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Koster, M.A.L.; Tijs, S.H.; Borm, P.E.M.

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Serial cost sharing methods for multi-commodity situations

Maurice Koster*, Stef Tijs, Peter Borm

Department of Econometrics and Center for Economic Research, Tilburg University, PO Box 90153, NL-5000 LE, Tilburg, Netherlands

Abstract

We consider ways to extend the serial cost sharing method (Moulin, H., Shenker, S., Serial cost sharing, Econometrica 60 (1992) 1009–1037; Moulin, H., Shenker, S., Average cost pricing versus serial cost sharing: An axiomatic comparison, Journal of Economic Theory 64 (1992) 178–201) to a setting where agents have a demand for bundles of heterogeneous goods. We introduce a class of serial extensions which is based on the use of preorderings. The class is characterized by the properties of Equal Treatment of Preordering Equivalents and the Radial Serial Principle. The preordering that judges agents on their stand alone costs leads us to a serial extension that is a generalization of the axial rule (Sprumont, 1996). A characterization of this Radial Serial Rule is provided. As a consequence we find that, in contrary to the more restrictive model of Sprumont (1996), in our setting the properties of Independence of Null Agents, Rank Independence of Irrelevant Agents together with Ordinality and the Serial Principle are incompatible.

Keywords: Serial cost sharing; Serial extensions; Equal treatment; Preordering equivalents; Radical serial principle

1. Introduction

In many real-life situations where individuals work together in a joint project, joint costs or surpluses occur which have to be shared. The central problem of cost sharing is the allocation of costs in a 'just' or 'fair' way among the participants. Examples include the allocation of joint overhead costs of a firm among its divisions (e.g. Shubik, 1962), setting fees for the use of a common facility such as a communication network, an airport or transit system (e.g. Young, 1985a; 1985b; 1994; Billera and Heath, 1982; Tijs and Driessen, 1986) and determining cost shares for a joint production of private or public goods (e.g. Moulin, 1994; Moulin and Shenker, 1992a,b). Although in practice

*Corresponding author. Tel.: +31-13-466-9111; fax: +31-13-466-3072; e-mail: koster@kub.nl

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mostly simple costing formulas and criteria are used to solve such problems, there is a growing demand for more sophisticated devices. One reason for this is the advancing notion of the context dependency of both solution and fairness concepts.

In this paper we focus on the situation where a group of agents jointly own a production facility for several heterogeneous divisible goods. We will assume that the costs associated to each level of output is summarized by a cost function c. Now, a cost sharing problem arises when a bundle of the goods in question has to be produced just to fulfil the individual needs of the owners of the enterprise. In general it will be far from justice just to split costs equally among the agents simply because they have equal access to the technology. Instead, we focus on a *cost sharing mechanism* that induces a much stronger relationship between individual demands and cost shares. The extra (objective) information this mechanism needs are the costs associated with *any* potential level of output.

In case returns to scale are not constant, the agents impose an externality on each other. For a one-output technology, the *serial cost sharing mechanism* (Moulin and Shenker, 1992a) proposes a solution to the problem of sharing this externality. In short we will explain how the serial mechanism determines the individual cost shares for the situation of a production facility for one divisible good that is shared by a fixed group of agents $N:=\{1, 2, \ldots, n\}$. Suppose $c:\mathbb{R}_+ \to \mathbb{R}_+$ is the function that relates the possible levels of output with its costs. It is assumed that c is continuous and $c(0)=0$. The demand of the individual agents is given by a vector $q_N=(q_1, q_2, \ldots, q_n) \in \mathbb{R}_+^N$. Without loss of generality we assume that the demands of the agents are ordered such that: $q_i \le q_j \Leftrightarrow i \le j$. Then agent i's cost share according to the serial mechanism ξ^s is given by

$$\xi_i^s(q_N, c) = \sum_{k=1}^{i} \frac{c(x_k) - c(x_{k-1})}{n+1-k}$$

Here the different x_k's represent intermediate production stages towards the total of production $\bar{q}_N = \Sigma_{k \in N} q_k$, starting with $x_0 = 0$ and for $k \in N$,

$$x_k = \sum_{r=1}^{k-1} q^r + (n+1-k)q^k$$

The serial mechanism captures the idea that in case of decreasing returns to scale the agents with relative small demands should enjoy protection against arbitrarily high cost shares due to excessive, inordinate demands of others. On the contrary, sharing costs proportional to the individual demands does not share this equity feature. The serial cost sharing rule has been singled out for the nice normative and strategic features in Moulin and Shenker (1992b); Moulin (1996). This is what motivates us to aim for serial-like generalizations.

In this paper we consider situations in which there is a set M of divisible output goods which result from a production process that is jointly owned by a fixed group of agents. The different agents each have a demand for a mixed bundle of those goods. If $q_i \in \mathbb{R}_+^M$ represents the demand of agent i, the total costs for production $c(\Sigma_{i \in N} q_i)$ have to be allocated among the group of participating agents.

