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Allocation Rules for Cooperative Games with Graph and Hypergraph Communication Structure

Guang Zhang

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PROEFSCHRIFT

ter verkrijging van de graad van doctor aan Tilburg University op gezag van de rector magnificus, prof. dr. E.H.L. Aarts, in het openbaar te verdedigen ten overstaan van een door het college voor promoties aangewezen commissie in de Ruth First zaal van de Universiteit op dinsdag 28 augustus 2018 om 14.00 uur door

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Guang Zhang

Shanghai, July 2018

Contents

Notation

$\mathcal{G}_N$ set of TU games, page 9

$\mathcal{H}_N$ set of hypergraphs, page 12

$\mathit{\Gamma}_N$ set of graphs, page 12

H_i hyperlinks containing i in (N, H), page 13

H_{-i} hyperlinks not containing i in (N, H), page 13

$\mathcal{H}_N^c$ set of connected hypergraphs, page 13

$\mathcal{H}_N^{cf}$ set of cycle-free hypergraphs, page 14

$\mathcal{H}_N^t$ set of hypertrees, page 14

$N(C)$ set of nodes in chain C, page 14

$C^H(S)$ set of connected subsets of S in (N, H), page 14

S/H set of components of S in (N, H), page 14

$\mathcal{D}_N$ set of digraphs, page 15

$S^D(i)$ set of successors of i in (N, D), page 15

$\widehat{S}^D(i)$ set of immediate successors of i in (N, D), page 15

$\bar{S}^D$ set of successors of i in (N, D) including i, page 15

N/D set of components of N in (N, D), page 15

$\mathcal{T}^\Gamma(K)$ set of admissible rooted spanning trees of $(K, \Gamma(K))$, page 15

$\mathcal{G}_N^{\mathcal{H}}$ set of hypergraph games, page 16

$\widehat{\mathcal{G}}_N^{\mathcal{H}}$ set of zero-normalized hypergraph games, page 16

$\mathcal{G}_N^{\mathcal{H}^{cf}}$ set of cycle-free hypergraph games, page 16

$\mathcal{G}_N^{\mathcal{H}^{t}}$ set of hypertree games, page 16

$\mathcal{G}_N^{\mathcal{H}^{c}}$ set of connected hypergraph games, page 16

$\mathcal{G}_N^{\Gamma}$ set of graph games, page 16

$\mathcal{G}_N^{\mathcal{D}}$ set of digraph games, page 18

$\mathcal{G}_N^{\mathcal{F}}$ set of rooted forest games, page 18

$\mathcal{G}_N^{\Gamma,\mathcal{M}}$ set of graph games with main players, page 25

$\mathcal{G}_N^{\Gamma^{cf},\mathcal{M}}$ set of cycle-free graph games with main players, page 25

$\mathcal{G}_N^{\Gamma^{c},\mathcal{M}}$ set of connected graph games with main players, page 25

$\mathcal{G}_N^{\Gamma^{t},\mathcal{M}}$ set of connected cycle-free graph games with main players, page 25

$\mathcal{G}_N^{\Gamma,\mathcal{M}^1}$ subset of $\mathcal{G}_N^{\Gamma,\mathcal{M}}$ where each component contains one main player, page 25

$\mathcal{G}_N^{\Gamma^{cf},\mathcal{M}^1}$ set of $\mathcal{G}_N^{\Gamma,\mathcal{M}^1} \cap \mathcal{G}_N^{\Gamma^{cf},\mathcal{M}}$, page 25

$\mathcal{G}_N^{\Gamma^{c},\mathcal{M}^1}$ set of $\mathcal{G}_N^{\Gamma,\mathcal{M}^1} \cap \mathcal{G}_N^{\Gamma^{c},\mathcal{M}}$, page 25

$\mathcal{G}_N^{\Gamma^{t},\mathcal{M}^1}$ set of $\mathcal{G}_N^{\Gamma,\mathcal{M}^1} \cap \mathcal{G}_N^{\Gamma^{t},\mathcal{M}}$, page 25

$\mathcal{T}_M^{\Gamma}(K)$ subset of $\mathcal{T}^{\Gamma}(K)$ with roots from $M \cap K$, page 26

$\mathcal{G}_N^{\Gamma^{cc},\mathcal{M}^1}$ subset of $\mathcal{G}_N^{\Gamma,\mathcal{M}^1}$ where the underlying graph is cycle, page 34

$\mathcal{B}^H(K)$ set of admissible collections of coalitions on K in (N, H), page 53

$\mathcal{G}_N^{\mathcal{C}}$ set of cycle hypergraph games, page 70

$E^H(T)$ set of extreme nodes of T in (N, H), page 70

$\mathcal{G}_N^{\mathcal{H}^{qcf}}$ set of quasi-cycle-free hypergraph games, page 85

$\mathcal{G}_N^{\mathcal{H}^{scf}}$ set of semi-cycle-free hypergraph games, page 91

$\widehat{\mathcal{G}}_N^{\mathcal{H}^{cf}}$ set of cycle-free zero-normalized hypergraph games, page 107

$\widehat{\mathcal{G}}_N^{\mathcal{H}^{u}}$ set of uniform zero-normalized hypergraph games, page 118

Chapter 1

Introduction

Game theory is well-known for describing and analyzing social interactive decision situations. The social interactive actions can be competition or cooperation among decision makers. Therefore, game theory is defined as a "study of mathematical models of conflict and cooperation between intelligent rational decision-makers" (Myerson (1991)). For the spirit of game theory, it can be traced back to Zermelo (1913) and Borel (1921). Game theory as a uniform theory is firstly introduced in the seminal book "Theory of Games and Economic Behaviors" by von Neumann and Morgenstern (1944). In this book, two fundamentally different approaches are determined in this field. The first approach in terms of strategic or non-cooperative game theory is based on the absolute absence of any binding commitments between the decision makers. The second approach is known as coalitional or cooperative game theory and it allows decision makers to make binding and enforceable agreements. In this respect, game theory can be roughly divided by commitments whether they exist or not (Harsanyi (1966)).

Instead of focusing on the details of how coalitions form, such as negotiations to reach agreements or searching for partners to cooperate, cooperative games draw more attention to joint outcomes of groups of decision makers, where decision makers are usually named players in terms of participation in a game and groups of players are often called coalitions. The essential issue of a cooperative game is how to distribute the joint revenues from cooperation to the players in a suitable way. The assumption that the outcomes of cooperation in coalitions can be freely assigned to the players without loss of utility leads to models of cooperative games with transferable utility, or simply TU games.

A TU game consists of a set of players and a characteristic function which assigns to each coalition of players its worth, i.e., what the coalition can achieve by cooperation of its members without participation of the other players outside the coalition. The main question when playing a TU game is how to divide the joint revenue obtained by the total cooperation among the players. To deal with this question, reasonable solutions are desired by focusing on the allocation of the worth achieved by the grand coalition, while taking into account the worth of any coalition. Therefore, a solution of TU games is a mapping that assigns to every TU game certain payoff vectors that determine the individual payoffs of the players.

In TU games there are two types of solution concepts, set-valued solutions and single-valued solutions. The most well-known set-valued solution is the core, introduced in Gillies (1953, 1959). The core of a TU game consists of all efficient payoff vectors satisfying group rationality. A payoff vector is efficient if it distributes the worth of the grand coalition among its players, and group rationality means that every coalition receives at least its worth. Hence, an element of the core is stable in a sense that no coalition has an incentive to leave the grand coalition. Single-valued solution concepts are typically characterized by a set of axioms. For example, in Shapley (1953) the Shapley value is introduced axiomatically by efficiency, additivity, the null-player property, and symmetry. For the Shapley value, there exist several other axiomatic characterizations, e.g., see Young (1985), Hart and Mas-Colell (1989), and van den Brink (2002).

In classical cooperative game theory, it is assumed that any coalition of players may form and by cooperation obtain its worth. However, in many practical situations not all coalitions are able to form. For example, two researchers may not apply for a grant by a joint proposal if they do not know each other, and some staff members from different departments are not allowed to work together without permission of the heads of the departments. Therefore, in many real life situations, the set of feasible coalitions is restricted by some social, economical, hierarchical, communicational, or other structure.

The study of TU games with limited cooperation is initiated in Aumann and Drèze (1974), where the limited cooperation is specified by a partition of the player set, later on extended in Owen (1977). Under the assumption that the players may get a better bargaining position if they join a coalition before realizing the cooperation of the grand coalition, a coalition structure is proposed and embedded into TU games. Hereafter, cooperative games with coalition structure have received much attention. Several solutions are known in the literature, such as the Aumann and Dréze value, the Owen value, and the two-step Shapley value (Kamijo (2009)).

Another model of a game with limited cooperation is introduced in Myerson (1977), in which the restriction of cooperation is given by an undirected graph (communication) structure. Graphs can be used to model many practical problems, especially in dealing with relations and processes. When graphs are used in sociology, such as rumor spreading, a friendship graph describes whether people know each other, and rumors can spread among the people along the friendship graph. If a rumor can spread from one person to another, it indicates the two individuals can communicate mutually, in other words, they are connected. Similar to the friendship graph, cooperative behavior typically takes place between people who know each other. Following this idea, the class of TU games with communication structure is based on the assumption that only connected coalitions can cooperate.

A communication structure can also be specified by a hypergraph, or conference, structure containing a set of hyperlinks. While the links of an undirected graph represent bilateral relationships between pairs of players, a hyperlink in a hypergraph structure as an extension of an undirected graph structure, may contain more than two players, which can model a club, an organization, or a committee. In a conference structure (Myerson (1980)) or hypergraph structure (van den Nouweland et al. (1992)), it is presumed that all players of a conference or hyperlink have to be present for communication.

Concerning solutions of TU games with undirected graph communication structure, shortly graph games, many solution concepts are known in the literature. For a graph game, the Myerson value introduced in Myerson (1977) is defined as the Shapley value of a modified TU game of the graph game, where the modified TU game is the Myerson (graph) restricted game that assigns to each coalition a worth that is equal to the sum of the worths of all its maximal connected subcoalitions, called its components. Another value for graph games, the position value introduced in Meessen (1988), highlights the bilateral relationships between players represented by the links in the underlying graph. The position value is defined in two steps. It first assigns to each link a Shapley payoff on the so-called link game, and then every player receives half of the payoffs of all his related links. In the link game, the worth of each subset of links equals the sum of the worths of all its components. Another value, the average tree solution introduced in Herings et al. (2008) is defined as the average of the marginal contributions with respect to certain rooted spanning trees (or partial orderings) of the underlying graph. It is different from the Myerson value which is based on the marginal contributions with respect to permutations (or linear orderings). Moreover, from the view points of egalitarity and cooperative ability, Béal et al. (2012a) study egalitarian solutions and Shan et al. (2016) propose several degree-based values, respectively.

In addition, the Myerson value and the position value are extended to TU games with hypergraph structures in van den Nouweland et al. (1992), as well as to union stable systems, see Algaba et al. (2000) and Algaba et al. (2001). As another type of TU games with restricted cooperation, union stable systems model a class of sets of feasible coalitions satisfying that if the intersection of two feasible coalitions is not empty, then their union is also feasible. The class of union stable systems is a more general case of restricted cooperation structure. It not only includes undirected graph structures and hypergraph structures, but also permission structures (Gilles et al. (1992) and van den Brink and Gilles (1996), also see van den Brink (1997)), precedence constraints (Faigle and Kern (1992)), antimatroids (Algaba et al. (2004)), augmenting systems (Bilbao (2003)), and building sets (Koshevoy and Talman (2014)). Furthermore, limited cooperation can be also represented by a composite structure, such as TU games with both coalition and graph structure as in Vázquez-Brage et al. (1996), Alonso-Meijide et al. (2009), and van den Brink et al. (2015), and with a two-level (or layered) communication structure as in Kongo (2011), Khmelnitskaya (2014), and van den Brink et al. (2016).

This thesis is devoted to the study of TU games with restricted cooperation represented by a communication structure. Chapter 2 gives an introduction to the main concepts, definitions, and notation about TU games, communication structures, and cooperative games with communication structure.

Chapter 3 considers TU games with a cooperation structure represented by an undirected communication graph, in which some of the players are selected a priori as main players. Such asymmetry between players is similar to a two-level hierarchical player partition, see Slikker and van den Nouweland (2000). Assuming that the redistribution of the total rewards obtained by the grand coalition is under control of the main players, we introduce a value for graph games with main players which generalizes the average tree value for graph games introduced in Herings et al. (2008, 2010). On the class of connected cycle-free graph games the new value is a random tree solution introduced in Béal et al. (2010). Subsequently, we provide axiomatic characterizations for two special cases of graph games with main players. For cycle-free graph games with main players, the average tree value is uniquely determined by component efficiency and main players component fairness, where the latter axiom is equivalent to component fairness in Herings et al. (2008) if all the players are main players and it is equivalent to successor equivalence for rooted-tree games in Khmelnitskaya (2010) if in each component only one player is the main player. We notice that main players component fairness can be also regarded as a special case of weighted component fairness introduced in Béal et al. (2012b). Another special case is cycle graph games with a unique main player. On the class of cycle graph games with unique main player, a characterization is obtained by efficiency, linearity, the restricted null-player property, and veto players equal treatment. We further follow the idea of classification of hub-spoke networks and divide the class of connected graphs with main players into two classes. On the class of graph games with main players in which the underlying structure is of a single allocation type, we prove that the average tree value can be obtained by a two-step distribution procedure. A connected graph with a set of main players is of a single allocation type if main players partition the player set into unions and each union contains exactly one main player and some ordinary players. Besides, in each union it also requires that any path between an ordinary player and the unique main player does not contain any other main player. Therefore, on this class of graph games with main players, the average tree value can be obtained by applying first the average tree value for graph games between the unions, and then using the average tree value for graph games with main players within each union. As an application, the two-step approach can be used to calculate the average tree value on the class of graph games with main players, where the underlying graph structure is a single allocation hub-spoke network in which hub nodes are considered as main players.

The following chapters aim at providing some new solutions for TU games with hypergraph communication structure and then characterizing those proposed solutions. Since the class of hypergraph communication structures can be used to model the multilateral relations among agents, there is a wide range of applications for TU games with hypergraph communication structure within the cooperation of clubs, parties, or associations, if the interaction between two organizations has at least one member in common. A sport situation about football associations is an example considered in van den Nouweland et al. (1992).

Chapter 4 studies limited cooperation represented by a hypergraph communication structure. For TU games with hypergraph communication structure, shortly hypergraph games, we usually assume that all the players in a hyperlink have to be present, otherwise the communication will not take place among such group of players. Therefore, in a hypergraph game, only the connected coalitions are feasible. The aim is to define the average tree value for hypergraph games and to provide several characterizations. Similar to Herings et al. (2010), first the class of admissible collections of coalitions with one top-player in a hypergraph structure is introduced, and then the average tree value for a player is defined as the average of his marginal contributions in all these collections. Axiomatizations of the average tree value are given for three particular cases, cycle-free hypergraph games, hypertree games, and cycle hypergraph games. A hypertree is a connected cycle-free hypergraph and a cycle hypergraph is linear and contains one cycle including all hyperlinks. By generalizing component fairness defined for graph games to the case of hypergraph games, we characterize the average tree value on the class of cycle-free hypergraph games by component efficiency and component fairness in the spirit of Herings et al. (2008). For the cases of hypertree games and cycle hypergraph games, the axiomatizations we obtain are extensions of the corresponding results for the average tree value for graph games in Mishra and Talman (2010) and Selçuk et al. (2013), respectively.

In Chapter 5, an average-tree-type solution concept, called the two-step average tree value, is introduced for graph and hypergraph games by applying a different interpretation of the set of admissible collections of coalitions. Since in a component and any admissible collection of coalitions the top-player is rewarded the best in this component by assigning the full dividend of the component coalition to this player, a situation that every player of a hypergraph game has equal probability to be a top-player is worthwhile to study. Based on this idea, the two-step average tree value is defined as follows. In the first step, the average payoffs of all players are calculated among the set of admissible collections of coalitions, in which a specific player is chosen as top-player, and the second step deals with the expected payoffs by considering all players having equal probability to be top-player. The two-step average tree value is different from the average tree value, because the latter has the presumption that every admissible collection of coalitions appears with equal probability. However, an interesting result shows that the two values coincide if the underlying hypergraph is a linear cactus, being a hypergraph with non-intersecting cycles. This is because in any linear cactus all players are top-players the same number of times. We notice that both cycle-free hypergraphs and cycle hypergraphs are special cases of linear cacti. We further study the two-step average tree value on different subclasses of hypergraph games. We prove that the two-step average tree value satisfies component fairness if the underlying hypergraph is quasi-cycle-free, where the class of quasi-cycle-free hypergraph is developed from cycle-free hypergraphs and may contain some cycles, which fails linearity. Moreover, on the class of semi-cycle-free hypergraph games, the new solution can be characterized by component efficiency, component fairness, and balanced contributions for interactive players, where the class of semi-cycle-free hypergraphs is a special case of quasi-cycle-free hypergraphs which also fails linearity and may contain cycles. The axiom of balanced contributions for interactive players can be traced back to the property of balanced contributions in Myerson (1980) and van den Nouweland et al. (1992).

Chapter 6 introduces the degree value for hypergraph games. The degree value highlights the importance of a player's degree in terms of the number of hyperlinks he belongs to. In both graph and hypergraph structures, the degrees of nodes represent the communicational ability of nodes in terms of the cooperative ability of players in graph games and hypergraph games. Similar to the Myerson value and the position value, the degree value is also introduced by applying the Shapley value on a modified game. In the modified game, called the degree game of a hypergraph game, a player is a degree, consisting of a player in the original game together with one of the hyperlinks he belongs to, and will be distributed a Shapley payoff. The degree value assigns to each player the sum of all Shapley payoffs of the degrees that are associated with this player. Three characterizations of the degree value are provided. On the class of cycle-free hypergraph games, the degree value is characterized by component efficiency, additivity, the superfluous conference property, and the degree property or the degree measure property. Another characterization is provided on the class of all hypergraph games by employing component efficiency and balanced conference contributions. The latter axiom is a natural extension of balanced link contributions introduced in Slikker (2005) and used to characterize the position value on the class of graph games. Although the degree value and the position value for hypergraph games are different, they coincide with each other if the underlying hypergraph is uniform, where in a uniform hypergraph all hyperlinks have the same number of nodes. This implies that in a graph game the degree value and the position value allocate the same payoffs to the players. The chapter concludes by comparing the properties of the different values with each other.

Chapter 2

Preliminaries

This chapter provides some basic concepts, definitions, and notation that we use throughout this thesis. In Section 2.1 and Section 2.2 we formally introduce transferable utility games and communication structures, respectively. In Section 2.3, we discuss several classes of transferable utility games with communication structure.

2.1 Transferable utility games

A *cooperative game with transferable utility* or *transferable utility game* (TU *game*) is a pair (N, v), where $N = \{1, \ldots, n\}$ is a finite set of players and $v : 2^N \to \mathbb{R}$ is a *characteristic function* assigning to every *coalition* of players $S \in 2^N$ its *worth* $v(S)$, with $v(\emptyset) = 0$. The members of S can obtain total joint payoff $v(S)$ by agreeing to cooperate, which can be freely distributed among the members of S. The class of TU games with fixed player set N is denoted by $\mathcal{G}_N$. Throughout this thesis, when we refer to a game we mean a TU game. For a game $(N, v) \in \mathcal{G}_N$ and a nonempty coalition $Q \subseteq N$, the *subgame* of (N, v) with respect to coalition Q is represented by the pair (Q, v_Q), where $v_Q : 2^Q \to \mathbb{R}$ is the characteristic function defined by $v_Q(S) = v(S)$, for all $S \in 2^Q$. For a nonempty $Q \subseteq N$, we denote by $\mathcal{G}_Q$ the class of subgames of all games in $\mathcal{G}_N$ with respect to coalition Q. We denote the cardinality of a given set A by $|A|$.

A game $(N, v) \in \mathcal{G}_N$ is *zero-normalized* if for every $i \in N$, $v(\{i\}) = 0$. A game $(N, v) \in \mathcal{G}_N$ is *superadditive* if $v(S) + v(T) \leq v(S \cup T)$, for any $S, T \in 2^N \setminus \{\emptyset\}$ with $S \cap T = \emptyset$. The *unanimity game* for a coalition $S \in 2^N \setminus \{\emptyset\}$ is the game

$(N, u_S) \in \mathcal{G}_N$, where u_S is defined by

$$u_S(Q) = \begin{cases} 1, & \text{if } S \subseteq Q, \\ 0, & \text{otherwise.} \end{cases}$$

It is well-known that unanimity games introduced in Shapley (1953) form a linear basis in $\mathcal{G}_N$.

For any game $(N, v) \in \mathcal{G}_N$, we have

$$v = \sum_{S \in 2^N \setminus \{\emptyset\}} \Delta_v(S) u_S, \tag{2.1.1}$$

where the coefficient $\Delta_v(S) \in \mathbb{R}$ is called *dividend* (see Harsanyi (1959, 1963)) of a coalition $S \in 2^N \setminus \{\emptyset\}$ in game (N, v), defined by

$$\Delta_v(S) = \sum_{Q \subseteq S} (-1)^{|S|-|Q|} v(Q). \tag{2.1.2}$$

A *payoff vector* $x \in \mathbb{R}^n$ is an n-dimensional vector assigning a payoff $x_i \in \mathbb{R}$ to player $i \in N$. A *solution* on $\mathcal{G}_N$ is a mapping F that assigns to every game $(N, v) \in \mathcal{G}_N$ a set of payoff vectors $F(N, v) \subseteq \mathbb{R}^n$. If for every $(N, v) \in \mathcal{G}_N$ it holds that $|F(N, v)| = 1$ and $F(N, v) = \{\xi(N, v)\}$, then ξ is called a *value* or *allocation rule* on $\mathcal{G}_N$.

The best well-known solution concept on the class of TU games is the Shapley value introduced in Shapley (1953). The Shapley value has different representations. Particularly, it can be defined as an allocation rule that assigns to each player the average marginal contributions of this player corresponding to all permutations on the player set.

A permutation $\sigma : N \to N$ is a bijective function, where $\sigma(i)$, $i \in N$, denotes the position of player i in permutation σ. Let $\Pi(N)$ denote the set of all permutations on N, then $|\Pi(N)| = n!$. For a given permutation $\sigma \in \Pi(N)$, the set of *predecessors* of $i \in N$ with respect to σ is defined as

$$P^\sigma(i) = \{j \in N : \sigma(j) < \sigma(i)\}.$$

We denote by $\bar{P}^\sigma(i) = P^\sigma(i) \cup \{i\}$ the set of predecessors of i with respect to σ including player i.

For a game $(N, v) \in \mathcal{G}_N$ and a permutation $\sigma \in \Pi(N)$ on N, the *marginal* *contribution vector* with respect to σ and (N, v) is the payoff vector $m^\sigma(N, v) \in \mathbb{R}^n$ defined by

$$m_i^\sigma(N, v) = v(\bar{P}^\sigma(i)) - v(P^\sigma(i)), \text{ for all } i \in N. \tag{2.1.3}$$

For any game $(N, v) \in \mathcal{G}_N$, the *Shapley value* is defined by

$$Sh(N, v) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(N, v). \tag{2.1.4}$$

The Shapley value can also be represented by means of dividends as

$$Sh_i(N, v) = \sum_{S \subseteq N : S \ni i} \frac{\Delta_v(S)}{|S|}, \text{ for all } i \in N. \tag{2.1.5}$$

In Shapley (1953) the Shapley value is introduced by the following four axioms.[1] Let ξ be a value on $\mathcal{G}_N$.

- **Efficiency**: For any $(N, v) \in \mathcal{G}, \mathcal{G} \subseteq \mathcal{G}_N$, it holds that

$$\sum_{i \in N} \xi_i(N, v) = v(N).$$

- **Additivity**: For any $(N, v), (N, w) \in \mathcal{G}, \mathcal{G} \subseteq \mathcal{G}_N$, it holds that

$$\xi(N, v + w) = \xi(N, v) + \xi(N, w).$$

- **Null-player property**: For any $(N, v) \in \mathcal{G}, \mathcal{G} \subseteq \mathcal{G}_N$, and null-player $i \in N$, it holds that $\xi_i(N, v) = 0$, where player $i \in N$ is a *null-player* in (N, v) if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \setminus \{i\}$;

- **Symmetry**: For any $(N, v) \in \mathcal{G}, \mathcal{G} \subseteq \mathcal{G}_N$, and symmetric players $i, j \in N$, it holds that $\xi_i(N, v) = \xi_j(N, v)$, where two players $i, j \in N$ are *symmetric* in (N, v) if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$.

Another well-known solution is the core introduced in Gillies (1953, 1959).

[1] The axioms in Shapley (1953) are formulated in a slightly different but equivalent way.

For a given game $(N, v) \in \mathcal{G}_N$, the *core* is defined as

$$C(N,v) = \Big\{ x \in \mathbb{R}^n : \sum_{i\in N} x_i = v(N), \sum_{i\in S} x_i \geq v(S), \text{ for all } S \subsetneq N \Big\}.$$

If a payoff vector belongs to the core, no coalition has an incentive to leave the grand coalition. Hence, elements of the core are stable payoff vectors.

2.2 Communication structures

In this thesis, a communication structure on a set of players is specified by an undirected graph, a directed graph, or a hypergraph. Since an undirected graph is a special type of hypergraph, we first consider them together and then consider directed graphs.

A *hypergraph* is a pair (N, H), where $N = \{1, \ldots, n\}$ is a finite set of nodes and $H \subseteq \{e \in 2^N : |e| \geq 2\}$ is a collection of sets of nodes. An element e in H is called a *hyperlink* or *conference*. Let $\mathcal{H}_N$ denote the set of hypergraphs with node set N. A hypergraph $(N, H) \in \mathcal{H}_N$ is *k-uniform*, for some $k \geq 2$, if all hyperlinks contain exactly k nodes, i.e., $|e| = k$ for all $e \in H$. A 2-uniform hypergraph is an *undirected graph*. We denote the set of undirected graphs on N by $\mathit{\Gamma}_N$. For any $(N, \Gamma) \in \mathit{\Gamma}_N$, we have $\Gamma \subseteq \Gamma^N = \{\{i,j\} : i,j \in N, i \neq j\}$, where $\{i,j\} \in \Gamma$ is called a *link*, or *edge*, and (N, Γ^N) is called the *complete* (*undirected*) *graph* which contains every pair of different nodes as a link. In what follows, all notions for hypergraphs also apply to undirected graphs.

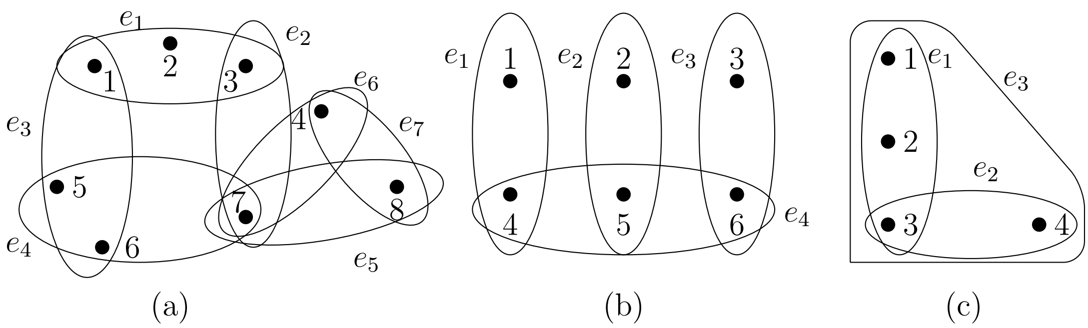


Figure 2.1: Three hypergraphs.

A hypergraph $(N, H) \in \mathcal{H}_N$ is *simple* if no hyperlink in H is a proper subset of another hyperlink in H, i.e., $e \subseteq e'$, for some $e, e' \in H$, implies $e = e'$. The hypergraph depicted in Figure 2.1 (c) is not simple because $e_1 \subsetneq e_3$. A hypergraph

$(N, H) \in \mathcal{H}_N$ is *linear* if any two distinct hyperlinks $e, e' \in H$ have at most one common node, i.e., $|e \cap e'| \leq 1$. The hypergraphs (a) and (c) in Figure 2.1 are not linear, since $|e_3 \cap e_4| = 2$ in (a) and $|e_1 \cap e_3| = 3$ in (c), while hypergraph (b) is linear.

For a hypergraph $(N, H) \in \mathcal{H}_N$ and node $i \in N$, we denote by $H_i = \{e \in H : i \in e\}$ the set of hyperlinks in (N, H) containing i and by $H_{-i} = \{e \in H : i \notin e\}$ the set of hyperlinks in (N, H) not containing i, where $|H_i|$ is called the *degree* of i in (N, H). For example, in Figure 2.1 (a), we have $H_7 = \{e_2, e_4, e_5, e_6\}$ and so $|H_7| = 4$ is the degree of node 7; moreover, $H_{-7} = \{e_1, e_3, e_7\}$.

