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The chromatic index of strongly regular graphs

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Abstract

We determine (partly by computer search) the chromatic index (edge-chromatic number) of many strongly regular graphs (SRGs), including the SRGs of degree $k \leq 18$ and their complements, the Latin square graphs and their complements, and the triangular graphs $T(m)$ with $m \not\equiv 0 \mod 4$, and their complements. Moreover, using a recent result of Ferber and Jain it is shown that an SRG of even order n, which is not the block graph of a Steiner 2-design or its complement, has chromatic index k, when n is big enough. Except for the Petersen graph, all investigated connected SRGs of even order have chromatic index equal to k, i.e., they are class 1, and we conjecture that this is the case for all connected SRGs of even order.

Keywords: strongly regular graph, chromatic index, edge coloring, 1-factorization. AMS subject classification: 05C15, 05E30.

1 Introduction

An *edge-coloring* of a graph G is a coloring of its edges such that intersecting edges have different colors. Thus a set of edges with the same colors (called

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a color class) is a matching. The *edge-chromatic number* $\chi'(G)$ (also known as the *chromatic index*) of G is the minimum number of colors in an edge-coloring. By Vizing's famous theorem [22], the chromatic index of a graph G of maximum degree Δ is Δ or $\Delta + 1$. A graph with maximum degree Δ is called class 1 if $\chi'(G) = \Delta$ and is called class 2 if $\chi'(G) = \Delta + 1$. It is also known that determining whether a graph G is class 1 is an NP-complete problem [16]. If G is regular of degree k, then G is class 1 if and only if G has an edge coloring such that each color class is a perfect matching. A perfect matching is also called a 1-*factor*, and a partition of the edge set into perfect matchings is called a 1-*factorization*. So being regular and class 1 is the same as having a 1-factorization (being 1-factorable), and requires that the graph has even order.

A graph G is called a *strongly regular graph* (SRG) with parameters (n, k, λ, μ) if it has n vertices, is k-regular $(0 < k < n - 1)$, any two adjacent vertices of G have exactly λ common neighbors and any two distinct non-adjacent vertices of G have exactly μ common neighbors. The complement of a strongly regular graph with parameters (n, k, λ, μ) is again strongly regular, and has parameters $(n, n - k - 1, n - 2k + \mu - 2, n - 2k + \lambda)$. An SRG G is called *imprimitive* if G or its complement is disconnected, and *primitive* otherwise. An imprimitive strongly SRG must be ℓK_m $(\ell, m \geq 2)$, the disjoint union of ℓ cliques of order m, or its complement $\overline{\ell K_m}$, the complete ℓ-multipartite graph with color classes of size m. It is well-known that K_m $(m \geq 2)$, and hence also ℓK_m, is class 1 if and only if m is even. The complement of ℓK_m is a regular complete multipartite graph which is known to be class 1 if and only if the order is even [15].

A *vertex coloring* of G is a coloring of the vertices of G such that adjacent vertices have different colors. The *chromatic number* $\chi(G)$ of G is the minimum number of colors in a vertex coloring. For the chromatic number there exist bounds in terms of the eigenvalues of the adjacency matrix, which turn out to be especially useful for strongly regular graphs (see for example [5]). These bounds imply that there exist only finitely many primitive SRGs with a given chromatic number, and made it possible to determine all SRGs with chromatic number at most four (see [13]). Motivated by these results, Alex Rosá asked the third author whether eigenvalue techniques can give information on the chromatic index of an SRG. There exist useful spectral conditions for the existence of a perfect matching (see [4, 8]), and Brouwer and Haemers [4] have shown that every regular graph of even order, degree k and second largest eigenvalue θ_2 contains at least $\lfloor \frac{k-\theta_2+1}{2} \rfloor$ edge disjoint perfect matchings. From this it follows that every connected SRG of even order has a perfect matching. Moreover, Cioabă and Li [9] proved that any matching of order $k/4$ of a primitive SRG of valency k and even order, is contained in a perfect matching. These authors conjectured that $k/4$ can be replaced by $\lceil k/2 \rceil - 1$ which would be best possible. Unfortunately, we found no useful eigenvalue tools for determining the chromatic index. However, the following recent result of Ferber and Jain [12] gives an asymptotic condition for being class 1 in terms of the eigenvalues.

