is, however, an unsolved problem. Sufficient conditions in literature on this subject are reported by e.g. Abou-Kandil et al. (1986), which result was generalized and proved in a rigorous way by Feucht in (1994).

III. The solutions for the algebraic Riccati equation

In this section we consider the set of solutions satisfying the set of so-called algebraic Riccati equations corresponding to (6,7)

$$\left.\begin{array}{l} 0 = -A^T K_1 - K_1 A - Q_1 + K_1 S_1 K_1 + K_1 S_2 K_2; \\ 0 = -A^T K_2 - K_2 A - Q_2 + K_2 S_2 K_2 + K_2 S_1 K_1; \end{array}\right\} \quad (ARE)$$

MacFarlane (1963) and Potter (1966) independently discovered that there exists a relationship between the stabilizing solution of the algebraic Riccati equation and the eigenvectors of a related Hamiltonian matrix in linear quadratic regulator problems. We will follow their approach and formulate similar results for our problem (1). Abou-Kandil et al. in (1993) already pointed out the existence of such a similar relationship. One of their results is that if the planning horizon t_f in (1) tends to infinity, under some technical conditions on the matrix M, the solution of the above mentioned set

of Riccati differential equations converges to a solution of the set of (ARE). This solution can be calculated from the eigenspaces of matrix M. We will reformulate their result geometrically. This makes it possible to understand better their technical assumptions (and to give a less technical proof). On the other hand this approach makes it possible to relate easily results on the converged finite-planning horizon equilibrium solution to equilibrium solutions of the infinite-planning horizon game.
Moreover, we elaborate on the relationship between solutions of (ARE) and matrix M in detail. We present both necessary and sufficient conditions in terms of the matrix M under which (ARE) has (a) real solution(s). In particular we will see that all solutions of (ARE) can be calculated from the invariant subspaces of M. Furthermore we will see that the eigenvalues of the associated closed-loop system, obtained by applying the control (8,9), are completely determined by the eigenvalues of matrix M. As a corollary from these results we obtain both necessary and sufficient conditions for the existence of a stabilizing control of this type. This result will also be used in the next section.
In our analysis the set of all M-invariant subspaces plays a crucial role. Therefore we introduce a separate notation for this set:

$$\mathcal{M}^{inv} := \{\mathcal{T} \mid M\mathcal{T} \subset \mathcal{T}\}.$$

It is well-known (see e.g. Lancaster and Tismenetsky (1985)) that this set contains only a finite number of (distinct) elements if and only if all eigenvalues of M have a geometric multiplicity one.
The set of possible solutions for the algebraic Riccati equation can, as will be shown in the next theorem, be calculated directly from the following collection of M-invariant subspaces:

$$\mathcal{K}^{pos} := \left\{\mathcal{K} \in \mathcal{M}^{inv} | \mathcal{K} \oplus Im \begin{pmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{pmatrix} = \mathbb{R}^{3n}\right\}.$$

Here the symbol $\oplus$ is used to denote the orthogonal sum of subspaces.
Note that elements in the set $\mathcal{K}^{pos}$ can be calculated using the set of matrices

$$K^{pos} := \left\{K \in \mathbb{R}^{3n \times n} | \mathrm{ImK} \oplus Im \begin{pmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{pmatrix} = \mathbb{R}^{3n}\right\}.$$

The exact result on how all solutions of (ARE) can be calculated is given in the next theorem. Here we use the notation $M|_{\mathcal{K}}$ to denote the restriction of the linear transformation induced by M to the subspace $\mathcal{K}$ (see e.g. Lancaster et al. (1985, p.142)). Furthermore we use the notation $\sigma(X)$ to denote the spectrum of a matrix X. The proof of this theorem can be found in the appendix.

Theorem 6:
(ARE) has a real solution (K_1, K_2) if and only if $K_1 = YX^{-1}$ and $K_2 = ZX^{-1}$ for some $\mathcal{K} =: Im \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \in \mathcal{K}^{pos}$.
Moreover, if the control functions $u_i^*(t) = -R_{ii}^{-1} B_i^T K_i \Phi(t) x_0$ are used to control the system (1), the spectrum of the closed-loop matrix $A - S_1K_1 - S_2K_2$ coincides with $\sigma(-M|_{\mathcal{K}})$. $\square$

From the above theorem a number of interesting properties concerning the solvability of (ARE) follow. First of all we observe that every element of $\mathcal{K}^{pos}$ defines exactly one solution of (ARE). Consequently, (ARE) will have no real solution if and only if $\mathcal{K}^{pos}$ is empty. Furthermore, this set contains a finite number of elements if the geometric multiplicities of all eigenvalues of M is one. So, in that case we immediately conclude that (ARE) will have at most a finite number of solutions.
Another immediate conclusion is that whenever matrix M has an invariant subspace $\mathcal{K} \in \mathcal{K}^{pos}$, such that all eigenvalues of $(M|_{\mathcal{K}})$ are positive, then all eigenvalues of the corresponding closed-loop matrix $A - S_1K_1 - S_2K_2$ will be negative. That the reverse statement also holds is easily verified too. So, we have

Corollary 7:
(ARE) will have a set of solutions (K_1, K_2) stabilizing the closed-loop system matrix $A - S_1K_1 - S_2K_2$ if and only if there exists an M-invariant subspace $\mathcal{K}$ in $\mathcal{K}^{pos}$ such that $Re\ \lambda > 0$ for all $\lambda \in \sigma(M|_{\mathcal{K}})$. $\square$

To illustrate some of the above mentioned properties, reconsider example 4.

Example 4 (continued):
It can be shown analytically that both $\frac{-5}{22}$ and $\frac{-1}{2}$ are eigenvalues of M with algebraic multiplicities 2 and 1, respectively, whereas both their geometric multiplicity is 1. Numerical calculations show that the other eigenvalues of M are $-1.8810, -0.1883$ and 1.7966. Rearranging the eigenvalues as

$\{\frac{-5}{22}, -1.8810, -0.1883, \frac{-1}{2}, 1.7966\}$ we have that M has the following corresponding (generalized) eigenspaces:

$$\begin{aligned}
\mathcal{T}_{11} &= \text{Span}\{T_{11}\} \text{ where } T_{11} = (0\ 0\ 0\ 0\ 0\ 1)^T, \\
\mathcal{T}_{12} &= \text{Span}\{T_{11}\ T_{12}\} \text{ where } T_{12} = (-0.2024\ 0.6012\ -0.2620\ -0.0057\ 0.5161\ 0)^T; \\
\mathcal{T}_{2} &= \text{Span}\{T_{2}\} \text{ where } T_{2} = (-0.3726\ -0.2006\ 0.4229\ 0\ 0.6505\ 0.4679)^T; \\
\mathcal{T}_{3} &= \text{Span}\{T_{3}\} \text{ where } T_{3} = (0.0079\ -0.0234\ 0.0097\ 0\ -0.0191\ -0.9995)^T; \\
\mathcal{T}_{4} &= \text{Span}\{T_{4}\} \text{ where } T_{4} = (0.0580\ -0.1596\ 0.1160\ 0\ -0.2031\ 0.9573)^T;
\end{aligned}$$

and

$$\mathcal{T}_{5} = \text{Span}\{T_{5}\} \text{ where } T_{5} = (-0.7274\ -0.1657\ -0.2601\ 0\ -0.3194\ -0.5232)^T.$$

Note that $\mathcal{T}_{12} \notin \mathcal{K}^{pos}$ since it violates the rank condition. For the same reason it is clear that no invariant subspace of $\mathcal{K}^{pos}$ can contain $\mathcal{T}_{11}$. Therefore, only $\begin{pmatrix} 4 \\ 2 \end{pmatrix} = 6$ different 2-dimensional M-invariant subspaces remain as candidate elements of $\mathcal{K}^{pos}$. Obviously, none of these candidate solutions will stabilize the closed-loop system matrix.