Though in the literature there is consensus about what the minimal quality of a serial extension should be, beyond this point there is only general disagreement; no proposed extension has been singled out like the Moulin and Shenker (1992b) rule. This is all due to the fact that we run in a fundamental problem by going to the heterogeneous case. The serial cost sharing mechanism uses interpersonal comparison of demand to specify the final allocation of the costs. Where in the one-good case it is clear how to compare the individual demands, it is not evident for higher dimensions. For instance, consider a two-good production technology owned by the agents 1 and 2. Suppose agent 1's demand for the two goods is summarized by (10,20) and that of agent 2 by (30,10), then apart from knowing the cost structure there is not a unique canonical way of comparing these demands. This especially holds in case the cost function is known to be asymmetric with respect to the different goods, a problem which in general can not be solved by simply rescaling the units in which the different goods are measured. The problem with extending the serial cost sharing mechanism to higher dimensions is that there is no natural complete ordering of the demand space. This is already recognized as a hard problem by Kolpin (1994), and also for a more restrictive model where each of the agents is associated with one particular good (see Friedman and Moulin, 1995; Moulin, 1995a,b; Sprumont, 1996).

In this paper we show that with losing the natural ordering of the demand space, we are left with a large class of serial extensions just by brutally ordering the demand profile through various 'reasonable' preorderings. This class of cost sharing methods can easily be characterized by the properties Equal Treatment of Preordering Equivalents and the Radial Serial Property. In particular, we focus on the preordering which compares different demands by their stand alone production costs. It results in the Radial Serial Rule, that can be seen as an extension of the *axial rule* (Sprumont, 1996). Sprumont's characterization result of the axial rule is translated to a characterization of the Radial Serial Rule by a similar set of properties, Equal Treatment of Equals, Radial Ordinality, Independence of Null Agents, Rank Independence of Irrelevant Agents and the Radial Serial Principle. As Kolpin (1997) already shows, we can not be too demanding with respect to serial extensions. In this respect, here the characterization of the Radial Serial Rule directly implies the incompatibility of the properties Equal Treatment of Equals, Rank Independence of Irrelevant Agents, Independence of Null Agents, Ordinality and the Serial Principle.

2. The model

Throughout this paper we will concentrate on a fixed and finite group of agents $N=\{1, 2, \dots, n\}$, jointly owning a production facility for a fixed and finite set of heterogeneous and divisible commodities, which we will denote by M. Without loss of generality we assume that $M=\{1, 2, \dots, m\}$ for some $m \in \mathbb{N}$. A particular level of output then can be described by a vector $q \in \mathbb{R}^M_+$; the jth coordinate q_j stands for the level of commodity j. For all $q', q'' \in \mathbb{R}^M_+$, $q' \leq q''$ if and only if $q'_j \leq q''_j$ for all $j \in M$. We will use $[q', q'']$ as an abbreviation for the set $\{q \in \mathbb{R}^M_+ | q' \leq q \leq q''\}$.

We assume that all information about the costs involved with bringing production up to a certain level is given by a cost function $c:\mathbb{R}_+^M \to \mathbb{R}_+$. We assume that $c(0)=0$, i.e. there are no fixed costs. The cone of all real-valued non-decreasing continuous cost functions is denoted by $\mathscr{C}$. We define $\mathscr{C}_+$ as the subspace of $\mathscr{C}$ of all monotone increasing elements of $\mathscr{C}$. This means that $\mathscr{C}_+$ consists of those $c \in \mathscr{C}$ such that $x \leq y$, $x \neq y$ implies $c(x)<c(y)$ for all x, $y \in \mathbb{R}_+^M$. A cost function c is said to be *homogeneous* if there is a function $c_0:\mathbb{R}_+ \to \mathbb{R}_+$ such that $c(q)=c_0(\Sigma_{i \in M} q_i)$ for all $q \in \mathbb{R}_+^M$. Then we will call $((\Sigma_{j \in M} q_{ij})_{i \in N}, c_0)$ the *reduced homogeneous equivalent for* (q_N, c).

Each individual agent $i \in N$ has a particular demand for each of the goods of M, that is summarized by $q_i \in \mathbb{R}_+^M$, where q_{ij} is the demand of agent i for good j. The profile of revealed demands then is $q_N := (q_i)_{i \in N}$. Then the aggregate of these demands, $\bar{q}_N = \Sigma_{i \in N} q_i$ is produced and the corresponding production costs $c(\bar{q}_N)$ have to be shared.