Theorem 1.1. *There exist universal constants n_0 and k_0, such that the following holds. If G is a connected k-regular graph of even order n with eigenvalues $k = \theta_1 > \theta_2 \geq \ldots \geq \theta_n$, and $n > n_0$, $k > k_0$ and $\max\{\theta_2, -\theta_n\} < k^{0.9}$, then G is class 1.*

If G has diameter 2 (as is the case for a connected SRG), then $n \leq k^2 + 1$. This implies that for an SRG we do not need to require that $k > k_0$ when we take $n_0 \geq k_0^2 + 1$. Theorem 1.1 enables us to show that, except for one family of SRGs, all connected SRGs of even order n are class 1, provided n is large enough. In addition, we present a number of sufficient conditions for an SRG to be class 1. By computer, using SageMath [20], we verified that all primitive SRGs of even order and degree $k \leq 18$ and their complements are class 1, except for he Petersen graph, which has parameters $(10, 3, 0, 1)$ and edge-chromatic number 4 (see [18, 22] for example). We also determine the chromatic index of several other primitive SRGs of even order, and all are class 1. Therefore we believe:

Conjecture 1.2. *Except for the Petersen graph, every connected SRG of even order is class 1.*

2 Sufficient conditions for being class 1

A well known conjecture (first stated by Chetwynd and Hilton [7], but attributed to Dirac) states that every k-regular graph of even order n with $k \geq n/2$ is 1-factorable. Cariolaro and Hilton [6] proved that the conclusion holds when $k \geq 0.823n$, and Csaba, Kühn, Lo, Osthus, and Treglown [10], proved the following result.

Theorem 2.1. *There exists a universal constant n_0, such that if n is even, $n > n_0$ and if $k \geq 2\lceil n/4 \rceil - 1$, then every k-regular graph of order n has chromatic index k.*

König [17] proved that every regular bipartite graph of positive degree has a 1-factorization. This result can be generalized in the following way:

Theorem 2.2. *Let $G = (V, E)$ be a connected regular graph of even order n, and let $\{V_1, V_2\}$ be a partition of V such that $|V_1| = |V_2| = n/2$.*
(i) If the graphs induced by V_1 and V_2 are 1-factorable, then so is G.
(ii) If V_1 (and hence V_2) is a clique or a coclique, then G is class 1.

Proof. Partition the edge set E into two classes E_1 and E_2, where E_1 contains all edges with both endpoints in the same vertex set V_1 or V_2, and the edges of E_2 have one endpoint in V_1 and the other endpoint in V_2.
(i) If the graphs induced by V_1 and V_2 are 1-factorable, then both have the same degree, and therefore also (V, E_1) is 1-factorable. By König's theorem (V, E_2) is 1-factorable, therefore G is class 1.
(ii) If V_1 is a coclique, then so is V_2 and we have the theorem of König. If V_1 is a clique, then so is V_2. If $n/2$ is even, then the result is proved in (i). If $n/2$ is odd, then we move a 1-factor F of (V, E_2) from E_2 to E_1 (here we use that G is connected). Let E'_1 and E'_2 be the resulting edge sets. Then (V, E'_2) is 1-factorable (or has no edges), and (V, E'_1) consists of two cliques of order $n/2$ and the 1-factor F. Thus F gives a bijection between V_1 and V_2. We color the edges of both cliques with $n/2$ colors, such that the bijection F preserves the edge colors. Now for each edge e of F, the two sets of colored edges that intersect at an endpoint of e use the same set of $n/2 - 1$ colors. So we can color e with the remaining color. □