As an example consider $\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} := (\mathcal{T}_2\ \mathcal{T}_3)$. This yields the solution

$K_1 = YX^{-1} = \begin{pmatrix} 0.4229 & 0.0097 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -0.3726 & 0.0079 \\ -0.2006 & -0.0234 \end{pmatrix}^{-1}$ and

$K_2 = ZX^{-1} = \begin{pmatrix} 0.6505 & -0.0191 \\ 0.4679 & -0.9995 \end{pmatrix} \begin{pmatrix} -0.3726 & 0.0079 \\ -0.2006 & -0.0234 \end{pmatrix}^{-1}$.

The eigenvalues of the closed-loop system (1) using the open-loop strategies $u_i^*(t) = -R_{ii}^{-1} B_i^T K_i \Phi(t) x_0, i = 1, 2$, are $\{1.8810, 0.1883\}$. It is easily verified that the rank of the first two rows of every other candidate solution is also two, so we conclude that (ARE) has six solutions, none of which is stabilizing. $\square$

IV. Convergence results

As argued in the introduction, it is interesting to see how the open-loop equilibrium solution changes when the planning horizon t_f tends to infinity. To study convergence properties of the equilibrium solution for the game, it seems reasonable to require that problem (1) has a properly defined solution for every finite planning horizon. Therefore in this section we will make the following well-posedness assumption (see theorem 5)

$$H(t_f) \text{ is invertible for all } t_f < \infty. \tag{15}$$

Of course, this assumption is difficult to verify in practice. It stresses once

more the need to find general conditions under which the set of Riccati differential equations (6,7) will have a solution on $(0, \infty)$.
Furthermore, to derive general convergence results, we will assume that the eigenstructure of matrix M satisfies an additional property, which we define first.

<u>Definition 8:</u>
M is called dichotomically separable if there exist subspaces V_1 and V_2 such that $M V_i \subset V_i, i = 1,2,\ V_1 \oplus V_2 = \mathbb{R}^{3n}$ and $V_1 \oplus Im \begin{pmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{pmatrix} = \mathbb{R}^{3n}$.
Moreover, V_i should be such that $Re\ \lambda > Re\ \mu$ for all $\lambda \in \sigma(M|_{V_1}), \mu \in \sigma(M|_{V_2})$. $\square$

From theorem 5 we know that assumption (15) implies that, to study the convergence of the open-loop Nash equilibrium solution, we can restrict ourselves to the study of $K_i(0, t_f)$, if t_f increases (see also remark 2.3). Note that $K_i(0, t_f)$ can be viewed as the solution $k(t)$ of an autonomous vector differential equation $\dot{k} = f(k)$, with $k(0) = k_0$ for some fixed k_0, and where f is a smooth function. Elementary analysis shows then that $K_i(0, t_f)$ converges to a limit $\bar{k}$ only if this limit $\bar{k}$ satisfies $f(\bar{k}) = 0$. Therefore, we immediately deduce from theorem 6 the following necessary condition for convergence.

<u>Lemma 9:</u>
$K_i(0, t_f)$ can only converge to a limit $\bar{K}_i(0)$ if the set $\mathcal{K}^{pos}$ is nonempty. $\square$

Note that dichotomic separability of M implies that $\mathcal{K}^{pos}$ is nonempty. On the other hand it is not difficult to construct an example where $\mathcal{K}^{pos}$ is nonempty, whereas M is not dichotomically separable.
In the appendix we give an elementary proof of the following result (see also Abou-Kandil et al (1993, section 4))

<u>Theorem 10:</u>
Assume that the well-posedness assumption (15) holds.
Then, if M is dichotomically separable and Span $\begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix} \oplus V_2 = \mathbb{R}^{3n}$,

$$K_1(0, t_f) \to Y_0 X_0^{-1}, \text{ and } K_2(0, t_f) \to Z_0 X_0^{-1}.$$

Here X_0, Y_0, Z_0 are defined by (using the notation of definition 8) $V_1 =: \text{Span}(X_0^T\ Y_0^T\ Z_0^T)^T$.

Combination of the results from theorem 10 and corollary 7 yields then

Corollary 11:
Assume that the planning horizon t_f in the differential game (1) tends to infinity and the following conditions are satisfied

1. all conditions mentioned in theorem 10
2. $Re\ \lambda > 0, \forall \lambda \in \sigma(M|_{V_1})$.

Then the unique open-loop Nash equilibrium solution converges to a (stationary) strategy

$$u_i^*(t) = -R_{ii}^{-1} B_i^T K_i \Phi(t,0) x_0, i = 1, 2,$$

where $\Phi(t,0)$ satisfies the transition equation $\dot{\Phi}(t,0) = (A - S_1 K_1 - S_2 K_2)\Phi(t,0)$; $\Phi(0,0) = I$.
In these equations the constant matrices $K_i, i = 1, 2$, can be calculated from the eigenspaces of matrix M (see theorem 10), and the strategies will stabilize the closed-loop system. □

V. The infinite-planning horizon case

In this section we assume that the performance criterion player $i = 1, 2$ aims to minimize is:

$$\lim_{t_f \to \infty} J_i(u_1, u_2),$$

where

$$J_1(u_1, u_2) := \frac{1}{2} \int_0^{t_f} \{x(t)^T Q_1 x(t) + u_1(t)^T R_{11} u_1(t)\} dt,$$

and

$$J_2(u_1, u_2) := \frac{1}{2} \int_0^{t_f} \{x(t)^T Q_2 x(t) + u_2(t)^T R_{22} u_2(t)\} dt.$$

Here matrix R_{ii} is positive definite, Q_i is semi-positive definite and additionally is positive definite w.r.t. the controllability subspace $< A, B_i >$, $i = 1, 2$. Compared to (1), note that each player's performance criterion does not include the cross term, penalizing the control efforts of the other player, anymore. We dropped this term here since it does not play a role in the analysis and therefore only would unnecessarily complicate the presented proofs.
The information structure is similar to the finite-planning horizon case. Each player only knows the initial state of the system and has to choose a control for the entire infinite time horizon. So, the actions are now described as functions of time, where time runs from zero to infinity. Since we only

like to consider those outcomes of the game that yield a finite cost to both players, we restrict ourselves to consider only control functions belonging to the following set

$$U := \{\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, t \in [0,\infty) | \lim_{t_f \to \infty} J_i(u_1, u_2) < \infty,\ i = 1, 2.\}$$

The basic result of this section is summarized in the next theorem:

Theorem 12:
The infinite-planning horizon two-player linear quadratic differential game has for every initial state an open-loop Nash equilibrium strategy $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ if and only if there exist K_1 and K_2 that are solutions of the algebraic Riccati equations (ARE) satisfying the additional constraint that the eigenvalues of $A_{cl} := A - S_1K_1 - S_2K_2$ are all situated in the left half complex plane.
In that case, the strategy

$$u_i(t) = -R_{ii}^{-1}B_i^T K_i \Phi(t,0)x_0, i = 1, 2,$$

where $\Phi(t,0)$ satisfies the transition equation $\dot{\Phi}(t,0) = A_{cl}\Phi(t,0)$; $\Phi(0,0) = I$, is an open-loop Nash equilibrium strategy.
Moreover, the costs obtained by using this strategy for the players are

$$\int_0^\infty \{(e^{A_{cl}t}x_0)^T(Q_i + K_i^T S_i K_i)e^{A_{cl}t}x_0 dt, i = 1, 2.$$

$\square$

Remark 13:

1) Parts of the proof of this theorem can be substituted by using the results of Haurie et al. (1984, lemma 5.1). This requires, however, the introduction of the concept of weak overtaking optimality. In this framework it is not required that the state or the performance criterion converge (see Halkin (1974)). Since we like to stay in the framework of bounded performance criteria, we choose to give an elementary self-contained proof of this theorem.