Definition 2.1. A *cost sharing mechanism* for $S \subseteq \mathscr{C}$ is a mapping $\xi:(\mathbb{R}_+^M)^N \in S \to \mathbb{R}_+^N$, associating to each demand profile q_N and cost function $c \in S$ a vector of cost shares $\xi(q_N, c) \in \mathbb{R}_+^N$, thereby clearing total costs for production, i.e.

$$\sum_{i \in N} \xi_i(q_N, c) = c(\bar{q}_N)$$

Definition 2.2. A cost sharing mechanism ξ is called a *serial extension* on $S \subseteq \mathscr{C}$ if for all *homogeneous problems* (q_N, c) with $c \in S$, ξ computes the individual cost according to the serial mechanism for the corresponding reduced homogeneous equivalent. More formally, a serial extension ξ on S satisfies

$$\xi(q_N, c) = \xi^s\left(\left(\sum_{j \in M} q_{ij}\right)_{i \in N}, c_0\right)$$

for all homogeneous problems (q_N, c) where $c \in S$ and where c_0 is the function such that $c(q)=c_0(\Sigma_{j \in M} q_j)$ for all $q \in \mathbb{R}_+^M$.

If units of the different goods in M are fully comparable and costs depend only on the total of units requested, then essentially we are left with a one-good problem, the reduced homogeneous equivalent problem. The above definition states that a serial extension should obey the principles of serial cost sharing in the sense that the allocation is just the same for all reductions from homogeneous problems. This is also the most common approach in Friedman and Moulin (1995); Sprumont (1996); Kolpin (1994), (1997).

3. A class of preordering-based serial extensions

As Kolpin (1994) already showed there is no unique serial extension; he presents a class that contains already uncountably many. This section discusses the possibility of extending the serial rule using some 'suitable' complete preordering that tells us how to compare different demand bundles. The methodology to be presented gives rise to a new class of serial extensions, the class of *preordering-based serial extensions*.

In order to respond to very asymmetrical cost structures we expect that the more sensible preordering will depend on the cost structure at hand. Fix $c \in \mathscr{C}$. In the following $R(c)$ denotes a complete preordering on $\mathbb{R}_+^M$ that is possibly related to the cost structure that is induced by c. We will write $xE(c)y$ for elements x, $y \in \mathbb{R}_+^M$ that are equivalent with respect to $R(c)$, i.e. $xR(c)y$ and $yR(c)x$. We will make the following reasonable assumptions on $R(c)$. For all demands x, $y \in \mathbb{R}_+^M$,

$$x \leq y \Rightarrow xR(c)y \tag{1}$$

This means that a preordering satisfying property of Eq. (1) leaves the natural partial ordering on $\mathbb{R}_+^M$ intact. Also we will not allow for $R(c)$'s having *thick* indifference classes, which boils down to

$$x \leq y, x \neq y \Rightarrow xR(c)y \wedge \neg(yR(c)x) \tag{2}$$

Furthermore we will assume that in case of a homogeneous cost function $R(c)$ should compare only the total units represented by different demand profiles. So for homogeneous cost functions c,

$$xR(c)y \Leftrightarrow \sum_{i \in M} x_i \leq \sum_{i \in M} y_i \tag{3}$$

We launch one more property that we will expect a preordering $R(c)$ to have,

$$xR(c)y \Rightarrow \exists \beta \in [0, 1] \text{ s.t. } xE(c)\beta y \tag{4}$$

By property of Eq. (2) the β in the above definition is unique. The condition of Eq. (4) can be read as a continuity condition; if the bundle y is valued to have a larger impact on total costs than bundle x, then there is a bundle proportional to y that is $R(c)$-indifferent to x. Let $R(S) := \{R(c) | c \in S\}$ be a family of preorderings. Then $R(S)$ is called *admissible* if for all $c \in S$, $R(c)$ has the properties of Eqs. (1)–(4). For the rest of this section we will fix $S \subseteq \mathscr{C}$ and $R(S)$ is an admissible family of preorderings. Since no confusion will arise, we will also denote the family by R.

If the agents agree upon a family of preorderings $R(S)$ for the interpersonal comparison of demands, then following the most basic equity principle in the cost sharing literature, that of *Equal Treatment of Equals* (ETE), would imply that agents with equivalent demands should pay the same. More formally, we have the following definition.

Definition 3.1. A cost sharing mechanism ξ satisfies *Equal Treatment of Preordering Equivalents w.r.t.* $R(S)$ (ETPE($R(S)$)) if for $c \in S \subseteq \mathscr{C}$ and i, $j \in N$ the following holds,

$$q_i E(c) q_j \Rightarrow \xi_i(q_N, c) = \xi_j(q_N, c)$$

It is clear that ETPE($R(S)$) implies equal treatment of equal demanders, i.e. if agent i's demand is equal to that of agent j, then this is enough information to charge them equally for that.

However, essentially the most typical characteristic of the serial cost sharing mechanism is that small demanders are protected against larger demanders. This characteristic is summarized by the property *Independence of Size of Larger Demands* (see Moulin and Shenker, 1992b). It allows the agents with demands larger than that of agent i to vary their demands and, as long as they remain larger, these changes do not affect agent i's cost share. But then, of course, translating this idea to the multi-good case is just as problematic as comparing the cost impact of different demands bundles in higher dimensions. This is why Sprumont (1996) argues that the only thing one still can do is direct intercomparison of cost shares. His terminology can still be used to extend the notion of the *Serial Principle* to our case, using the following definition.