There exist several SRGs that have the partition of case (i). The Gewirtz graph is the unique SRG with parameters $(56, 10, 0, 2)$, and admits a partition into two Coxeter graphs (see [3]). The Coxeter graph is known to be 1-factorable (see [2]), therefore the Gewirtz graph is class 1. The same holds for the point graph of the generalized quadrangle $GQ(3, 9)$ (the unique SRG(112,30,2,10)), which admits a partition into two Gewirtz graphs, and for the Higman-Sims graph (the unique SRG(100,22,0,6)), which can be partitioned into two copies of the Hoffman-Singleton graph (the unique SRG with parameters(50,7,0,1)), which has chromatic index 7 (see next section).

Suppose $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$ are the eigenvalues of a graph G of order n. Hoffman (see [5, Theorem 3.6.2] for example) proved that the chromatic number of G is at least $1 + \frac{\theta_1}{-\theta_n}$. A vertex coloring that meets this bound is called a *Hoffman coloring*. For k-regular graphs, the color classes of a Hoffman coloring are cocliques of which the size meets Hoffman's coclique bound $\frac{n(-\theta_n)}{k-\theta_n}$. This implies (see [5] for example) that all the color classes have equal size, and any vertex v of G has exactly $-\theta_n$ neighbors in each color class different from the color class of v.

Theorem 2.3. *Suppose $G = (V, E)$ is a primitive k-regular graph with an even chromatic number that meets Hoffman's bound. Then both G and its complement are class 1.*

Proof. Let $S_1, \ldots, S_{2t}$ be the color classes in a Hoffman coloring of G. This implies that each S_i is a coclique attaining equality in the Hoffman ratio bound, which means that each vertex outside S_i has exactly $-\theta_n$ neighbors in S_i. Hence, each subgraph induced by two distinct cocliques S_i and S_j is a bipartite regular graph of valency $-\theta_n$. A 1-factorization of K_{2t} corresponds to a partition $E_1, \ldots, E_{2t-1}$ of E, such that each (V, E_i) consists of t disjoint regular bipartite graphs of degree $-\theta_n = k/(2t-1)$. By König's theorem it follows that each (V, E_i) is 1-factorable, and therefore G is class 1.
For the complement $\overline{G} = (V, F)$ of G, a similar approach works. We can partition the edge set $F_0, F_1, \ldots, F_{2t-1}$, such that (V, F_i) is the disjoint union of t regular bipartite graphs of the same degree for $i = 1, \ldots, 2t-1$. But now there is an additional graph (V, F_0) consisting of $2t$ disjoint cliques. We combine F_0 and F_1. Then $(V, F_0 \cup F_1)$ is the disjoint union of t complements of regular incomplete bipartite graphs with the same degree, and therefore has a 1-factorization by Theorem 2.2. Since for $i = 2, \ldots, 2t-1$ (V, F_i) has a 1-factorization, it follows that $\overline{G}$ is 1-factorable. □

For an SRG the complement of a Hoffman coloring is called a *spread* (see [14]). As a consequence of this result, it follows that any primitive strongly regular graph with a spread with an even number of cliques, or a Hoffman coloring with an even number of colors is class 1. Among such SRGs are the Latin square graphs. Consider a set of t ($t \geq 0$) mutually orthogonal Latin squares of order m ($m \geq 2$). The vertices of the Latin square graph are the m^2 entries of the Latin squares, and two distinct entries are adjacent if they lie in the same row, the same column, or have the same symbol in one of the squares. If $t = m-1$ we obtain the complete graph K_{m^2}, and if $t = m-2$ we have a complete multipartite graph. Otherwise the Latin square graph is a primitive SRG with parameters $(m^2, (t+2)(m-1), m-2+t(t+1), (t+1)(t+2))$. If $t = 0$ we only have the rows and columns, then the Latin square graph is better known as the Lattice graph $L(m)$. If $m \neq 4$, the Lattice graph is determined by the parameters. The m rows of a