2) In the above theorem the costs for the individual players are expressed as an integral. In fact, analogously to the optimal LQ regulator theory, we have that the costs can be obtained indirectly by solving the following associated Lyapunov equations[1]

$$A_{cl}^T M_i + M_i A_{cl} + Q_i + K_i^T S_i K_i = 0, \tag{16}$$

[1] I like to thank Arie Weeren for pointing out this to me

where $A_{cl} := A - S_1K_1 - S_2K_2$, i=1,2.
Note that since all eigenvalues of A_{cl} are in the left half complex plane and $Q_i + K_i^T S_i K_i \geq 0$, this equation has a unique positive semi-definite solution M_i. Therefore, using the notation $x(t) := e^{A_{cl}t}x_0$, we have $J_i(\bar{u}_1, \bar{u}_2) = \int_0^\infty x(t)^T(Q_i + K_i^T S_i K_i)x(t)dt = -\int_0^\infty x(t)^T(A_{cl}^T M_i + M_i A_{cl})x(t)dt = -\int_0^\infty \frac{d}{dt}[x(t)^T M_i x(t)]dt = x_0^T M_i x_0 - \lim_{t\to\infty} x(t)^T M_i x(t)$ $= x_0^T M_i x_0$.

3) It is clear now that the control strategies presented in corollary 11 will also be equilibrium strategies for the infinite planning horizon game.

So, the study of the equilibria of our LQ game boils down to the study of all M-invariant subspaces $\mathcal{K}$ in $\mathcal{K}^{pos}$ for which $Re\ \lambda > 0$ for all $\lambda \in \sigma(M|_{\mathcal{K}})$.
The next example shows that in general the infinite-planning horizon problem may have more than one open-loop Nash equilibrium. Moreover it shows that it is possible that the finite-planning horizon has no solution whereas a solution to the infinite-planning horizon exists.

Example 14:
Reconsider example 4 with matrix A replaced by $A = \begin{pmatrix} 1 & 0 \\ 0 & 5/22 \end{pmatrix}$.
Then, the eigenvalues of matrix M are $\{\frac{5}{22}, 1.8810, 0.1883, \frac{1}{2}, -1.7966\}$. Numerical calculation of the corresponding eigenspaces shows that $\mathcal{K}^{pos}$ has 3 different stabilizing elements. So, according the previous theorem, the infinite-planning horizon game has 3 open-loop Nash equilibria.
On the other hand it can be shown, by constructing final cost matrices K_{1f} and K_{2f} using the same procedure as in example 4, that matrix $H(t)$ at e.g. $t = 0.1$ is not always invertible. In other words, though the infinite-planning horizon problem has solutions, a solution to the corresponding finite-planning horizon problem may fail to exist. □

On the other hand one might hope that, if the finite-planning horizon always has a unique solution, then this solution always converges to a solution of the infinite-planning horizon case. That this conjecture is false is illustrated by the next example:

Example 15:
Reconsider example 4 (continued) with the following notation: $T := (T_{11}\ T_{12}\ T_2\ T_3\ T_4\ T_5)$.
Using this notation we have that $e^{Mt} = Te^{Jt}T^{-1}$, where

$$J = \begin{pmatrix} \frac{-5}{22} & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{-5}{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & -1.8810 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.1883 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.7966 \end{pmatrix}.$$

With $K_{1f} := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, and $K_{2f} := \begin{pmatrix} -\frac{T^{-1}(3,1)}{T^{-1}(3,5)} & -\frac{T^{-1}(3,2)}{T^{-1}(3,5)} \\ -\frac{T^{-1}(3,2)}{T^{-1}(3,5)} & 1 \end{pmatrix}$,
we have that both K_{1f} and K_{2f} are semi-positive definite. Numerical calculation shows that with these choices for the final cost, the determinant of $H(t)$ equals approximately $-0.02907 * e^{-0.6883t} + 2.419 * e^{1.6083t} - 1.390 * e^{1.2966t}$. From this we see that $H(t)$ is invertible for every positive t. So, the finite-planning horizon problem has a unique equilibrium for every t_f. On the other hand, it is clear from theorem 10 that the equilibrium solution converges to the solution that can be calculated from $(X_0^T\ Y_0^T\ Z_0^T)^T := (T_3\ T_5)$ (that is the eigenspaces corresponding with the two largest eigenvalues of M, -0.1883 and 1.7966). So, the converged equilibrium solution is not a "stabilizing solution". In other words the equilibrium solution does not converge to an equilibrium solution of the infinite-planning horizon game. $\square$

By replacing in example 4 matrix A by $\begin{pmatrix} -0.1 & 0 \\ 0 & -2 \end{pmatrix}$, numerical computations show that we obtain an infinite-planning horizon problem which has three equilibria. This, although matrix A is stable. We conclude this section by showing that if both players discount their future welfare loss by a factor that is large enough, this phenomenon does not occur anymore. That is, in that case the game has generically a unique Nash equilibrium.
So, we consider next the case that the performance criterion player $i = 1, 2$ likes to minimize is given by:

$$\lim_{t_f \to \infty} J_i(u_1, u_2) := \frac{1}{2} \int_0^{t_f} e^{-rt}\{x(t)^T Q_i x(t) + u_i(t)^T R_{ii} u_i(t)\} dt,$$

where $r \geq 0$ is the discount factor.
Then, introducing $\tilde{x}(t) := e^{-\frac{1}{2}rt}x(t)$ and $\tilde{u}_i(t) := e^{-\frac{1}{2}rt}u_i(t)$, we see that the above minimization problem can be rewritten as:

$$\min_{\tilde{u}_i} \lim_{t_f \to \infty} \frac{1}{2} \int_0^{t_f} \{\tilde{x}(t)^T Q_i \tilde{x}(t) + \tilde{u}_i(t)^T R_{ii} \tilde{u}_i(t)\} dt, \tag{17}$$

subject to

$$\dot{\tilde{x}} = (A - \frac{1}{2}rI)\tilde{x} + B_1\tilde{u}_1 + B_2\tilde{u}_2,\ \tilde{x}(0) = x_0. \tag{18}$$

The above conclusion follows then directly by noting that matrix $\tilde{M} :=$

$\begin{pmatrix} -(A-\frac{1}{2}rI) & S_1 & S_2 \\ Q_1 & (A^T-\frac{1}{2}rI) & 0 \\ Q_2 & 0 & (A^T-\frac{1}{2}rI) \end{pmatrix}$, has $2n$ stable eigenvalues and n unstable ones if r is chosen large enough.

VI. The scalar case and an economic example

We start this section by showing that matrix $H(t_f)$, as mentioned in theorem 5, is always invertible if the dimensions of both the state and the input variables in system (1) equal one. This implies that for this kind of systems the usually stated assertion that the open-loop Nash strategy is given by (8,9) is correct and, moreover, that the associated Riccati equations yield the appropriate solution. Note that this case may be also considered as a special situation as considered by Abou-Kandil et al (1986) and Feucht (1994), where they show directly that the set of associated Riccati equations has a solution.
To prove the invertibility of matrix $H(t_f)$ we first calculate the exponential of matrix M. To stress the fact that we are dealing with the scalar case, we will put the system parameters in lower case, so e.g. a instead of A.