In a hypergraph $(N, H) \in \mathcal{H}_N$, a node $i \in N$ is *connective* if its degree is greater than or equal to two, i.e., $|H_i| \geq 2$, and we denote by $C(N, H)$ the set of connective nodes in (N, H). The isolated nodes, with $|H_i| = 0$, and the 1-degree nodes, with $|H_i| = 1$, are called *non-connective* nodes in (N, H). For a hyperlink $e \in H$, a node $i \in N$ is *incident* with e if e contains i. Two distinct nodes $i, j \in N$ are *adjacent* in (N, H) if there is a hyperlink $e \in H$ containing both i and j. For instance, in Figure 2.1 (b), both 1 and 4 are incident with e_1 and therefore 1 and 4 are adjacent.

Two distinct nodes $i, j \in N$ are *connected* in a hypergraph $(N, H) \in \mathcal{H}_N$ if there exists a sequence of different nodes $(i_1, i_2, \ldots, i_k)$, $k \geq 2$, such that $i_1 = i$, $i_k = j$, and i_h is adjacent to i_{h+1} in (N, H) for $h = 1, 2, \ldots, k-1$. In an undirected graph $(N, \Gamma) \in \mathit{\Gamma}_N$ such a sequence is called a *path* between i_1 and i_k. A hypergraph $(N, H) \in \mathcal{H}_N$ is *connected* if any two distinct nodes of N are connected in (N, H). We denote by $\mathcal{H}_N^c$ the set of connected hypergraphs with node set N.

A sequence $(i_1, e_1, i_2, e_2, \ldots, i_{k-1}, e_{k-1}, i_k)$, with $k \geq 2$, is a *chain* in a hypergraph $(N, H) \in \mathcal{H}_N$ between node i_1 and node i_k if it satisfies the following conditions: (i) $i_1, i_2, \ldots, i_{k-1}$ are distinct nodes in N, (ii) $i_2, i_3, \ldots, i_k$ are distinct nodes in N, (iii) $e_1, e_2, \ldots, e_{k-1}$ are distinct hyperlinks in H, and (iv) $\{i_t, i_{t+1}\} \subseteq e_t$ for all $t \in \{1, \ldots, k-1\}$. Note that in a hypergraph $(N, H) \in \mathcal{H}_N$, if nodes $i, j \in N$ are connected, then there is a chain between these two nodes. For example, 2 and 8 are connected in Figure 2.1 (a) due to sequence $(2, 3, 7, 8)$ in (a), and there is a chain $(2, e_1, 3, e_2, 7, e_5, 8)$ in (a) between 2 and 8.

A chain $(i_1, e_1, i_2, e_2, \ldots, i_{k-1}, e_{k-1}, i_k)$, with $k \geq 3$, in a hypergraph $(N, H) \in \mathcal{H}_N$ is a *cycle* in (N, H) if $i_1 = i_k$. Note that $(1, e_1, 3, e_2, 7, e_4, 5, e_3, 1)$ is a cycle in Figure 2.1 (a). A hypergraph is *cycle-free* or a *hyperforest* if it does not contain any cycle (see Figure 2.1 (b)). A connected cycle-free hypergraph is called a *hypertree* (also see Figure 2.1 (b)). Let $\mathcal{H}_N^{cf}$ and $\mathcal{H}_N^t$ denote the collection of cycle-free hypergraphs and the collection of hypertrees, respectively, with fixed node set N. Note that $\mathcal{H}_N^t \subseteq \mathcal{H}_N^{cf}$ and, moreover, each cycle-free hypergraph $(N,H) \in \mathcal{H}_N^{cf}$ is linear, because $\{i_1,i_2\} \subseteq e \cap e'$, for some distinct $e,e' \in H$, implies that (i_1,e,i_2,e',i_1) is a cycle in (N,H).

For a chain $C = (i_1,e_1,i_2,e_2,\ldots,i_{k-1},e_{k-1},i_k)$ in hypergraph $(N,H) \in \mathcal{H}_N$, $N(C) = \{i \in e_t : t \in \{1,2\ldots,k-1\}\}$ denotes the set of nodes contained in C.

A hypergraph $(N,H) \in \mathcal{H}_N$ is a *cactus* if any two distinct cycles in (N,H) have at most one node in common, that is, for any two distinct cycles C,C' in (N,H), it holds that $|N(C) \cap N(C')| \leq 1$. Note that both (a) and (b) in Figure 2.1 are cacti, but (c) is not. This is because for the two cycles $C = (1,e_1,3,e_3,1)$ and $C' = (1,e_1,3,e_2,4,e_3,1)$ in (c), it holds that $N(C) \cap N(C') = \{1,3\}$.

For a hypergraph $(N,H) \in \mathcal{H}_N$ and nonempty $S \subseteq N$, $(S,H(S))$ is the *subhypergraph* of (N,H) on node set S, where $H(S) = \{e \in H : e \subseteq S\}$. If $(S,H(S))$ is connected, then $S \subseteq N$ is *connected* in (N,H). For any $(N,H) \in \mathcal{H}_N$ and $S \subseteq N$, let $C^H(S)$ denote the set of connected subsets of S in (N,H).

A subset $K \in C^H(S)$ is a *component* of $S \subseteq N$ in a hypergraph $(N,H) \in \mathcal{H}_N$, if K is a maximal connected subset of S in (N,H), i.e., K is connected in (N,H) and for every $i \in S \setminus K$, $K \cup \{i\}$ is not connected in (N,H). We denote S/H as the set of components of S in (N,H), and, for $i \in N$, $(S/H)_i$ denotes the unique component of S in (N,H) that contains node i.

For a hypergraph $(N,H) \in \mathcal{H}_N$, a hyperlink $e \in H$ is a *bridge* in (N,H) if the hypergraph $(N,H \setminus \{e\})$ has more components than (N,H), that is, $|N/H| < |N/(H \setminus \{e\})|$. Note that all hyperlinks in Figure 2.1 (b) are bridges.

Before introducing directed graphs we discuss the relationship between hypergraphs and union stable systems.

A set system $\mathcal{F} \subseteq 2^N$ on N is *union stable* if, for all $A,B \in \mathcal{F}$ with $A \cap B \neq \emptyset$, $A \cup B \in \mathcal{F}$. Let $\mathcal{F}$ be a union stable system and $\mathcal{G} \subseteq \mathcal{F}$, then the following families of set systems are defined recursively

$$\mathcal{G}^{(0)} = \mathcal{G}, \quad \mathcal{G}^{(m)} = \{S \cup Q : S,Q \in \mathcal{G}^{(m-1)}, S \cap Q \neq \emptyset\}, \quad m = 1,2,\ldots.$$

Note that $\mathcal{G}^{(0)} \subseteq \mathcal{G}^{(m-1)} \subseteq \mathcal{G}^{(m)} \subseteq \mathcal{F}$ for all $m \in \mathbb{N}$. Let $\bar{\mathcal{G}} = \mathcal{G}^{(k)}$, where k is the smallest integer such that $\mathcal{G}^{(k+1)} = \mathcal{G}^{(k)}$. In order to obtain the basis of a union stable system, let

$$\mathcal{E}(\mathcal{F}) = \{G \in \mathcal{F} : G = A \cup B, A \neq G, B \neq G, A, B \in \mathcal{F}, A \cap B \neq \emptyset\}.$$

The set $\mathcal{B}(\mathcal{F}) = \mathcal{F} \setminus \mathcal{E}(\mathcal{F})$ is called the *basis* of $\mathcal{F}$ and the elements of $\mathcal{B}(\mathcal{F})$ are called *supports* of $\mathcal{F}$. Note that $\mathcal{B}(\mathcal{F})$ is the minimal subset of $\mathcal{F}$ such that $\overline{\mathcal{B}(\mathcal{F})} = \mathcal{F}$. Finally, let $\mathcal{C}(\mathcal{F}) = \{B \in \mathcal{B}(\mathcal{F}) : |B| \geq 2\}$. Then, for any hypergraph $(N, H) \in \mathcal{H}_N$, $C^H(N)$ is a union stable system on N, and, on the other hand, for any union stable system $\mathcal{F}$ on N, $(N, \mathcal{C}(\mathcal{F}))$ is a hypergraph on N.

A *directed graph*, or *digraph*, on N is a pair (N, D) where $D \subseteq \{(i, j) : i, j \in N, i \neq j\}$ is a set of *directed links*, or *arcs*. Let $\mathcal{D}_N$ denote the set of digraphs with node set N. For a digraph $(N, D) \in \mathcal{D}_N$, a sequence of distinct nodes $(i_1, i_2 \ldots, i_k)$, $k \geq 2$, is a *directed path* in (N, D) from node i_1 to node i_k if $(i_h, i_{h+1}) \in D$ for $h = 1, \ldots, k-1$. For a digraph $(N, D) \in \mathcal{D}_N$, if for $i, j \in N$ there exists a directed path in (N, D) from i to j, then j is a *successor* of i and i is a *predecessor* of j in (N, D). If $(i, j) \in D$, then j is an *immediate successor* of i and i is an *immediate predecessor* of j. For $i \in N$, $S^D(i)$ and $\widehat{S}^D(i)$ denote the set of successors and the set of immediate successors of node i in (N, D), respectively, and $\bar{S}^D = S^D(i) \cup \{i\}$. In addition, for a digraph $(N, D) \in \mathcal{D}_N$, the undirected graph (N, Γ_D) on N associated with D is defined by $\Gamma_D = \{\{i, j\} : (i, j) \in D\}$. A subset $K \subseteq S$ is a *component* of $S \subseteq N$ in $(N, D) \in \mathcal{D}_N$ if K is a component of S in (N, Γ_D). Similarly, N/D denotes the set of components of N in (N, D).

A digraph $(N, T) \in \mathcal{D}_N$ is a *rooted tree* if it has a unique node without predecessors, called the *root* of (N, T) and denoted by $r(T)$, and for every other node in N there is a unique directed path in (N, T) from $r(T)$ to that node. A digraph $(N, D) \in \mathcal{D}_N$ is a *rooted forest* if $(K, D(K))$ is a rooted tree for all $K \in N/D$, where $D(K) = \{(i, j) \in D : i, j \in K\}$. A rooted tree $(N, T) \in \mathcal{D}_N$ is a *rooted spanning tree* of a connected undirected graph $(N, \Gamma) \in \mathit{\Gamma}_N^c$ if every $(i, j) \in T$ implies $\{i, j\} \in \Gamma$. A rooted spanning tree $(N, T) \in \mathcal{D}_N$ of a connected undirected graph $(N, \Gamma) \in \mathit{\Gamma}_N^c$ is *admissible* if $(i, j) \in T$ implies that the set of successors of j in (N, T) together with j is a component, in (N, Γ), of the set of successors of i in (N, T), i.e., $\bar{S}^T(j) \in S^T(i)/\Gamma$ (see Figure 2.2). The set of admissible rooted spanning trees of $(K, \Gamma(K))$, $K \in N/\Gamma$, in (N, Γ) is denoted by $\mathcal{T}^\Gamma(K)$, and for every $i \in K$, $\mathcal{T}_i^\Gamma(K)$ denotes the set of admissible rooted spanning trees having i as root.

In the sequel, an undirected graph is called a graph and a directed graph is

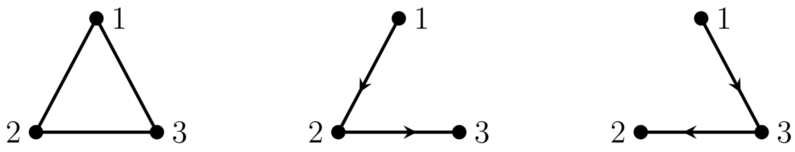


Figure 2.2: An undirected graph and the admissible rooted spanning trees having node 1 as the root.

called a digraph.

2.3 Games with communication structure

A *game with hypergraph communication structure*, or simply a *hypergraph game*, is a triple (N, v, H), where $(N, v) \in \mathcal{G}_N$ is a TU game on player set N and $(N, H) \in \mathcal{H}_N$ is a hypergraph on N. In particular, if the underlying structure is an (undirected) graph, games with graph communication structure are called *graph games*, i.e., (N, v, Γ) is a graph game, where $(N, v) \in \mathcal{G}_N$ and $(N, \Gamma) \in \mathit{\Gamma}_N$. Let $\mathcal{G}_N^{\mathcal{H}}$ denote the class of all hypergraph games with fixed player set N, and let $\widehat{\mathcal{G}}_N^{\mathcal{H}}$, $\mathcal{G}_N^{\mathcal{H}^{cf}}$, $\mathcal{G}_N^{\mathcal{H}^{t}}$, and $\mathcal{G}_N^{\mathcal{H}^{c}}$ denote the subclasses of zero-normalized, cycle-free, connected cycle-free, and connected hypergraph games on N, respectively. A hypergraph game $(N, v, H) \in \mathcal{G}_N^{\mathcal{H}}$ is zero-normalized if the game (N, v) is zero-normalized. Similarly, let $\mathcal{G}_N^{\mathit{\Gamma}}$ denote the class of all graph games on N, and let $\mathcal{G}_N^{\mathit{\Gamma}^{cf}}$, $\mathcal{G}_N^{\mathit{\Gamma}^{t}}$, and $\mathcal{G}_N^{\mathit{\Gamma}^{c}}$ denote the subclasses of cycle-free, connected cycle-free, and connected graph games on N, respectively.

For the solution for graph and hypergraph games, we first introduce the concept of the core. For any hypergraph game $(N, v, H) \in \mathcal{G}_N^{H}$, the *core* is defined as

$$C(N, v, H) = \left\{ x \in \mathbb{R}^n : \begin{array}{ll} \sum\limits_{i \in K} x_i = v(K), & \text{for all } K \in N/H \\ \sum\limits_{i \in S} x_i \geq v(S), & \text{for all } S \in C^H(N) \end{array} \right\}.$$

Similar to the core defined on TU games, the elements in $C(N, v, H)$ satisfy a kind of stability that no connected coalition has an incentive to leave the component the coalition is a subset of.

A *value* or *allocation rule* on a subclass of hypergraph games $\mathcal{G} \subseteq \mathcal{G}_N^{\mathcal{H}}$ is a mapping $\xi : \mathcal{G} \to \mathbb{R}^n$ that assigns to every hypergraph game $(N, v, H) \in \mathcal{G}$ a payoff vector $\xi(N, v, H) \in \mathbb{R}^n$.

Following Myerson (1977), we assume that for any graph or hypergraph game cooperation is possible only among connected players. The Myerson value is introduced first on graph games in Myerson (1977), and then extended to cooperative games with conference structures in Myerson (1980). Later on, the Myerson value is formally introduced on hypergraph games in van den Nouweland et al. (1992). For hypergraph games, the Myerson value is defined as the Shapley value applied to a modified game deduced from the hypergraph game. Formally, for any $(N, v, H) \in \mathcal{G}_N^{\mathcal{H}}$, the *Myerson value* is defined by

$$\mu(N, v, H) = Sh(N, v^H), \tag{2.3.1}$$

where $(N, v^H) \in \mathcal{G}_N$ is the *hypergraph-restricted game*, or *point game*, of the original hypergraph game (N, v, H), given by

$$v^H(S) = \sum_{K \in S/H} v(K), \text{ for all } S \subseteq N.$$

Particularly, for a graph game $(N, v, \Gamma) \in \mathcal{G}_N^{\Gamma}$, the Myerson value and the graph-restricted game $(N, v^\Gamma) \in \mathcal{G}_N$ are given by

$$\mu(N, v, \Gamma) = Sh(N, v^\Gamma), \text{ and } v^\Gamma(S) = \sum_{K \in S/\Gamma} v(K), \text{ for all } S \subseteq N.$$

The position value for graph games is introduced in Meessen (1988) and extended to hypergraph games in van den Nouweland et al. (1992). The position value assigns first to each hyperlink a Shapley payoff of a deduced game on the set of hyperlinks, and then the payoff to each hyperlink is distributed equally among its incident players. Formally, for any $(N, v, H) \in \mathcal{G}_N^{\mathcal{H}}$, the *position value* is defined as

$$\pi_i(N, v, H) = \sum_{e \in H_i} \frac{1}{|e|} Sh_e(H, v^N), \text{ for all } i \in N, \tag{2.3.2}$$

where $(H, v^N) \in \mathcal{G}_H$ is the *hyperlink game*, or *conference game*, of the original hypergraph game (N, v, H), given by

$$v^N(A) = \sum_{K \in N/A} v(K) \text{ for all } A \subseteq H.$$

Particularly, for a graph game $(N, v, \Gamma) \in \mathcal{G}_N^{\Gamma}$, the position value and the link game (Γ, v^N) are given by

$$\pi_i(N, v, \Gamma) = \frac{1}{2} \sum_{\ell \in \Gamma_i} Sh_\ell(N, v^N), \text{ for all } i \in N,$$

and

$$v^N(A) = \sum_{K \in N/A} v(K), \text{ for all } A \subseteq \Gamma.$$

The average tree value, or the average tree solution, is introduced in Herings et al. (2008) for cycle-free graph games and extended to the class of all graph games in Herings et al. (2010). Given a connected graph game, the average tree value assigns to each player as payoff the average of his marginal contributions to his successors in all admissible rooted spanning trees of the underlying graph. Formally, for any $(N, v, \Gamma) \in \mathcal{G}_N^{\Gamma c}$, the *average tree value* is defined by

$$AT(N, v, \Gamma) = \frac{1}{|\mathcal{T}^\Gamma(N)|} \sum_{T \in \mathcal{T}^\Gamma(N)} m^T(N, v, \Gamma), \tag{2.3.3}$$

where $m^T(N, v, H)$ is the *marginal contribution vector* corresponding to (N, v, Γ) and $T \in \mathcal{T}^\Gamma(N)$, given by

$$m_i^T(N, v, \Gamma) = v(\bar{S}^T(i)) - \sum_{K \in S^T(i)/\Gamma} v(K), \text{ for all } i \in N. \tag{2.3.4}$$

A *game with digraph communication structure*, or *digraph game*, is a triple (N, v, D), where $(N, v) \in \mathcal{G}_N$ is a TU game on player set N and $(N, D) \in \mathcal{D}_N$ is a digraph on N. Let $\mathcal{G}_N^{\mathcal{D}}$ denote the class of digraph games with fixed player set N, and let $\mathcal{G}_N^{\mathcal{F}}$ denote the subclass of rooted forest games on N. On a subclass of digraph games $\mathcal{G} \subseteq \mathcal{G}_N^{\mathcal{D}}$, a *value* is a mapping $\xi : \mathcal{G} \to \mathbb{R}^n$ that assigns to every digraph game $(N, v, D) \in \mathcal{G}$ a payoff vector $\xi(N, v, D) \in \mathbb{R}^n$.

For a digraph game, if the underlying digraph is a rooted forest, Demange (2004) and Khmelnitskaya (2010) study the tree value that assigns to each player as payoff the player's marginal contribution in the game to his successors in the rooted forest. Formally, for any rooted forest game $(N, v, D) \in \mathcal{G}_N^{\mathcal{F}}$, the *tree value* is defined by

$$t_i(N, v, D) = v(\bar{S}^D(i)) - \sum_{j \in \widehat{S}^D(i)} v(\bar{S}^D(j)), \qquad \text{for all } i \in N. \tag{2.3.5}$$

We conclude this section with a summary of axioms for games with communication structure. Let ξ be a value on $\mathcal{G} \subseteq \mathcal{G}_N^{\mathcal{H}} \bigcup \mathcal{G}_N^{\mathcal{D}}$, then we have the following axioms, where G is Γ, H, or D if the underlying structure is a graph, a hypergraph, or a digraph, respectively.

- **Component efficiency (CE)**: For any $(N, v, G) \in \mathcal{G}$ and component $K \in N/G$, it holds that
$$\sum_{i \in K} \xi_i(N, v, G) = v(K).$$
- **Efficiency (E)**: For any $(N, v, G) \in \mathcal{G}$, it holds that
$$\sum_{i \in K} \xi_i(N, v, G) = v(N).$$
- **Fairness (F)**: For any $(N, v, H) \in \mathcal{G}$, $e \in H$, and $i, j \in e$, it holds that
$$\xi_i(N, v, H) - \xi_i(N, v, H \setminus \{e\}) = \xi_j(N, v, H) - \xi_j(N, v, H \setminus \{e\}).$$
- **Additivity (A)**: For any $(N, v, G), (N, w, G) \in \mathcal{G}$, it holds that
$$\xi(N, v + w, G) = \xi(N, v, G) + \xi(N, w, G).$$
- **Linearity (L)**: For any $(N, v, G), (N, w, G) \in \mathcal{G}$ and $\alpha, \beta \in \mathbb{R}$, it holds that
$$\xi(N, \alpha v + \beta w, G) = \alpha\xi(N, v, G) + \beta\xi(N, w, G).$$
- **The superfluous conference property (SCP)**: For any $(N, v, H) \in \mathcal{G}$ and superfluous $e \in H$, it holds that
$$\xi(N, v, H) = \xi(N, v, H \setminus \{e\}),$$
where a hyperlink $e \in H$ is *superfluous* in $(N, v, H) \in \mathcal{G}$ if
$$v^N(A \setminus \{e\}) = v^N(A), \text{ for all } A \subseteq H.$$

- **The influence property (IP)**: For any conference anonymous $(N, v, H) \in \mathcal{G}$ there exists $\alpha \in \mathbb{R}$ such that

$$\xi_i(N, v, H) = \alpha I_i(N, H), \text{ for all } i \in N,$$

where $I_i(N, H)$ is the *influence* of player $i \in N$ in $(N, H) \in \mathcal{H}_N$ given by

$$I_i(N, H) = \sum_{e \in H_i} |e|^{-1},$$

and a hypergraph game $(N, v, H) \in \mathcal{G}$ is called *conference anonymous* if there exists a function $f : \{0, 1, \ldots, |H|\} \to \mathbb{R}$ such that

$$v^N(A) = f(|A|), \text{ for all } A \subseteq H.$$

- **Balanced link contributions (BLC)**: For any $(N, v, \Gamma) \in \mathcal{G}$ and $i, j \in N$, it holds that

$$\sum_{e \in \Gamma_j} \big(\xi_i(N, v, H) - \xi_i(N, v, \Gamma \setminus \{e\})\big) = \sum_{e \in \Gamma_i} \big(\xi_j(N, v, \Gamma) - \xi_j(N, v, \Gamma \setminus \{e\})\big).$$

- **Partial balanced conference contributions (PBCC)**: For any $(N, v, H) \in \mathcal{G}$ and $i, j \in N$, it holds that

$$\sum_{e \in H_j} \frac{1}{|e|}(\xi_i(N, v, H) - \xi_i(N, v, H \setminus \{e\})) = \sum_{e \in H_i} \frac{1}{|e|}(\xi_j(N, v, H) - \xi_j(N, v, H \setminus \{e\})).$$

- **Component fairness (CF)**: For any $(N, v, \Gamma) \in \mathcal{G}$ and $\{i, j\} \in \Gamma$, it holds that

$$\begin{aligned} & \frac{1}{|K^i|} \sum_{h \in K^i} \big(\xi_h(N, v, \Gamma) - \xi_h(N, v, \Gamma \setminus \{i, j\})\big) \\ = \; & \frac{1}{|K^j|} \sum_{h \in K^j} \big(\xi_h(N, v, \Gamma) - \xi_h(N, v, \Gamma \setminus \{i, j\})\big), \end{aligned}$$

where K^i and K^j are the components of N in $(N, \Gamma \setminus \{i, j\})$ containing player i and player j, respectively.

- **Successor equivalence (SE)**: For any $(N, v, D) \in \mathcal{G}$ and $(i, j) \in D$, it holds that

$$\xi_h(N, v, D \setminus \{(i,j)\}) = \xi_h(N, v, D), \text{ for all } h \in \bar{S}^D(j). \tag{2.3.6}$$

Note that if the underlying communication structure is connected then component efficiency reduces to efficiency. Moreover, if the underlying hypergraph is a graph, then partial balanced conference contributions reduces to balanced link contributions.

Myerson (1977) and van den Nouweland et al. (1992) show that on the class of (hyper)graph games the Myerson value is the unique solution that satisfies CE and F. In van den Nouweland et al. (1992), it is shown that on the class of cycle-free hypergraph games, the position value is the unique solution that satisfies CE, A, SCP, and IP. We notice that in the latter characterization additivity can be replaced by linearity. Slikker (2005) and Shan and Zhang (2016) show that on the class of (hyper)graph games the position value is the unique solution that satisfies CE and BLC (or PBCC). In Herings et al. (2008), it is shown that on the class of cycle-free graph games the average tree value is the unique solution that satisfies CE and CF. Moreover, in Khmelnitskaya (2010), it is shown that on the class of rooted forest games the tree value is the unique solution that satisfies CE and SE.

Chapter 3

The average tree value for graph games with main players

3.1 Introduction

In this chapter we investigate games with graph communication structure, in which some of the players are considered to be main players. The cooperative model of a graph game with main players is inspired by cooperative situations in which some participants chosen as 'main players' are playing more important roles than others, for example, the managers in an organization, the servers in an internet system, or the hubs in a transportation network. Due to the importance of these participants, they usually are treated in a special way when the total rewards of cooperation are allocated to the individual participants.

On the class of graph games with main players we introduce a solution concept that takes into account that the main players should be rewarded better than non-main players by adapting the ideas laying behind the average tree value for graph games introduced in Herings et al. (2008, 2010). In fact, the average tree value assigns to each player a payoff equal to the average of the player's tree value payoffs in the digraph games with the admissible rooted spanning trees of the given communication graph as digraphs. In a rooted tree digraph game, and therefore in any digraph game determined by a rooted spanning tree of a graph, the tree value, originally introduced in Demange (2004) under the name of the vector of hierarchical outcomes, rewards the root the best. For the average tree value, it is assumed that all players are of equal importance and therefore all can be chosen as a root. For this reason, the entire set of admissible rooted spanning trees is considered in the average tree value for graph games. However, when the players are not homogeneous and a group of selected main players is a priori given, we propose to take into account only those admissible spanning trees, for which the roots are given by the main players. We call this solution the average tree value for graph games with main players.

We also obtain several characterizations of the average tree value for graph games with main players. On the class of cycle-free graph games with main players, the average tree value can be characterized by component efficiency and the new axiom of main players component fairness saying that, when a link in the underlying graph is deleted, the changes of the total payoffs in the two resulting components are proportional to the number of main players in these two components. Another characterization of the average tree value is proposed on the class of cycle graph games with unique main player and is given by efficiency, linearity, the restricted null-player property, and veto players equal treatment. This characterization follows Shapley's approach in Shapley (1953).

Inspired by the hub-spoke networks, we discuss another classification of graphs with main players, single allocation type and multiple allocation type, and focus on the single allocation type. A graph with main players is of single allocation type if each ordinary (non-main) player is allocated to exactly one main player, i.e., every path between the two players contains no other main players. In such a graph, each main player together with the ordinary players, that are allocated to this player, can be treated as a union. In this way, on the class of graph games with main players in which the underlying structure is of single allocation type, the average tree value can be seen as a two-step distribution procedure similar to two-step procedures that determine the two-level graph game values for games with two-level communication structures studied in Khmelnitskaya (2014). In our approach, first, each union gets its total average tree value payoff in the graph game given by the Owen quotient game (Owen (1977)) determined by the partition of the entire player set by these unions and the graph being the subgraph on the main players. Second, the total payoff of every union is distributed among its members by the average tree value applied to the graph game with unique main player, where the game is given by the subgame on the union in which the worth of the entire coalition is replaced by the total payoff to the union obtained at the first step, and the graph is the subgraph on the union. Finally, as possible application of the average tree value for graph games with main players, we provide an alternative way to calculate the average tree value when the underlying structure of a graph game with main players is a classical single allocation hub-spoke network by regarding hubs as main players.

This chapter is based on a working paper Khmelnitskaya et al. (2018) and the structure of this chapter is as follows. In Section 3.2 we introduce the model of a graph game with main players and the solution concept of average tree value. Section 3.3 provides the axiomatic characterizations of the average tree value for cycle-free graph games with main players and for cycle graph games with unique main player in Subsection 3.3.1 and Subsection 3.3.2, respectively. Finally, as application of the average tree value for graph games with main players, a two-step distribution procedure of the average tree value is proposed based on an alternative classification of graph games with main players inspired by hub-spoke networks.

3.2 Modification of the average tree value for graph games

In this section we consider a class of graph games, in which some of the players are selected a priori as main players. It is assumed that main players have a more important role in the game than the other players and therefore deserve to be rewarded better.