Definition 3.2. A cost sharing mechanism ξ satisfies the *Serial Principle* (SP) for $c\in\mathscr{C}$ if for all demand profiles $q_N, q'_N\in(\mathbb{R}^M_+)^N$ and all $i\in N$, $\xi_i(q_N, c)=\xi_i(q'_N, c)$ is implied by the following two statements,

(i) $q'_j=q_j$ for $j=i$ and for all $j\in N\backslash\{i\}$ with $\xi_j(q_N, c)\leq\xi_i(q_N, c)$;
(ii) $q'_j\geq q_j$ for all $j\in N\backslash\{i\}$ such that $\xi_j(q_N, c)\geq\xi_i(q_N, c)$.

For our purposes we will, most of all, be concerned with a weaker version of SP, where the demand of an agent j with the higher cost share may be varied only on the radial through 0 and q_j,

Definition 3.3. A cost sharing mechanism ξ satisfies the *Radial Serial Principle* (RSP) for $c\in\mathscr{C}$ if for all demand profiles $q_N, q'_N\in(\mathbb{R}^M_+)^N$ and all $i\in N$, $\xi_i(q_N, c)=\xi_i(q'_N, c)$ is implied by the following two statements,

(i) $q'_j=q_j$ for $j=i$ and for all $j\in N\backslash\{i\}$ with $\xi_j(q_N, c)\leq\xi_i(q_N, c)$;
(ii) $q'_j=\beta_j q_j$ for some $\beta_j\in[1,\infty)$ for all $j\in N\backslash\{i\}$ such that $\xi_j(q_N, c)\geq\xi_i(q_N, c)$.

We will say that ξ satisfies (R)SP on $S\subseteq\mathscr{C}$ if ξ satisfies (R)SP for all $c\in S$.

Definition 3.4. For all ordered triples $(c, x, y)\in S\times\mathbb{R}^M_+\times\mathbb{R}^M_+$ such that $xR(c)y$ we define $\rho^R(c, x, y):=\beta y$ where $\beta\in[0, 1]$ is the scalar with $xE(c)\beta y$.

If $xR(c)y$ then the function ρ^R determines for each cost function $c\in S$ a proportional part of the demand y that is $R(c)$-equivalent with the demand bundle x. This function is introduced as a tie-breaking rule for choosing between bundles in the same equivalence class. It enables us to construct a serial extension in the following way.

Take a cost sharing problem $(q_N, c)\in\mathbb{R}^M_+\times S$. First we rank the agents in accordance with the ranking of their demands w.r.t. $R(c)$. Without loss of generality we assume that $q_1R(c)q_2R(c)\ldots R(c)q_n$. Then, like the procedure for the serial cost sharing mechanism in Moulin and Shenker (1992a), we define *intermediate stages of production* $x_0, x_1, \ldots, x_n\in[0, \bar{q}_N]$. The first level is the one where nothing is produced, $x_0=0$. At the next intermediate level x_1 we will satisfy each of the agents up to an amount as part of their demand that is equivalent with the $R(c)$ smallest demand, q_1. In general each of the equivalence classes $\{q\in[0, q_k]|qE(c)q_1\}$ will be quite large. Here ρ^R enters as the tie-breaking rule. For all $k\in N$, at the second stage of production agent k is allocated $\rho^R(c, q_1, q_k)$ as part of his demand. Then set $x_1=q_1+\Sigma^n_{k=2}\ \rho^R(c, q_1, q_k)$. Now agent 1 is completely satisfied, and she is not interested in any further production. We repeat the same procedure in order to determine the other stages of production. The third stage involves reaching a level of production x_2, such that agent 2 is completely satisfied and any other agent $k>2$ enjoys an amount as part of his demand that is equivalent with q_2. Again, with ρ^R as the tie-breaking rule this amounts to the production of a bundle $\rho^R(c, q_2, q_k)$ for all agents $k>2$. In that case, $x_2=q_1+q_2+\Sigma_{k>2}\ \rho^R(c, q_2, q_k)$. Now agent 2 is completely served. We go on by an additional production as much as is needed just to satisfy agent 3 and all the larger demanders up to a level equivalent with q_3 as prescribed by ρ^R. Then $x_3=q_1+q_2+q_3+\Sigma_{k>3}\ \rho^R(c, q_3, q_k)$. In general we get the following formula,

$$\begin{cases} x_0 = 0 \\ x_k = \sum_{r=1}^{k} q^r + \sum_{r=k+1}^{n} \rho^R(c, q_k, q_r) \end{cases}$$