Lemma 16:
Consider matrix M in (4). The exponential of matrix M, e^{Ms}, is given by

$$V \begin{pmatrix} e^{-\mu s} & 0 & 0 \\ 0 & e^{as} & 0 \\ 0 & 0 & e^{\mu s} \end{pmatrix} V^{-1}, \tag{19}$$

where

$$V = \begin{pmatrix} a+\mu & 0 & a-\mu \\ -q_1 & -s_2 & -q_1 \\ -q_2 & s_1 & -q_2 \end{pmatrix}$$

and its inverse

$$V^{-1} = \frac{1}{detV} \begin{pmatrix} (s_1q_1+s_2q_2) & s_1(a-\mu) & s_2(a-\mu) \\ 0 & -2q_2\mu & 2q_1\mu \\ -(s_1q_1+s_2q_2) & -s_1(a+\mu) & -s_2(a+\mu) \end{pmatrix},$$

with the determinant of V, $detV = 2\mu(s_1q_1+s_2q_2)$, and $\mu = \sqrt{a^2+s_1q_1+s_2q_2}$.

Proof:
Straightforward multiplication shows that we can factorize M as $M = V\text{diag}(a, \mu, -\mu)V^{-1}$. So (see e.g. Lancaster et al (1985, theorem 9.4.3)), the exponential of matrix M, e^{Ms}, is as stated above. $\square$

Next consider the matrix $H(t_f)$ from theorem 5 for an arbitrarily chosen $t_f \in [0, t_1]$. Obviously, $H(t_f) = (1\ 0\ 0)e^{Mt_f} \begin{pmatrix} 1 \\ k_{1f} \\ k_{2f} \end{pmatrix}$. Using the expressions in lemma 17 for V and V^{-1} we find

$$H(t_f) = \frac{1}{detV}[(s_1q_1+s_2q_2)\{(\mu-a)e^{\mu t_f}+(a+\mu)e^{-\mu t_f}\}+(\mu^2-a^2)(e^{\mu t_f}-e^{-\mu t_f})(s_1k_{1f}+s_2k_{2f})].$$

Clearly, $H(t_f)$ is positive for every $t_f \geq 0$. This implies in particular that $H(t_f)$ differs from zero for every $t_f \in [0, t_1]$, whatever $t_1 > 0$ is. So from theorem 5 we now immediately have the following conclusion.

Theorem 17:
Problem (1) has a unique open-loop Nash equilibrium solution:

$$u_1^*(t) = -\frac{1}{r_{11}}b_1k_1(t)\Phi(t,0)x_0$$

$$u_2^*(t) = -\frac{1}{r_{22}}b_2k_2(t)\Phi(t,0)x_0.$$

Here $k_1(t)$ and $k_2(t)$ are the solutions of the coupled asymmetric Riccati-type differential equations

$$\begin{aligned} \dot{k}_1 &= -ak_1 - k_1a - q_1 + k_1^2s_1 + k_1s_2k_2;\ k_1(t_f) = k_{1f} \\ \dot{k}_2 &= -ak_2 - k_2a - q_2 + k_2^2s_2 + k_2s_1k_1;\ k_2(t_f) = k_{2f}. \end{aligned}$$

Furthermore, $\Phi(t,0)$ satisfies the transition equation

$$\dot{\Phi}(t,0) = (a - s_1k_1 - s_2k_2)\Phi(t,0);\ \Phi(0,0) = 1.$$

Here $s_i = \frac{1}{r_{ii}}b_i^2, i = 1, 2.$ □

The next theorem shows that in the scalar case the equilibrium solution always converges.

Theorem 18:
Assume that $s_1q_1 + s_2q_2 > 0$.
Then, the open-loop Nash equilibrium solution from theorem 17 converges to the (stationary) strategy:

$$u_1^*(t) = -\frac{1}{r_{11}}b_1k_1e^{(a-s_1k_1-s_2k_2)}x_0$$

$$u_2^*(t) = -\frac{1}{r_{22}}b_2k_2e^{(a-s_1k_1-s_2k_2)}x_0$$

where $k_1 = \frac{(a+\mu)q_1}{s_1q_1+s_2q_2}$ and $k_2 = \frac{(a+\mu)q_2}{s_1q_1+s_2q_2}$.
Moreover, these strategies are the unique solution to the infinite-planning horizon open-loop problem.

Proof:
Since $s_1q_1 + s_2q_2 > 0$, it is clear from (19) that M is dichotomically separable. Furthermore we showed above that the well-posedness assumption is always satisfied in the scalar case. Note that $\mu > 0$. So, according to corollary 11 the open-loop Nash strategies converge to a stationary strategy whenever $k_{if},\ i = 1,2$, are such that $s_1q_1 + s_2q_2 + s_1(a-\mu)k_{1f} + s_2(a-\mu)k_{2f} \neq 0$. Now consider the case that $s_1q_1 + s_2q_2 + s_1(a-\mu)k_{1f} + s_2(a-\mu)k_{2f} = 0$. To study this case, reconsider (23) and (24) for $t_f \to \infty$. Elementary spelling out of these formulas, using (19), shows that also in this case both $k_1(0,t_f)$ and $k_2(0,t_f)$ converge to the limits as advertised above. Finally, note that $\mathcal{K}^{pos}$ contains, independent of the sign of a, exactly one stabilizing solution. Using the results of theorem 12 this concludes the proof. $\square$

We conclude this section by illustrating the computational advantages of our approach in an economic example. The example is taken from van Aarle et al. (1995). In this paper they analyze a differential game on government debt stabilization. They assume that government debt accumulation ($\dot{d}$) is the sum of interest payments on government debt ($rd(t)$) and primary fiscal deficits ($f(t)$) minus the seignorage (i.e., the issue of base money) ($m(t)$):

$$\dot{d}(t) = rd(t) + f(t) - m(t), d(t_0) = d_0. \tag{20}$$

Here $d(t), f(t)$ and $m(t)$ are expressed as fractions of GDP and r represents the rate of interest on outstanding government debt minus the growth rate of output. r is assumed to be exogenous. They assume that fiscal and monetary policies are controlled by different institutions, the fiscal authority and the monetary authority, respectively, which have different objectives. The objective of the fiscal authority is to minimize a sum of time profiles of the primary fiscal deficit, base-money growth and government debt:

$$L^F(t_0) = \frac{1}{2}\int_{t_0}^{\infty} \{(f(t)-\bar{f})^2 + \eta(m(t)-\bar{m})^2 + \lambda(d(t)-\bar{d})^2\}e^{-\delta(t-t_0)}dt. \tag{21}$$

Whereas the monetary authorities set the growth of base money so as to minimize the loss function:

$$L^M(t_0) = \frac{1}{2}\int_{t_0}^{\infty} \{(m(t)-\bar{m})^2 + \kappa(d(t)-\bar{d})^2\}e^{-\delta(t-t_0)}dt. \tag{22}$$