Latin square give a partition of the vertex set of the Latin square graph into cliques, which is a spread. Thus we have:

Corollary 2.4. *If G is a Latin square graph of even order, then both G and its complement are 1-factorable.*

3 Asymptotic results

A *Steiner 2-design* (or *2-$(m, \ell, 1)$ design*) consists of a point set $\mathcal{P}$ of cardinality m, together with a collection of subsets of $\mathcal{P}$ of size ℓ ($\ell \geq 2$), called *blocks*, such that every pair of points from $\mathcal{P}$ is contained in exactly one of the blocks. The *block graph* of a Steiner 2-design is defined as follows. The blocks are the vertices, and two vertices are adjacent if the blocks intersect in one point. If $m = \ell^2 - \ell + 1$, the Steiner 2-design is a projective plane, and the block graph is K_m. Otherwise the block graph is an SRG with parameters $(m(m-1)/\ell(\ell-1), \ell(m-\ell)/(\ell-1), (\ell-1)^2 + (m-2\ell+1)/(\ell-1), \ell^2)$.

Theorem 3.1. *There exist an integer n_0, such that every primitive strongly regular graph of even order $n > n_0$, which is not the block graph of a Steiner 2-design or its complement, is class 1.*

Proof. Suppose G is a primitive (n, k, λ, μ)-SRG with eigenvalues $k = \theta_1 > \theta_2 \geq \ldots \geq \theta_n$. Assume that G nor its complement $\overline{G}$ is the block graph of a Steiner 2-design or a Latin square graph, then (see Neumaier [19])

$$\theta_2 \leq \theta_n(\theta_n + 1)(\mu + 1)/2 - 1.$$

Another result of Neumaier [19] (the so-called μ-bound) gives $\mu \leq \theta_n^3(2\theta_n+3)$. Therefore $\theta_2 \leq \theta_n^6$. Next we apply the same result to $\overline{G}$, and obtain $1 - \theta_n \leq (1-\theta_2)^6$, which yields $-\theta_n \leq \theta_2^6$. By use of the identity $k + \theta_2\theta_n = \mu > 0$, and the above inequalities we get $\max\{\theta_2, -\theta_n\} \leq k^{6/7}$. Now we apply the result of Ferber and Jain and conclude that G is class 1 when n is large enough.

If G is a Latin square graph of even order then by Corollary 2.4 both G and its complement $\overline{G}$ are class 1. □

A set of m/ℓ disjoint blocks of a Steiner 2-design is called a *parallel class*, and a partition of the block graph into parallel classes is a *parallelism*. A parallelism of a Steiner 2-design gives a Hoffman coloring in the block graph, so if the number of parallel classes is even, the block graph has a 1-factorization.

The block graph of a Steiner 2-design with block size 2 is the triangular graph, which is investigted in Section 5. If the block size equals 3, the design is better known as a Steiner triple system. The chromatic index of the block graph of a Steiner triple system is investigated in [11]. The paper contains several sufficient conditions for such a graph to be class 1, and the authors conjecture that all these graphs are class 1 when the order is even.

In many cases the complement of the block graph of a Steiner 2-design has $k > n/2$, so it will have a 1-factorization by Theorem 2.1, provided n is even and large enough. The following result follows straightforwardly from the mentioned result of Cariolaro and Hilton [6].