A graph game $(N, v, \Gamma) \in \mathcal{G}_N^{\Gamma}$ and a nonempty set of players $M \subseteq N$ constitute a *graph game with main players* (N, v, Γ, M). We assume that, for each component $K \in N/\Gamma$, there exists at least one main player in K, and without main players a component cannot function. The players in $N \backslash M$ are called *ordinary players*. By $\mathcal{G}_N^{\Gamma,\mathcal{M}}$ we denote the class of graph games with main players on player set N, and by $\mathcal{G}_N^{\Gamma^{cf},\mathcal{M}}$, $\mathcal{G}_N^{\Gamma^{c},\mathcal{M}}$, and $\mathcal{G}_N^{\Gamma^{t},\mathcal{M}}$ its subclasses with cycle-free graphs, with connected graphs, and with connected cycle-free graphs, respectively. When each component contains exactly one main player, i.e., $|M \cap K| = 1$ for all $K \in N/\Gamma$, the corresponding subclasses are denoted by $\mathcal{G}_N^{\Gamma,\mathcal{M}^1}$, $\mathcal{G}_N^{\Gamma^{cf},\mathcal{M}^1}$, $\mathcal{G}_N^{\Gamma^{c},\mathcal{M}^1}$, and $\mathcal{G}_N^{\Gamma^{t},\mathcal{M}^1}$, respectively. Note that, since each component contains at least one main player, $\mathcal{G}_N^{\Gamma^{c},\mathcal{M}^1}$ and $\mathcal{G}_N^{\Gamma^{t},\mathcal{M}^1}$ indicate $|M| = 1$. Similar to graph games, on a subclass of graph games with main players $\mathcal{G} \subseteq \mathcal{G}_N^{\Gamma,\mathcal{M}}$, a function $\xi : \mathcal{G} \to \mathbb{R}^n$ is a *value*, or *allocation rule*, that assigns to every $(N, v, \Gamma, M) \in \mathcal{G}$ a payoff vector $\xi(N, v, \Gamma, M) \in \mathbb{R}^n$.

From (2.3.4) and (2.3.5), we see that the marginal contribution of the average tree value corresponding to an admissible rooted spanning tree is defined similar to the tree value for a rooted forest game. In Khmelnitskaya (2010) it is shown that the tree value for rooted forest games rewards the best the player located at the root of the tree by assigning the full dividend of the grand coalition to this player. Therefore, in order to take into account the importance of main players, we introduce a modification of the average tree value for graph games, which exploits the latter property of the tree value, by assuming that only the main players can be the root among admissible rooted spanning trees.

For any $(N, v, \Gamma, M) \in \mathcal{G}_N^{\Gamma,\mathcal{M}}$, let $\mathcal{T}_M^\Gamma(K)$ denote the set of admissible rooted spanning trees of subgraph $(K, \Gamma(K))$, $K \in N/\Gamma$, with roots from the set $M \cap K$. If no ambiguity appears, we use the set of arcs when we refer to a rooted tree or rooted forest. Therefore, $\mathcal{T}_M^\Gamma(K) = \{T \in \mathcal{T}^\Gamma(K) : r(T) \in M \cap K\}$. Note that we assume that $|M \cap K| \neq 0$ and if (N, Γ) is connected then $K = N$. Then, we define for a graph game with main players the average tree value for a player as the average of his marginal contributions according to the admissible rooted spanning trees of the underlying graph, the roots of which are the main players.

Definition 3.2.1. For any $(N, v, \Gamma, M) \in \mathcal{G}_N^{\Gamma,\mathcal{M}}$ and $i \in K$, $K \in N/\Gamma$, the *average tree value* is given by

$$ATM_i(N, v, \Gamma, M) = \frac{1}{|\mathcal{T}_M^\Gamma(K)|} \sum_{T \in \mathcal{T}_M^\Gamma(K)} m_i^T(N, v, \Gamma). \tag{3.2.1}$$

The following example illustrates Definition 3.2.1.

Example 3.2.1. Consider the graph game with main players $(N, v, \Gamma, M) \in \mathcal{G}_N^{\Gamma,\mathcal{M}}$, where $N = \{1, \ldots, 5\}$, $M = \{1, 3\}$, $v(S) = (|S| - 1)^2$ for all $S \subseteq N$, and $\Gamma = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}\}$. The structure is displayed in Figure 3.1.

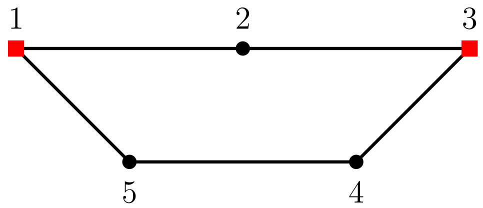


Figure 3.1: The underlying structure in Example 3.2.1. The squares indicate the main players and the circles indicate the ordinary players

There are four admissible rooted spanning trees, namely T_1^1, T_1^2, T_3^1, and T_3^2, i.e., $\mathcal{T}_M^\Gamma(N) = \{T_1^1, T_1^2, T_3^1, T_3^2\}$, where $T_1^1 = \{(1,2),(2,3),(3,4),(4,5)\}$, $T_1^2 = \{(1,5),(5,4),(4,3),(3,2)\}$, $T_3^1 = \{(3,4),(4,5),(5,1),(1,2)\}$, and $T_3^2 = \{(3,2),(2,1),(1,5),(5,4)\}$. Then, from (2.3.4), we have

$$m^{T_1^1}(N,v,\Gamma) = (7,5,3,1,0), \quad m^{T_1^2}(N,v,\Gamma) = (7,0,1,3,5),$$
$$m^{T_3^1}(N,v,\Gamma) = (1,0,7,5,3), \quad m^{T_3^2}(N,v,\Gamma) = (3,5,7,0,1).$$

Therefore, from (3.2.1), it follows that

$$ATM(N,v,\Gamma,M) = (\frac{9}{2}, \frac{5}{2}, \frac{9}{2}, \frac{9}{4}, \frac{9}{4}).$$

□

From Definition 3.2.1, we notice that when all players are main players, the average tree value for graph games with main players by definition coincides with the average tree value for graph games, in the sense that $ATM(N,v,\Gamma,M) = AT(N,v,\Gamma)$ if $M = N$. Moreover, if the underlying graph structure $(N,\Gamma) \in \mathit{\Gamma}_N$ is cycle-free, then we can rewrite (3.2.1) as

$$ATM_i(N,v,\Gamma,M) = \frac{1}{|M \cap K|} \sum_{r \in M \cap K} m_i^{T_r}(N,v,\Gamma), \tag{3.2.2}$$

where $T_r \in \mathcal{T}_M^\Gamma(K)$, $r \in M \cap K$, is the unique admissible rooted spanning tree on $(K, \Gamma(K))$ with root r.

If $|M \cap K| = 1$ and $(K, \Gamma(K))$ is cycle-free for some $K \in N/\Gamma$, then we have $ATM_i(N,v,\Gamma,M) = t_i(K, v_K, T)$, for all $i \in K$, see (2.3.5), where T is the unique admissible rooted spanning tree of $\mathcal{T}_M^\Gamma(K)$ on K. In addition, on the class of tree games with main players, the average tree value is a random tree solution[1] introduced in Béal et al. (2010). Specifically, for any $(N,v,\Gamma,M) \in \mathcal{G}_N^{\Gamma^t,\mathcal{M}}$, we have $ATM(N,v,\Gamma,M) = RT(N,v,\Gamma)$, whenever $a_r = \frac{1}{|M|}$, $r \in M$, and $a_r = 0$, otherwise.

From Definition 3.2.1 we see that the new allocation rule highlights the main players compared with the ordinary players, which differs from the average tree value for graph games, because in the latter value every player can be the root in an admissible rooted spanning tree.

[1]For any $(N,v,\Gamma) \in \mathcal{G}_N^{\Gamma^t}$, a *random tree* solution is defined as $RT(N,v,\Gamma) = \sum_{r \in N} a_r m^{T_r}(N,v,\Gamma)$, where $\sum_{r \in N} a_r = 1$ and $a_r \geq 0$ for all $r \in N$.

3.3 Axiomatizations

In this section we examine two classes of graph games with main players and provide characterizations of the average tree value on these classes.

3.3.1 Cycle-free graph games with main players

We first focus on cycle-free graph games with main players and give an axiomatic characterization of the average tree value for such kind of games.

In Herings et al. (2008) it is shown that on the class of cycle-free graph games, the average tree value for graph games is characterized by two axioms, component efficiency and component fairness (see Section 2.3). Now we show that on the class of cycle-free graph games with main players the new value can be characterized similarly by two axioms, component efficiency and a new deletion link property.

The first property, component efficiency, on graph games with main players is similar to component efficiency defined on graph games and states that the total payoffs of the players of any component equals the worth of that component.

Component efficiency: For any $(N, v, \Gamma, M) \in \mathcal{G}, \mathcal{G} \subseteq \mathcal{G}_N^{\Gamma,\mathcal{M}}$, and $K \in N/\Gamma$, it holds that

$$\sum_{i \in K} \xi_i(N, v, \Gamma, M) = v(K).$$

The second property as a new deletion link property deals with the payoff changes between the two resulting components when a link is removed, which is given as follows.

Main players component fairness: For any $(N, v, \Gamma, M) \in \mathcal{G}_N^{\Gamma^{cf},\mathcal{M}}$ and $\{i, j\} \in \Gamma$, it holds that

$$\begin{aligned} & |M \cap K^j| \sum_{h \in K^i} \big(\xi_h(N, v, \Gamma, M) - \xi_h(N, v, \Gamma_{-ij}, M_{-ij})\big) \\ = \; & |M \cap K^i| \sum_{h \in K^j} \big(\xi_h(N, v, \Gamma, M) - \xi_h(N, v, \Gamma_{-ij}, M_{-ij})\big), \end{aligned} \tag{3.3.1}$$

where $\Gamma_{-ij} = \Gamma \setminus \{\{i, j\}\}$, K^i and K^j are the two components of N in (N, Γ_{-ij}) containing player i and j, respectively, and $M_{-ij} = M$ if $K^k \cap M \neq \emptyset$ for all $k \in \{i, j\}$, and $M_{-ij} = M \cup \{k\}$ if $K^k \cap M = \emptyset$ for some $k \in \{i, j\}$.

Our motivation for the definition of the set M_{-ij} is based on the following consideration. Deletion of link $\{i,j\} \in \Gamma$ in a cycle-free graph splits the component K into components K^i and K^j and it might happen that one of the two resulting components, say K^k, contains no main players. In such a case in the original graph all the paths between main players and members of K^k go through node k, and so, after deletion of the link this node becomes a main player for the graph (N, Γ_{-ij}). Note that, in this case, there is at most one such k.

Main players component fairness states that the deletion of a link in the graph implies that the changes of the total payoffs in both resulting components are proportional to the numbers of main players in these two components. The main players in fact determine the weight coefficients of the two induced components. In this sense, this property is a special case of weighted component fairness introduced in Béal et al. (2012b) by setting the weights of the two induced components K^i and K^j as $\frac{|M \cap K^j|}{|M \cap K|}$ and $\frac{|M \cap K^i|}{|M \cap K|}$, respectively. Moreover, in case all players are main players main players component fairness is equivalent to component fairness, in other words, a payoff vector satisfying main players component fairness implies that this payoff vector satisfies component fairness, if all players are main players.

Next, we provide a characterization of the average tree value for cycle-free graph games with main players. First of all, we show that the average tree value on cycle-free graph games with main players satisfies the two axioms above.

Lemma 3.3.1. *On the class of cycle-free graph games with main players, the average tree value satisfies component efficiency and main players component fairness.*

Proof. Take any $(N, v, \Gamma, M) \in \mathcal{G}_N^{\Gamma^{cf}, \mathcal{M}}$. Let $K \in N/\Gamma$ and $T_r \in \mathcal{T}_M^{\Gamma}(K)$ be the unique admissible rooted spanning tree of $(K, \Gamma(K))$ with root $r \in M \cap K$. Then $\bar{S}^{T_r}(r) = K$ and therefore

$$\sum_{i \in K} m_i^{T_r}(N, v, \Gamma) = \sum_{i \in \bar{S}^{T_r}(r)} m_i^{T_r}(N, v, \Gamma) = v(\bar{S}^{T_r}(r)) = v(K),$$

where the second equality follows from (2.3.4).

Hence, from (3.2.2), we have

$$\begin{aligned}\sum_{i\in K} ATM_i(N,v,\Gamma,M) &= \sum_{i\in K}\frac{1}{|M\cap K|}\sum_{r\in M\cap K} m_i^{T_r}(N,v,\Gamma)\\ &= \frac{1}{|M\cap K|}\sum_{r\in M\cap K}\sum_{i\in K} m_i^{T_r}(N,v,\Gamma)\\ &= \frac{1}{|M\cap K|}\sum_{r\in M\cap K} v(K) \;=\; v(K),\end{aligned}$$

which shows that ATM satisfies component efficiency on $\mathcal{G}_N^{\Gamma^{cf},\mathcal{M}}$.

Take any $(N,v,\Gamma,M)\in\mathcal{G}_N^{\Gamma^{cf},\mathcal{M}}$ and $\{i,j\}\in\Gamma(K)$, $K\in N/\Gamma$. Let K^i and K^j be the two components of N in (N,Γ_{-ij}) containing players i and j, respectively. Since ATM satisfies component efficiency, it holds that $\sum\limits_{k\in K^h} ATM_k(N,v,\Gamma_{-ij},M_{-ij}) = v(K^h)$ for $h\in\{i,j\}$.

From (3.2.2), for any $r\in M\cap K$, we have that

$$\sum_{h\in K^i} m_h^{T_r}(N,v,\Gamma) = v(K^i), \qquad \text{if } r\in M\cap K^j, \tag{3.3.2}$$

$$\sum_{h\in K^i} m_h^{T_r}(N,v,\Gamma) = v(K)-v(K^j), \quad \text{if } r\in M\cap K^i. \tag{3.3.3}$$

Hence, from (3.2.2), we obtain

$$\sum_{h\in K^i} ATM_h(N,v,\Gamma,M) = \frac{1}{|M\cap K|}\big(|M\cap K^j|v(K^i)+|M\cap K^i|(v(K)-v(K^j))\big).$$

Because $|M\cap K| = |M\cap K^i|+|M\cap K^j|$, it follows that

$$\begin{aligned}&\sum_{h\in K^i}\big(ATM_h(N,v,\Gamma,M)-ATM_h(N,v,\Gamma_{-ij},M_{-ij})\big)\\ =\;& \frac{1}{|M\cap K|}\big(|M\cap K^j|v(K^i)+|M\cap K^i|(v(K)-v(K^j))\big)-v(K^i)\\ =\;& \frac{|M\cap K^i|}{|M\cap K|}\big(v(K)-v(K^j)-v(K^i)\big).\end{aligned}$$

Therefore, we have

$$\begin{aligned} & |M \cap K^j| \sum_{k\in K^i} \big(ATM_k(N,v,\Gamma,M)) - ATM_k(N,v,\Gamma_{-ij},M_{-ij})\big) \\ = & \frac{|M\cap K^i|\cdot|M\cap K^j|}{|M\cap K|}\big(v(K)-v(K^j)-v(K^i)\big). \end{aligned}$$

By interchanging the roles of i and j, it follows that ATM satisfies main players component fairness on $\mathcal{G}_N^{\Gamma^{cf},\mathcal{M}}$. □

The next theorem shows that component efficiency together with main players component fairness uniquely determine a solution.

Lemma 3.3.2. *On the class of cycle-free graph games with main players, there is a unique value that satisfies component efficiency and main players component fairness.*

Proof. Assume that on the class of cycle-free graph games with main players ξ satisfies component efficiency and main players component fairness.

Take any $(N,v,\Gamma,M) \in \mathcal{G}_N^{\Gamma^{cf},\mathcal{M}}$ and $K \in N/\Gamma$. Let $\{i,j\} \in \Gamma(K)$. Then component efficiency implies

$$\sum_{h\in K} \xi_h(N,v,\Gamma,M) = v(K), \tag{3.3.4}$$

and for all $h \in \{i,j\}$

$$\sum_{k\in K^h} \xi_k(N,v,\Gamma_{-ij},M_{-ij}) = v(K^h), \tag{3.3.5}$$

where K^h, $h \in \{i,j\}$, is the component of N in (N,Γ_{-ij}) containing player h. Then, main players component fairness implies

$$\begin{aligned} & |M\cap K^j|\big(\sum_{h\in K^i}\xi_h(N,v,\Gamma,M)-v(K^i)\big) \\ = & |M\cap K^i|\big(\sum_{h\in K^j}\xi_h(N,v,\Gamma,M)-v(K^j)\big). \end{aligned} \tag{3.3.6}$$

Combined with (3.3.4), (3.3.6) can be reorganized as

$$\sum_{h\in K^i}\xi_h(N,v,\Gamma,M) = \frac{|M\cap K^i|}{|M\cap K|}\big(v(K)-v(K^j)\big)+\frac{|M\cap K^j|}{|M\cap K|}v(K^i), \tag{3.3.7}$$

where $|M \cap K| = |M \cap K^i| + |M \cap K^j| \neq 0$.

Take any $r \in M \cap K$ and let T_r be the unique admissible rooted spanning tree in $\mathcal{T}_M^{\Gamma}(K)$ with root r. Then there are $|\Gamma(K)| = |K| - 1$ equations of type (3.3.7) satisfying $K^i = \bar{S}^{T_r}(i)$, for all $i \in K \setminus \{r\}$. Combined with (3.3.4), they form a system of $|K|$ linear equations with $|K|$ unknowns. Now, we consider an ordering of the players on K such that, for any different $j_1, j_2 \in K$, $\sigma(j_1) < \sigma(j_2)$ implies $|\bar{S}^{T_r}(j_1)| \leq |\bar{S}^{T_r}(j_2)|$. According to ordering σ, we obtain the following system:

$$\sum_{\sigma(h) \in K^{\sigma(i)}} \xi_{\sigma(h)}(N, v, \Gamma, M) = \begin{cases} b(K^{\sigma(i)}), & \text{if } i \in K \setminus \{r\}, \\ v(K), & \text{if } i = r, \end{cases}$$

where $b(K^{\sigma(i)}) = \frac{|M \cap K^{\sigma(i)}|}{|M \cap K|}\big(v(K) - v(K \setminus K^{\sigma(i)})\big) + \frac{|M \cap (K \setminus K^{\sigma(i)})|}{|M \cap K|} v(K^{\sigma(i)})$.

The coefficient matrix associated to this system has a nonzero determinant, since it is lower triangular with each diagonal term equal to 1. Therefore, the $|K|$ equations in this system are linearly independent and uniquely determine $\xi_i(N, v, \Gamma, M)$, $i \in K$, on component K.

By applying the same process on all components of N in (N, Γ), $\xi(N, v, \Gamma, M)$ is uniquely determined among all players. □

From Lemma 3.3.1 and Lemma 3.3.2, we have the following result.

Theorem 3.3.1. *On the class of cycle-free graph games with main players, the average tree value is the unique allocation rule that satisfies component efficiency and main players component fairness.*

Next, we consider the class of cycle-free graph games with main players, where each component contains exactly one main player. In this case, there is a unique admissible rooted spanning tree for each component and main players component fairness reduces to that, for any $(N, v, \Gamma, M) \in \mathcal{G}_N^{\Gamma^{cf}, \mathcal{M}^1}$ and any $\{i, j\} \in \Gamma$,

$$\sum_{k \in K^h} \xi_k(N, v, \Gamma, M) = \sum_{k \in K^h} \xi_k(N, v, \Gamma_{-ij}, M_{-ij}), \tag{3.3.8}$$

where K^h is the unique component of N in (N, Γ_{-ij}) containing the player $h \in \{i, j\}$ for which $K^h \cap M = \emptyset$.

Since each game of $\mathcal{G}_N^{\Gamma^{cf}, \mathcal{M}^1}$ is equivalent to a rooted forest game having in each component its unique main player as the root, the equality as in (3.3.8) can be used to formulate the following corresponding axiom for digraph games, which states that if an arc is removed from a digraph game then the total payoff of all successors of the arc's head including himself will not change.

Successor equivalence in total: For any $(N, v, D) \in \mathcal{G}, \mathcal{G} \subseteq \mathcal{G}_N^{\mathcal{D}}$, and $(i, j) \in D$, it holds that

$$\sum_{h \in \bar{S}^D(j)} \xi_h(N, v, D \setminus \{(i, j)\}) = \sum_{h \in \bar{S}^D(j)} \xi_h(N, v, D).$$

Note that this property is different from the property of successor equivalence (see Section 2.3). However, for rooted forest games, we have the following result.

Lemma 3.3.3. *On the class of rooted forest games, successor equivalence coincides with successor equivalence in total.*

Proof. It is easy to check that if a value ξ on $\mathcal{G}_N^{\mathcal{F}}$ satisfies successor equivalence then it satisfies successor equivalence in total. Now we show the opposite direction.

Let ξ on $\mathcal{G}_N^{\mathcal{F}}$ satisfy successor equivalence in total. Take any $(N, v, T) \in \mathcal{G}_N^{\mathcal{F}}$ and let $(i, j) \in T$, then it holds that

$$\sum_{h \in \bar{S}^T(j)} \xi_h(N, v, T \setminus \{(i, j)\}) = \sum_{h \in \bar{S}^T(j)} \xi_h(N, v, T). \tag{3.3.9}$$

We prove successor equivalence of ξ by induction on $|\bar{S}^T(j)|$. If $|\bar{S}^T(j)| = 1$, then (3.3.9) implies that $\xi_j(N, v, T \setminus \{(i, j)\}) = \xi_j(N, v, T)$, which shows successor equivalence.

Assume that (3.3.9) implies that

$$\xi_h(N, v, T \setminus \{(i, j)\}) = \xi_h(N, v, T), \quad h \in \bar{S}^T(j),$$

for all $|\bar{S}^T(j)| < t \leq n$.

Let $|\bar{S}^T(j)| = t$. Since (N, T) is a rooted forest, (3.3.9) is equivalent to

$$\begin{aligned} &\sum_{j' \in N: (j, j') \in T} \sum_{h \in \bar{S}^T(j')} \xi_h(N, v, T \setminus \{(i, j)\}) + \xi_j(N, v, T \setminus \{(i, j)\}) \\ = &\sum_{j' \in N: (j, j') \in T} \sum_{h \in \bar{S}^T(j')} \xi_h(N, v, T) + \xi_j(N, v, T). \end{aligned}$$

According to the assumption and since $|\bar{S}^T(j')| < t$ for all $j' \in N$ such that

$(j, j') \in T$, we have that for each such j'

$$\xi_h(N, v, T \setminus \{(i, j)\}) = \xi_h(N, v, T), \quad \text{for all } h \in \bar{S}^T(j'),$$

which is equivalent to

$$\xi_h(N, v, T \setminus \{(i, j)\}) = \xi_h(N, v, T), \quad \text{for all } h \in S^T(j).$$

Furthermore, it holds that $\xi_j(N, v, T \setminus \{(i, j)\}) = \xi_j(N, v, T)$.

Therefore, (3.3.9) shows that ξ satisfies successor equivalence, i.e., for any $(i, j) \in T$ it holds that

$$\xi_h(N, v, T \setminus \{(i, j)\}) = \xi_h(N, v, T), \quad \text{for all } h \in \bar{S}^T(j).$$

□

From this lemma, we observe that, for any $(N, v, \Gamma, M) \in \mathcal{G}_N^{\Gamma^{cf}, \mathcal{M}^1}$, main players component fairness of a solution on (N, v, Γ, M) as in Lemma (3.3.2) implies successor equivalence of that solution on (N, v, T), where $T(K) \in \mathcal{T}_K^{\Gamma}(N)$ is the unique admissible rooted spanning tree of $K \in N/\Gamma$ in (N, Γ, M). On the other hand, for any $(N, v, T) \in \mathcal{G}_N^{\mathcal{F}}$, successor equivalence of a solution on (N, v, T) implies main players component fairness of that solution on (N, v, Γ, M), where M is the set of roots in the rooted forest and $\Gamma = \{\{i, j\} : (i, j) \in T\}$.

From Lemma 3.3.3 and Theorem 1 in Khmelnitskaya (2010), a characterization of the tree value by component efficiency and successor equivalence, we obtain the following theorem.

Theorem 3.3.2. *On the class of rooted forest games, the tree value is the unique allocation rule that satisfies component efficiency and successor equivalence in total.*

3.3.2 Cycle graph games with unique main player

In this subsection we consider the class of cycle graph games with unique main player, where a cycle graph consists of one cycle. For cycle graph games without main players, we refer to Selçuk et al. (2013).

Let $\mathcal{G}_N^{\Gamma^{cc}, \mathcal{M}^1}$ denote the set of cycle graph games with unique main player with fixed player set N. For $(N, v, \Gamma, M) \in \mathcal{G}_N^{\Gamma^{cc}, \mathcal{M}^1}$, without loss of generality, we use

$a \in N$ to represent the unique main player, i.e., $M = \{a\}$. For any cycle graph game with unique main player, there are only two admissible rooted spanning trees having the main player as root, each going in one of the two directions along the cycle.

Now we introduce some axioms on graph games with main players. The first two axioms are adapted from two standard properties on TU games, efficiency and linearity (see Section 2.1).

Efficiency: For any $(N, v, \Gamma, M) \in \mathcal{G}, \mathcal{G} \subseteq \mathcal{G}_N^{\Gamma^c, \mathcal{M}}$, it holds that

$$\sum_{i \in N} \xi_i(N, v, \Gamma, M) = v(N).$$

Linearity: For any two $(N, v, \Gamma, M), (N, w, \Gamma, M) \in \mathcal{G}, \mathcal{G} \subseteq \mathcal{G}_N^{\Gamma, \mathcal{M}}$, and $\alpha, \beta \in \mathbb{R}$, it holds that

$$\xi(N, \alpha v + \beta w, \Gamma, M) = \alpha \xi(N, v, \Gamma, M) + \beta \xi(N, w, \Gamma, M).$$

It is obvious that on the class of connected graph games with main players, the average tree value satisfies efficiency and linearity.

The next two axioms are adapted from the null-player property and the property of symmetry for TU games (see Section 2.1).

A player $i \in N$ is a *restricted null-player* on $(N, v, \Gamma, \{a\}) \in \mathcal{G}_N^{\Gamma^{cc}, \mathcal{M}^1}$ if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \setminus \{i, a\}$ such that $S, S \cup \{i\} \in C^\Gamma(N)$. Observe that the unique main player can also be a restricted null-player.

Restricted null-player property: For any $(N, v, \Gamma, \{a\}) \in \mathcal{G}_N^{\Gamma^{cc}, \mathcal{M}^1}$ and restricted null-player $i \in N$, it holds that $\xi_i(N, v, \Gamma, \{a\}) = 0$.

The axiom states that if a player contributes nothing to any connected coalition not containing the main player, then this player gets a zero payoff.

A player $i \in N$ is a *veto player* on $(N, v, \Gamma, \{a\}) \in \mathcal{G}_N^{\Gamma^{cc}, \mathcal{M}^1}$ if $v(S) = 0$ for all $S \in C^\Gamma(N \setminus \{i\})$. Let $V(N, v, \Gamma)$ be the set of veto players on $(N, v, \Gamma, \{a\}) \in \mathcal{G}_N^{\Gamma^{cc}, \mathcal{M}^1}$. In $(N, v, \Gamma, \{a\}) \in \mathcal{G}_N^{\Gamma^{cc}, \mathcal{M}^1}$, two veto players $i, j \in V(N, v, \Gamma) \setminus \{a\}$, $i \neq j$, are *symmetric* if there exists $\gamma \in \mathbb{R}$ such that $v(S) = \gamma$ for every $S \in C^\Gamma(N \setminus \{a\})$ satisfying $\{i, j\} \subseteq S$.

Veto players equal treatment: For any $(N, v, \Gamma, \{a\}) \in \mathcal{G}_N^{\Gamma^{cc}, \mathcal{M}^1}$ and symmetric veto players $i, j \in N \setminus \{a\}$, it holds that $\xi_i(N, v, \Gamma, \{a\}) = \xi_j(N, v, \Gamma, \{a\})$.

The property of veto players equal treatment states that if two veto players are symmetric in a cycle graph game with unique main player, then they will be treated equally, where the symmetry between two veto players requires that the worths of all connected coalitions containing the two veto players but not the main player are the same.

Note that both the restricted null-player property and veto players equal treatment are defined only for cycle graph games with unique main player. From linearity and the restricted null-player property, we have the following result.

Lemma 3.3.4. *If a value* $\xi : \mathcal{G}_N^{\Gamma^{cc},\mathcal{M}^1} \rightarrow \mathbb{R}^n$ *satisfies linearity and the restricted null-player property, then it holds that* $\xi(N, v, \Gamma, \{a\}) = \xi(N, v^\Gamma, \Gamma, \{a\})$ *for all* $(N, v, \Gamma, \{a\}) \in \mathcal{G}_N^{\Gamma^{cc},\mathcal{M}^1}$.