Here $\frac{1}{\kappa}$ can be interpreted as a measure for the conservatism of the central bank w.r.t. the money growth. Furthermore all variables denoted with a bar

are assumed to be fixed targets which are given a priori.
Introducing $x_1(t) := (d(t) - \bar{d})e^{-\frac{1}{2}\delta t}, x_2(t) := (r\bar{d} + \bar{f} - \bar{m})e^{-\frac{1}{2}\delta t}, u_1(t) := (f(t) - \bar{f})e^{-\frac{1}{2}\delta t}$ and $u_2(t) := (m(t) - \bar{m})e^{-\frac{1}{2}\delta t}$ the above game can be rewritten in our notation (1) with:
$A = \begin{pmatrix} r - \frac{1}{2}\delta & 1 \\ 0 & -\frac{1}{2}\delta \end{pmatrix}, B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, B_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, Q_1 = \begin{pmatrix} \kappa\ 0 \\ 0\ 0 \end{pmatrix}, Q_2 = \begin{pmatrix} \lambda\ 0 \\ 0\ 0 \end{pmatrix}$, $R_{11} = 1$ and $R_{22} = 1$.
It is not difficult to see that the eigenvalues of M are: $\{-\frac{1}{2}\delta, -\frac{1}{2}\delta, \frac{1}{2}\delta, r - \frac{1}{2}\delta, l, -l\}$, where $l := \sqrt{\kappa + \lambda + (r - \frac{1}{2}\delta)^2}$. The corresponding eigenspaces are:

$$\begin{aligned}
\mathcal{T}_1 &= \text{Span}\{T_1\} \text{ where } T_1 = (0\ 0\ 0\ 0\ 0\ 1)^T, \\
\mathcal{T}_2 &= \text{Span}\{T_2\} \text{ where } T_2 = (0\ 0\ 0\ 1\ 0\ 0)^T, \\
\mathcal{T}_3 &= \text{Span}\{T_3\} \text{ where } T_3 = (-(r-\delta)\delta\ \ \delta(\lambda + \kappa + r(r-\delta))\ \ \kappa\delta\ \ \kappa\ \ \delta\lambda\ \ \lambda)^T, \\
\mathcal{T}_4 &= \text{Span}\{T_4\} \text{ where } T_4 = (0\ 0\ \ -r\ \ -1\ r\ 1)^T, \\
\mathcal{T}_5 &= \text{Span}\{T_5\} \text{ where } T_5 = (\frac{1}{2}\delta - r + l\ \ 0\ \ \kappa\ \ \frac{\kappa}{\frac{1}{2}\delta + l}\ \ \lambda\ \ \frac{\lambda}{\frac{1}{2}\delta + l})^T, \text{ and} \\
\mathcal{T}_6 &= \text{Span}\{T_6\} \text{ where } T_6 = ((\frac{1}{2}\delta - r - l)(\frac{1}{2}\delta - l)\ \ 0\ \ \kappa(\frac{1}{2}\delta - l)\ \ \kappa\ \ \lambda(\frac{1}{2}\delta - l)\ \ \lambda)^T.
\end{aligned}$$

So, the only stabilizing equilibrium strategy according theorem 12 is obtained by considering the eigenspaces corresponding to the eigenvalues $\frac{1}{2}\delta$ and l. This gives rise to $u_i = -B_i^T K_i \Phi(t,0)x_0$, with:

$$K_1 := \kappa \begin{pmatrix} \delta & 1 \\ 1 & \frac{1}{\frac{1}{2}\delta + l} \end{pmatrix} \begin{pmatrix} -(r-\delta)\delta & \frac{1}{2}\delta - r + l \\ \delta(\lambda + \kappa + r(r-\delta)) & 0 \end{pmatrix}^{-1} \text{ and } K_2 := \frac{\lambda}{\kappa} K_1.$$

In particular this implies that the equilibrium strategies satisfy the relationship $u_2(t) = -\frac{\lambda}{\kappa} u_1(t)$. Or, stated differently, $m(t) = \bar{m}(t) - \frac{\lambda}{\kappa}(f(t) - \bar{f}(t))$. Substitution of the equilibrium strategies into the system equation yields the closed-loop system

$$\dot{x}(t) = \begin{pmatrix} -l & p \\ 0 & -\frac{1}{2}\delta \end{pmatrix} x(t),$$

where $p = \frac{(\delta - r)(l - \frac{1}{2}\delta)}{\lambda + \kappa - r(\delta - r)}$. Note that we implicitly assumed here that $\lambda + \kappa + r(r - \delta)$ differs from zero; a technical assumption which is not crucial.
The advantages of this approach are clear. First, it gives more insight into the problem. That is, we obtain in an elementary way the optimal strategies and thus the closed-loop dynamics of the problem. This makes it e.g. additionally possible to state an exact condition (i.e. $\frac{1}{2}\delta - l < 0$) under which the analysis performed by van Aarle et al. holds and what happens if this

condition is violated. Second, the approach can be straightforwardly generalized to multi-dimensional systems and is numerically easily implementable.

VII. Concluding remarks

In this paper we reconsidered the existence and asymptotic behaviour of open-loop Nash equilibrium solutions in the two-player linear quadratic game. Furthermore, we analyzed the infinite-planning horizon game in a rigorous mathematical way. We formulated both necessary and sufficient conditions for this open problem, which can be computationally easily verified. To link the results on the infinite-planning horizon game with convergence results for the finite-planning horizon game we reformulated and proved some existing results on the finite-planning horizon game in a geometric way.
The finite-planning horizon game was analyzed starting from its basics: the Hamiltonian equations. We derived necessary and sufficient conditions for the existence of a unique open-loop Nash equilibrium solution in terms of a full rank condition on a modified fundamental matrix. We showed by means of an example that in general a solution to the set of associated differential Riccati equations may fail to exist whereas an open-loop Nash equilibrium solution exists. Furthermore, we showed that solvability of these Riccati equations is equivalent to existence of a Nash solution at every point of time during a fixed time interval. To study convergence of the open-loop equilibrium solution, if the planning horizon is extended to infinity, we therefore studied the asymptotic behavior of the Riccati differential equations. To that end we first considered the existence of real solutions for the corresponding algebraic Riccati equations. We showed how every real solution to (ARE) can be calculated from the invariant subspaces of the matrix $M = \begin{pmatrix} -A & S_1 & S_2 \\ Q_1 & A^T & 0 \\ Q_2 & 0 & A^T \end{pmatrix}$.
Furthermore we showed how the eigenvalues of the system, if the corresponding feedback control strategies are used in (1), are related to the eigenvalues of this matrix.
The results on the existence of real solutions to (ARE) were used to show that if the dimension of the direct sum of the invariant subspaces corresponding with the n largest eigenvalues (counted with algebraic multiplicities) equals n, then generically the finite-planning horizon solution of the game converges to a solution which can be directly calculated from this direct sum. Moreover we showed that, if this solution stabilizes the closed-loop system, this strategy will also be an equilibrium for the infinite-planning horizon game. By means of two examples we illustrated that convergence of the finite-planning horizon strategy is neither a necessary nor a sufficient condition for existence of an infinite-planning horizon equilibrium strategy.
The solution structure of all infinite-planning horizon equilibrium strategies

is completely determined by those invariant subspaces which have the above mentioned stabilization property. In an example we illustrated that, even if matrix A is stable, the game may have more than one equilibrium. In case a discounting factor is included in the performance function, that is large enough, we showed that there will be only one equilibrium (generically).
Since there are a number of applications which just involve scalar systems we concluded the paper by a detailed analysis of that case. We showed that for those systems, the unique open-loop Nash equilibrium solution can always be found by solving the associated set of Riccati differential equations. Moreover, this solution converges to a stationary strategy which is the unique solution of the infinite-planning horizon game.
It will be clear that there are still a number of open problems in this area. In particular it remains a challenge for future research to get a better intuition and understanding why sometimes solutions to this open-loop problem exist whereas under some slight modifications they fail to exist. Finally we note that the obtained results can be straightforwardly generalized to the N player game.
We believe that the presented results are of interest in the analysis of, e.g., policy coordination problems. In particular our analysis shows that the question which equilibrium to choose, a well-known problem if one deals with feedback-strategies, also applies for open-loop problems. We saw that in principle one can proceed in two ways to get (sometimes) around this problem. Either by discounting future welfare loss by a factor that is "large enough", or by considering the converged finite-planning horizon strategy. However, neither of both approaches guarantees existence of a unique open-loop strategy for the infinite-planning game.