Next we determine the individual cost shares by equally splitting the incremental costs involved with any subsequent intermediate level of production among the agents that are not satisfied before that stage. So the incremental costs involved with the second production level $c(x_1)-c(0)=c(x_1)$ are equally shared by the group of agents N. The incremental costs $c(x_2)-c(x_1)$ are equally shared by the agents of $N\backslash\{1\}$, $c(x_3)-c(x_2)$ by $N\backslash\{1, 2\}$, etc. In this way the above procedure determines a cost sharing mechanism ξ^R for problems (q_N, c) with $c\in S$. To be more precise, for all $c\in S$ and $q_N\in(\mathbb{R}_+^M)^N$

$$\xi_i^R(q_N, c) = \sum_{k=1}^{i} \frac{c(x_k) - c(x_{k-1})}{n+1-k}$$

It is immediately clear that by condition of Eq. (3) on the family of preorderings R, this mechanism defines indeed a serial extension on S. By varying over all possible admissible families of preorderings on S we get a whole class of serial extensions which we will call the class of *preordering-based serial extensions on S.*

The class of *preordering-based serial extensions on* $S\subseteq\mathbb{C}$ is easily characterized by the two properties ETPE($R(S)$) and RSP on S. A proof includes arguments that are very similar to those in Moulin and Shenker (1992b), only now one has to lower 'larger' demands through the radials and use ETPE($R(S)$) instead of ETE. So we obtain the following theorem.

Theorem 3.5. *Let* $R=\{R(c)|c\times S\}$ *be an admissible family of preorderings. Then* ξ^R *is the unique cost sharing mechanism on* $(\mathbb{R}_+^M)^N\times S$ *satisfying* ETPE($R(S)$) *and* RSP *on* S.

Example 3.6. For any $c\in\mathscr{C}$ define the preordering $R(c)$ by

$$xR(c)y \Leftrightarrow \sum_{j\in M} x_j \le \sum_{j\in M} y_j$$

This preordering is natural in case mutual comparison of the different units in which the goods are measured makes sense. Note that $R(c)$ does not depend on c. Then obviously all the four properties for preorderings, Eqs. (1)–(4) are satisfied for arbitrary $c \in \mathscr{C}$. Then for $xR(c)y$,

$$\rho^R(c, x, y) = \begin{cases} 0 & \text{if } y = 0 \\ \left(\sum_{i \in M} x_i\right) \cdot \left(\sum_{i \in M} y_i\right)^{-1} & \text{if } y \neq 0 \end{cases}$$

Then ξ^R is a preordering-based serial extension on $\mathscr{C}$.

Remark 3.7. *The above exposition should give the reader a clue that the same technique can be used to enlarge the set of cost mechanisms that are in fact a serial extension. Suppose that P is a set of increasing and continuous functions $p:[0, \infty) \to \mathbb{R}_+^M$ with $p(0) = 0$, such that each $x \in \mathbb{R}_+^M$, $x \neq 0$ is within the range of exactly one such $p \in P$. In this way, each $x \in \mathbb{R}_+^M$, $x \neq 0$ corresponds unambiguously to a $p_x \in P$. Suppose that instead of property of* Eq. (4) *for a preordering R(c) we have the following,*

$$xR(c)y \Rightarrow \text{there is } y' \leq y,\ y' \in p_y([0, \infty)) \text{ such that } y'E(c)x \tag{5}$$

Then for triples (c, x, y) with $xR(c)y$ and y' as above, we define $\rho^R(c, x, y) := y'$. Exactly the same determination of intermediate production stages and consecutive division of incremental costs leads to a serial extension again. By varying the set of admissible preorderings, where the condition of Eq. (4) *is replaced with that of* Eq. (5), *this gives rise to a new class of preordering-based serial extensions, that is characterized again by ETPE(R(S)) and a weakening of the Serial Principle requiring that the agent with the lower original cost share should pay the same in case an agent with a higher initial cost share increases his demand along the 'path' $p_x([0, \infty))$ corresponding to his original demand x. Thus, by varying the collections of 'path' functions p_x, we derive a larger class of preordering-based serial extensions. Observe that in the above exposition the above 'paths' are just the radials through the origin. These are generated by the set of path functions $P = \{p_x | x \in \mathbb{R}_+^M \backslash \{0\}\}$ where for $z \in \mathbb{R}_+^M \backslash \{0\}$, $p_z(t) = t(z/\|z\|)$ for all $t \in \mathbb{R}_+$, with $\|\ldots\|$ as the Euclidian norm on $\mathbb{R}_+^M$.*

Remark 3.8. *Another well known characterizing property in the cost sharing literature is that of scale invariance, that requires robustness of the cost sharing mechanism against changes of unit scale of the different goods. It is easy to see that the mechanism ξ^R preserves this property if only the following holds for all x, $y \in \mathbb{R}_+^M$, $\lambda \in \mathbb{R}_{++}^M$,*

$$\lambda' x R(c) \lambda' y \Rightarrow x R(c \circ \lambda) y \tag{6}$$

4. The radial serial rule

Suppose that, for $c \in \mathscr{C}_+$, the cost impact of a bundle x is evaluated higher than that of a bundle y if and only if the costs for producing solely x are lower than the costs of producing y. This determines a preordering $R(c)$ by

$$xR(c)y \Leftrightarrow c(x) \leq c(y)$$

the different equivalence classes for this relation correspond to the different ırves. It is easily seen that this preordering satisfies the properties Eqs. (1)–(3). ıce c is an increasing continuous function, for all demand bundles this ıg satisfies condition of Eq. (4). So the preordering $R(c)$ is admissible for all 'hen by using the same techniques as in the former section, and with ρ^R as in ı 3.4, we get a serial extension on C_+. For this particular choice of R we will he *radial serial rule* and denote it by ξ^{RS}. Below, we will look at this m in more detail.