Proposition 3.2. *If G is the complement of the block graph of a Steiner 2-$(m, \ell, 1)$ design with $6\ell^2 < m$, then G is class 1, provided G has even order.*

4 SRGs of degree at most 18

According to the list of Brouwer [1] all SRGs of even order and degree at most 18 are known (one only has to check the parameter sets up to $n = 18^2 + 1 = 325$). The parameters of the primitive ones are given in Table 1; the number next to each parameter set gives the number of non-isomorphic SRGs with those parameters. The graph with parameter set a is the Petersen graph,

a	(10,3,0,1)	1	f	(36,10,4,2)	1	k	(50,7,0,1)	1
b	(16,5,0,2)	1	g	(36,14,4,6)	180	l	(56,10,0,2)	1
c	(16,6,2,2)	2	h	(36,14,7,4)	1	m	(64,14,6,2)	1
d	(26,10,3,4)	10	i	(36,15,6,6)	32548	n	(64,18,2,6)	167
e	(28,12,6,4)	4	j	(40,12,2,4)	28	o	(100,18,8,2)	1

Table 1: Primitive SRGs with n even and $k \leq 18$

which is class 2. The complement of the Petersen graph is the triangular graph $T(5)$ which is class 1 (see next section). For the parameter sets f, m and o there is a unique SRG, the so called Lattice graph. This SRG belongs to the Latin square graphs, and by Corollary 2.4 the graph is class 1, and so is its complement. Case k and l are the Hoffman-Singleton graph, and the Gewirtz graph, respectively. Both graphs and their complements are class 1 by Theorem 2.2, as we saw in Section 2. All other graphs are tested by computer (we actually tested all graphs in Table 1). Using SageMath[20], we wrote a computer program that searches for an edge coloring in a k-regular graph with k colors. In each step we look (randomly) for a perfect matching, remove all its edges and continue until the remaining graph has no perfect matching. If there are still edges left we start again. We run this algorithm repeatedly until an edge coloring is found. By use of this approach we found a 1-factorization in all graphs of Table 1, and in their complements, except for the Petersen graph. Thus we found:

Theorem 4.1. *With the single exception of the Petersen graph, a primitive SRG of even order and degree at most 18 is class 1 and so is its complement.*

For the description of the graphs we used the website of Spence [21]. This website also contains several SRGs with parameters $(50, 21, 8, 9)$. We also ran the search for these graphs. All are class 1.

It is surprising that in all cases our straightforward heuristic finds a 1-factorization. The heuristic is fast. It took about one hour to find a 1-factorization in each of the 32548 SRGs with parameter set i

5 The triangular graph

The parameter sets e and h of the above table belong to the triangular graphs $T(8)$ and $T(9)$. The triangular graphs $T(m)$ is the line graphs of the complete graph K_m, and when $m \geq 4$ it is an SRG with parameters $(m(m-1)/2, 2(m-2), m-2, 4)$. Clearly the order of $T(m)$ is even if $m \equiv 0$, or $1 \mod 4$. The triangular graph is also the block graph of a Steiner 2-design with block size 2, so $T(m)$ belongs to the exception in Theorem 3.1. Therefore we tested a few more triangular graphs ($m \leq 32$, $m \equiv 0$ or $1 \mod 4$) and their complements ($m \leq 21$, $m \equiv 0$ or $1 \mod 4$). The complement of $T(5)$ is the Petersen graph, which is class 2. All others turn out to be class 1. Note that Proposition 3.2 implies that for n even and $m \geq 24$ the complement of $T(m)$ is class 1. Thus we can conclude:

Theorem 5.1. *For $m \equiv 0$, or $1 \mod 4$ and $m \neq 5$, the complement of the triangular graph $T(m)$ is class 1.*

We tried hard to prove that $T(m)$ is class 1 if the order is even with partial success.

Theorem 5.2. *If $m \equiv 1 \mod 4$, then the triangular graph $T(m)$ is class 1.*

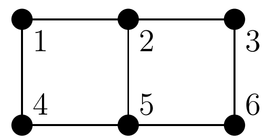

Figure 1: Subgraph used in Lemma 5.3

To prove this theorem, we need a lemma.