Appendix

Proof of Theorem 6:
" $\Rightarrow$ " Assume (K_1, K_2) solve (ARE). Then simple calculations show that

$$M \begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} -A + S_1K_1 + S_2K_2 \\ Q_1 + A^TK_1 \\ Q_2 + A^TK_2 \end{pmatrix} = \begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix} (-A + S_1K_1 + S_2K_2).$$

Now, introducing $X := I, Y := K_1$, and $Z := K_2$, we see that $M \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} J$, for some matrix J and matrix X invertible, which completes this part of the proof.
" $\Leftarrow$ " This part has been proved in a more general context by Meyer in

(1976). However, since the last statement of the theorem can be immediately deduced from the following proof, we present this part of the proof here too. Let $\mathcal{K} \in \mathcal{K}^{pos}$. Then there exist K_1 and K_2 such that $\mathcal{K} = Im \begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix}$, and a matrix J such that

$$M \begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix} J.$$

Spelling out the left hand side of this equation gives

$$\begin{pmatrix} -A + S_1K_1 + S_2K_2 \\ Q_1 + A^TK_1 \\ Q_2 + A^TK_2 \end{pmatrix} = \begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix} J,$$

which immediately yields that $J = -A + S_1K_1 + S_2K_2$. Substitution of this equality into the right hand side of the equality shows then that $Q_1 + A^TK_1 = K_1(-A + S_1K_1 + S_2K_2)$ and $Q_2 + A^TK_2 = K_2(-A + S_1K_1 + S_2K_2)$, or stated differently, K_1, K_2 satisfy (ARE). This proves the second part of the theorem. As already noted above, the last statement of the theorem concerning the spectrum of the matrix $-A + S_1K_1 + S_2K_2$ follows directly from the above arguments. If we choose as a basis for $\mathbb{R}^{3n}$ $\begin{pmatrix} I & 0 & 0 \\ K_1 & I & 0 \\ K_2 & 0 & I \end{pmatrix}$, matrix M has the block-triangular structure $\begin{pmatrix} -A + S_1K_1 + S_2K_2 & S_1 & S_2 \\ 0 & A^T - K_1S_1 & -K_1S_2 \\ 0 & -K_2S_1 & A^T - K_2S_2 \end{pmatrix}$, which completes the proof. $\square$

<u>Proof of Theorem 10</u>

To study the convergence of $K_i(0, t_f)$ we reconsider (13) and (14). From these formulas we have that

$$K_1(0, t_f) = (0\ I\ 0)e^{Mt_f} \begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix} \left((I\ 0\ 0)e^{Mt_f} \begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix} \right)^{-1}, \text{ and} \tag{23}$$

$$K_2(0, t_f) = (0\ 0\ I)e^{Mt_f} \begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix} \left((I\ 0\ 0)e^{Mt_f} \begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix} \right)^{-1}. \tag{24}$$

Now, choose $\begin{pmatrix} I & 0 & 0 \\ K_{1f} & I & 0 \\ K_{2f} & 0 & I \end{pmatrix}$ as a basis for $\mathbb{R}^{3n}$. Then, because

$$\mathrm{Span}\begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix} \oplus V_2 = \mathbb{R}^{3n},$$

there exists an invertible matrix $V_{22} \in \mathbb{R}^{2n\times 2n}$ such that $V_2 = \mathrm{Span}\begin{pmatrix} 0 \\ V_{22} \end{pmatrix}$. Moreover, because M is dichotomically separable, there exist matrices J_1, J_2 such that

$$M = V\begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} V^{-1},$$

where

$$V = \begin{pmatrix} X_0 & 0 \\ \begin{pmatrix} Y_0 \\ Z_0 \end{pmatrix} & V_{22} \end{pmatrix},$$

and $\sigma(J_i) = \sigma(M|_{V_i})$, $i = 1, 2$.

Using this, we can rewrite $K_1(0, t_f)$ and $K_2(0, t_f)$ in (23,24) as $\tilde{G}_i(t_f)\tilde{H}^{-1}(t_f)$, $i = 1, 2$, where

$$\begin{aligned}
\tilde{G}_1(t_f) &= (0\ I\ 0)Ve^{-\lambda_n t_f}\begin{pmatrix} e^{J_1 t_f} & 0 \\ 0 & e^{J_2 t_f} \end{pmatrix} V^{-1}\begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix}, \\
\tilde{G}_2(t_f) &= (0\ 0\ I)Ve^{-\lambda_n t_f}\begin{pmatrix} e^{J_1 t_f} & 0 \\ 0 & e^{J_2 t_f} \end{pmatrix} V^{-1}\begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix}, \\
\tilde{H}(t_f) &= (I\ 0\ 0)Ve^{-\lambda_n t_f}\begin{pmatrix} e^{J_1 t_f} & 0 \\ 0 & e^{J_2 t_f} \end{pmatrix} V^{-1}\begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix}.
\end{aligned}$$

Here λ_n is the element of $\sigma(M|\,V_1)$ which has the smallest real part.

Next, consider $\tilde{G}_1(t_f) - Y_0X_0^{-1}\tilde{H}(t_f)$.
Simple calculations show that this matrix can be rewritten as

$$e^{-\lambda_n t_f}(-Y_0X_0^{-1}\ I\ 0)V\begin{pmatrix} e^{J_1 t_f} & 0 \\ 0 & e^{J_2 t_f} \end{pmatrix} V^{-1}\begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix}, \qquad (25)$$

Since $(-Y_0X_0^{-1}\ I\ 0)\left(X_0^T\ Y_0^T\ Z_0^T\right)^T = 0$, (25) equals

$$e^{-\lambda_n t_f}\begin{pmatrix} I & 0 \end{pmatrix} V_{22}e^{J_2 t_f}V_{22}^{-1}\begin{pmatrix} K_{1f} - Y_0X_0^{-1} \\ K_{2f} - Z_0X_0^{-1} \end{pmatrix}.$$

As $e^{-\lambda_n t_f} e^{J_2 t_f}$ converges to zero for $t_f \to \infty$, it is obvious now that $\tilde{G}_1(t_f) - Y_0 X_0^{-1} \tilde{H}(t_f)$ converges to zero for $t_f \to \infty$. Similarly it can be shown that also $\tilde{G}_2(t_f) - Z_0 X_0^{-1} \tilde{H}(t_f)$ converges to zero for $t_f \to \infty$. To conclude from this that $K_1(0, t_f) \to Y_0 X_0^{-1}$, and $K_2(0, t_f) \to Z_0 X_0^{-1}$, it suffices to show that $\tilde{H}^{-1}(t_f)$ remains bounded for $t_f \to \infty$. This follows, however, directly by spelling out $\tilde{H}(t_f)$ as

$$\tilde{H}(t_f) = e^{-\lambda_n t_f} X_0 e^{J_1 t_f} X_0^{-1}. \square$$

<u>Proof of Theorem 12</u>

To prove this theorem we first need a property on the structure of U. Note that U depends on the initial state of the system . For simplicity of notation we omit, however, this dependency. Furthermore, it is clear that $u_i(.) \in L^2$, the set of square integrable functions, but that U is not a linear subspace of L^2. First, since the zero-function will in general not belong to U and, second, in general with $v, w \in U$, $v + w \notin U$. However, U does satisfy the following important property:

<u>Lemma:</u>
Assume that both v and $v + w$ are an element of U. Then for any real ϵ also $v + \epsilon w \in U$.