4.1. Consider the situation where there are three agents jointly owning a n facility for two goods, say $N=\{1, 2, 3\}$ and $M=\{1, 2\}$. Suppose that the costs ction can be expressed by the function $c \in \mathscr{C}_+$, $c(x, y)=x+2\cdot y+\max\{x, y\}$. ts 1, 2 and 3, respectively, have demands $q_1=(1, 1)$, $q_2=(2, 1)$, respectively).

$c(q)$
4
7
12

'(c) as the preordering that compares stand alone costs for production, q_1, q_2 'e ordered as $q_1R(c)q_2R(c)q_3$. In order to compute the individual cost shares to ξ^{RS}, we have to calculate the intermediate production levels first:

$$x_0 = 0$$
$$= q_1 + \rho^R(c, q_1, q_2) + \rho^R(c, q_1, q_3) = (1, 1) + (\tfrac{4}{7}, \tfrac{8}{7}) + (\tfrac{4}{3}, \tfrac{2}{3}) = (\tfrac{61}{21}, \tfrac{59}{21})$$
$$x_2 = q_1 + q_2 + \rho^R(c, q_2, q_3) = (1, 1) + (1, 2) + (2\tfrac{1}{3}, 1\tfrac{1}{6}) = (4\tfrac{1}{3}, 4\tfrac{1}{6})$$
$$x_3 = q_1 + q_2 + q_3 = (1, 1) + (1, 2) + (4, 2) = (6, 5)$$

shows how the intermediate production levels are determined. ıe final cost shares according to ξ^{RS} are determined by:

$$\xi_1^{RS}(q_N; c) = \frac{c(x_1)}{3} = \frac{80}{71}$$
$$\xi_2^{RS}(q_N; c) = \frac{c(x_1)}{3} + \frac{c(x_2) - c(x_1)}{2} = \frac{277}{42}$$
$$(q_N; c) = \frac{c(x_1)}{3} + \frac{c(x_2) - c(x_1)}{2} + \frac{c(x_3) + c(x_2)}{1} = \frac{277}{42} + 5$$

serial rule ξ^{RS} can be seen as an extension of the axial rule (Sprumont, 1996), ılies for the model where agents are identified with goods. It shares some of the racteristics of the axial rule as well. For instance it satisfies the following property, which states that zero-demanders can be entirely removed from the vithout altering the outcome for the others in the remaining problem.

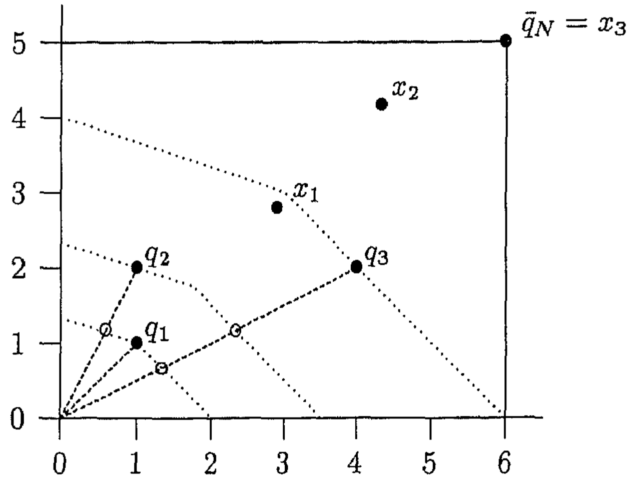

Fig. 1. Determining intermediate production levels.

Definition 4.2. A cost sharing mechanism satisfies *Independence of Null Agents* (INA) if for all problems (q_N, c) and every $i \in N$, $q_i = 0 \Rightarrow \xi_j(q_{N \setminus \{i\}}, c) = \xi_j(q_N, c)$ for all $j \in N \setminus \{i\}$.

Another property of the radial serial rule is that the ranking of two agents cost shares depends on their demands only. In particular, though possibly it causes changes to all individual cost shares, a change in an agent's demand does not affect the interpersonal ranking of the other cost shares. The axiom exactly expresses the fundamental idea that if an agent i pays more than another agent j, then it must be because we judge that agent i's demand is larger than that of agent j. In this respect, the information of what actually is demanded by the other agents should be of no importance. More formally, we have the following definition.