Lemma 5.3. *Let G be a regular bipartite graph of order 2ℓ and degree 4. Suppose G contains the subgraph (not necessarily induced) given in Figure 1, then G has a perfect matching that contains the edges $\{1, 4\}$ and $\{2, 5\}$.*

Proof. Suppose not. Then the subgraph G' obtained from G by deleting the vertices 1, 2, 4 and 5, has no perfect matching. By Hall's marriage theorem, G' contains a coclique C of size $\ell - 1$. In G, C has $4(\ell - 1)$ outgoing edges and together with the four edges $\{1, 2\}$, $\{2, 5\}$, $\{4, 1\}$ and $\{4, 5\}$ we obtain all edges in G. This implies that the edges $\{2, 3\}$ and $\{5, 6\}$ are outgoing edges of C, but then 3 and 6 are not adjacent, which is a contradiction. □

Now we prove Theorem 5.2.

Proof. Write $m = 4k + 1$. The edges of the complete graph K_m can be partitioned onto $2k$ m-cycles $C_1, \ldots, C_{2k}$ as follows. Label the vertices of K_m with $0, 1, \ldots, 4k - 1, \infty$, define

$$C_1 = (\infty, 0, 4k - 1, 1, 4k - 2, 2, \ldots, 2k - 1, 2k, \infty),$$

and $C_i = C_1 + i \mod 4k$ for $i = 0, \ldots, 2k - 1$. This gives a partition of the vertex set V of $T(m)$ into $2k$ m-cycles $C'_1, \ldots, C'_{2k}$ (C'_i consists of the edges in C_i). This partition is equitable, which means that every vertex in one cycle has exactly 4 neighbors in each other cycle. Let $F_1, \ldots, F_{2k-1}$ be a 1-factorization of K_{2k}. We partition the edge set E of $T(m)$ as follows: $E = E_0 \cup E_1 \cup \cdots \cup E_{2k-1}$, where E_0 contains all the edges in the m-cycles $C'_1, \ldots, C'_{2k}$, and for $i \geq 1$ E_i consists of all edges between the cycles C'_a and C'_b for which $\{a, b\} \in F_i$. Then, for $i = 1, \ldots, 2k - 1$, (V, E_i) has k components each of which is bipartite and regular of degree 4. Therefore each (V, E_i) ($i \neq 0$) can be edge-colored with 4 colors. We do so for $i = 2, \ldots, 2k - 1$ (using $4(2k - 2)$ colors). Next we need to color the edges of $E_0 \cup E_1$ with 6 colors. We assume that $F_1 = \{\{1, 2\}, \{3, 4\}, \ldots, \{2k - 1, 2k\}\}$, so that E_1 consists of the edges between C'_a and C'_{a+1} for odd $a \in [1, 2k - 1]$. The graph $(V, E_0 \cup E_1)$ has k regular components of degree 6, which ar mutually isomorphic. It suffices to prove that one component H (say) consisting of the cycles C_1' and C_2' and the edges between these cycles, is class 1. Let G be the bipartite graph obtained from H by deleting the edges of C_1' and C_2'. Then G is regular of degree 4, and the following six vertices of G induce a subgraph in G containing the graph of Figure 1:

$$\{2k, \infty\}, \{\infty, 1\}, \{4k-1, 1\}, \{2k+1, \infty\}, \{\infty, 0\}, \{0, 1\}.$$

Now Lemma 5.3 implies that G has a perfect matching that contains the edges $\{\{2k, \infty\}, \{2k+1, \infty\}\}$ and $\{\{\infty, 1\}, \{\infty, 0\}\}$. Therefore the edges of G can be colored with four colors such that these two edges get the same color (say red). Next color the edges of C_1' and C_2' except $\{\{\infty, 0\}, \{\infty, 2k\}\}$ and $\{\{\infty, 1\}, \{\infty, 2k+1\}\}$ with two colors (say blue and green), such that $\{\{\infty, 0\}, \{0, 4k-1\}\}$ and $\{\{\infty, 1\}, \{0, 1\}\}$ get the same color, and color $\{\{\infty, 0\}, \{\infty, 2k\}\}$ and $\{\{\infty, 1\}, \{\infty, 2k+1\}\}$ red. Finally, we change the color of $\{\{\infty, 0\}, \{\infty, 1\}\}$ and $\{\{\infty, 2k\}, \{\infty, 2k+1\}\}$ into the unique admissible color, blue or green. Thus we obtain a coloring of the edges of H with six colors. □