<u>Proof:</u>
First we introduce some notation.
Let x_u denote the state trajectory obtained by using the control function u, that is, $x_u(t) := e^{At} x_0 + \int_0^t e^{A(t-\tau)}(B_1\ B_2) u(\tau) d\tau$.
Since by assumption both v and $v + w$ belong to U, $x_v(t)$ and $x_{v+w}(t)$ converge to zero if $t \to \infty$. So, $x_v(t) - x_{v+w}(t) = \int_0^t e^{A(t-\tau)}(B_1\ B_2) w(\tau) d\tau \to 0$, if $t \to \infty$. Moreover, since both x_v and x_{v+w} are square integrable, also the righthandside of the above equation is square integrable. Now, consider $x_{v+\epsilon w}(t)$. Elementary calculation shows that $x_{v+\epsilon w}(t) = x_{v+w}(t) - (1 - \epsilon) \int_0^t e^{A(t-\tau)}(B_1\ B_2) w(\tau) d\tau$. So, using the above result, it is clear that $x_{v+\epsilon w}(t)$ is square integrable. Moreover, since both v and $v + w$ are square integrable it follows that w has to be square integrable too. From this follows then immediately that also $v + \epsilon w$ is square integrable.
Combining both results gives then that $\lim_{t_f \to \infty} J_i(v + \epsilon w) < \infty,\ i = 1, 2$. Which implies that $v + \epsilon w \in U$. $\square$

The proof of the theorem reads now as follows

" $\Rightarrow$ " To prove this part we use the variational approach (see e.g. Friedman (1971) and Lukes and Russell (1971)).
Suppose that $\bar{u}_1, \bar{u}_2$ are a Nash solution. That is,

$$J_1(u_1, \bar{u}_2) \geq J_1(\bar{u}_1, \bar{u}_2) \text{ and } J_2(\bar{u}_1, u_2) \geq J_2(\bar{u}_1, \bar{u}_2). \tag{26}$$

Then for any control function $\begin{pmatrix} w \\ 0 \end{pmatrix}$ such that $\begin{pmatrix} \bar{u}_1 + w \\ \bar{u}_2 \end{pmatrix} \in U$ we have, according the above stated lemma, that for any real number ϵ

$$J_1(\epsilon) := J_1(\bar{u}_1 + \epsilon w, \bar{u}_2) \geq J_1(\bar{u}_1, \bar{u}_2). \tag{27}$$

Let $x_{\bar{u}}(t)$ and $x_{\bar{u}+\epsilon w}(t)$ be the solutions to (1) corresponding to the controls $\begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \end{pmatrix}$ and $\begin{pmatrix} \bar{u}_1 + \epsilon w \\ \bar{u}_2 \end{pmatrix}$, respectively. Then it is easily verified that (see also proof of the lemma)

$$x_{\bar{u}+\epsilon w}(t) = x_{\bar{u}}(t) + \epsilon g(t), \tag{28}$$

where $g(t) := \int_0^t e^{A(t-s)} B_1 w(s) ds$ is a square integrable function.
So, $J_1(\epsilon)$ can be rewritten as:

$$\frac{1}{2} \int_0^\infty f(t, \epsilon) dt,$$

where

$$f(t, \epsilon) := (x_{\bar{u}}(t) + \epsilon g(t))^T Q_1 (x_{\bar{u}}(t) + \epsilon g(t)) + (\bar{u}_1(t) + \epsilon w(t))^T R_{11} (\bar{u}_1(t) + \epsilon w(t)).$$

Note that $f(t, \epsilon)$ is differentiable w.r.t. ϵ for every $t \in (0, \infty)$. Simple calculations show that

$$\frac{\partial f}{\partial \epsilon} = 2\epsilon(g^T(t) Q_1 g(t) + w^T(t) R_{11} w(t)) + 2(g^T(t) Q_1 x_{\bar{u}}(t) + w^T(t) R_{11} \bar{u}_1(t))$$

Using the facts that $g(t), w(t)$ and $\bar{u}_1(t)$ are square integrable, it is obvious now that $\frac{\partial f}{\partial \epsilon}$ is integrable for, e.g., all $\epsilon \in [-1, 1]$. Using standard arguments we have then that $J_1(\epsilon)$ is differentiable on $(-1, 1)$ and that

$$\frac{dJ_1(\epsilon)}{d\epsilon} = \int_0^\infty \{\epsilon(g^T(t) Q_1 g(t) + w^T(t) R_{11} w(t)) + (g^T(t) Q_1 x_{\bar{u}}(t) + w^T(t) R_{11} \bar{u}_1(t))\} dt$$

From (27) we get

$$\frac{dJ_1(\epsilon)}{d\epsilon}|_{\epsilon=0} = 0. \tag{29}$$

So, we get:

$$\int_0^\infty \{g^T(t) Q_1 x_{\bar{u}}(t) + w^T(t) R_{11} \bar{u}_1(t)\} dt = 0.$$

Substitution of the expression for $g(t)$ into this equation and then interchanging the order of integration yields:

$$\int_0^\infty \{\int_s^\infty (e^{A(t-s)}B_1w(s))^TQ_1x_{\bar{u}}(t)dt\}ds + \int_0^\infty w^T(t)R_{11}\bar{u}_1(t)dt = 0.$$

Which can be restated as:

$$\int_0^\infty \{w^T(t)B_1^Te^{-A^Tt}\int_t^\infty e^{A^Ts}Q_1x_{\bar{u}}(s)ds\}dt + \int_0^\infty w^T(t)R_{11}\bar{u}_1(t)dt = 0.$$

Now, choose in the above expression, consecutively,

$$w(t) := sgn[e_i^T(B_1^T\int_t^\infty e^{A^T(s-t)}Q_1x_{\bar{u}}(s)ds + R_{11}\bar{u}_1(t))]e^{-\lambda t}e_i, i = 1, .., n,$$

where λ is an arbitrary real number larger than the spectral radius of matrix A and e_i is the i-th standard basis vector in $\mathbb{R}^n$. Then it is clear that for every choice of $w(t)$, $\bar{u}_1 + w(t) \in U$. Consequently it follows that

$$\bar{u}_1(t) = -R_{11}^{-1}B_1^T\int_t^\infty e^{A^T(s-t)}Q_1x_{\bar{u}}(s)ds. \tag{30}$$

Similarly, it can be shown that $\bar{u}_2(t)$ necessarily satisfies:

$$\bar{u}_2(t) = -R_{22}^{-1}B_2^T\int_t^\infty e^{A^T(s-t)}Q_2x_{\bar{u}}(s)ds \tag{31}$$

Next, we introduce the vector $v := (v_1^T\ v_2^T\ v_3^T)^T$ as follows: $v_1(t) := x_{\bar{u}}(t)$, $v_2(t) := \int_t^\infty e^{A^T(s-t)}Q_1x_{\bar{u}}(s)ds$ and $v_3(t) := \int_t^\infty e^{A^T(s-t)}Q_2x_{\bar{u}}(s)ds$. Then $\bar{u}_1 = -R_{11}^{-1}B_1^Tv_2(t)$ and $\bar{u}_2 = -R_{22}^{-1}B_2^Tv_3(t)$. Moreover, it is easily verified by differentiation of $v_2(t)$ and $v_3(t)$, that $v(t)$ satisfies

$$\dot{v}(t) = -\begin{pmatrix} -A & S_1 & S_2 \\ Q_1 & A^T & 0 \\ Q_2 & 0 & A^T \end{pmatrix} v(t), \text{ with } v_1(0) = x_0. \tag{32}$$

Since by assumption for arbitrary x_0, $v(t)$ converges to zero, it follows that there exist $K_1,\ K_2$ and a stable matrix Λ such that $-\begin{pmatrix} -A & S_1 & S_2 \\ Q_1 & A^T & 0 \\ Q_2 & 0 & A^T \end{pmatrix}\begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} I \\ K_1 \\ K_2 \end{pmatrix}\Lambda$. Writing out these equations yields then the advertised result.