Proof. Consider the game $(N, w, \Gamma, \{a\}) \in \mathcal{G}_N^{\Gamma^{cc},\mathcal{M}^1}$ where $w = v - v^\Gamma$. For any $i \in N$ and $S \subseteq N \setminus \{i, a\}$ satisfying $S, S \cup \{i\} \in C^\Gamma(N)$, it holds that $w(S \cup \{i\}) = w(S) = 0$. So, every player is a restricted null-player in $(N, w, \Gamma, \{a\})$ and receives zero payoff, i.e., $\xi_i(N, w, \Gamma, \{a\}) = 0$ for all $i \in N$. Then by linearity and $w = v - v^\Gamma$, we have $\xi(N, v, \Gamma, \{a\}) = \xi(N, w, \Gamma, \{a\}) + \xi(N, v^\Gamma, \Gamma, \{a\}) = \xi(N, v^\Gamma, \Gamma, \{a\})$. □

The following lemma shows that on the class of cycle graph games with unique main player the average tree value satisfies both the restricted null-player property and veto players equal treatment.

Lemma 3.3.5. *On the class of cycle graph games with unique main player, the average tree value satisfies the restricted null-player property and veto players equal treatment.*

Proof. Take any $(N, v, \Gamma, \{a\}) \in \mathcal{G}_N^{\Gamma^{cc},\mathcal{M}^1}$. There are exactly two admissible rooted spanning trees in (N, Γ). Let $\mathcal{T}_{\{a\}}^\Gamma(N) = \{T_1, T_2\}$, where $r(T_1) = r(T_2) = a$.

If a player $i \in N$ is a restricted null-player in $(N, v, \Gamma, \{a\})$, then, for every $T \in \{T_1, T_2\}$, it holds that $v(\bar{S}^T(i)) = v(S^T(i))$ since both $\bar{S}^T(i)$ and $S^T(i)$ are connected in cycle graph (N, Γ). Hence, we have

$$m_i^T(N, v, \Gamma) = v(\bar{S}^T(i)) - v(S^T(i)) = 0.$$

Therefore, by (3.2.1), it turns out that

$$ATM_i(N, v, \Gamma, \{a\}) = \frac{1}{2}(m_i^{T_1}(N, v, \Gamma) + m_i^{T_2}(N, v, \Gamma)) = 0,$$

which shows the restricted null-player property.

Consider two symmetric veto players $i, j \in N \setminus \{a\}$ with $i \neq j$. Since (N, Γ) is a cycle graph and $M = \{a\}$, it follows that, for all $T \in \{T_1, T_2\}$, either $j \in \bar{S}^T(i)$ or $i \in \bar{S}^T(j)$. Suppose without loss of generality that $j \in \bar{S}^{T_1}(i)$ and therefore $i \in \bar{S}^{T_2}(j)$, then there exists $\gamma \in \mathbb{R}$ such that

$$v(\bar{S}^{T_1}(i)) = v(\bar{S}^{T_2}(j)) = \gamma.$$

According to (2.3.4), we have

$$m_i^{T_1}(N, v, \Gamma) = v(\bar{S}^{T_1}(i)) - v(S^{T_1}(i)) = \gamma,$$

since $i \in V(N, v, \Gamma)$ is a veto player and therefore $v(S^{T_1}(i)) = 0$, and

$$m_i^{T_2}(N, v, \Gamma) = v(\bar{S}^{T_2}(i)) - v(S^{T_2}(i)) = 0,$$

since $j \in V(N, v, \Gamma)$ is a veto player and $j \notin \bar{S}^{T_2}(i)$. Hence, the average tree value of player i is equal to

$$ATM_i(N, v, \Gamma, \{a\}) = \frac{1}{2}(m_i^{T_1}(N, v, \Gamma) + m_i^{T_2}(N, v, \Gamma)) = \frac{\gamma}{2}.$$

Analogously, the average tree value of player j is equal to

$$ATM_j(N, v, \Gamma, \{a\}) = \frac{1}{2}(m_j^{T_1}(N, v, \Gamma) + m_j^{T_2}(N, v, \Gamma)) = \frac{\gamma}{2},$$

which shows veto players equal treatment. □

Owen (1986) shows that, for a graph game $(N, v, \Gamma) \in \mathcal{G}_N^\Gamma$, the dividends of disconnected coalitions on the graph-restricted game are equal to zero, i.e., $\Delta_{v^\Gamma}(S) = 0$, for all $S \notin C^\Gamma(N)$. According to (2.1.1), it follows that

$$v^\Gamma = \sum_{S \in C^\Gamma(N)} \Delta_{v^\Gamma}(S) u_S, \tag{3.3.10}$$

which implies that the set of unanimity games with respect to a connected coalition $\{(N, u_S) : S \in C^\Gamma(N)\}$ forms a basis for the space of graph-restricted games.

Based on Lemma 3.3.4 and (3.3.10), we immediately obtain the following result on cycle graph games with unique main player.

Lemma 3.3.6. *If a value* $\xi : \mathcal{G}_N^{\Gamma^{cc},\mathcal{M}^1} \to \mathbb{R}^n$ *satisfies linearity and the restricted null-player property, then, for any* $(N, v, \Gamma, M) \in \mathcal{G}_N^{\Gamma^{cc},\mathcal{M}^1}$*, it holds that*

$$\xi(N, v, \Gamma, M) = \sum_{S \in C^\Gamma(N)} \Delta_{v^\Gamma}(S) \cdot \xi(N, u_S, \Gamma, M).$$

The following theorem provides a characterization of the average tree value on the class of cycle graph games with unique main player.

Theorem 3.3.3. *On the class of cycle graph games with unique main player, the average tree value is the unique solution that satisfies efficiency, linearity, the restricted null-player property, and veto players equal treatment.*

Proof. It is easy to check that the average tree value satisfies efficiency and linearity. From Lemma 3.3.5, it follows that the average tree value satisfies the other two axioms on the class of cycle graph games with unique main player. Next we show uniqueness.

Let ξ be an allocation rule that satisfies the four axioms on cycle graph games with unique main player. Due to Lemma 3.3.6 it is sufficient to consider cycle graph unanimity games defined on connected coalitions.

For any $(N, u_Q, \Gamma, \{a\}) \in \mathcal{G}_N^{\Gamma^{cc},\mathcal{M}^1}$, $Q \in C^\Gamma(N)$, we consider the following three cases:

1). If $a \in Q$, then each $i \in N \setminus \{a\}$ is a restricted null player.

For any $S \subseteq N \setminus \{i, a\}$ such that $S, S \cup \{i\} \in C^\Gamma(N)$, since the unique main player $a \in N$ does not belong to both S and $S \cup \{i\}$, it follows that

$$u_Q(S \cup \{i\}) = u_Q(S) = 0.$$

By the restricted null-player property, we have

$$\xi_i(N, u_Q, \Gamma, \{a\}) = 0, \quad \text{for all } i \in N \setminus \{a\}.$$

Then, by efficiency, we obtain that $\xi_a(N, u_Q, \Gamma, \{a\}) = 1$.

Due to (3.2.1), we have

$$ATM_i(N, u_Q, \Gamma, \{a\}) = \begin{cases} 1, & i = a, \\ 0, & i \in N \setminus \{a\}. \end{cases}$$

Therefore, we have $\xi(N, u_Q, \Gamma, \{a\}) = ATM(N, u_Q, \Gamma, \{a\})$ for all $Q \in C^{\Gamma}(N)$ such that $Q \ni a$.

2). If $a \notin Q$ and $|Q| = 1$, then each $i \in N \setminus Q$ is a restricted null player.

For any $S \subseteq N \setminus \{i, a\}$ such that $S, S \cup \{i\} \in C^{\Gamma}(N)$, we have

$$u_Q(S \cup \{i\}) = u_Q(S) = \begin{cases} 1, & \text{if } Q \subseteq S, \\ 0, & \text{if } Q \nsubseteq S. \end{cases}$$

Note that the main player a is also a restricted null-player.

By the restricted null-player property, it follows that

$$\xi_i(N, u_Q, \Gamma, \{a\}) = 0, \quad \text{for all } i \in N \setminus Q.$$

Then, by efficiency, we obtain that $\xi_j(N, u_Q, \Gamma, \{a\}) = 1$, where $Q = \{j\}$.

By the definition of the average tree value (3.2.1), we obtain

$$ATM_i(N, u_Q, \Gamma, \{a\}) = \begin{cases} 1, & \text{if } i \in Q, \\ 0, & \text{if } i \in N \setminus Q. \end{cases}$$

Therefore, $\xi(N, u_Q, \Gamma, \{a\}) = ATM(N, u_Q, \Gamma, \{a\})$ for all $Q \in C^{\Gamma}(N)$ such that $Q \not\ni a$ and $|Q| = 1$.

3). If $a \notin Q$ and $|Q| \geq 2$, then there are two extreme players of Q, say $i, j \in Q$, such that $Q \setminus \{i, j\}$ is connected and $\{i, j\} \notin \Gamma$ if $|Q| > 2$, and let $Q = \{i, j\}$ if $|Q| = 2$, since $(Q, \Gamma(Q))$ is a subgraph of a cycle and is connected. Next, we show that each $k \in N \setminus \{i, j\}$ is a restricted null-player and i, j are symmetric veto players.

For any $k \in Q \setminus \{i, j\}$ and $S \subseteq N \setminus \{k, a\}$ such that $S, S \cup \{k\} \in C^{\Gamma}(N)$, we have

$$u_Q(S \cup \{k\}) = u_Q(S) = 0,$$

since both $Q \nsubseteq S$ and $Q \nsubseteq S \cup \{k\}$, while, for any $k \in N \setminus Q$ and $S \subseteq N \setminus \{k, a\}$ such that $S, S \cup \{k\} \in C^{\Gamma}(N)$, we have

$$u_Q(S \cup \{k\}) = u_Q(S) = \begin{cases} 1, & \text{if } Q \subseteq S, \\ 0, & \text{if } Q \nsubseteq S. \end{cases}$$

Hence, each $k \in N \setminus \{i, j\}$ is a restricted null-player.

By the restricted null-player property, it turns out that

$$\xi_k(N, u_Q, \Gamma, \{a\}) = 0, \quad \text{for all } k \in N \setminus \{i, j\}.$$

Moreover, each $k \in Q$ is a veto player in $(N, u_Q, \Gamma, \{a\})$, since $u_Q(S) = 0$ for all $S \in C^\Gamma(N \setminus \{k\})$.

The players i and j are symmetric veto players in $(N, u_Q, \Gamma, \{a\})$, since $u_Q(S) = 1$ for all $S \in C^\Gamma(N \setminus \{a\})$ satisfying $\{i, j\} \subseteq S$. Thus, by veto players equal treatment, we have

$$\xi_i(N, u_Q, \Gamma, \{a\}) = \xi_j(N, u_Q, \Gamma, \{a\}).$$

Combined with efficiency, it follows that

$$\xi_i(N, u_Q, \Gamma, \{a\}) = \xi_j(N, u_Q, \Gamma, \{a\}) = \frac{1}{2},$$

which shows that $\xi_i(N, u_Q, \Gamma, \{a\})$ is uniquely determined for all $i \in N$.

From (3.2.1), we have

$$ATM_k(N, u_Q, \Gamma, \{a\}) = \begin{cases} \frac{1}{2}, & \text{if } k \in \{i, j\}, \\ 0, & \text{if } k \in N \setminus \{i, j\}, \end{cases}$$

which coincides with ξ on $(N, u_Q, \Gamma, \{a\})$ where $Q \in C^\Gamma(N)$ is such that $a \notin Q$ and $|Q| \geq 2$.

Hence, for any $Q \in C^\Gamma(N)$, it holds that

$$\xi(N, u_Q, \Gamma, \{a\}) = ATM(N, u_Q, \Gamma, \{a\}).$$

Then according to Lemma 3.3.6 we complete the proof. □

The following example illustrates that the four axioms in Theorem 3.3.3 are logical independent.

Example 3.3.1. The following four values show the independence of efficiency, linearity, the restricted null-player property, and veto players equal treatment on the class of cycle graph games with unique main player.

(1) Let ξ^1 be an allocation rule on $\mathcal{G}_N^{\Gamma^{cc}, \mathcal{M}^1}$ given by

$$\xi^1_i(N, v, \Gamma, \{a\}) = 0, \quad \text{for all } i \in N.$$

This solution satisfies all four axioms except efficiency.

(2) Let ξ^2 be an allocation rule on $\mathcal{G}_N^{\Gamma^{cc},\mathcal{M}^1}$ given by

$$\xi_i^2(N, v, \Gamma, \{a\}) = \frac{v(N)}{n}, \quad \text{for all } i \in N.$$

This solution satisfies all four axioms except the restricted null-player property.

(3) Let ξ^3 be an allocation rule on $\mathcal{G}_N^{\Gamma^{cc},\mathcal{M}^1}$ given by

$$\xi^3(N, v, \Gamma, \{a\}) = m^T(N, v, \Gamma), \quad \text{for some } T \in \mathcal{T}_a^\Gamma(N).$$

This solution only fails veto players equal treatment because the two extreme nodes, say i and j, of some connected coalition $Q \subseteq N \setminus \{a\}$ with $|Q| \geq 2$ receive different payoffs in game $(N, u_Q, \Gamma, \{a\})$, that is $\xi_i^3(N, u_Q, \Gamma, \{a\}) = 1$ and $\xi_j^3(N, u_Q, \Gamma, \{a\}) = 0$ if $j \in S^T(i)$.

(4) Let ξ^4 be an allocation rule on $\mathcal{G}_N^{\Gamma^{cc},\mathcal{M}^1}$ given by

$$\xi^4(N, v, \Gamma, \{a\}) = \begin{cases} ATM(N, u_Q, \Gamma, \{a\}), & \text{if } v = u_Q \text{ for some } Q \in C^\Gamma(N), \\ \varphi(N, v, \Gamma, \{a\}), & \text{otherwise,} \end{cases}$$

where

$$\varphi_i(N, v, \Gamma, \{a\}) = \begin{cases} 0, & \text{if } i \in Nu(N, v, \Gamma, \{a\}), \\ \frac{v(N)}{n - |Nu(N,v,\Gamma,\{a\})|}, & \text{if } i \notin Nu(N, v, \Gamma, \{a\}), \end{cases}$$

in which $Nu(N, v, \Gamma, \{a\})$ is the set of restricted null-players in $(N, v, \Gamma, \{a\})$. This solution satisfies efficiency, the restricted null-player property, and veto players equal treatment, since both ATM and φ satisfy these properties. This solution fails linearity. For example, consider $v = u_{Q_1} + u_{Q_2}$ where $Q_1 \neq Q_2$ and $a \in Q_1 \cap Q_2$, then $\varphi_i(N, v, \Gamma, \{a\}) = \frac{2}{|Q_1 \cup Q_2|}$, for all $i \in Q_1 \cup Q_2$, while $ATM_i(N, u_{Q_k}, \Gamma, \{a\}) = 0$, for all $i \neq a$, $k \in \{1, 2\}$. It is easy to check that in general $\varphi_i(N, v, \Gamma, \{a\}) \neq ATM_i(N, u_{Q_1}, \Gamma, \{a\}) + ATM_i(N, u_{Q_2}, \Gamma, \{a\})$ for all $i \in Q_1 \cup Q_2$. Therefore, the solution ξ^4 does not satisfy linearity.

□

3.4 Application to the single allocation type model

In this section we introduce a classification of graph games with main players which depends on the positions of both the main players and the ordinary players. The idea is motivated by hub-spoke networks. The classical hub and spoke structure, called hub-spoke network, is usually described as that spokes have to be connected to hubs while hubs are fully connected, which means the subgraph related to the hubs is a complete graph. Considering the hubs as the main players and the spokes as the ordinary players, graph games with main players can be used to model cooperative situations with hub and spoke structure. Therefore, the average tree value for graph games with main players can be a solution to solve the allocation problem raised in hub-spoke networks.

Hub and spoke structures are widely used in practice, such as telecommunication systems, transportation systems, city planning, and the organization of a firm. For example, Matsubayashi et al. (2005) study the cost allocation problem, Lin (2013) investigates an airport privatization issue, and Adler (2005) analyzes the design issue of hub-spoke networks. According to the literature, there are two types of hub-spoke networks – single allocation and multiple allocation. In single allocation hub-spoke networks, each spoke is connected to exactly one hub, while multiple allocation networks allow spokes to be connected to more than one hub. Figure 3.2 illustrates a single allocation type of hub-spoke network.

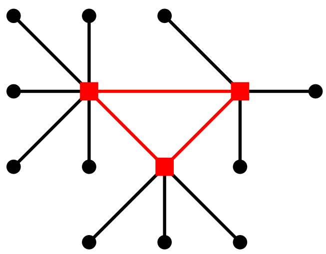


Figure 3.2: A single allocation type of hub-spoke network. The squares indicate the hubs and the circles indicate the spokes.

Similarly, graphs with main players may be of two types – with single and with multiple allocation of ordinary players to main players in the underlying graph, where an ordinary player is allocated to a main player if for some path between them no other main players belong to this path. More precisely, given a graph with main players (N, Γ, M) where $M \subseteq N$, player $i \in N \backslash M$ is *allocated* to main player $k \in M$ if there is a path $P = (i, \dots, k)$ in (N, Γ) such that $N(P) \cap (M \setminus \{k\}) = \emptyset$, where $N(P)$ is the set of nodes contained in path P. A connected graph with main players is of a single allocation type if each ordinary player is allocated to exactly one main player, otherwise the structure is of a multiple allocation type. For the single allocation type, we have the formal definition.

Definition 3.4.1. Let (N, Γ, M) be a structure where $(N, \Gamma) \in \mathit{\Gamma}_N^c$ is a connected graph and $M \subseteq N$ is the set of main players. Then (N, Γ, M) is of a *single allocation type* if $\{N_k : k \in M\}$ is a partition of N, where, for any $k \in M$, N_k consists of k and all $h \in N \setminus M$ such that h is allocated to k.

Figure 3.3 illustrates Definition 3.4.1. Note that the class of single allocation type of connected graphs with main players is more general than the class of single allocation type of hub-spoke networks. Firstly, the path between a main player and one of his allocated ordinary player may contain other ordinary players. Secondly, the subgraph with respect to the main players is not necessarily a complete graph.

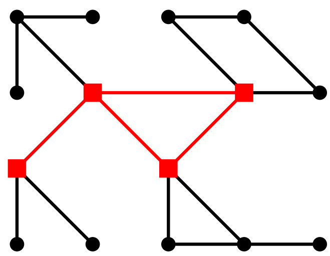


Figure 3.3: A single allocation type of connected graph with main players. The squares indicate the main players and the circles indicate the ordinary players.

In this section, we concentrate on the class of graph games with main players where the underlying structure is of a single allocation type. For this class of graph games with main players, we provide a two-step procedure to compute the average tree value.

To explain the two-step procedure, we first discuss the concept of coalition structures introduced in Aumann and Drèze (1974) and Owen (1977).

A *coalition structure* is a pair $(N, \mathcal{P})$, where N is the player set and $\mathcal{P} = \{N_1, \ldots, N_\ell\}$ is a partition of N, i.e., $N = \bigcup_{i=1}^{\ell} N_i$ and $N_k \cap N_h = \emptyset$ for $k \neq h$, and each N_k, $k = 1, \ldots, \ell$, is called a *union*. A triple $(N, v, \mathcal{P})$ constitutes a *game with coalition structure*, where $(N, v) \in \mathcal{G}_N$ is a TU game and $(N, \mathcal{P})$ is a coalition structure. Owen (1977) defines a game $v^{\mathcal{P}}$, called a *quotient game*, on

$L = \{1, \ldots, \ell\}$, in which each union N_h, $h \in L$, acts as a player, defined by

$$v^{\mathcal{P}}(Q) = v(\bigcup_{h \in Q} N_h), \quad \text{for all } Q \subseteq L.$$

Form Definition 3.4.1 we know that each single allocation type (N, Γ, M) induces a coalition structure $(N, \mathcal{P})$, where $\mathcal{P} = \{N_k : k \in M\}$ such that N_k consists of k and all $h \in N \setminus M$ allocated to k. Moreover, for any $i \in N$, let $k(i) \in M$ be such that $i \in N_{k(i)}$. According to this observation and the setting, we have the following theorem.

Theorem 3.4.1. *For any $(N, v, \Gamma, M) \in \mathcal{G}_N^{\Gamma c, \mathcal{M}}$, if (N, Γ, M) is of a single allocation type, it holds that*

$$ATM_i(N, v, \Gamma, M) = ATM_i(N_{k(i)}, \bar{v}_{N_{k(i)}}, \Gamma(N_{k(i)}), \{k(i)\}), \text{ for all } i \in N, \quad (3.4.1)$$

where, for every $k \in M$, the game $(N_k, \bar{v}_{N_k}) \in \mathcal{G}_{N_k}$ is given by

$$\bar{v}_{N_k}(S) = \begin{cases} v(S), & \text{if } S \subsetneq N_k, \\ AT_k(M, v^{\mathcal{P}}, \Gamma(M)), & \text{if } S = N_k. \end{cases}$$

Proof. This proof will be completed in two steps. First, we show that the total ATM payoff of each union is equal to his AT payoff on the corresponding graph quotient game. And then we prove that for each $k \in M$ the ATM payoffs of the players in N_k are the same in the original game (N, v, Γ, M) and in the graph game with one main player $(N_k, \bar{v}_{N_k}, \Gamma(N_k), \{k\})$.

Firstly, we prove

$$\sum_{i \in N_k} ATM_i(N, v, \Gamma, M) = AT_k(M, v^{\mathcal{P}}, \Gamma(M)), \text{ for all } k \in M. \quad (3.4.2)$$

Let g be a mapping from $\mathcal{T}_M^{\Gamma}(N)$ to $\mathcal{T}^{\Gamma}(M)$ such that, for every two main players $h, k \in M$ and $T \in \mathcal{T}_M^{\Gamma}(N)$,

$$h \in \bar{S}^{g(T)}(k) \text{ if and only if } h \in \bar{S}^{T}(k).$$

Then, for every $T \in \mathcal{T}_M^\Gamma(N)$ and $k \in M$, by (2.3.4), we have

$$\begin{aligned}
\sum_{i \in N_k} m_i^T(N, v, \Gamma) &= v(\bar{S}^T(k)) - \sum_{K \in S^T(k)/\Gamma : K \cap N_k = \emptyset} v(K) \\
&= v\Big(\bigcup_{h \in (\bar{S}^T(k) \cap M)} N_h\Big) - \sum_{K \in S^T(k)/\Gamma : K \cap N_k = \emptyset} v\Big(\bigcup_{h \in K \cap M} N_h\Big) \\
&= v\Big(\bigcup_{h \in \bar{S}^{g(T)}(k)} N_h\Big) - \sum_{K \in S^{g(T)}(k)/\Gamma(M)} v\Big(\bigcup_{h \in K} N_h\Big) \\
&= v^{\mathcal{P}}(\bar{S}^{g(T)}(k)) - \sum_{K \in S^{g(T)}(k)/\Gamma(M)} v^{\mathcal{P}}(K) \\
&= m_k^{g(T)}(M, v^{\mathcal{P}}, \Gamma(M)),
\end{aligned}$$

where the first two equalities follow from $N_h \cap M = \{h\}$ for all $h \in M$, the third equality follows from the definition of the mapping g, and the fourth equality follows from the definition of $v^{\mathcal{P}}$.

Note that $g(\mathcal{T}_M^\Gamma(N)) = \mathcal{T}^\Gamma(M)$, so some admissible rooted spanning trees of $\mathcal{T}_M^\Gamma(N)$ are mapped by g into the same admissible rooted spanning tree of $\mathcal{T}^\Gamma(M)$ and for any $T' \in \mathcal{T}^\Gamma(M)$ the number of admissible rooted spanning trees $T \in \mathcal{T}_M^\Gamma(N)$ that satisfy $g(T) = T'$ equals $|\mathcal{T}_M^\Gamma(N)|/|\mathcal{T}^\Gamma(M)|$. Then, for every $k \in M$, we have

$$\frac{1}{|\mathcal{T}_M^\Gamma(N)|} \sum_{T \in \mathcal{T}_M^\Gamma(N)} \sum_{i \in N_k} m_i^T(N, v, \Gamma) = \frac{1}{|\mathcal{T}^\Gamma(M)|} \sum_{T' \in \mathcal{T}^\Gamma(M)} m_k^{T'}(M, v^{\mathcal{P}}, \Gamma(M)).$$

By the definitions of the average tree values as in (2.3.3) and in (3.2.1), we obtain (3.4.2).

Secondly, for every $k \in M$, let $\bar{v}_{N_k}(S) = v(S)$ for $S \subsetneq N_k$ and $\bar{v}_{N_k}(N_k) = AT_k(M, v^{\mathcal{P}}, \Gamma(M))$.

Take any $k \in M$. Let f be a mapping from $\mathcal{T}_M^\Gamma(N)$ to $\mathcal{T}_k^\Gamma(N_k)$ such that, for any two players $i, j \in N_k$ and $T \in \mathcal{T}_M^\Gamma(N)$,

$$i \in \bar{S}^{f(T)}(j) \text{ if and only if } i \in \bar{S}^T(j).$$

Then, for every $T \in \mathcal{T}_M^\Gamma(N)$ and $i \in N_k \setminus \{k\}$, we have

$$\begin{aligned} m_i^T(N, v, \Gamma) &= v(\bar{S}^T(i)) - \sum_{K \in S^T(i)/\Gamma} v(K) \\ &= \bar{v}_{N_k}(\bar{S}^T(i)) - \sum_{K \in S^T(i)/\Gamma} \bar{v}_{N_k}(K) \\ &= \bar{v}_{N_k}(\bar{S}^{f(T)}(i)) - \sum_{K \in S^{f(T)}(i)/\Gamma} \bar{v}_{N_k}(K) \\ &= m_i^{f(T)}(N_k, \bar{v}_{N_k}, \Gamma(N_k)). \end{aligned}$$

Again, some admissible rooted spanning trees of $\mathcal{T}_M^\Gamma(N)$ are mapped by f into the same admissible rooted spanning tree of $\mathcal{T}_k^\Gamma(N_k)$ and for any $T' \in \mathcal{T}_k^\Gamma(N_k)$ the number of admissible rooted spanning trees $T \in \mathcal{T}_M^\Gamma(N)$ that satisfy $f(T) = T'$ equals $|\mathcal{T}_M^\Gamma(N)|/|\mathcal{T}_k^\Gamma(N_k)|$. Therefore, for every $i \in N_k \setminus \{k\}$, we have

$$\frac{1}{|\mathcal{T}_M^\Gamma(N)|} \sum_{T \in \mathcal{T}_M^\Gamma(N)} m_i^T(N, v, \Gamma) = \frac{1}{|\mathcal{T}_k^\Gamma(N_k)|} \sum_{T' \in \mathcal{T}_k^\Gamma(N_k)} m_i^{T'}(N_k, \bar{v}_{N_k}, \Gamma(N_k)).$$

From (3.2.1), it follows that

$$ATM_i(N, v, \Gamma, M) = ATM_i(N_k, \bar{v}_{N_k}, \Gamma(N_k), \{k\}), \text{ for all } i \in N_k \setminus \{k\}. \quad (3.4.3)$$

Since $(N_k, \Gamma(N_k))$ is connected, we have

$$\sum_{i \in N_k} ATM_i(N_k, \bar{v}_{N_k}, \Gamma(N_k), \{k\}) = \bar{v}_{N_k}(N_k).$$

From the last two equalities, we obtain

$$ATM_k(N, v, \Gamma, M) = ATM_k(N_k, \bar{v}_{N_k}, \Gamma(N_k), \{k\}),$$

which completes the proof. □

Theorem 3.4.1 provides a two-step approach to calculate the average tree value on a special class of graph games with main players.

Next, we discuss a special case of a single allocation type, which is motivated by the requirement of a hub-spoke network that the subgraph on the set of hubs is complete. Together with the work of Herings et al. (2010) that the average tree value of a complete graph coincides with the Shapley value, we have the following result from Theorem 3.4.1.

Corollary 3.4.1. *For any* $(N, v, \Gamma, M) \in \mathcal{G}_N^{\Gamma c, \mathcal{M}}$, *if* (N, Γ, M) *is of a single allocation type with* $(M, \Gamma(M))$ *being a complete graph, then it holds that*

$$ATM_i(N, v, \Gamma, M) = ATM_i(N_{k(i)}, \bar{v}_{N_{k(i)}}, \Gamma(N_{k(i)}), \{k(i)\}), \;\; \textit{for all } i \in N,$$

where, for every $k \in M$, *the game* $(N_k, \bar{v}_{N_k}) \in \mathcal{G}_{N_k}$ *is given by*

$$\bar{v}_{N_k}(S) = \begin{cases} v(S), & \textit{if } S \subsetneq N_k, \\ Sh_k(M, v^{\mathcal{P}}), & \textit{if } S = N_k. \end{cases}$$

If the underlying structure of a graph game with main players is a classical single allocation hub-spoke network, where the subgraph with respect to main players is a complete graph and each ordinary player connects to a main player as his unique neighbor (see Figure 3.2), then we have the following result.