Definition 4.3. A cost sharing mechanism satisfies *Rank Independence of Irrelevant Agents* (RIIA) if for two cost sharing problems (q_N, c) and (q'_N, c) for which $q_i = q'_i$ and $q_j = q'_j$ with $i, j \in N$, then

$$\xi_i(q_N, c) \leq \xi_j(q_N, c) \Leftrightarrow \xi_i(q'_N, c) \leq \xi_j(q'_N, c)$$

Despite its attractiveness, as Sprumont (1996) points out, many well known rules violate this property.

As expected the extension also incorporates all of the negative spirit that is attached with the axial rule. In the case of one output externality imposed on others is related to the stand alone costs. This is what makes the serial cost sharing mechanism such a good choice there. However, for going to the multi-good model we lose this relationship. Consider the following somewhat contrived, but certainly striking example.[1] Let $N = \{1,$

[1] This example was given by an anonymous referee after reading an earlier version of this paper.

$^r=\{1, 2\}$ and $c(x, y)=2x+y+2000\max\{xy-0.5, 0\}$. Suppose that the individual are given by $q_1=(1, 0)$ and $q_2=(0, 1)$. Then the cost of serving both agents is 1003. Agent 1 imposes a stand alone cost of 2, whereas agent 2 imposes a stand st of 1. The radial serial rule determines the cost shares $\xi_1^{RS}(q_N, c)=1002$ and $c)=1$. But this seems not to be the normatively appropriate allocation here. t this example carries over to the more restrictive model where the goods are 1 with the agents, thereby illustrating the same deficiency for the axial rule.

theless, with all its shortcomings, Sprumont (1996) shows that the axial rule is ue cost sharing mechanism obeying the principles involved with RIIA, INA, and an additional property which he refers to as *Ordinality* (ORD). Ordinality at cost shares should be invariant under all transformations that preserve the f the problem under consideration. Before giving a more formal definition of y, we will specify what kind of transformations will be allowed.

se that a cost sharing mechanism is defined for a set of cost functions $S\subseteq\mathscr{C}$. Let $\mathbb{R}_+^M$ be a bijection. Then for each $c\in S$ we define the function $c^f:\mathbb{R}_+^M\rightarrow\mathbb{R}_+$ by $f(x))$ for all $x\in\mathbb{R}_+^M$. Then f is an ordinal transformation on S if $c^f\in S$ for all) for $S=\mathscr{C}_+$, a bijection f is an ordinal transformation if and only if f is g and continuous.

se that f has the additional property, that it leaves the radials intact in the sense each $q\in\mathbb{R}_+^M$, $q\neq 0$ there is a $q'\in\mathbb{R}_+^M$ such that

$$\alpha q|\alpha\in\mathbb{R}_+\})=f(\{\alpha q'|\alpha\in\mathbb{R}_+\})$$

s called a *radially ordinal transformation*. Now, two problems (q_N, c) and $(q_N',$ called *(radially) ordinally equivalent* if there exists an (radially) ordinal nation f such that $c'=c^f$ and $q=f(q')$. For q' in that case we will write q^f. Now) ordinality reads as in the following definition.

n 4.4. Suppose ξ is a cost mechanism on $(\mathbb{R}_+^M)^N\times S$. Then ξ is called *(Radially)* ((RORD) ORD) if for all (radially) ordinally equivalent problems (q_N, c) and in $(\mathbb{R}_+^M)^N\times S$, it holds that $\xi(q_N, c)=\xi(q_N', c')$.

y, the property ORD, and even RORD, is much more involved than scale e. But together with the weak form of the serial principle RSP and the ng ETE, RIIA, INA it leaves us not much freedom to choose from cost sharing sms that satisfy all of those.

1 4.5. *There is only one cost sharing mechanism that satisfies* ETE, IIA, INA, ! RORD *on* $\mathscr{C}_+$, *and that one is* ξ^{RS}.

Essentially, the proof follows the same lines of that proposed in Sprumont First of all, it is easy to show that the radial serial rule satisfies the enlisted s. Now, suppose ξ is a cost sharing mechanism with the same properties. Then ing RSP and ETE to homogeneous problems it follows that ξ must be a serial 1. Consider a cost sharing problem $(q_N, c)\in(\mathbb{R}_+^M)^N\times\mathscr{C}_+$. Then the properties

RIIA, INA are used to show that the ranking of the cost shares of two agents i and j in the problem (q_N, c) is the same as the ranking of the cost shares of i and j in the reduced problem $(q_{\{i,j\}}, c)$. Next, define $f:\mathbb{R}_+^M \to \mathbb{R}_+^M$, by

$$f(q) := c(q)\frac{q}{\sum_{k \in M} q_k} \text{ for all } q \in \mathbb{R}_+^M$$