"$\Leftarrow$" Let K_1, K_2 be any pair of solutions satisfying the algebraic Riccati equations (ARE) and the additional constraint that the eigenvalues of $A -$

$S_1K_1 - S_2K_2$ are all situated in the left half complex plane. We will show next that

$$\min_{u_1} \lim_{t_f\to\infty} J_1(u_1, u_2^*),$$

where $u_2^*(t) = -R_{22}^{-1}B_2^TK_2e^{(A-S_1K_1-S_2K_2)t}x_0$, is obtained by choosing $u_1(t) = u_1^*(t) := -R_{11}^{-1}B_1^TK_1e^{(A-S_1K_1-S_2K_2)t}x_0$. Since a similar reasoning shows that $\min_{u_2}\lim_{t_f\to\infty} J_2(u_1^*, u_2) \geq \lim_{t_f\to\infty} J_1(u_1^*, u_2^*)$, we have by definition that (u_1^*, u_2^*) is an open-loop Nash equilibrium.
To prove this claim, we first note that by substituting u_2^* into (1), we have that

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}[B_1u_1(\tau) - S_2K_2e^{(A-S_1K_1-S_2K_2)\tau}x_0]d\tau$$

Now, consider $\int_0^t e^{At}\frac{d}{d\tau}[e^{-A\tau}e^{(A-S_1K_1-S_2K_2)\tau}]x_0d\tau$. Evaluating this expression by on the one hand carrying out the differentiation w.r.t. τ and on the other hand calculating the integral yields the following equality:

$$\int_0^t e^{A(t-\tau)}(-A)e^{(A-S_1K_1-S_2K_2)\tau}x_0d\tau + \int_0^t e^{A(t-\tau)}(A-S_1K_1-S_2K_2)e^{(A-S_1K_1-S_2K_2)\tau}x_0d\tau$$

$$= e^{(A-S_1K_1-S_2K_2)t}x_0 - e^{At}x_0.$$

Some elementary rewriting of this equality gives that:

$$e^{At}x_0 - \int_0^t e^{A(t-\tau)}S_2K_2e^{(A-S_1K_1-S_2K_2)\tau}x_0d\tau =$$

$$e^{(A-S_1K_1-S_2K_2)t}x_0 + \int_0^t e^{A(t-\tau)}S_1K_1e^{(A-S_1K_1-S_2K_2)\tau}x_0d\tau.$$

So, we can rewrite $x(t)$ as:

$$x(t) = e^{(A-S_1K_1-S_2K_2)t}x_0 + \int_0^t e^{A(t-\tau)}B_1\{R_{11}^{-1}B_1^TK_1e^{(A-S_1K_1-S_2K_2)\tau}x_0 + u_1(\tau)\}d\tau$$

Therefore, using the notation $v(t) := u_1^*(t) + u_1(t)$,

$$\lim_{t_f\to\infty} J_1(u_1, u_2^*) = \int_0^\infty \{(e^{(A-S_1K_1-S_2K_2)t}x_0 + \int_0^t e^{A(t-\tau)}B_1v(\tau)d\tau)^TQ_1(e^{(A-S_1K_1-S_2K_2)t}x_0 +$$

$$\int_0^t e^{A(t-\tau)}B_1v(\tau)d\tau) + u_1^T(t)R_{11}u_1(t)\}dt \quad (i)$$

Now, consider

$$s := \int_0^\infty (\int_0^t e^{A(t-\tau)}B_1v(\tau)d\tau)^TQ_1e^{(A-S_1K_1-S_2K_2)t}x_0 - v^T(t)B_1^TK_1e^{(A-S_1K_1-S_2K_2)t}x_0)dt.$$

Since, by assumption, K_1 and K_2 satisfy (ARE) we can rewrite Q_1 as:

$$Q_1 = -A^T K_1 - K_1(A - S_1K_1 - S_2K_2)$$

Substitution into s yields:

$$\begin{aligned} s &= -\int_0^\infty (\int_0^t e^{A(t-\tau)}B_1 v(\tau)d\tau)^T(A^TK_1 + K_1(A - S_1K_1 - S_2K_2))e^{(A-S_1K_1-S_2K_2)t}x_0 \\ &\quad +v^T(t)B_1^TK_1e^{(A-S_1K_1-S_2K_2)t}x_0)dt \\ &= -\int_0^\infty \frac{d}{dt}(\int_0^t e^{A(t-\tau)}B_1v(\tau)d\tau)^TK_1e^{(A-S_1K_1-S_2K_2)t}x_0)dt \\ &= -(\lim_{t_f\to\infty}\int_0^{t_f} e^{A(t_f-\tau)}B_1v(\tau)d\tau)^TK_1e^{(A-S_1K_1-S_2K_2)t_f}x_0). \end{aligned}$$

Due to our assumption that all eigenvalues of $A - S_1K_1 - S_2K_2$ are stable and the fact that $J_1(u_1^*, u_2^*) < \infty$, which implies that we may assume without loss of generality (note that $Q_1 > 0 \forall x \in < A|B_1 >$ in J_1) that $\lim_{t_f\to\infty}\int_0^{t_f} e^{A(t_f-\tau)}B_1v(\tau)d\tau$ exists, we have from the last equation that $s = 0$. Rewriting the equation gives:

$$\int_0^\infty (\int_0^t e^{A(t-\tau)}B_1v(\tau)d\tau)^TQ_1e^{(A-S_1K_1-S_2K_2)t}x_0dt = \int_0^\infty v^T(t)B_1^TK_1e^{(A-S_1K_1-S_2K_2)t}x_0dt.$$

The rest of the proof follows now by completion of squares. Substitution of the last expression into the formula (i) for J_1 shows that we can rewrite J_1 as:

$$\int_0^\infty \{(e^{(A-S_1K_1-S_2K_2)t}x_0)^TQ_1e^{(A-S_1K_1-S_2K_2)t}x_0 + 2v^T(t)R_{11}u_1^*(t) + u_1^T(t)R_{11}u_1(t) +$$

$$(\int_0^t e^{A(t-\tau)}B_1v(\tau)d\tau)^TQ_1\int_0^t e^{A(t-\tau)}B_1v(\tau)d\tau\}dt$$

$$= \int_0^\infty \{(e^{(A-S_1K_1-S_2K_2)t}x_0)^TQ_1e^{(A-S_1K_1-S_2K_2)t}x_0 + (u_1(t)+u_1^*(t))^TR_{11}(u_1(t)+u_1^*(t)) +$$

$$(u_1^*(t))^TR_{11}u_1^*(t) + (\int_0^t e^{A(t-\tau)}B_1(u_1^*(\tau)+u_1(\tau))d\tau)^TQ_1\int_0^t e^{A(t-\tau)}B_1(u_1^*(\tau)+u_1(\tau))d\tau\}dt.$$

Using standard arguments it follows now immediately from this formula that $\lim_{t_f\to\infty} J_1(u_1, u_2^*)$ is minimal by choosing $u_1(t) = u_1^*(t)$. Moreover, its minimal value is $\int_0^\infty \{(e^{(A-S_1K_1-S_2K_2)t}x_0)^T(Q_1 + K_1^TS_1K_1)e^{(A-S_1K_1-S_2K_2)t}x_0dt$. $\square$

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