{06, 67}	{06, 16}	{02, 06}	{06, 56}	{05, 06}	{06, 07}	{03, 06}	{06, 36}	{06, 46}	{01, 06}	{06, 26}	{04, 06}
{01, 03}	{03, 37}	{03, 23}	{03, 04}	{03, 34}	{03, 05}	{07, 67}	{03, 07}	{03, 36}	{03, 13}	{03, 35}	{02, 03}
{27, 57}	{46, 67}	{57, 67}	{16, 67}	{26, 67}	{36, 67}	{17, 57}	{17, 67}	{56, 67}	{37, 67}	{27, 67}	{47, 67}
{34, 35}	{56, 57}	{34, 47}	{35, 57}	{45, 57}	{47, 57}	{13, 34}	{37, 57}	{07, 57}	{25, 57}	{15, 57}	{05, 57}
{16, 26}	{34, 45}	{26, 56}	{34, 46}	{24, 25}	{34, 37}	{23, 26}	{04, 34}	{23, 34}	{24, 34}	{34, 36}	{14, 34}
{05, 25}	{26, 36}	{12, 25}	{25, 26}	{02, 04}	{24, 26}	{25, 56}	{02, 26}	{12, 26}	{26, 46}	{02, 25}	{26, 27}
{02, 07}	{25, 35}	{05, 45}	{05, 07}	{23, 36}	{25, 45}	{04, 05}	{25, 27}	{15, 25}	{05, 35}	{12, 23}	{23, 25}
{13, 23}	{02, 23}	{04, 14}	{01, 02}	{14, 15}	{02, 27}	{14, 46}	{23, 24}	{02, 05}	{02, 12}	{14, 17}	{13, 17}
{12, 17}	{12, 14}	{17, 37}	{23, 37}	{17, 47}	{23, 35}	{02, 24}	{01, 14}	{14, 45}	{23, 27}	{45, 56}	{35, 56}
{15, 56}	{07, 17}	{24, 46}	{14, 47}	{46, 56}	{13, 14}	{01, 12}	{05, 56}	{16, 17}	{14, 16}	{24, 47}	{24, 45}
{14, 24}	{04, 24}	{01, 16}	{17, 27}	{01, 07}	{01, 17}	{45, 47}	{46, 47}	{24, 27}	{15, 17}	{01, 05}	{01, 15}
{04, 47}	{01, 13}	{13, 15}	{12, 24}	{12, 27}	{16, 56}	{15, 35}	{15, 16}	{01, 04}	{36, 56}	{16, 46}	{36, 46}
{45, 46}	{05, 15}	{07, 27}	{15, 45}	{13, 16}	{12, 15}	{27, 37}	{12, 13}	{37, 47}	{07, 47}	{13, 37}	{12, 16}
{36, 37}	{27, 47}	{35, 36}	{13, 36}	{35, 37}	{04, 46}	{16, 36}	{35, 45}	{13, 35}	{04, 45}	{04, 07}	{07, 37}

Table 2: a 1-factorization of T(8)

Unfortunately we were not able to prove that $T(m)$ is class 1 when m is a multiple of 4. Still we strongly believe that it is true, since for each $m \leq 32$

($m \equiv 0 \mod 4$) the computer search found a 1-factorization in less than a second. In Table 2 we give a 1-factorization of $T(8)$, which was found in 66 milliseconds (each column is a color class; the vertex ab of $T(8)$ corresponds to the edge $\{a, b\}$ of K_8).

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