Then f can be seen as a function that *straightens* isocost curves, such that each element q of a specific cost curve is mapped onto βq for some $\beta \in \mathbb{R}_+$. Especially, this f is a radially ordinal transformation. Furthermore, for all x, $y \in \mathbb{R}_+^M$ we have

$$\sum_{k \in M} f(x)_k \leq \sum_{k \in M} f(y)_k \Leftrightarrow c(x) \leq c(y) \tag{7}$$

Thus we see that the two person cost sharing problem $(q_{\{ij\}}, c)$ is radially ordinally equivalent with the homogeneous problem $(q^f_{\{ij\}}, c^f)$. Then by RORD we have $\xi_i(q_{\{ij\}}, c) = \xi_i(q^f_{\{ij\}}, c^f)$ and $\xi_j(q_{\{ij\}}, c) = \xi_j(q^f_{\{ij\}}, c^f)$. So, using that ξ is a serial extension and the equivalence relation of Eq. (7) gives us

$$\xi_i(q^f_{\{ij\}}, c^f) \leq \xi_j(q^f_{\{ij\}}, c^f) \Leftrightarrow c(q_i) \leq c(q_j)$$

Thus, as a consequence

$$\xi_i(q_N, c) \leq \xi_j(q_N, c) \Leftrightarrow c(q_i) \leq c(q_j)$$

Then, essentially the rest of the proof is no different from that presented by Sprumont (1996). Only we use RSP instead of SP. Assume without loss of generality that $c(q_i) \leq c(q_j)$, whenever $i \leq j$. Then define for $k \in N$, $q(k) \in (\mathbb{R}_+^M)^N$ such that

$$q_j(k) = \begin{cases} q_j & \text{if } j \leq k \\ \rho^R(c, q_k, q_j) & \text{if } j > k \end{cases}$$

Let $x_0, x_1, \ldots, x_n$ represent the same intermediate production stages as defined in the former section. Then $\xi_i(q(1), c) = [c(x_1)/n]$. As a consequence of RSP we have next $\xi_1(q(2), c) = \xi_1(q(1), c) = [c(x_1)/n]$ and since the ranking of the cost shares can be represented by the ranking of the stand alone costs of the different agents, $\xi_i(q(2), c) = [c(x_1)/n] + [c(x_2) - c(x_1)/n - 1]$ for all $i \in N \backslash \{1\}$. Proceeding in this way gives $\xi(q_N, c) = \xi(q(n), c) = \xi^{RS}(q(n), c) = \xi^{RS}(q_N, c)$. □

Suppose we stick on to our belief in the properties INA, RIIA and ETE. We will see that, opposed to Sprumont's result, combining these with SP and ORD leads to the following impossibility result.

Theorem 4.6. *There is no cost sharing mechanism that satisfies* ETE, RIIA, INA, SP *and* ORD *on* $\mathscr{C}_+$.

We present only a sketch of the proof. Suppose that there is such a cost sharing mechanism ξ. Then ξ certainly satisfies the characterizing properties as listed in Theorem 4.5. But Theorem 4.5 leaves no other possibility than that ξ is the radial serial rule ξ^{RS}. Then the proof can simply be completed by showing that ξ^{RS} does not satisfy one of the properties SP, ORD. In fact, both properties are not compatible with ξ^{RS}.

To see this, first we take $i, j \in N$ with $\xi_i^{\mathrm{RS}}(q_N, c) \leq \xi_j^{\mathrm{RS}}(q_N, c)$ for $i, j \in N$. Then SP would imply that agent i's cost share does not change for any increase of agent j's demand. But in the generic case, agent j will affect all intermediate production levels by affecting the terms $\rho^R(q_k, q_j)$. Therefore, it will in general also cause a change in the level of costs that agent i is equally held responsible for. But then agent i's cost share is about to change, which gives a contradiction.

In general, the radials through 0 as part of the demand space will not stay intact in the sense that they are not mapped onto radials under an ordinal transformation. In that case ξ^{RS} will, by persisting on a radial approach, determine other intermediate levels of production for the ordinally equivalent problem. So, this will involve some change in the individual cost shares, in other words ξ^{RS} is not an ordinal cost sharing mechanism.

Notice that the formulation of Theorem 4.6 can be tightened, since both the combinations of properties ETE, RIIA, INA, SP, RORD and ETE, RIIA, INA, RSP, ORD are incompatible.

5. Concluding remark

Note that we discussed only standard situations, where for each of the goods in M, the total demand is determined as the sum of the individual demands. By relaxing this basic assumption, a more general model would include also those situations where the demands for the different goods are aggregated differently, and where possibly the aggregated demand for a specific good is related to the demands for the other goods. From this point of view, the more complete model incorporates an additional compound, that is the type of aggregation or *aggregation rule*. Tijs and Koster (in press) proposes a generalized serial rule for one-good cases with other aggregation rules. We can introduce the notion of a serial extension for multi-good situations with non-standard aggregation rules, just by requiring that an extension should behave like the generalized serial rule in all homogeneous cases. Then serial extensions are constructed by similar techniques as proposed in this paper.

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