Corollary 3.4.2. *For any* $(N, v, \Gamma, M) \in \mathcal{G}_N^{\Gamma c, \mathcal{M}}$, *if* (N, Γ, M) *is of a single allocation type with* $(M, \Gamma(M))$ *being a complete graph and* $\{i, j\} \in \Gamma$ *and* $i \in N \setminus M$ *imply* $j \in M$, *then it holds that*

$$ATM_i(N, v, \Gamma, M) = \begin{cases} v(\{i\}), & \textit{if } i \in N \setminus M, \\ Sh_i(M, v^{\mathcal{P}}) - \sum_{j \in N_i \setminus \{i\}} v(\{j\}), & \textit{if } i \in M. \end{cases}$$

Chapter 4

The average tree value for hypergraph games

4.1 Introduction

A hypergraph communication structure consists of a set of nodes and a set of hyperlinks defined on the node set, where each hyperlink contains at least two nodes which can represent a club, a conference, an organization, or a committee. In cooperative perspective, we usually assume that all players in a hyperlink have to be present before communication can take place. Hypergraph communication structure, as a natural extension of undirected graph communication, is first studied in Myerson (1980). In Myerson (1980), a conference structure is proposed by assuming that each conference is a group of players containing at least two players, similar to a hyperlink in a hypergraph structure. In this sense, a conference structure is equivalent to a hypergraph. Cooperative games with hypergraph communication structure, or shortly hypergraph games, are formally introduced in van den Nouweland et al. (1992). Even though the class of hypergraph games was proposed more than two decades ago, the study of allocation rules for hypergraph games is rather limited compared with graph games.

Algaba et al. (2004) discuss the relationship between games on union stable systems and games with hypergraph communication structure. They point out that the collection of connected coalitions induced by a hypergraph game forms a union stable system, and on the other hand, a union stable system may correspond to several hypergraph structures. However, the reason causing the difference between the two structures is due to the diversity of hypergraphs that a hyperlink may be a proper subset of another hyperlink and the different treatment of individual players that every individual player is a connected (feasible) coalition. For any union stable system we can determine a unique hypergraph with minimal hyperlinks which corresponds to the set of supports of this union stable system in which all the sizes of supports are greater than or equal to two.

This chapter continues studying the average tree value. As mentioned in the previous chapter, the average tree value is proposed in Herings et al. (2008) for cycle-free graph games and then generalized to the class of all graph games in Herings et al. (2010) as well as in Baron et al. (2011). Later on, the solution is studied on the class of games with permission tree structure in van den Brink et al. (2015). Different from some solutions defined on the marginal contributions associated to linear orderings, the average tree value takes into account the partial orderings which are derived from the underlying structure. As an important solution concept in the field of games with communication structure, the average tree value is worthwhile being studied on the class of hypergraph games.

In this chapter we first focus on proposing the concept of the average tree value on the class of all hypergraph games, which is a generalization from graph games to hypergraph games. Because the class of graph games is a special case of the class of hypergraph games, the average tree value for hypergraph games reduces to the average tree value for graph games if the underlying hypergraph is a graph. Secondly, we study characterizations of the average tree value for cycle-free hypergraph games, hypertree games, and cycle hypergraph games.

On the class of cycle-free hypergraph games, the average tree value can be characterized by component efficiency and component fairness. Component fairness includes the case of cycle-free graph games and says that after deleting a hyperlink the average payoff difference is the same for every induced component. When a hyperlink is deleted, more than two components may result, while in a cycle-free graph always two components are obtained. An alternative characterization on the class of hypertree games is provided by efficiency, linearity, the restricted null-player property, weak symmetry, and independence in hypertree unanimity games. For hypergraph games with cycles, we characterize the average tree value on the class of cycle hypergraph games. The approach is similar to the case of hypertree games by applying efficiency, linearity, the restricted null-player property, and two other axioms, symmetry in cycle unanimity games and independence in cycle unanimity games.

This chapter is based on a working paper Kang et al. (2018a) and the rest of this chapter is organized as follows. In Section 4.2 we first define the class of admissible collections of coalitions on hypergraph structures and then, with the help of admissible collections of coalitions, propose the average tree value for hypergraph games. Section 4.3 contains four subsections and deals with characterizations of the average tree value. The average tree value is characterized in Subsection 4.3.1 for cycle-free hypergraph games, in Subsection 4.3.2 for hypertree games, and in Subsection 4.3.3 for cycle hypergraph games. The logical independence among the axioms in the provided characterizations is analyzed in Subsection 4.3.4.

4.2 Generalization of the average tree value

In this section, we introduce the average tree value for hypergraph games. The solution is in the spirit of Herings et al. (2010), where the average tree value is defined for graph games.

In games with hypergraph communication structure, it is assumed that only connected coalitions can be formed and realize their worth. This motivates us to investigate the structures of connected coalitions and hypergraphs. We notice that, except singleton players, any connected coalition is a hyperlink or the union of some hyperlinks. For a connected hypergraph $(N, H) \in \mathcal{H}_N^c$ and a node $r \in N$, when all hyperlinks containing this node are removed, the resulting hypergraph (N, H_{-r}) consists of components, each component containing a node adjacent to r. Then we may remove all hyperlinks containing these nodes, and so on, until no hyperlinks are left. Based on this observation we propose the average tree value for hypergraph games by deleting hyperlinks in a specific way. Before defining the value, we first define admissible collections of coalitions in a given hypergraph.

Definition 4.2.1. Given a hypergraph $(N, H) \in \mathcal{H}_N$ and component $K \in N/H$, a collection of coalitions $B = \{B_i : i \in K\}$ is *admissible* for K if it satisfies:

(1) For each $i \in K$, it holds that $B_i \subseteq K$, $i \in B_i$, and for some $r \in K$, it holds that $B_r = K$;

(2) For each $i \in K$ and component $K' \in (B_i \setminus \{i\})/H$, it holds that $K' = B_j$ and $\{i, j\} \subseteq e$ for some $j \in K$ and $e \in H(B_i)$.

Similar to Definition 1 in Herings et al. (2010) for a graph, the definition states that for an admissible B for a component $K \in N/H$ in a hypergraph $(N, H) \in \mathcal{H}_N$, any B_i, $i \in K$, is a connected coalition of K due to $B_i = K$ or $B_i \in (B_j \setminus \{j\})/H$ for some $j \in K$. If $|B_i| \geq 2$, each component of $B_i \setminus \{i\}$ is equal to B_h for some $h \in K$ satisfying $\{i, h\} \subseteq e$ for some $e \in H(B_i)$.

Moreover, we obtain the following properties of admissible collections of coalitions for hypergraphs, which is similar to Lemma 1 in Herings et al. (2010) for graph structures.

Lemma 4.2.1. *For a given hypergraph $(N, H) \in \mathcal{H}_N$ and component $K \in N/H$, let $B = \{B_i : i \in K\}$ be an admissible collection of coalitions for K. Then the following properties hold:*

(1) There exists a unique player $r \in K$ such that $B_r = K$;

(2) For all $i, j \in K$, $i \neq j$, it holds that either $B_i \subseteq B_j \setminus \{j\}$, or $B_j \subseteq B_i \setminus \{i\}$, or both $B_i \cap B_j = \emptyset$ and $B_i \cup B_j$ is not connected;

(3) The digraph (K, t^B) is a rooted tree on K, where $t^B = \{(i, j) : B_j \in (B_i \setminus \{i\})/H, i \in K\}$ satisfying that $\{i, j\} \subseteq e$ for some $e \in H(B_i)$ if $(i, j) \in t^B$.

Proof. By condition (1) of Definition 4.2.1 it holds that $B_r = K$ for some $r \in K$. From condition (2) of Definition 4.2.1 it follows that for every $K' \in (B_r \setminus \{r\})/H$ there exists $i \in K$ and $e \in H$ such that $K' = B_i$ and $\{r, i\} \subseteq e$, which leads to the arc (r, i) of t^B. Then we continue this procedure with every i chosen in the previous step for which $|B_i| \geq 2$ and proceed in this way until all remaining B_j, $j \in K$, are singletons. Then (K, t^B) is a rooted tree satisfying the conditions in (3). Moreover, the root r of (K, t^B) is the unique node satisfying $B_r = K$, which shows (1).

To prove (2), consider two nodes $i, j \in K$. Because (K, t^B) is a rooted tree, either $B_i \subseteq B_j \setminus \{j\}$, or $B_j \subseteq B_i \setminus \{i\}$, or $B_i \cap B_j = \emptyset$. So, in the last case we need to show that $B_i \cup B_j$ is not connected. Let B_h, $h \in K$ and $h \neq i, j$, be the minimal set in B satisfying $B_i \subseteq B_h$ and $B_j \subseteq B_h$. Since $B_i \cap B_j = \emptyset$, it follows that B_i and B_j belong to different components of $(B_h \setminus \{h\})/H$, which implies that $B_i \cup B_j$ is not connected. $\square$

Given an admissible collection of coalitions $B = \{B_i : i \in K\}$ for a component $K \in N/H$, the unique player $r \in K$ in property (1) satisfying $B_r = K$ is called the *top-player* for B. Property (3) states that B induces a rooted tree (K, t^B) on K. This rooted tree satisfies that, for every arc $(i,j) \in t^B$, both i and j belong to at least one of the hyperlinks in $(K, H(K))$. In this sense, the set of players B_i, for any $i \in K$, can be regarded as the set of subordinates of player i in B including himself. Moreover, if $r \in K$ is the top-player for B, then r is the root of rooted tree (K, t^B). From property (2) we see that each admissible collection of coalitions induces a hierarchical structure, and therefore, compared with a permutation on the same node set, an admissible collection of coalitions can be interpreted as a partial ordering. Property (3) further implies that if $i \in e$ and $e \subseteq B_i$ for some $i \in K$ and $e \in H(B_i)$, then all the other nodes in e become i's immediate successors, i.e., $(i,j) \in t^B$ for all $j \in e \setminus \{i\}$, and therefore, for any $j_1, j_2 \in e \setminus \{i\}$, (j_1, j_2) cannot be an arc of t^B.

The following example illustrates both Definition 4.2.1 and Lemma 4.2.1.

Example 4.2.1. Let $(N, H) \in \mathcal{H}_N$ be a hypergraph given by $N = \{1, \ldots, 8\}$ and $H = \{e_1, e_2 \ldots, e_5\}$, where $e_1 = \{1,2,3\}, e_2 = \{3,4,7\}, e_3 = \{1,5,6\}, e_4 = \{5,6,7\}$, and $e_5 = \{7,8\}$, as depicted in Figure 4.1.

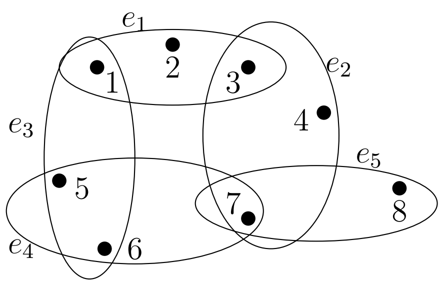


Figure 4.1: Hypergraph (N, H) in Example 4.2.1.

Note that (N, H) is connected. There are three admissible collections of coalitions for which node 1 is the top-player, given by B^1, B^2, and B^3, where $B^1_1 = B^2_1 = B^3_1 = N$, $B^1_2 = \{2\}$, $B^1_3 = \{3,4,5,6,7,8\}$, $B^1_4 = \{4\}$, $B^1_5 = \{5\}$, $B^1_6 = \{6\}$, $B^1_7 = \{5,6,7,8\}$, $B^1_8 = \{8\}$; $B^2_2 = \{2\}$, $B^2_3 = \{3\}$, $B^2_4 = \{4\}$, $B^2_5 = \{3,4,5,6,7,8\}$, $B^2_6 = \{6\}$, $B^2_7 = \{3,4,7,8\}$, $B^2_8 = \{8\}$; $B^3_2 = \{2\}$, $B^3_3 = \{3\}$, $B^3_4 = \{4\}$, $B^3_5 = \{5\}$, $B^3_6 = \{3,4,5,6,7,8\}$, $B^3_7 = \{3,4,7,8\}$, $B^3_8 = \{8\}$. The induced rooted trees (N, t^{B^1}), (N, t^{B^2}), and (N, t^{B^3}) are depicted in Figure 4.2. □

For a hypergraph $(N, H) \in \mathcal{H}_N$ and $K \in N/H$, we denote $\mathcal{B}^H(K)$ as the set of admissible collections of coalitions on component K. The following lemma shows a property for admissible collections of coalitions on cycle-free hypergraphs.

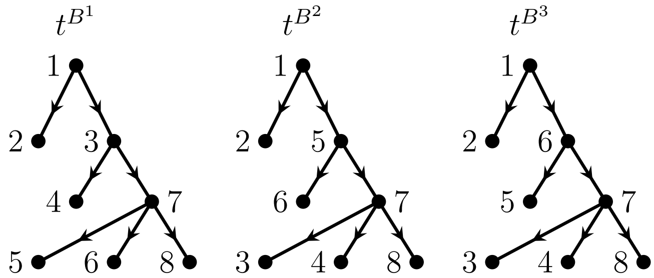


Figure 4.2: The induced rooted trees having node 1 as the top-player in Example 4.2.1.

Lemma 4.2.2. *Let $(N, H) \in \mathcal{H}_N^{cf}$ be a cycle-free hypergraph, then for any $K \in N/H$ and $r \in K$ there exists a unique $B \in \mathcal{B}^H(K)$ such that $B_r = K$. Moreover, $|\mathcal{B}^H(K)| = |K|$ for all $K \in N/H$.*

Proof. We prove the statement by contradiction. Let $B, B' \in \mathcal{B}^H(K)$ be two distinct admissible collections of coalitions on K having $r \in K$ as their top-player, i.e., $B_r = B'_r = K$. Then there exists $k \in K \setminus \{r\}$ such that $B_k \neq B'_k$.

According to condition (2) of Definition 4.2.1, there exists a chain $C_B = (i_1, e_1, \ldots, e_{t-1}, i_t)$ between $i_1 = r$ and $i_t = k$, such that $B_{i_{h+1}} \in (B_{i_h} \setminus \{i_h\})/H$ and $\{i_h, i_{h+1}\} \subseteq e_h$ for $h = 1, \ldots, t-1$. Similarly, there exists a chain $C_{B'} = (j_1, f_1, \ldots, f_{t'-1}, j_{t'})$ between $j_1 = r$ and $j_{t'} = k$ such that $B_{j_{h+1}} \in (B_{j_h} \setminus \{j_h\})/H$ and $\{j_h, j_{h+1}\} \subseteq f_h$ for $h = 1, \ldots, t'-1$. Because $B_k \neq B'_k$, it holds that $C_B \neq C_{B'}$. Let $C'_B = (i_\ell, e_\ell, \ldots, e_{t-1}, i_t)$ and $C'_{B'} = (j_{\ell'}, f_{\ell'}, \ldots, f_{t'-1}, j_{t'})$ be the shortest sub-chains of C_B and $C_{B'}$ with $i_t = j_{t'} = k$, respectively, such that either (a) $e_\ell = f_{\ell'}$ and the nodes $i_{\ell+1}, \ldots, i_{t-1}, j_{\ell'+1}, \ldots, j_{t'-1}$ are all different, or (b) except for $i_\ell = j_{\ell'}$ and k, all other nodes and all hyperlinks in C'_B are different from all other nodes and all hyperlinks in $C'_{B'}$. In case (a) the chain $(k, e_{t-1}, \ldots, e_{\ell+1}, i_{\ell+1}, e_\ell, j_{\ell'+1}, f_{\ell'+1}, \ldots, f_{t'-1}, k)$ is a cycle in (N, H) and in case (b) the chain $(i_\ell, e_\ell, \ldots, e_{t-1}, k, f_{t'-1}, \ldots, f_{\ell'}, j_{\ell'})$ is a cycle in (N, H), both contradicting that (N, H) is cycle-free. Hence, for each $r \in K$, there is unique $B \in \mathcal{B}^H(K)$ such that $B_r = K$. This also implies that $|\mathcal{B}^H(K)| = |K|$ for all $K \in N/H$. □

In order to introduce the average tree value on hypergraph games, we first define the concept of marginal contributions for hypergraph games corresponding to an admissible collection of coalitions.

Definition 4.2.2. For any hypergraph game $(N, v, H) \in \mathcal{G}_N^{\mathcal{H}}$, and component $K \in N/H$, the *marginal contribution* of player $i \in K$ corresponding to $B \in \mathcal{B}^H(K)$ is given by

$$m_i^B(N, v, H) = v(B_i) - \sum_{K' \in (B_i \setminus \{i\})/H} v(K'). \tag{4.2.1}$$

The marginal contribution of a player $i \in K$ corresponding to some $B \in \mathcal{B}^H(K)$ of a hypergraph game $(N, v, H) \in \mathcal{G}_N^{\mathcal{H}}$ in component $K \in N/H$ is equal to the worth of connected coalition B_i minus the sum of the worths of all the components of the set of his subordinates $B_i \setminus \{i\}$.

Based on the marginal contributions as defined in (4.2.1), we define the average tree value for hypergraph games as follows.

Definition 4.2.3. On the class of hypergraph games, the *average tree value* assigns to every $(N, v, H) \in \mathcal{G}_N^{\mathcal{H}}$, a payoff vector $AT(N, v, H)$ given by

$$AT_i(N, v, H) = \frac{1}{|\mathcal{B}^H(K)|} \sum_{B \in \mathcal{B}^H(K)} m_i^B(N, v, H), \quad i \in K,\ K \in N/H. \tag{4.2.2}$$

The average tree value assigns to every player the average of his marginal contributions corresponding to all admissible collections of coalitions in the component he belongs to.

From Definition 4.2.1 and Definition 4.2.3, we can easily check that if the underlying hypergraph is a graph the proposed solution for hypergraph games reduces to the average tree value for graph games as in Herings et al. (2010). Hence, the extension of the average tree value is realized from graph games to hypergraph games.

In order to illustrate the average tree value for hypergraph games, we take the following example.

Example 4.2.2. Consider the hypergraph game $(N, v, H) \in \mathcal{G}_N^{\mathcal{H}}$, where $N = \{1, 2, 3, 4, 5\}$ is the player set, v is the unanimity game $u_{\{1,5\}}$, and $H = \{e_1, e_2, e_3\}$ is the set of hyperlinks, where $e_1 = \{1, 2, 3\}, e_2 = \{2, 3, 4\}, e_3 = \{4, 5\}$. The hypergraph is displayed in Figure 4.3.

There are eight admissible collections of coalitions. Let $\{B^1, \ldots, B^8\}$ be the set of admissible collections of coalitions in (N, H) satisfying $B_1^1 = B_1^2 = B_2^3 = B_3^4 = B_4^5 = B_4^6 = B_5^7 = B_5^8 = N$, $B_2^1 = B_3^2 = N \setminus \{1\}$, and $B_2^5 = B_3^6 = B_2^7 =$

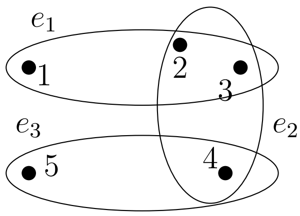


Figure 4.3: The underlying hypergraph in Example 4.2.2.

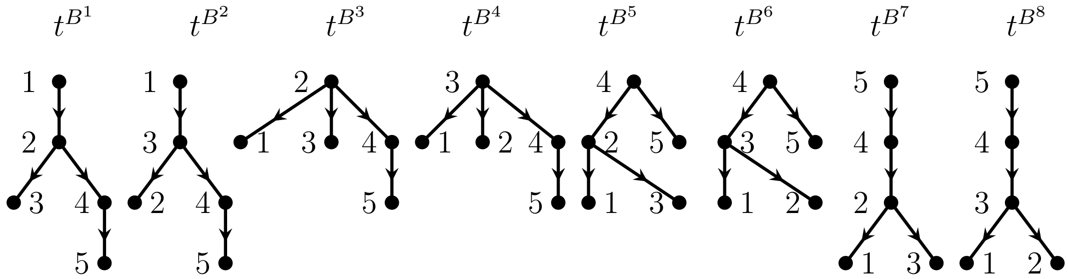


Figure 4.4: The induced rooted trees in Example 4.2.2.

$B_3^8 = \{1, 2, 3\}$. Their corresponding induced rooted trees are displayed in Figure 4.4. From (4.2.1) we obtain

$$
\begin{aligned}
m^{B^1}(N, v, H) &= m^{B^2}(N, v, H) = (1, 0, 0, 0, 0),\\
m^{B^3}(N, v, H) &= (0, 1, 0, 0, 0),\\
m^{B^4}(N, v, H) &= (0, 0, 1, 0, 0),\\
m^{B^5}(N, v, H) &= m^{B^6}(N, v, H) = (0, 0, 0, 1, 0),\\
m^{B^7}(N, v, H) &= m^{B^8}(N, v, H) = (0, 0, 0, 0, 1).
\end{aligned}
$$

Finally, from (4.2.2), it follows that

$$AT(N, v, H) = (\frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{4}).$$

□

Now we study the stability of the average tree value for hypergraph games. The following theorem shows that on the class of cycle-free hypergraph games if the underlying game is superadditive then the average tree value is an element of the core.

Theorem 4.2.1. *For any $(N, v, H) \in \mathcal{G}_N^{\mathcal{H}^{cf}}$, if (N, v) is superadditive, then it holds that $AT(N, v, H) \in C(N, v, H)$.*

Proof. If (N, v) is superadditive, then it holds that

$$v(S \cup T) \geq v(S) + v(T), \text{ for all } S, T \in 2^N \setminus \{\emptyset\} \text{ and } S \cap T = \emptyset,$$

which is equivalent to

$$v(S \cup \bigcup_{t=1}^{m} T_t) \geq v(S) + \sum_{t=1}^{m} v(T_t), \tag{4.2.3}$$

where $\{T_1, \ldots, T_m\}$ is a partition of T, i.e., $T = \bigcup_{t=1}^{m} T_t$ and, for all $i \neq j$, $T_i \cap T_j = \emptyset$.

Take any $S \in C^H(K)$ and $B \in \mathcal{B}^H(K)$ for some $K \in N/H$. We first show that there exists a player $i \in S$ satisfying $S \subseteq B_i$. If not, then there exist some players in S, say $\{j_1, \ldots, j_k\}$, such that $S \subseteq \bigcup_{t=1}^{k} B_{j_t}$ and $B_{j_t} \cap B_{j_{t'}} = \emptyset$, for all $t \neq t'$, which contradicts that S is a connected coalition of a cycle-free hypergraph.

Let $I(S) = \{j \in B_i \setminus S : (h, j) \in t^B, h \in S\}$ be the set of immediate successors of S in the tree (K, t^B). Since (N, H) is cycle-free, it holds that S is connected in (K, t^B) and that B_i is partitioned by the sets S and B_j, $j \in I(S)$. Therefore, we have

$$\begin{aligned}
\sum_{h \in S} m_h^B(N, v, H) &= \sum_{h \in S} \big(v(B_h) - \sum_{K' \in (B_h \setminus \{h\})/H} v(K')\big) \\
&= \sum_{h \in B_i \setminus (\bigcup_{j \in I(S)} B_j)} \big(v(B_h) - \sum_{K' \in (B_h \setminus \{h\})/H} v(K')\big) \\
&= v(B_i) - \sum_{j \in I(S)} v(B_j) \\
&= v(S \cup (\bigcup_{j \in I(S)} B_j)) - \sum_{j \in I(S)} v(B_j) \\
&\geq v(S),
\end{aligned}$$

where the first equality follows from (4.2.1), the second and fourth equalities follow from the fact that $B_i = S \cup (\bigcup_{j \in I(S)} B_j)$, the third equality follows because all terms for $h \in S \setminus \{i\}$ cancel, and the inequality follows from (4.2.3).

Combined with $\sum_{h \in K} m_h^B(N, v, H) = v(K)$, it follows that $(m_i^B(N, v, H))_{i \in K} \in C(K, v_K, H(K))$ for any $B \in \mathcal{B}^H(K)$ and $K \in N/H$. Therefore, from Definition 4.2.3, we have $(AT_i(N, v, H))_{i \in K} \in C(K, v_K, H(K))$ since $C(K, v_K, H(K))$ is a convex set and $(AT_i(N, v, H))_{i \in K}$ is a convex combination of $(m_i^B(N, v, H))_{i \in K}$

over all $B \in \mathcal{B}^H(K)$.

Finally, because the payoffs in a component by using the average tree value do not affect the payoffs in other components, it follows that $AT(N, v, H) \in C(N, v, H)$, which completes the proof. □

4.3 Axiomatizations

This section aims to provide characterizations of the average tree value for hypergraph games. In the first three subsections, we characterize the average tree value on the class of cycle-free hypergraph games, on the class of hypertree games, and on the class of cycle hypergraph games, respectively. The studies in this section generalize the works in Herings et al. (2008), Mishra and Talman (2010), and Selçuk et al. (2013), regarding the average tree value for graph games. In the fourth subsection, the logical independence of the axiomatizations is shown.

4.3.1 Cycle-free hypergraph games

In this subsection, we show that on the class of cycle-free hypergraph games the average tree value can be characterized by component efficiency and component fairness, the latter being generalized from graph games to hypergraph games.

Before presenting the characterization, we introduce a simple expression of the average tree value, which can be considered as an extension from cycle-free graph games in Herings et al. (2008) to cycle-free hypergraph games. From Lemma 4.2.2, the average tree value for cycle-free hypergraph games can be simplified to the following form. For any $(N, v, H) \in \mathcal{G}_N^{\mathcal{H}^{cf}}$, we have

$$AT_i(N, v, H) = \frac{1}{|K|} \sum_{r \in K} m_i^{B^r}(N, v, H), \quad i \in K,\ K \in N/H, \tag{4.3.1}$$

where, for every $r \in K$, $B^r = \{B_i^r : i \in K\}$ is the unique element in $\mathcal{B}^H(K)$ such that $B_r^r = K$.

Let ξ be a solution defined on $\mathcal{G}_N^{\mathcal{H}^{cf}}$. We discuss the axioms related to the average tree value for cycle-free hypergraph games. For component efficiency, we refer to Section 2.3. The following axiom is a generalization of component fairness introduced in Herings et al. (2008) from cycle-free graph games to cycle-free hypergraph games.

Component fairness: For any cycle-free hypergraph game $(N, v, H) \in \mathcal{G}, \mathcal{G} \subseteq \mathcal{G}_N^{\mathcal{H}}$, and hyperlink $e \in H$, it holds that

$$\begin{aligned} &\frac{1}{|K^1|} \sum_{h \in K^1} \big(\xi_h(N, v, H) - \xi_h(N, v, H \setminus \{e\})\big) \\ = \; &\frac{1}{|K^2|} \sum_{h \in K^2} \big(\xi_h(N, v, H) - \xi_h(N, v, H \setminus \{e\})\big), \end{aligned}$$

for every $K^1, K^2 \in N/(H \setminus \{e\})$ satisfying $K^1 \cap e \neq \emptyset$ and $K^2 \cap e \neq \emptyset$.

Component fairness for cycle-free hypergraph games says that if a hyperlink is broken, the average payoff difference is the same for each induced component. Note that this axiom is more general than the one in Herings et al. (2008), because in a cycle-free hypergraph more than two components may be induced by breaking a hyperlink, while in a cycle-free graph always two components are induced.

Based on the above axiom and component efficiency, we get a characterization of the average tree value for cycle-free hypergraph games. First, we show that the average tree value satisfies component efficiency and component fairness.

Lemma 4.3.1. *On the class of cycle-free hypergraph games, the average tree value satisfies component efficiency and component fairness.*

Proof. Take any cycle-free hypergraph game $(N, v, H) \in \mathcal{G}_N^{\mathcal{H}^{cf}}$ and component $K \in N/H$. By Lemma 4.2.2, for every $r \in K$ there is exactly one admissible collection of coalitions $B^r = \{B_h^r : h \in K\}$ in $\mathcal{B}^H(K)$ such that $B_r^r = K$. Then, by Definition 4.2.2, for every $r \in K$ it holds that

$$\sum_{h \in K} m_h^{B^r}(N, v, H) = \sum_{h \in B_r^r} m_h^{B^i}(N, v, H) = v(B_r^r) = v(K).$$

Hence, we have

$$\begin{aligned} \sum_{h \in K} AT_h(N, v, H) &= \sum_{h \in K} \frac{1}{|K|} \sum_{r \in K} m_h^{B^r}(N, v, H) \\ &= \frac{1}{|K|} \sum_{r \in K} \sum_{h \in K} m_h^{B^r}(N, v, H) \\ &= \frac{1}{|K|} \sum_{r \in K} v(K) \; = \; v(K). \end{aligned}$$

Therefore, the average tree value satisfies component efficiency on $\mathcal{G}_N^{\mathcal{H}^{cf}}$.

To show component fairness, for a given $(N, v, H) \in \mathcal{G}_N^{\mathcal{H}^{cf}}$ and $K \in N/H$, let $e \in H(K)$, with $|e| = m$. Note that $m \geq 2$. Since hypergraph (N, H) is cycle-free, by deleting e, the component K is split into m components in the resulting hypergraph $(N, H \setminus \{e\})$, denoted by $\{K^1, \ldots, K^m\}$, which is a partition of K. Take any $p \in \{1, \ldots, m\}$. Then for every $r \in K$ we have

$$\sum_{h \in K^p} m_h^{B^r}(N, v, H) = v(K^p), \quad \text{if } r \notin K^p,$$

and

$$\sum_{h \in K^p} m_h^{B^r}(N, v, H) = v(K) - \sum_{q \in \{1,\ldots,m\} \setminus \{p\}} v(K^q), \quad \text{if } r \in K^p.$$

Since each player in K is only once a top-player in component K, the number of top-players $r \notin K^p$ for which coalition K^p receives total payoff $v(K^p)$ in the corresponding marginal contributions is equal to $|K \setminus K^p|$ and the number of top-players $r \in K^p$ for which coalition K^p receives total payoff $v(K) - \sum_{q \neq p} v(K^q)$ in the corresponding marginal contributions is equal to $|K^p|$.

Consequently, we have

$$\sum_{h \in K^p} AT_h(N, v, H) = \frac{|K^p|\big(v(K) - \sum_{q \neq p} v(K^q)\big) + |K \setminus K^p| v(K^p)}{|K|}.$$

Therefore, we obtain that

$$\frac{1}{|K^p|}\big(\sum_{h \in K^p} AT_h(N, v, H) - v(K^p)\big) = \frac{v(K) - \sum_{q=1}^m v(K^q)}{|K|}.$$

Since the average tree value satisfies component efficiency, it holds that

$$v(K^p) = \sum_{h \in K^p} AT_h(N, v, H \setminus \{e\}),$$

from which it follows that

$$\frac{1}{|K^p|}\big(\sum_{h \in K^p} AT_h(N, v, H) - AT_h(N, v, H \setminus \{e\})\big) = \frac{v(K) - \sum_{q=1}^m v(K^q)}{|K|},$$

which is the same for every $p \in \{1, \ldots, m\}$. Therefore, the average tree value satisfies component fairness on $\mathcal{G}_N^{\mathcal{H}^{cf}}$. $\square$

Before we give a characterization of the average tree value, we need a property about cycle-free hypergraphs, which is also a new property in hypergraph theory.

Lemma 4.3.2. *For any component $K \in N/H$ of a cycle-free hypergraph $(N, H) \in \mathcal{H}_N^{cf}$ with $|K| \geq 2$, it holds that $\sum_{e \in H(K)}(|e| - 1) = |K| - 1$.*

Proof. We prove the statement by induction on $|H(K)|$. If $|H(K)| = 1$, i.e., $H(K) = \{e\}$ for some $e \in H$, then $K = e$. So, we have $|e| - 1 = |K| - 1$.

Assume that the assertion is true for any component of a cycle-free hypergraph with less than ℓ hyperlinks for some $\ell \geq 2$. Let K be a component of a cycle-free hypergraph (N, H) with $|H(K)| = \ell$. Let $P = (i_1, e_1, i_2, e_2, \ldots, i_{k-1}, e_{k-1}, i_k)$ be a longest chain in $(K, H(K))$. Then $k \geq 3$ and each node in e_{k-1}, except node i_{k-1}, has degree one, because if a node $j \in e$, $j \neq i_{k-1}$, has degree more than 1, there exists another hyperlink $e \in H(K)$, $e \neq e_{k-1}$, such that $j \in e$. It contradicts that P is the longest chain in $(K, H(K))$ if $e \neq e_h$ for every $h \in \{1, \ldots, k-2\}$, and it contradicts that (N, H) is cycle-free if $e = e_h$ for some $h \in \{1, \ldots, k-2\}$. Hence, $K' = K \setminus (e_{k-1} \setminus \{i_{k-1}\})$ is a component of the cycle-free hypergraph $(N, H \setminus \{e_{k-1}\})$ with $|H(K')| = \ell - 1$. It holds that $H(K) = H(K') \cup \{e_{k-1}\}$ and $|K| = |K'| + |e_{k-1}| - 1$. By the induction hypothesis, we have

$$\sum_{e \in H(K')} (|e| - 1) = |K'| - 1.$$

Therefore, $\sum_{e \in H(K)}(|e| - 1) = |K'| - 1 + (|e_{k-1}| - 1) = |K| - 1$. □

The lemma reveals the relation between the number of nodes and the number of hyperlinks in cycle-free hypergraph structures. In fact, there is a similar well-known result for cycle-free graphs (see Corollary 1.5.3 in Diestel (2000)): In a component of a cycle-free graph, the number of links is equal to the number of nodes minus one. Indeed, Lemma 4.3.2 includes the result for the graph case.

The following result is about uniqueness, saying that component efficiency and component fairness determine exactly one solution for cycle-free hypergraph games.

Lemma 4.3.3. *On the class of cycle-free hypergraph games, there is a unique solution that satisfies component efficiency and component fairness.*

Proof. Suppose that on the class of cycle-free hypergraph games a value ξ satisfies component efficiency and component fairness. For any cycle-free hypergraph game

$(N, v, H) \in \mathcal{G}_N^{\mathcal{H}^{cf}}$ and component $K \in N/H$, we show that there are $|K|$ linearly independent equations which are derived from component fairness and component efficiency.

For any hyperlink $e \in H(K)$, let $\{K_e^1, \ldots, K_e^{m_e}\}$ be the m_e components of K in hypergraph $(N, H \setminus \{e\})$. Component efficiency implies that

$$\sum_{h \in K} \xi_h(N, v, H) = v(K), \tag{4.3.2}$$

and

$$\sum_{h \in K_e^p} \xi_h(N, v, H \setminus \{e\}) = v(K_e^p), \quad \text{for all } p \in \{1, \ldots, m_e\}. \tag{4.3.3}$$

Therefore, component fairness implies

$$\frac{1}{|K_e^p|}\Big(\sum_{h \in K_e^p} \xi_h(N, v, H) - v(K_e^p)\Big) = \frac{1}{|K_e^q|}\Big(\sum_{h \in K_e^q} \xi_h(N, v, H) - v(K_e^q)\Big),$$

for every $p, q \in \{1, \ldots, m_e\}$. Let $\alpha_e = (v(K) - \sum_{p=1}^{m_e} v(K_e^p))/|K|$, then

$$\frac{1}{|K_e^p|}\Big(\sum_{h \in K_e^p} \xi_h(N, v, H) - v(K_e^p)\Big) = \alpha_e$$

and therefore

$$\sum_{h \in K_e^p} \xi_h(N, v, H) = |K_e^p|\alpha_e + v(K_e^p), \tag{4.3.4}$$

for all $p \in \{1, \ldots, m_e\}$.

Take any $B = \{B_i : i \in K\} \in \mathcal{B}^H(K)$ with $B_r = K$. Without loss of generality we assume that for every $h, k \in K$ it holds that $|B_h| \leq |B_k|$ whenever $h < k$. Moreover, for $j \in K \setminus \{r\}$, let $i(j)$ be the unique node $i \in K$ such that $B_j \in (B_i \setminus \{i\})/H$ and let $e(j)$ be the unique hyperlink in $H(B_{i(j)})$ containing both $i(j)$ and j. Then, for every $j \in K \setminus \{r\}$ there are unique $e \in H(B_{i(j)})$ and $p \in \{1, \ldots, m_e\}$ such that $B_j = K_e^p$. Note that $B_j = K_{e(j)}^p$ and that $r \notin K_{e(j)}^p$. Conversely, it holds that for every K_e^p not containing r there exists a unique $j \in K \setminus \{r\}$ satisfying $B_j = K_e^p$.

Hence, according to Lemma 4.3.2, there are $|K| - 1 = \sum_{e \in H(K)}(|e| - 1)$ equations of type (4.3.4) satisfying $K_e^p = B_j$ for some $j \in K \setminus \{r\}$. Combined with

(4.3.2), they form the following system of $|K|$ linear equations with $|K|$ unknowns,

$$\sum_{h\in B_j} \xi_h(N, v, H) = \begin{cases} |B_j|\alpha_{e(j)} + v(B_j), & \text{if } j \in K \setminus \{r\}, \\ v(K), & \text{if } j = r. \end{cases}$$

The coefficient matrix associated to this system has a nonzero determinant since it is lower triangular with each diagonal element equal to 1. Therefore, the $|K|$ equations in the system are linearly independent and uniquely determine $\xi_h(N, v, H)$ for all $h \in K$. □

From Lemma 4.3.1 and Lemma 4.3.3 we obtain the following theorem.

Theorem 4.3.1. *On the class of cycle-free hypergraph games, the average tree value is the unique solution that satisfies component efficiency and component fairness.*

4.3.2 Hypertree games

From the Definition 4.2.3, we see that the payoffs in a component by using the average tree value do not affect the payoffs in other components, which implies that there is no loss of generality to consider a hypertree game instead of a cycle-free hypergraph game. Therefore, in this subsection we focus on hypertree games and provide another characterization of the average tree value.

Let $\mathcal{G}_N^{\mathcal{H}^t}$ be the collection of hypertree games with fixed player set N, let ξ be a solution defined on $\mathcal{G}_N^{\mathcal{H}^t}$, and let $\mathcal{B}^H$ denote the set of admissible collections of coalitions in connected hypergraph $(N, H) \in \mathcal{H}_N^t$. Note that $\mathcal{H}_N^t \subseteq \mathcal{H}_N^c$.

Before stating the characterization, we introduce some axioms. Some of them are used in graph games or hypergraph games, such as efficiency and linearity, and others are adapted axioms, for example, from the null-player property and the symmetry property. For efficiency we refer to Section 2.3. Now we give the axiom of linearity as follows.

Linearity: For any (N, v, H), $(N, w, H) \in \mathcal{G}, \mathcal{G} \subseteq \mathcal{G}_N^{\mathcal{H}}$, and $a, b \in \mathbb{R}$, it holds that

$$\xi(N, av + bw, H) = a\xi(N, v, H) + b\xi(N, w, H).$$

The following axioms are adapted properties from the null-player property and the symmetry property.

A player $i \in N$ is a *restricted null-player* in hypergraph game $(N, v, H) \in \mathcal{G}_N^{\mathcal{H}}$ if $v(S) = \sum_{K \in (S \setminus \{i\})/H} v(K)$ for all $S \in C^H(N)$ satisfying that $i \in S$.

Restricted null-player property: For any $(N, v, H) \in \mathcal{G}, \mathcal{G} \subseteq \mathcal{G}_N^{\mathcal{H}}$, and restricted null-player $i \in N$, it holds that $\xi_i(N, v, H) = 0$.

The restricted null-player property states that if a player in a hypergraph game contributes nothing to any connected coalition, then this player gets zero payoff.

Weak symmetry: For any $(N, v, H) \in \mathcal{G}, \mathcal{G} \subseteq \mathcal{G}_N^{\mathcal{H}}$ with $v(S) = 0$ for all $S \in C^H(N) \setminus \{N\}$, it holds that $\xi_i(N, v, H) = \xi_j(N, v, H)$ for all $i, j \in N$.

Weak symmetry states that if a hypergraph game allocates every connected coalition, except the grand coalition, a zero worth, then all players get the same payoff.

From linearity and restricted null-player property, we have a similar result as Lemma 3.3.4.

Lemma 4.3.4. *If a value $\xi : \mathcal{G} \to \mathbb{R}^n$, $\mathcal{G} \subseteq \mathcal{G}_N^{\mathcal{H}}$, satisfies linearity and the restricted null-player property, it holds that $\xi(N, v, H) = \xi(N, v^H, H)$ for all $(N, v, H) \in \mathcal{G}_N^{\mathcal{H}}$.*

Proof. Consider the game $(N, w, H) \in \mathcal{G}_N^{\mathcal{H}}$ where $w = v - v^H$. Every player is a restricted null-player in this game since $w(S) = 0$ for all $S \in C^H(N)$, and therefore, each player receives zero payoff, i.e., $\xi_i(N, w, H) = 0$ for all $i \in N$. Then by linearity and $w = v - v^H$, we have $\xi(N, v, H) = \xi(N, w, H) + \xi(N, v^H, H) = \xi(N, v^H, H)$. □

The following two lemmas first show that, on the class of connected hypergraph games, the average tree value satisfies efficiency, linearity, and the restricted null-player property, and then illustrate that, if the underlying structure is a hypertree, the average tree value satisfies weak symmetry.

Lemma 4.3.5. *On the class of connected hypergraph games, the average tree value satisfies efficiency, linearity, and the restricted null-player property.*

Proof. From Theorem 4.3.1, it follows that the average tree value satisfies efficiency on the class of connected hypergraph games.

Concerning linearity, for any connected hypergraph games $(N, v, H), (N, w, H) \in \mathcal{G}_N^{\mathcal{H}^c}$, $a, b \in \mathbb{R}$, and $B \in \mathcal{B}^H$, by (4.2.1), we have

$$\begin{aligned} m_i^B(N, av+bw, H) &= (av+bw)(B_i) - \sum_{K\in(B_i\setminus\{i\})/H} (av+bw)(K) \\ &= av(B_i) + bw(B_i) - \sum_{K\in(B_i\setminus\{i\})/H} av(K) - \sum_{K\in(B_i\setminus\{i\})/H} bw(K) \\ &= a\big(v(B_i) - \sum_{K\in(B_i\setminus\{i\})/H} v(K)\big) + b\big(w(B_i) - \sum_{K\in(B_i\setminus\{i\})/H} w(K)\big) \\ &= am_i^B(N, v, H) + bm_i^B(N, w, H), \qquad \text{for all } i \in N. \end{aligned}$$

By (4.2.2), it follows that the average tree value satisfies linearity on $\mathcal{G}_N^{\mathcal{H}^c}$.

If player $i \in N$ is a restricted null-player in connected hypergraph game $(N, v, H) \in \mathcal{G}_N^{\mathcal{H}^c}$, then for every $B \in \mathcal{B}^H$ it follows that

$$m_i^B(N, v, H) = v(B_i) - \sum_{K\in(B_i\setminus\{i\})/H} v(K) = 0.$$

Recall that $B_i \in C^H(N)$ for all $i \in N$ and $B \in \mathcal{B}^H$. Hence, by (4.2.2), it turns out that $AT_i(N, v, H) = 0$. □

Lemma 4.3.6. *On the class of hypertree games, the average tree value satisfies weak symmetry.*

Proof. If a hypertree game $(N, v, H) \in \mathcal{G}_N^{\mathcal{H}^t}$ satisfies $v(S) = 0$ for all $S \subsetneq N$, then for every $B \in \mathcal{B}^H$ satisfying $B_r = N$, $r \in N$, we have

$$m_i^B(N, v, H) = \begin{cases} v(N), & \text{if } i = r, \\ 0, & \text{otherwise.} \end{cases}$$

Since the underlying hypergraph is a hypertree, by (4.3.1), we see that the average tree value satisfies weak symmetry on $\mathcal{G}_N^{\mathcal{H}^t}$. □

Now, we examine the Harsanyi dividends on a hypergraph-restricted game. The following result shows that only connected coalitions have non-zero dividends, which generalizes the work in Owen (1986) for graph games.

Lemma 4.3.7. *For any $(N, v, H) \in \mathcal{G}_N^{\mathcal{H}}$, it holds that $\Delta_{v^H}(S) = 0$, for all $S \notin C^H(N)$.*

Proof. We prove the property by induction on the cardinality of S. Since every singleton player is connected, the initial step is for $|S| = 2$. Suppose that $S = \{i,j\} \notin C^H(N)$, then $\Delta_{v^H}(S) = v^H(S) - v(\{i\}) - v(\{j\}) = 0$.

Assume that $S \notin C^H(N)$ with $|S| \geq 3$ and $\Delta_{v^H}(T) = 0$ whenever $T \notin C^H(N)$ with $|T| < |S|$, then we show that $\Delta_{v^H}(S) = 0$.

By (2.1.1), we have $v^H(S) = \sum_{T \subseteq S} \Delta_{v^H}(T)$. So, it turns out that

$$\begin{aligned}
\Delta_{v^H}(S) &= v^H(S) - \sum_{T \subsetneq S} \Delta_{v^H}(T) \\
&= \sum_{K \in S/H} v(K) - \sum_{T \in C^H(S)} \Delta_{v^H}(T) \\
&= \sum_{K \in S/H} \sum_{T \in C^H(K)} \Delta_{v^H}(T) - \sum_{T \in C^H(S)} \Delta_{v^H}(T) \\
&= 0,
\end{aligned}$$

where the second equality follows from the definition of v^H and the hypothesis, the third equality follows from (2.1.1) and the fact that S is not connected, and the last equality holds because $\{T \in C^H(K) : K \in S/H\} = C^H(S)$. □

According to Lemma 4.3.7 and (2.1.1), we obtain that on the class of hypergraph games the set of unanimity games with respect to a connected coalition forms a linear basis for the hypergraph-restricted games, that is, for any hypergraph $(N,H) \in \mathcal{H}_N$, $\{(N,u_S) : S \in C^H(N)\}$ can fully express any TU game restricted by (N,H). For any $(N,v,H) \in \mathcal{G}_N^{\mathcal{H}}$, we have

$$v^H = \sum_{S \in C^H(N)} \Delta_{v^H}(S) u_S, \tag{4.3.5}$$

which implies that

$$v^H(T) = \sum_{S \in C^H(T)} \Delta_{v^H}(S), \quad \text{for all } T \in 2^N \setminus \{\emptyset\}.$$

Based on Lemma 4.3.4 and (4.3.5), we immediately obtain the following result.

Lemma 4.3.8. *If a value $\xi : \mathcal{G} \to \mathbb{R}^n$, $\mathcal{G} \subseteq \mathcal{G}_N^{\mathcal{H}}$, satisfies linearity and the restricted null-player property, then, for any $(N,v,H) \in \mathcal{G}$, it holds that*

$$\xi(N,v,H) = \sum_{S \in C^H(N)} \Delta_{v^H}(S) \cdot \xi(N,u_S,H). \tag{4.3.6}$$

Lemma 4.3.8 states that if a value satisfies linearity and the restricted null-player property, this value of any connected hypergraph game can be decomposed into this value of certain unanimity games with the same hypergraph structure.

Next, we specify the expression of the average tree value on hypertree unanimity games, which is similar to Herings et al. (2008) on cycle-free graph games.

First, we need to introduce some notation. For hypertree $(N, H) \in \mathcal{H}_N^t$, $S \in C^H(N)$, and $j \in S$, let $P_S^H(j)$ be the set of players outside S represented by player j, where player $j \in S$ represents player $h \in N \setminus S$ if h is connected to j and the hyperlinks of the unique chain between j and h do not contain any player in S except player j. Specifically,

$$P_S^H(j) = \{k \in K : K \in (N \setminus S)/H, h \in K, \{j, h\} \subseteq e, e \in H\}.$$

Lemma 4.3.9. *For any hypertree unanimity game* $(N, u_T, H) \in \mathcal{G}_N^{\mathcal{H}^t}$ *and* $T \in C^H(N)$*, it holds that*

$$AT_i(N, u_T, H) = \begin{cases} \frac{1+|P_T^H(i)|}{n}, & \text{if } i \in T, \\ 0, & \text{if } i \in N \setminus T. \end{cases} \tag{4.3.7}$$

Proof. Note that $|T| + \sum_{j \in T} |P_T^H(j)| = n$. From Definition 4.2.2, for any $B \in \mathcal{B}^H$ such that $B_r = N$ for some $r \in N$, we have

$$m_i^B(N, u_T, H) = \begin{cases} 1, & \text{if } i = r \text{ or } r \in P_T^H(i), \\ 0, & \text{otherwise.} \end{cases}$$

Then, by (4.3.1), we obtain (4.3.7). □

From (4.3.7), we notice that two hypertree unanimity games assign the same payoff to a player if the players he represents are the same in the two games, that is, in hypertree unanimity games $(N, u_S, H), (N, u_T, H) \in \mathcal{G}_N^{\mathcal{H}^t}$, it holds that $AT_i(N, u_S, H) = AT_i(N, u_T, H)$ if $P_S^H(i) = P_T^H(i)$.

Based on this observation, we extend an independence property introduced in Mishra and Talman (2010) to hypertree games.

Independence in hypertree unanimity games: For any hypertree game $(N, u_T, H) \in \mathcal{G}_N^{\mathcal{H}^t}$, $T \in C^H(N)$, and $e \in H \setminus H(T)$ satisfying $T \cup e \in C^H(N)$, it holds that $\xi_i(N, u_T, H) = \xi_i(N, u_{T \cup e}, H)$, for all $i \in T \setminus e$.

The independence in hypertree unanimity games states that when a group of players, who are adjacent to each other, join a feasible coalition, any player in the coalition not being adjacent to that group of players receives the same payoff.

Theorem 4.3.2. *On the class of hypertree games, the average tree value is the unique solution that satisfies efficiency, linearity, the restricted null-player property, weak symmetry, and independence in hypertree unanimity games.*

Proof. Take any $(N, u_T, H) \in \mathcal{G}_N^{\mathcal{H}^t}$, $T \in C^H(N)$, and $e \in H \setminus H(T)$ satisfying $T \cup e \in C^H(N)$. For every $i \in T \setminus e$ it holds that $P_T^H(i) = P_{T\cup e}^H(i)$. From (4.3.7) it then follows that $AT_i(N, u_T, H) = AT_i(N, u_{T\cup e}, H)$ for all $i \in T \setminus e$, which shows independence in hypertree unanimity games. From Lemma 4.3.5 and Lemma 4.3.6, it follows that on the class of hypertree games the average tree value satisfies all five axioms. Now we just need to verify uniqueness.

Let ξ be an allocation rule on the class of hypertree games that satisfies all five axioms. Due to Lemma 4.3.8, we just need to consider the hypertree unanimity games with respect to connected coalitions.

Let $(N, H) \in \mathcal{H}_N^t$ be a hypertree, we show that

$$\xi_i(N, u_S, H) = AT_i(N, u_S, H), \text{ for all } S \in C^H(N)$$

by induction on $|S|$.

In the initial step, we consider $S = N$. We have $u_N(S) = 1$ when $S = N$, and $u_N(S) = 0$ otherwise. Hence, by weak symmetry, it follows that $\xi_i(N, u_N, H) = \xi_j(N, u_N, H)$ for all $i, j \in N$. Then, by efficiency, we have

$$\xi_i(N, u_N, H) = \frac{1}{n}, \quad \text{for all } i \in N.$$

Additionally, from (4.3.7), it follows that

$$AT_i(N, u_N, H) = \frac{1}{n}, \quad \text{for all } i \in N.$$

Therefore, $\xi(N, u_N, H) = AT(N, u_N, H)$.

Now, take any $1 \leq t < n$ and suppose that $\xi(N, u_S, H) = AT(N, u_S, H)$ for all $S \in C^H(N)$ with $|S| > t$, then we show that $\xi(N, u_T, H) = AT(N, u_T, H)$ for every $T \in C^H(N)$ with $|T| = t$.

Since each $i \in N \setminus T$ is a restricted null-player in $(N, u_T, H) \in \mathcal{G}_N^{\mathcal{H}^t}$, it follows from the restricted null-player property that

$$\xi_i(N, u_T, H) = AT_i(N, u_T, H) = 0, \qquad \text{for all } i \in N \setminus T. \tag{4.3.8}$$

Since the grand coalition N is connected in (N, H) and $t < n$, there exists $e \in H \setminus H(T)$ such that $T \cup e \in C^H(N)$. Let $S = T \cup e$. Note that $|T \cup e| > t$. By the induction hypothesis, we have

$$\xi_i(N, u_S, H) = AT_i(N, u_S, H) = \frac{1 + |P_S^H(i)|}{n}, \qquad \text{for all } i \in S. \tag{4.3.9}$$

Moreover, because $S = T \cup e$ and $S \in C^H(N)$, by independence in hypertree unanimity games and (4.3.9), it holds that

$$\xi_i(N, u_T, H) = \xi_i(N, u_S, H) = \frac{1 + |P_S^H(i)|}{n}, \qquad \text{for all } i \in T \setminus e.$$

Since $P_T^H(i) = P_S^H(i)$ for every $i \in T \setminus e$, from (4.3.7) it follows that

$$\xi_i(N, u_T, H) = \frac{1 + |P_T^H(i)|}{n} = AT_i(N, u_T, H), \qquad \text{for all } i \in T \setminus e. \tag{4.3.10}$$

Since (N, H) is a hypertree, it holds that $|T \cap e| = 1$. Therefore, by (4.3.8), (4.3.10), and efficiency, we obtain that

$$\xi_i(N, u_T, H) = AT_i(N, u_T, H), \qquad \text{for all } i \in N.$$

Then according to Lemma 4.3.8, the proof is completed. □

4.3.3 Cycle hypergraph games

In the previous two subsections, we provide two characterizations of the average tree value for cycle-free hypergraph games. Now, we study the average tree value on a class of hypergraph games with cycles. First of all, we specify the class of cycle hypergraph games.

A linear connected hypergraph is a cycle hypergraph if it contains a unique cycle containing all hyperlinks. A cycle hypergraph is displayed in Figure 4.5.

Definition 4.3.1. A hypergraph $(N, H) \in \mathcal{H}_N$ is a *cycle hypergraph* if it satisfies the following conditions:

(i) (N,H) is connected;

(ii) (N,H) is linear;

(iii) (N,H) contains a unique cycle and this cycle contains all hyperlinks in H.

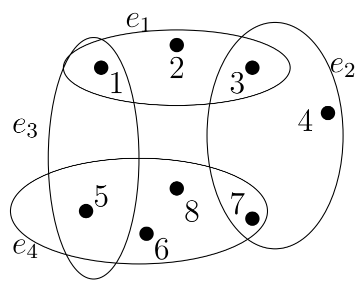


Figure 4.5: A cycle hypergraph.

Note that cycle-free hypergraphs are also linear but are not cycle hypergraphs, for example, see the hypertree displayed in Figure 2.1 (b). Moreover, if we add a hyperlink $e_5 = \{1,2,3\}$ in Figure 2.1 (b), then the new hypergraph is still linear but contains three cycles, which is also not a cycle hypergraph.

A hypergraph game $(N,v,H) \in \mathcal{G}_N^{\mathcal{H}}$ is called a *cycle hypergraph game* if the underlying hypergraph (N,H) is a cycle hypergraph. Let $\mathcal{G}_N^{\mathcal{C}}$ denote the set of cycle hypergraph games with fixed player set N.

For a given cycle hypergraph $(N,H) \in \mathcal{H}_N$ and connected coalition $T \in C^H(N)$ with $|T| \geq 2$, some players in T are adjacent to players not in T. A player $i \in T$ is an *extreme player* in (N,H) with respect to T, if there exists a hyperlink $e \in H \setminus H(T)$ such that $i \in e$. Let $E^H(T)$ denote the set of extreme players of T, $T \geq 2$, in cycle hypergraph $(N,H) \in \mathcal{H}_N$. Each player in $T \setminus E^H(T)$ is called an *inner player* in (N,H) with respect to T. Note that each $i \in T \setminus E^H(T)$ satisfies $(H(T))_i = H_i$, while each $j \in E^H(T)$ satisfies $(H(T))_j \subsetneq H_j$. Note that $|E^H(T)| = 2$ if $2 \leq |T| < n$, since (N,H) is linear.

To illustrate the concepts of extreme players and inner players with respect to a connected coalition, consider the cycle hypergraph (N,H) displayed in Figure 4.5. Let $T = e_1$, then $E^H(T) = \{1,3\}$ and the inner player with respect to T is 2. Let $T' = e_1 \cup e_2$, then $E^H(T') = \{1,7\}$ and the inner players with respect to T' are 2, 3, and 4. Moreover, $E^H(N) = \emptyset$ and all players in N are inner players with respect to N.

Note that in a cycle hypergraph which is not a graph, any admissible collection of coalitions corresponds to a partial ordering and not necessarily to a permutation. This is an essential difference with the case of cycle graph games in Selçuk et al. (2013), in which the underlying cycle graph always induces two permutations.

Based on the concepts introduced above, we have the following expression of the average tree value on cycle hypergraph unanimity games.

Lemma 4.3.10. *For any cycle hypergraph unanimity game* $(N, u_T, H) \in \mathcal{G}_N^{\mathcal{C}}$ *with respect to* $T \in C^H(N)$, *it holds that*

$$AT_i(N, u_T, H) = \begin{cases} 1, & \text{if } i \in T \text{ and } |T| = 1, \\ \frac{1}{n}, & \text{if } i \in T \setminus E^H(T), \\ \frac{n-|T|+2}{2n}, & \text{if } i \in E^H(T) \text{ and } |T| > 1, \\ 0, & \text{if } i \in N \setminus T. \end{cases} \tag{4.3.11}$$

Proof. In a cycle hypergraph game, each player $i \in N$ is two times a top-player among all admissible collections of coalitions. From Definition 4.2.1, for every $i \in N$, there is a unique component, say K with $|K| \geq 2$, within $(N \setminus \{i\})/H$, containing two players, say j_1 and j_2, such that $\{i, j_1\} \subseteq e$ and $\{i, j_2\} \subseteq e'$ for some $e, e' \in H$. Since $(N \setminus \{i\}, H(N \setminus \{i\}))$ is cycle-free, according to Lemma 4.2.2, j_h, $h \in \{1, 2\}$, is one time a top-player in this sub-hypergraph. Hence, it follows that $|\mathcal{B}^H| = 2n$.

If $|T| = 1$, according to the definition of marginal contribution in (4.2.1), it turns out that

$$m_i^B(N, u_T, H) = u_T(B_i) - u_T^H(B_i \setminus \{i\}) = \begin{cases} 1, & \text{if } i \in T, \\ 0, & \text{else,} \end{cases}$$

for all $B \in \mathcal{B}^H$. Note that $u_{\{i\}}^H(B_i) = 1$, while $u_{\{i\}}^H(B_i \setminus \{i\}) = 0$. Therefore, if $|T| = 1$, we have $AT_i(N, u_T, H) = 1$ if $i \in T$, and $AT_i(N, u_T, H) = 0$, otherwise.

If $|T| \geq 2$, for any $B^r \in \mathcal{B}^H$, $r \in N$, such that $B_r^r = N$, by Definition 4.2.2, we have the following two cases:

Case 1. If $r \in T$, then

$$m_i^{B^r}(N, u_T, H) = u_T(B_i^r) - u_T^H(B_i^r \setminus \{i\}) = \begin{cases} 1, & \text{if } i = r, \\ 0, & \text{else.} \end{cases}$$

Case 2. If $r \in N \setminus T$, then

$$m_i^{B^r}(N, u_T, H) = u_T(B_i^r) - u_T^H(B_i^r \setminus \{i\}) = \begin{cases} 1, & \text{if } i \in E^H(T) \text{ and } T \subseteq B_i^r, \\ 0, & \text{else.} \end{cases}$$

If $i \in N \setminus T$, both in Case 1 and in Case 2 player i receives a zero marginal contribution, therefore, it holds that $AT_i(N, u_T, H) = 0$. If $i \in T \setminus E^H(T)$, then only in Case 1 player i gets non-zero marginal contributions, and it takes place only twice. So, we have $AT_i(N, u_T, H) = \frac{2}{2n} = \frac{1}{n}$. Each player $i \in E^H(T)$ receives $|N \setminus T| + 2$ times 1 marginal contributions in Case 2. Hence, $AT_i(N, u_T, H) = \frac{n-|T|+2}{2n}$ since $|E^H(T)| = 2$. □

Before stating the characterization of the average tree value for cycle hypergraph games, we need to introduce two other axioms.

Symmetry in cycle unanimity games: For any $(N, u_T, H) \in \mathcal{G}_N^{\mathcal{C}}$ with $T \in C^H(N)$ and $|T| \geq 2$, it holds that $\xi_i(N, u_T, H) = \xi_j(N, u_T, H)$ if either $i, j \in T \setminus E^H(T)$ or $i, j \in E^H(T)$.

Symmetry in cycle unanimity games indicates two kinds of symmetry, inner players are symmetric and extreme players are symmetric.

Independence in cycle unanimity games: For any $(N, u_T, H) \in \mathcal{G}_N^{\mathcal{C}}$ with $T \in C^H(N)$, and $e \in H \setminus H(T)$ satisfying $T \cup e \in C^H(N)$, it holds that $\xi_i(N, u_T, H) = \xi_i(N, u_{T\cup e}, H)$, for all $i \in T \setminus E^H(T)$.

Independence in cycle unanimity games is similar to independence in hypertree unanimity games. The axiom states that the payoffs of the inner players associated to a connected coalition do not change if an adjacent group of players joins.

Next, we give a characterization of the average tree value on the class of cycle hypergraph games.

Theorem 4.3.3. *On the class of cycle hypergraph games, the average tree value is the unique solution that satisfies efficiency, linearity, the restricted null-player property, symmetry in cycle unanimity games, and independence in cycle unanimity games.*

Proof. From Lemma 4.3.5, it follows that the average tree value satisfies efficiency, linearity, and the restricted null-player property. Now we show that this value also satisfies symmetry in cycle unanimity games and independence in cycle unanimity games.

Take any $(N, u_T, H) \in \mathcal{G}_N^{\mathcal{C}}$ with $T \in C^H(N)$ and $|T| \geq 2$. From (4.3.11), we have $AT_i(N, u_T, H) = AT_j(N, u_T, H) = \frac{1}{n}$ if $i, j \in T \backslash E^H(T)$, while, $AT_i(N, u_T, H) = AT_j(N, u_T, H) = \frac{n-|T|+2}{2n}$ if $i, j \in E^H(T)$. Besides, if $T = N$, then $E^H(T) = \emptyset$ and $AT_i(N, u_T, H) = AT_j(N, u_T, H) = \frac{1}{n}$ for all $i, j \in N$. Therefore, the average tree value satisfies symmetry in cycle unanimity games on $\mathcal{G}_N^{\mathcal{C}}$.

Take any $(N, u_T, H) \in \mathcal{G}_N^{\mathcal{C}}$ with $T \in C^H(N)$, and $e \in H \setminus H(T)$ satisfying $T \cup e \in C^H(N)$. From (4.3.11), we have $AT_i(N, u_T, H) = AT_i(N, u_{T\cup e}, H) = \frac{1}{n}$ for all $i \in T \setminus E^H(T)$, since $i \in T \setminus E^H(T)$ implies that $i \in (T \cup e) \setminus E^H(T \cup e)$, which shows independence in cycle unanimity games on $\mathcal{G}_N^{\mathcal{C}}$.

Therefore, the average tree value satisfies all five axioms on cycle hypergraph games. Now we prove uniqueness.

Let ξ be an allocation rule satisfying all five axioms. By linearity and Lemma 4.3.8 we only have to consider cycle hypergraph unanimity games with respect to connected coalitions.

We first consider a cycle hypergraph game $(N, u_N, H) \in \mathcal{G}_N^{\mathcal{C}}$. By efficiency and symmetry in cycle unanimity games, it turns out that

$$\xi_i(N, u_N, H) = \frac{1}{n}, \qquad \text{for all } i \in N,$$

since $E^H(N) = \emptyset$, i.e., all players are inner players with respect to N in (N, H).

Additionally, according to (4.3.11), we have

$$AT_i(N, u_N, H) = \frac{1}{n}, \qquad \text{for all } i \in N.$$

Therefore, we obtain that $\xi(N, u_N, H) = AT(N, u_N, H)$.

Next, we consider a cycle hypergraph unanimity game $(N, u_T, H) \in \mathcal{G}_N^{\mathcal{C}}$ for some $T \in C^H(N) \setminus \{N\}$.

If $|T| = 1$, by efficiency and the restricted null-player property, we have $\xi_i(N, u_T, H) = 1 = AT_i(N, u_T, H)$, $i \in T$, and $\xi_i(N, u_T, H) = 0 = AT_i(N, u_T, H)$, for all $i \in N \setminus T$.

Now, let $|T| \geq 2$. Since $T, N \in C^H(N)$, there exists a nonempty subset of hyperlinks $A = \{e \in H : e \nsubseteq T\} \subsetneq H$ satisfying $T \cup \{i \in e : e \in A\} = N$. Therefore, by applying $|A|$ times independence in cycle unanimity games, we obtain

$$\xi_i(N, u_T, H) = \xi_i(N, u_N, H) = \frac{1}{n} = AT_i(N, u_T, H), \quad \text{for all } i \in T \setminus E^H(T).$$

In addition, because each $i \in N \setminus T$ is a restricted null-player in (N, u_T, H), by the restricted null-player property, it follows that $\xi_i(N, u_T, H) = 0$ for all $i \in N \setminus T$, as in the average tree value.

Since $2 \leq |T| < n$, it holds that $E^H(T) = 2$. Let $E^H(T) = \{i, j\}$. By symmetry in cycle unanimity games, it follows that $\xi_i(N, u_T, H) = \xi_j(N, u_T, H)$. Together with efficiency, $\xi_i(N, u_T, H) = \frac{1}{n}$ for all $i \in T \setminus E^H(T)$, and $\xi_i(N, u_T, H) = 0$ for all $i \in N \setminus T$, we obtain that

$$\xi_i(N, u_T, H) = \frac{n - |T| + 2}{2n} = AT_i(N, u_T, H), \quad \text{for all } i \in E^H(T).$$

Therefore, we have $\xi(N, u_T, H) = AT(N, u_T, H)$ for all $T \in C^H(N)$. By Lemma 4.3.8, this completes the proof. □

4.3.4 Logical independence of axioms

In this subsection, we show that the axioms in Theorem 4.3.2 and Theorem 4.3.3, which are used to characterize the average tree values, are logical independent for each characterization. For the two axioms in Theorem 4.3.1, it is clear that they are independent to each other.

We use an example to illustrate the independence among the axioms by providing different allocation rules.

Example 4.3.1. The following five allocation rules on $\mathcal{G}_N^{\mathcal{H}^t} \cup \mathcal{G}_N^{\mathcal{C}}$ show the independence of the axioms in Theorem 4.3.2 and Theorem 4.3.3.

- Let ξ^1 be an allocation rule given by

  $$\xi_i^1(N, v, H) = 0, \quad \text{for all } i \in N.$$

  This allocation rule satisfies all axioms of Theorem 4.3.2 and of Theorem 4.3.3, except efficiency.

- Let ξ^2 be an allocation rule given by

$$\xi_i^2(N,v,H) = \frac{v(N)}{n}, \qquad \text{for all } i \in N.$$

This allocation rule satisfies all axioms of Theorem 4.3.2 and of Theorem 4.3.3, except restricted null-player property.

- Let ξ^3 be an allocation rule given by

$$\xi^3(N,v,H) = \mu(N,v,H).$$

This allocation rule satisfies all axioms of Theorem 4.3.2 and of Theorem 4.3.3, except independence in hypertree unanimity games in Theorem 4.3.2 or independence in cycle unanimity games in Theorem 4.3.3. For efficiency, linearity, and the restricted null-player property, we refer to van den Nouweland et al. (1992). Consider a game $(N,v,H) \in \mathcal{G}_N^{\mathcal{H}^t} \cup \mathcal{G}_N^{\mathcal{C}}$, such that $v(S)=0$ for all $C^H(N)\setminus\{N\}$, then $\mu_i(N,v,H) = \frac{v(N)}{n}$ for all $i \in N$, which shows weak symmetry. Take any $(N,u_T,H) \in \mathcal{G}_N^{\mathcal{H}^t} \cup \mathcal{G}_N^{\mathcal{C}}$, with $T \in C^H(N)$. $\mu(N,u_T,H)$ allocates $\frac{1}{|T|}$ to players in T and 0 to players not in T, which shows symmetry in cycle unanimity games. Take any $e \in H \setminus H(T)$ such that $T \cup e \in C^H(N)$, then $\mu_i(N,u_{T\cup e},H)$ assigns $\frac{1}{|T\cup e|}$ to players in $T\cup e$ and 0 to players not in $(T\cup e)$. Since $\frac{1}{|T|} \neq \frac{1}{|T\cup e|}$, it shows that this allocation rule fails independence in hypertree unanimity games if (N,H) is a hypertree or fails independence in cycle unanimity games if (N,H) is a cycle hypergraph.

- Let ξ^4 be an allocation rule given by

$$\xi^4(N,v,H) = m^B(N,v,H), \qquad \text{for some } B \in \mathcal{B}^H.$$

This allocation rule only fails weak symmetry in Theorem 4.3.2 or symmetry in cycle unanimity games in Theorem 4.3.3. Consider a game $(N,u_N,H) \in \mathcal{G}_N^{\mathcal{H}^t} \cup \mathcal{G}_N^{\mathcal{C}}$, then $m_r^B(N,u_N,H)=1$ if $B_r = N$ and $m_i^B(N,u_N,H)=0$ for all $i \in N \setminus \{r\}$, which fails weak symmetry and symmetry in cycle unanimity games.

- Let ξ^5 be an allocation rule given by

$$\xi^5(N,v,H) = \begin{cases} AT(N,u_T,H), & \text{if } v = u_T \text{ for some } T \in C^H(N), \\ \mu(N,v,H), & \text{if } v \neq u_T \text{ for any } T \in C^H(N). \end{cases}$$

This allocation rule satisfies all axioms of Theorem 4.3.2 and of Theorem 4.3.3, except linearity. From the third solution in this example, the Myerson value fails independence in hypertree unanimity games if (N, H) is a hypertree and fails independence in cycle unanimity games if (N, H) is a cycle hypergraph. However, in this solution, if the underlying game is a unanimity game then it refers to the average tree value, which satisfies the two properties in their corresponding games, respectively. Consider a game $(N, v, H) \in \mathcal{G}_N^{\mathcal{H}^t} \cup \mathcal{G}_N^{\mathcal{C}}$ satisfying $v = u_{Q_1} + u_{Q_2}$ for some $Q_1, Q_2 \in C^H(N)$ such that $Q_1 \cap Q_2 \neq \emptyset$, then, for any $i \in Q_1 \cap Q_2$ with $|H_i| = 1$, we have $\mu_i(N, v, H) = \frac{1}{|Q_1|} + \frac{1}{|Q_2|}$, while $AT_i(N, u_{Q_1}, H) = AT_i(N, u_{Q_2}, H) = \frac{1}{n}$, which implies that this solution fails linearity.

□

Chapter 5

The two-step average tree value for graph and hypergraph games

5.1 Introduction

In this chapter, we continue focusing on graph and hypergraph games and propose an alternative allocation rule determined by the marginal contributions associated with the set of admissible collections of coalitions as defined in Chapter 4.

Following the research line of the average tree value in the previous chapter, in this chapter another allocation rule, named the two-step average tree value, is proposed for both hypergraph games and graph games. In a connected hypergraph game or graph game, the two-step average tree value assigns to each player first as payoff his average marginal contributions among all admissible collections of coalitions when any one of the players is chosen as the top-player or the root, and then assigning the average of all these payoffs. Contrary to the average tree value, each player has equal probability to be a top-player and therefore the two-step average tree value satisfies weak symmetry on the class of all hypergraph games.

The two-step average tree value is also an extension of the average tree solution for cycle-free graph games in Herings et al. (2008). If the underlying hypergraph is cycle-free, the two-step average tree value coincides with the average tree value. In fact, on the class of linear cactus hypergraph games the two values coincide, where a linear cactus hypergraph possesses the property that any two different cycles in the hypergraph have at most one node in common.

Moreover, we study the two-step average tree value on a class of hypergraph

graph games, called quasi-cycle-free hypergraph games, which is more general than the class of cycle-free hypergraph games and allows for some special cycles. By generalizing component fairness from cycle-free hypergraph games to quasi-cycle-free hypergraph games, we prove that the two-step average tree value satisfies the generalized property of component fairness and component efficiency.

In order to give a characterization of the two-step average tree value, we define a smaller class of hypergraphs than quasi-cycle-free, called semi-cycle-free, which is still more general than cycle-free and allows for some more specific cycles. On the class of semi-cycle-free hypergraph games, the two-step average tree value can be characterized by component efficiency, component fairness, and a property called balanced contributions for interactive players. The third property is developed from the balanced contributions property introduced in Myerson (1980) and van den Nouweland et al. (1992). Two players are called interactive if they are the common nodes of two distinct hyperlinks. Balanced contributions for interactive players says that the contributions of any two interactive players to each other are equal, where the contribution of a player to another player is the payoff difference of the second player by breaking all the hyperlinks of the first player. An example is provided to show that a quasi-cycle-free hypergraph game, in which the underlying hypergraph is not semi-cycle-free, fails to satisfy balanced contributions for interactive players.

This chapter is based on Kang et al. (2018b) and the organization of this chapter is as follows. Section 5.2 introduces the two-step average tree value and discusses the relationship between the two-step average tree value and the average tree value. Section 5.3 verifies that on the class of quasi-cycle-free hypergraph games the two-step average tree value satisfies component fairness and component efficiency. Section 5.4 gives on the class of semi-cycle-free hypergraph games a characterization of the two-step average tree value.

5.2 The two-step average tree value

This section introduces a two-step value for hypergraph games, which is developed from marginal contributions of players corresponding to admissible collections of coalitions similar to the average tree value. This value is proposed mainly for hypergraph games containing cycles and coincides with the average tree value for cycle-free hypergraph games. The idea is based on the fact that, in a component and any admissible collection of coalitions, the top-player is rewarded the best in this component by assigning the full dividend of the component coalition to this player. Therefore, the situation that every player of a hypergraph game has equal probability to be a top-player is worthwhile to study. This situation implies that the calculations of the payoffs of the players comprise two stages. The first stage is to calculate the average payoffs of all players when a specific player is the top-player, and the second stage deals with the expected payoffs by considering the distribution of the top-players with equal probability. Based on this idea, we propose the two-step average tree solution for hypergraph games and compare it to the average tree value introduced in Chapter 4.

In order to introduce the new value, we need some notation. For any hypergraph $(N, H) \in \mathcal{H}_N$ and component $K \in N/H$, let $\mathcal{B}_r^H(K)$, $r \in K$, be the set of all admissible collections of coalitions $B = \{B_i : i \in K\} \in \mathcal{B}^H(K)$ as in Definition 4.2.1 such that $B_r = K$. Particularly, if hypergraph $(N, H) \in \mathcal{H}_N$ is connected, we use $\mathcal{B}^H$ and $\mathcal{B}_r^H$, $r \in N$, when we refer to $\mathcal{B}^H(N)$ and $\mathcal{B}_r^H(N)$, respectively. Then, we introduce the following solution concept.

Definition 5.2.1. On the class of hypergraph games, the *two-step average tree value* assigns to every $(N, v, H) \in \mathcal{G}_N^{\mathcal{H}}$, a payoff vector $ATT(N, v, H)$ given by

$$ATT_i(N, v, H) = \frac{1}{|K|} \sum_{r \in K} \frac{1}{|\mathcal{B}_r^H(K)|} \sum_{B \in \mathcal{B}_r^H(K)} m_i^B(N, v, H), \quad i \in K,\ K \in N/H.$$

Note that the two-step average tree value is also a new solution for graph games. If the underlying hypergraph structure is a graph, we have the following corresponding expression: For any $(N, v, \Gamma) \in \mathcal{G}_N^{\Gamma}$, $K \in N/\Gamma$, and $i \in K$,

$$ATT_i(N, v, \Gamma) = \frac{1}{|K|} \sum_{r \in K} \frac{1}{|\mathcal{T}_r^{\Gamma}(K)|} \sum_{T \in \mathcal{T}_r^{\Gamma}(K)} m_i^T(N, v, \Gamma). \tag{5.2.1}$$

The two-step average tree value of a hypergraph game assigns to each player first as payoff his average marginal contributions when any of the players in the component he belongs to is chosen as the top-player or the root, and then assigning the average of all these payoffs. The value is defined by averaging twice on the marginal contributions with respect to admissible collection of coalitions for hypergraph games or to admissible rooted spanning trees for graph games, which differs from the average tree value as in Definition 4.2.3 and in (2.3.3) by averaging only once. Therefore, compared with the average tree value, the two-step average tree value reflects the idea that every player in a component is as important as any other player in that component. According to this fact, the two-step average tree value is not equal to the average tree value in general.

Example 5.2.1. Reconsider the hypergraph game as described in Example 4.2.2. Since the marginal contribution vectors corresponding to admissible collections of coalitions are already obtained, by Definition 5.2.1, we immediately get the two-step average tree value for this hypergraph game as follows,

$$ATT(N, v, H) = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}),$$

which is different from $AT(N, v, H) = (\frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{4})$. □

Moreover, we can interpret the difference between the average tree value and the two-step average tree value from a theoretical perspective. Based on the weak symmetry property introduced in Subsection 4.3.2, we have the following theorem.

Theorem 5.2.1. *On the class of connected hypergraph games, the two-step average tree value satisfies weak symmetry.*

Proof. For any $(N, v, H) \in \mathcal{G}_N^{\mathcal{H}^c}$ such that $v(S) = 0$ for all $S \in C^H(N) \setminus \{N\}$, and $B \in \mathcal{B}^H$ with $B_r = N$, by (4.2.1), we have

$$m_i^B(N, v, H) = \begin{cases} v(N), & \text{if } i = r, \\ 0, & \text{if } i \neq r. \end{cases}$$

From Definition 5.2.1, we obtain

$$ATT_i(N, v, H) = \frac{v(N)}{n}, \quad \text{for all } i \in N,$$

which shows ATT satisfying weak symmetry on $\mathcal{G}_N^{\mathcal{H}^c}$. □

Note that for any $(N, v, H) \in \mathcal{G}_N^{\mathcal{H}^c}$ such that $v(S) = 0$ for all $S \in C^H(N) \setminus \{N\}$, we have

$$AT_i(N, v, H) = \frac{|\mathcal{B}_i^H| v(N)}{|\mathcal{B}^H|}, \quad \text{for all } i \in N,$$

which shows that AT does not satisfy weak symmetry. In fact, Example 5.2.1 also shows this point.

From Example 5.2.1 and the discussions above, it is clear that on the class of hypergraph games the two-step average tree value differs from the average tree value. However, on some specific subclasses, the two values coincide.

Before showing the coincidence between the average tree value and the two-step average tree value, we need the following lemma regarding hypergraph structures. Recall that linear hypergraphs require that any two distinct hyperlinks have at most one common node and a cactus requires that any two distinct cycles have at most one node in common. Note that Figure 2.1 (a) is a cactus but not linear, while both Figure 2.1 (b) and Figure 4.5 are linear cacti.

Lemma 5.2.1. *For any connected hypergraph $(N, H) \in \mathcal{H}_N^c$ which is a linear cactus, it holds that $|\mathcal{B}_r^H| = 2^c$ for all $r \in N$, where $c \in \mathbb{N}$ is the number of cycles in (N, H). Moreover, it holds that $|\mathcal{B}^H| = 2^c \cdot n$.*

Proof. If (N, H) contains no cycles and therefore $c = 0$, it follows from Lemma 4.2.2 that $|\mathcal{B}_r^H| = 1 = 2^0$ for every $r \in N$ and therefore $|\mathcal{B}^H| = n$. If the linear cactus (N, H) contains one or more cycles, we prove the result by induction on the number of cycles. Suppose (N, H) contains only one cycle, denoted by C. For any $r \in N$ and $B \in \mathcal{B}^H$ such that $B_r = N$, let $B_i \in B$ be the unique minimal coalition containing all nodes in C, i.e., $N(C) \subseteq B_i$ and $N(C) \nsubseteq B_i \setminus \{i\}$. Note that $i \in N(C)$. Then, since (N, H) is linear and contains no other cycle, for any $j \in N \setminus \{r\}$ satisfying $B_i \subsetneq B_j$ or $B_i \cap B_j = \emptyset$, it holds that $E^H(B_j) = \{j\}$, which implies that no other node in B_j than node j is adjacent to some node not in B_j.

Since the subhypergraph $(B_i \setminus \{i\}, H(B_i \setminus \{i\}))$ is cycle-free, there exists a unique $K \in (B_i \setminus \{i\})/H$ such that $N(C) \setminus \{i\} \subset K$ and therefore $|E^H(K)| = 2$. Otherwise, if $|E^H(K)| = 1$ for all $K \in (B_i \setminus \{i\})/H$, then $(B_i, H(B_i))$ is cycle-free, which contradicts that $(B_i, H(B_i))$ contains a cycle; and if $|E^H(K)| > 2$ for some $K \in (B_i \setminus \{i\})/H$ or $|E^H(K)| = 2$ for several $K \in (B_i \setminus \{i\})/H$, then it contradicts that $(B_i, H(B_i))$ contains exactly one cycle.

Let $E^H(K) = \{h, k\}$, then according to Definition 4.2.1 either $K = B_h$ or $K = B_k$. Since the remaining hypergraph $(B_h, H(B_h))$ or $(B_k, H(B_k))$ is cycle-free, by Lemma 4.2.2, there exists a unique admissible collection of coalitions in $(B_h, H(B_h))$ and $(B_k, H(B_k))$ having h and k as top-players, respectively. Therefore, by picking any node as the top-player, a linear cactus containing one cycle induces two different admissible collections of coalitions. Hence, we have $|\mathcal{B}_r^H| = 2$ for all $r \in N$.

Assume the assertion is true for any linear cactus with less than c cycles for some $c > 1$. Now, we show that the assertion is also true for any linear cactus $(N, H) \in \mathcal{H}_N^c$ containing c cycles.

Let $\{C_1, \ldots, C_c\}$ denote the set of cycles in (N, H). For any $r \in N$ and $B \in \mathcal{B}^H$ such that $B_r = N$, let $B_i \in B$ be the unique minimal coalition containing the nodes of all c cycles. Note that $i \in C_h$ for some $h \in \{1, \ldots, c\}$. Suppose the subhypergraph $(B_i \setminus \{i\}, H(B_i \setminus \{i\}))$ contains ℓ cycles. Without loss of generality, let $\{C_1, \ldots, C_{c-\ell}\}$ be the set of cycles containing node i. Note that $0 \leq \ell < c$.

For any $K \in (B_i \setminus \{i\})/H$, there are three cases: a) K contains exactly one node, which is a one-degree node adjacent to i; b) $|K| \geq 2$ and $|E^H(K)| = 1$, which implies that $|K \cap N(C_h)| \leq 1$ for all $h \in \{1, \ldots, c - \ell\}$, and there is only one node in K adjacent to i; c) $|E^H(K)| = 2$, which implies that $|K \cap N(C_h)| \geq 2$ for some $h \in \{1, \ldots, c - \ell\}$, and there are two nodes in K adjacent to i.

Since for any $h, h' \in \{1, \ldots, c\}$ and $h \neq h'$ it holds that $|N(C_h) \cap N(C_{h'})| \leq 1$, there are $c - \ell$ different components K of $B_i \setminus \{i\}$ in $(B_i \setminus \{i\}, H(B_i \setminus \{i\}))$ such that $|E^H(K)| = 2$. Hence, similar to the initial step, these components yield in total $2^{c-\ell}$ different choices of nodes being adjacent to i in each of the components of $B_i \setminus \{i\}$. Together with the hypothesis that for the components of $B_i \setminus \{i\}$ the ℓ cycles in $(B_i \setminus \{i\}, H(B_i \setminus \{i\}))$ yield 2^ℓ different combinations, it follows that in total there are $2^c = 2^{c-\ell} \cdot 2^\ell$ admissible collections of coalitions in (N, H). This shows that $|\mathcal{B}_r^H| = 2^c$ for all $r \in N$ if (N, H) has c cycles.

Therefore, we have $|\mathcal{B}^H| = \sum_{r \in N} |\mathcal{B}_r^H| = 2^c \cdot n$, which completes the proof. □

Lemma 5.2.1 reveals an interesting property of linear cacti that each node is the same number of times a top-player. Therefore, combined with the definitions of the average tree value and the two-step average tree value, we have the following result.

Theorem 5.2.2. *The average tree solution coincides with the two-step average tree value if the underlying hypergraph is cycle-free, a linear cactus with cycles, or the complete graph.*

Proof. Take any $(N, v, H) \in \mathcal{G}_N^{\mathcal{H}^{cf}}$ and $K \in N/H$. From Lemma 4.2.2 it follows that $|\mathcal{B}_r^H(K)| = 1$ for all $r \in K$. Therefore, according to Definition 5.2.1 and (4.3.1), it holds that $ATT(N, v, H) = AT(N, v, H)$.

Now we examine the class of hypergraph games where the underlying hypergraph is a linear cactus with cycles. From Definition 5.2.1, the payoffs in a component by using the two-step average tree value do not affect the payoffs in other components. Therefore, without loss of generality, we consider connected hypergraph games. For any $(N, v, H) \in \mathcal{G}_N^{\mathcal{H}^c}$, if the underlying hypergraph

$(N, H) \in \mathcal{H}_N^c$ is a linear cactus with c cycles, from Definition 5.2.1 and Lemma 5.2.1, we obtain

$$\begin{aligned} ATT(N, v, H) &= \frac{1}{2^c \cdot n} \sum_{r \in N} \sum_{B \in \mathcal{B}_r^H} m^B(N, v, H) \\ &= \frac{1}{|\mathcal{B}^H|} \sum_{B \in \mathcal{B}^H} m^B(N, v, H) \\ &= AT(N, v, H). \end{aligned}$$

Note that the case of cycle-free is also included whenever $c = 0$.

For any $(N, v, \Gamma) \in \mathcal{G}_N^{\Gamma c}$, if the underlying structure is the complete graph on N, we have $|\mathcal{T}_r^\Gamma(N)| = (n-1)!$ for all $r \in N$ and $|\mathcal{T}^\Gamma(N)| = n!$. From (5.2.1), we obtain

$$\begin{aligned} ATT(N, v, \Gamma) &= \frac{1}{n} \sum_{r \in N} \frac{1}{(n-1)!} \sum_{T \in \mathcal{T}_r^\Gamma(N)} m^T(N, v, \Gamma) \\ &= \frac{1}{n!} \sum_{T \in \mathcal{T}^\Gamma(N)} m^T(N, v, \Gamma) \\ &= AT(N, v, \Gamma). \end{aligned}$$

□

In fact, if the underlying hypergraph is the complete graph, both the average tree value and the two-step average tree value coincide with the Shapley value, see also Herings et al. (2010). Moreover, on the class of cycle-free, hypertree, and cycle hypergraph graph games the characterizations of the average tree value are also valid for the two-step average tree values due to the fact that in those cases the two values coincide.

5.3 Quasi-cycle-free hypergraph games

This section examines the two-step average tree value on a subclass of hypergraph games. This subclass of hypergraphs is wider than the class of cycle-free hypergraphs, and we show that the two-step average tree value satisfies component efficiency and component fairness on games underlying such kind of hypergraphs.

First of all, we define the class of quasi-cycle-free hypergraphs.

Definition 5.3.1. A hypergraph $(N, H) \in \mathcal{H}_N$ is *quasi-cycle-free* if it is induced by a cycle-free hypergraph $(N', H') \in \mathcal{H}^{cf}_{N'}$ satisfying

(i) $N' \subseteq N$;

(ii) $|H'| = |H|$;

(iii) For every $e' \in H'$, there exists a unique $e \in H$ such that $e' = e \cap N'$, and for every $e'_1, e'_2 \in H'$ it holds that

$$e_1 \cap e_2 \neq \emptyset \implies e'_1 \cap e'_2 \neq \emptyset.$$

From Definition 5.3.1 we see that a cycle-free hypergraph is quasi-cycle-free. However, a quasi-cycle-free may contain cycles. Figure 5.1 shows a cycle-free hypergraph and one of its induced quasi-cycle-free hypergraphs. Note that since $|e_1 \cap e_2| = 2$ and $|e_1 \cap e_3| = 3$, the displayed quasi-cycle-free hypergraph in Figure 5.1 is not linear, which implies that it contains cycles. Moreover, from Figure 5.1 we also see that $|N/H| = |N'/H'|$, but $|N/(H \setminus \{e_1\})| \neq |N'/(H' \setminus \{e'_1\})|$.

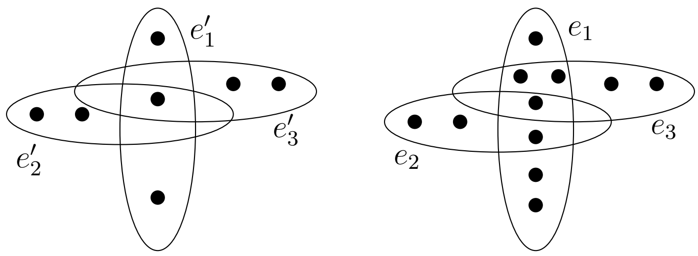


Figure 5.1: Two types of hypergraphs.

For quasi-cycle-free hypergraphs, we have the following property.

Lemma 5.3.1. *All hyperlinks of a quasi-cycle-free hypergraph are bridges, i.e., if $(N, H) \in \mathcal{H}_N$ is quasi-cycle-free, then, for every $e \in H$, it holds that $|N/H| < |N/(H \setminus \{e\})|$.*

Proof. If the lemma is not true, then there exist a quasi-cycle-free hypergraph $(N, H) \in \mathcal{H}_N$ and a hyperlink $e \in H$ such that $|N/H| = |N/(H \setminus \{e\})|$. That means that for any two distinct nodes $i, j \in e$, there exists a chain between i and j, say $(i_1, e_1, i_2, \ldots, i_{k-1}, e_{k-1}, i_k)$ satisfying $i_1 = i$, $i_k = j$, and $e_t \neq e$ for all $t \in \{1, \ldots, k-1\}$. By Definition 5.3.1, in the cycle-free hypergraph (N', H')

from which (N, H) is induced there exist a hyperlink $e' \in H'$ by which e is induced and a chain $(i'_1, e'_1, i'_2, \ldots, i'_{k-1}, e'_{k-1}, i'_k)$, where e_t is induced by e'_t for all $t \in \{1, \ldots, k-1\}$. Since $i \in e \cap e_1$ and $j \in e \cap e_{k-1}$, this implies that i'_1 and i'_k can be chosen such that $i'_1 \in e' \cap e'_1$ and $i'_k \in e' \cap e'_{k-1}$. Therefore, there exists a cycle $(i'_1, e'_1, \ldots, i'_{k-1}, e'_{k-1}, i'_k, e', i'_1)$ in (N', H'), which contradicts that (N', H') is cycle-free. □

From the lemma it follows that every hyperlink in a quasi-cycle-free hypergraph is a bridge. Therefore, we may generalize component fairness from cycle-free hypergraph games to quasi-cycle-free hypergraph games. Let $\mathcal{G}_N^{\mathcal{H}^{qcf}}$ denote the set of games with quasi-cycle-free hypergraph on player set N, and let ξ be a solution on $\mathcal{G}_N^{\mathcal{H}^{qcf}}$. Then we have the following axiom.

Component fairness: For any $(N, v, H) \in \mathcal{G}, \mathcal{G} \subseteq \mathcal{G}_N^{\mathcal{H}^{qcf}}$, and hyperlink $e \in H$, it holds that

$$\begin{aligned} & \frac{1}{|K^1|} \sum_{h \in K^1} \big(\xi_h(N, v, H) - \xi_h(N, v, H \setminus \{e\})\big) \\ = & \frac{1}{|K^2|} \sum_{h \in K^2} \big(\xi_h(N, v, H) - \xi_h(N, v, H \setminus \{e\})\big), \end{aligned}$$

for every $K^1, K^2 \in N/(H \setminus \{e\})$ such that $K^1 \cap e \neq \emptyset$ and $K^2 \cap e \neq \emptyset$.

The following lemma states that the two-step average tree value satisfies component efficiency and component fairness in case the underlying hypergraph is quasi-cycle-free.

Lemma 5.3.2. *On the class of quasi-cycle-free hypergraph games, the two-step average tree value satisfies component efficiency and component fairness.*

Proof. Take any $(N, v, H) \in \mathcal{G}_N^{\mathcal{H}^{qcf}}$ and $K \in N/H$. From Theorem 4.3.1, we obtain that

$$\sum_{h \in K} m_h^B(N, v, H) = v(B_r) = v(K), \text{ for all } B \in \mathcal{B}_r^H(K) \text{ and } r \in K.$$

Hence, it turns out that

$$\begin{aligned}
\sum_{h\in K} ATT_h(N,v,H) &= \sum_{h\in K}\frac{1}{|K|}\sum_{r\in K}\frac{1}{|\mathcal{B}_r^H(K)|}\sum_{B\in\mathcal{B}_r^H(K)} m_h^B(N,v,H)\\
&= \frac{1}{|K|}\sum_{r\in K}\frac{1}{|\mathcal{B}_r^H(K)|}\sum_{B\in\mathcal{B}_r^H(K)}\sum_{h\in K} m_h^B(N,v,H)\\
&= \frac{1}{|K|}\sum_{r\in K}\frac{1}{|\mathcal{B}_r^H(K)|}\sum_{B\in\mathcal{B}_r^H(K)} v(B_r) \;=\; v(K),
\end{aligned}$$

which shows component efficiency on $\mathcal{G}_N^{\mathcal{H}^{qcf}}$.

In order to show component fairness, we have to analyze the influence of deleting a hyperlink. Take any $e\in H(K)$, and let $e=\{i_1,\ldots,i_m\}$. Note that $m\geq 2$. Since all the hyperlinks of a semi-cycle-free hypergraph are bridges, by deleting hyperlink $e\in H(K)$, component K is split into several components in the resulting hypergraph $(N, H\setminus\{e\})$, denoted by $\{K^1,\ldots,K^\ell\}$ with $\ell\geq 2$ and $\ell\leq m$, which is a partition of K. For every $B\in\mathcal{B}_r^H(K)$, $r\in K$, and $p\in\{1,\ldots,\ell\}$, we have

$$\sum_{h\in K^p} m_h^B(N,v,H) = v(K^p), \qquad \text{if } r\notin K^p,$$

and

$$\sum_{h\in K^p} m_h^B(N,v,H) = v(K) - \sum_{q\neq p} v(K^q), \qquad \text{if } r\in K^p.$$

Since each player $r\in K$ can be chosen as the top-player in component K, coalition K^p receives $|\mathcal{B}_r^H(K)|v(K^p)$ by taking player $r\in K\setminus K^p$ as the top-player in component K, and this type of situation will take place $|K\setminus K^p|$ times. On the other hand, coalition K^p receives $|\mathcal{B}_r^H(K)|\big(v(K)-\sum_{q\neq p}v(K^q)\big)$ by taking player $r\in K^p$ as the top-player in component K, and this type of situation will take place $|K^p|$ times. Therefore, by the definition of ATT, for any $p\in\{1,\ldots,\ell\}$, we have

$$\sum_{h\in K^p} ATT_h(N,v,H) = \frac{|K^p|\big(v(K)-\sum_{q\neq p}v(K^q)\big) + |K\setminus K^p|v(K^p)}{|K|}.$$

Since the rest of the proof is similar to the proof of Theorem 4.3.1, we obtain that ATT satisfies component fairness on $\mathcal{G}_N^{\mathcal{H}^{qcf}}$. □

From Lemma 4.3.3 we see that, on the class of cycle-free hypergraph games, component efficiency and component fairness uniquely determine a solution. This is because cycle-free hypergraphs are linear, that is, for any two distinct hyperlinks there is at most one common node. However, since quasi-cycle-free hypergraphs may not be linear, these two properties are not sufficient to determine a solution for quasi-cycle-free hypergraph games. Even though we could not characterize the two-step average tree value on the class of all quasi-cycle-free hypergraph games, in the following section, a characterization of this value is provided on a subclass of quasi-cycle-free hypergraph games.

Now, we study the stability of the two-step average tree value for hypergraph games. The following result states that on the class of quasi-cycle-free hypergraph games if the underlying game is superadditive then the value is an element of the core.

Theorem 5.3.1. *For any $(N, v, H) \in \mathcal{G}_N^{\mathcal{H}^{qcf}}$, if (N, v) is superadditive, then it holds that $ATT(N, v, H) \in C(N, v, H)$. Moreover, $AT(N, v, H) \in C(N, v, H)$.*

Proof. Similar to Theorem 4.2.1, it is sufficient to prove that $(m_i^B(N, v, H))_{i\in K} \in C(K, v_K, H(K))$ for every $B \in \mathcal{B}^H(K)$ and $K \in N/H$.

From Definition 5.3.1, let $(N', H') \in \mathcal{H}_{N'}^{cf}$ be the cycle-free hypergraph from which (N, H) is induced. Take any $S \in C^H(N)$ and $B \in \mathcal{B}^H(K)$ for some $K \in N/H$. It holds that the coalition S is either a singleton player, i.e., $S = \{i\}$ for some $i \in K$, or is a union of some hyperlinks. Now we show that there exists at least one player $i \in S$ satisfying $S \subseteq B_i$. It trivially holds if $S = \{i\}$ for some $i \in K$. Now, we consider $|S| \geq 2$ and let $S' = S \cap N'$. Note that $S' \in C^{H'}(N')$ due to the third condition of Definition 5.3.1. If the assertion is not true, then there exist some players in S, say $j_1, \ldots, j_k$, such that $S \subseteq \bigcup_{t=1}^k B_{j_t}$ and $B_{j_t} \cap B_{j_{t'}} = \emptyset$, for all $t \neq t'$. It implies that there exist coalitions of N', say $Q_1, \ldots, Q_\ell$, $\ell \leq k$, such that $S' \subseteq \bigcup_{t=1}^\ell Q_t$ and $Q_t = N' \cap B_{j_t}$ for every $t \in \{1, \ldots, \ell\}$. Hence, $Q_t \cap Q_{t'} = \emptyset$ for all $t \neq t'$. Then it contradicts S' is a connected coalition of a cycle-free hypergraph. Therefore, there exists at least one player $i \in S$ satisfying $S \subseteq B_i$ for all $S \in C^H(N)$ in quasi-cycle-free hypergraph (N, H).

Since the rest of the proof is similar to the proof of Theorem 4.2.1, we obtain that, for every $B \in \mathcal{B}^H(K)$ and $K \in N/H$, $(m_i^B(N, v, H))_{i\in K} \in C(K, v_K, H(K))$. Therefore, from Definition 5.2.1 and Definition 4.2.3, it follows that $ATT(N, v, H)$ $\in C(N, v, H)$ and $AT(N, v, H) \in C(N, v, H)$. □

5.4 An axiomatization of the two-step average tree value

In this section, we provide a characterization of the two-step average tree value on a specific subclass of hypergraph games. Before stating the characterization, we define the class of hypergraph structures, which is a subclass of quasi-cycle-free hypergraphs.

Definition 5.4.1. A hypergraph $(N, H) \in \mathcal{H}_N$ is *semi-cycle-free* if it is induced by a cycle-free hypergraph $(N', H') \in \mathcal{H}^{cf}_{N'}$ by replacing each connective node $i \in C(N', H')$ with a nonempty set of nodes C_i in N, which satisfy the following conditions:

(i) For every $i \in C(N', H')$ and $h \in C_i$, it holds that $|H_h| = |H'_i|$ and $H_k = H_h$ for every $k \in C_i$;

(ii) $|H| = |H'|$ and for every $e \in H$, there exists a unique $e' \in H'$ such that $e = (e' \setminus C(N', H')) \bigcup \{i \in C_j : j \in e' \cap C(N', H')\}$.

A semi-cycle-free hypergraph is derived from a cycle-free hypergraph by replacing each connective node with a set of new nodes. In Figure 5.2, the node in $e'_1 \cap e'_2$ of a cycle-free hypergraph (N', H') is replaced by a set of two nodes in $e_1 \cap e_2$ of the semi-cycle-free hypergraph (N, H), and the node in $e'_1 \cap e'_3$ of (N', H') is replaced by a set of three nodes in $e_1 \cap e_3$ of (N, H). Note that the new nodes have the same degree as the replaced connective node in the original cycle-free hypergraph, where quasi-cycle-free hypergraphs do not have this restriction. Therefore, the new added players do not change the structure of the original cycle-free hypergraph. It is obvious that a cycle-free hypergraph is semi-cycle-free and a semi-cycle-free hypergraph is quasi-cycle-free, but the reverses are not true. Figure 5.3 shows the three types of hypergraphs.

From Definition 5.4.1, we obtain the following property, which reveals the relation between hyperlinks and nodes of a semi-cycle-free hypergraph.

Lemma 5.4.1. *For any semi-cycle-free hypergraph $(N, H) \in \mathcal{H}_N$ and $K \in N/H$, there is a unique partition $\{\{i_1\}, \ldots, \{i_p\}, C_{j_1}, \ldots, C_{j_q}\}$ of K, in which $i_1, \ldots, i_p$ are all non-connective nodes, and each C_{j_ℓ}, $\ell = 1, \ldots, q$, is the set of connective*

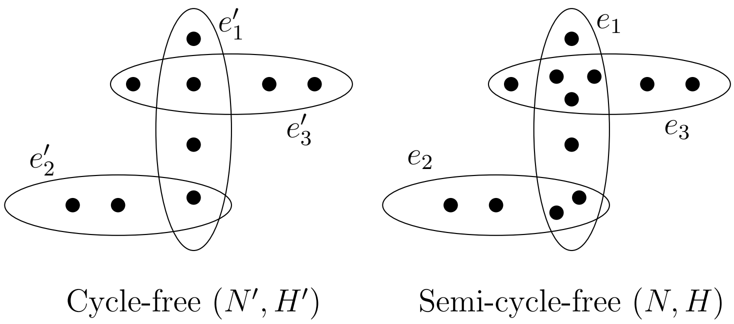


Figure 5.2: A cycle-free hypergraph and a semi-cycle-free hypergraph.

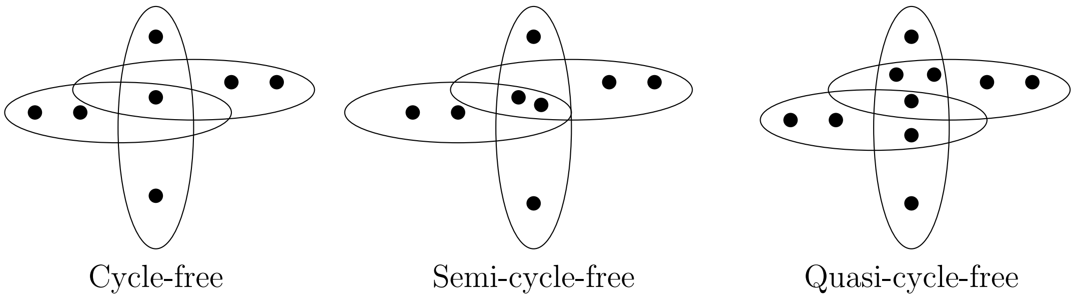


Figure 5.3: Three types of hypergraphs.

nodes in K satisfying $H_i = H_j$ for every $i, j \in C_{j_\ell}$. Moreover,

$$\sum_{e \in H(K)} (p_e + q_e - 1) = p + q - 1, \tag{5.4.1}$$

where, for every $e \in H(K)$, $p_e = |\{i_t \in e : t \in \{1, \dots, p\}\}|$ and $q_e = |\{C_{j_\ell} \subseteq e : \ell \in \{1, \dots, q\}\}|$.

Proof. It is clear that $\{\{i_1\}, \dots, \{i_p\}, C_{j_1}, \dots, C_{j_q}\}$ is the unique partition of K according to above settings. Each non-connective node and also any set of connective nodes having the same incident hyperlinks is a member of the partition.

Let $(N', H') \in \mathcal{H}^{cf}_{N'}$ be the cycle-free hypergraph from which (N, H) is induced and let K' be the component of (N', H') from which K is induced. From Definition 5.4.1 it follows that $K' = \{i_1, \dots, i_q, j_1, \dots, j_q\}$ and therefore $|K'| = p + q$.

For $e \in H(K)$, let $p_e = |\{i_t \in e : t \in \{1, \dots, p\}\}|$ and $q_e = |\{C_{j_\ell} \subseteq e : \ell \in \{1, \dots, q\}\}|$, then, for the corresponding $e' \in H'(K')$, it follows that $|e'| = p_e + q_e$. Since (N', H') is a cycle-free hypergraph, by Lemma 4.3.2, we obtain $\sum_{e' \in H'(K')} (|e'| - 1) = |K'| - 1$, which implies that $\sum_{e \in H(K)} (p_e + q_e - 1) = p + q - 1$. $\square$

Before stating the next lemma, we introduce some notation regarding admissible collections of coalitions. For a hypergraph $(N, H) \in \mathcal{H}_N$ and component $K \in N/H$, we denote $\mathcal{B}^r_{(i,j)} = \{B \in \mathcal{B}^H_r(K) : B_j \subseteq B_i \setminus \{i\}\}$ as the set of admissible collections of coalitions in which $j \in K$ is a subordinate of $i \in K$ when $r \in K$ is the top-player, and denote $\mathcal{B}^r_{\{i,j\}} = \{B \in \mathcal{B}^H_r(K) : B_j \cap B_i = \emptyset\}$ as the set of admissible collections of coalitions where there are no common members between the subordinates of i and j in K when $r \in K$ is the top-player. Note that by property (2) of Lemma 4.2.1 for every $i, j \in K$ it holds that $\{\mathcal{B}^r_{(i,j)}, \mathcal{B}^r_{(j,i)}, \mathcal{B}^r_{\{i,j\}}\}$ is a partition of $\mathcal{B}^H_r(K)$, for all $r \in K$.

Lemma 5.4.2. *For any hypergraph $(N, H) \in \mathcal{H}_N$ and distinct $i, j \in K$, $K \in N/H$, such that $H_i = H_j$, it holds that for each $B \in \mathcal{B}^r_{(i,j)}$, $r \in K$, there is exactly one $B' \in \mathcal{B}^H(K)$ satisfying $i \in B'_j$ and $B_h = B'_h$ for all $h \in K \setminus \{i, j\}$. Moreover, $|\mathcal{B}^i_{(i,j)}| = |\mathcal{B}^j_{(j,i)}|$ and $|\mathcal{B}^r_{(i,j)}| = |\mathcal{B}^r_{(j,i)}|$ for all $r \in K \setminus \{i, j\}$.*

Proof. If $r = j$, then $\mathcal{B}^j_{(i,j)} = \mathcal{B}^j_{\{i,j\}} = \emptyset$, so, we just consider the case $r \neq j$. For $B \in \mathcal{B}^r_{(i,j)}$, note that $B_j = \{j\}$, let the collection of coalitions $B' = \{B'_h : h \in K\}$ be given by

$$B'_h = \begin{cases} B_i, & \text{if } h = j, \\ \{i\}, & \text{if } h = i, \\ B_h, & \text{if } h \in K \setminus \{i, j\}. \end{cases} \tag{5.4.2}$$

According to Definition 4.2.1, since $H_i = H_j$, B' is admissible, i.e., $B' \in \mathcal{B}^H(K)$. Moreover, there is no other admissible $B' \in \mathcal{B}^H(K)$ satisfying $i \in B'_j$ and $B_h = B'_h$ for all $h \in K \setminus \{i, j\}$. Otherwise, if there exists another $\bar{B} \in \mathcal{B}^H(K)$ satisfying $i \in \bar{B}_j$ and $B_h = \bar{B}_h$ for all $h \in K \setminus \{i, j\}$, then $\bar{B} \neq B'$ implies $\bar{B}_i \neq B'_i = \{i\}$. Therefore, there exists some $k \in K \setminus \{i, j\}$ such that $k \in \bar{B}_i$ and $\bar{B}_k = B_k$. It contradicts that $\bar{B}_i \in (\bar{B}_j \setminus \{j\})/H$ is a singleton due to $H_i = H_j$.

In addition, if $B \in \mathcal{B}^r_{(i,j)}$, $r \in K \setminus \{i, j\}$, then B' defined in (5.4.2) is an element of $\mathcal{B}^r_{(j,i)}$, and vice versa. Therefore, $|\mathcal{B}^r_{(i,j)}| = |\mathcal{B}^r_{(j,i)}|$. Moreover, for any $B \in \mathcal{B}^i_{(i,j)}$, the collection of coalitions B' defined in (5.4.2) belongs to $\mathcal{B}^j_{(j,i)}$ and vice versa, which implies $|\mathcal{B}^i_{(i,j)}| = |\mathcal{B}^j_{(j,i)}|$. □

The lemma shows that for any hypergraph $(N, H) \in \mathcal{H}_N$ and two connective nodes i and j in a component $K \in N/H$ such that $H_i = H_j$, there are pairwise two admissible collections of coalitions B and B' in $\mathcal{B}^H(K)$ for which only the coalitions B_i and B'_i and also B_j and B'_j are different. Therefore, the number of admissible collections of coalitions in $\mathcal{B}^H(K)$ for which node i has node j as its subordinate equals the number of admissible collections of coalitions in $\mathcal{B}^H(K)$ for which j has i as its subordinate. Moreover, since $B_h = B'_h$ for all $h \in K \setminus \{i,j\}$, it implies that $(B_i \setminus \{i,j\})/H = (B'_j \setminus \{i,j\})/H$.

Let $\mathcal{G}_N^{\mathcal{H}^{scf}}$ denote the set of games with semi-cycle-free hypergraph on player set N. Now we examine a characterization of the two-step average tree value. As stated at the end of Section 5.3, we need an axiom to deal with the situation that two hyperlinks contain multiple common players. Let $\xi : \mathcal{G} \to \mathbb{R}^n$, $\mathcal{G} \subseteq \mathcal{G}_N^{\mathcal{H}}$, be a solution. For a given hypergraph $(N,H) \in \mathcal{H}_N$, two distinct nodes i and j in N are *interactive* if $i,j \in e \cap e'$ for some $e, e' \in H$, $e \neq e'$.

Balanced contributions for interactive players: For any $(N,v,H) \in \mathcal{G}, \mathcal{G} \subseteq \mathcal{G}_N^{\mathcal{H}}$, and interactive players $i,j \in N$, it holds that

$$\xi_i(N,v,H) - \xi_i(N,v,H_{-j}) = \xi_j(N,v,H) - \xi_j(N,v,H_{-i}).$$

This axiom is developed from balanced contributions introduced in Myerson (1980) for cooperative games with conference structure and in van den Nouweland et al. (1992) for hypergraph games. It says that the contributions of interactive players to each other are equal, where the contribution of a player to another player is the payoff difference of the latter player when breaking all hyperlinks of the former player. The axiom in fact reveals the equivalent influence between two interactive players.

The following lemma shows that the two-step average tree value satisfies component efficiency, component fairness, and the axiom above.

Lemma 5.4.3. *On the class of semi-cycle-free hypergraph games, the two-step average tree value satisfies component efficiency, component fairness, and balanced contributions for interactive players.*

Proof. Since the class of semi-cycle-free hypergaphs is a subclass of quasi-cycle-free hypergaphs, it follows that $\mathcal{G}_N^{\mathcal{H}^{scf}} \subseteq \mathcal{G}_N^{\mathcal{H}^{qcf}}$. Therefore, from Lemma 5.3.2, it follows that the two-step average tree value satisfies component efficiency and component fairness. Now, we just need to prove that this value satisfies balanced contributions for interactive players.

Take any $(N,v,H) \in \mathcal{G}_N^{\mathcal{H}^{scf}}$. Since (N,H) is a semi-cycle-free hypergraph, for any two interactive players $i,j \in K$ it holds that $H_i = H_j$. So, it is sufficient to prove that

$$ATT_i(N,v,H) - v(\{i\}) = ATT_j(N,v,H) - v(\{j\}),$$

because i is an isolated player in (N, H_{-j}) and j is an isolated player in (N, H_{-i}).

For any two interactive players $i, j \in e \cap e'$, $e, e' \in H(K)$, we consider the difference of marginal contributions between players i and j, i.e., $m_i^B(N,v,H) - m_j^B(N,v,H)$, for $B \in \mathcal{B}^H(K)$. From property (2) of Lemma 4.2.1 and since $i, j \in e \cap e'$, it follows that either $\{i\} = B_i \subseteq B_j \setminus \{j\}$, or $\{j\} = B_j \subseteq B_i \setminus \{i\}$, or both $B_i = \{i\}$ and $B_j = \{j\}$. Recall that $\{\mathcal{B}^r_{(i,j)}, \mathcal{B}^r_{(j,i)}, \mathcal{B}^r_{\{i,j\}}\}$ is a partition of $\mathcal{B}^H_r(K)$, that $\mathcal{B}^r_{(i,j)} = \emptyset$ and $\mathcal{B}^r_{\{i,j\}} = \emptyset$ if $r = j$, and that $\mathcal{B}^r_{(j,i)} = \emptyset$ and $\mathcal{B}^r_{\{i,j\}} = \emptyset$ if $r = i$.

From Lemma 5.4.2, we obtain

$$\sum_{B \in \mathcal{B}^r_{(i,j)} \cup \mathcal{B}^r_{(j,i)}} \big(m_i^B(N,v,H) - m_j^B(N,v,H)\big) = |\mathcal{B}^r_{(i,j)} \cup \mathcal{B}^r_{(j,i)}| \big(v(\{i\}) - v(\{j\})\big)$$

and

$$\sum_{B \in \mathcal{B}^r_{\{i,j\}}} \big(m_i^B(N,v,H) - m_j^B(N,v,H)\big) = |\mathcal{B}^r_{\{i,j\}}| \big(v(\{i\}) - v(\{j\})\big).$$

Therefore, according to Definition 5.2.1, we have

$$\begin{aligned}
& ATT_i(N,v,H) - ATT_j(N,v,H) \\
= & \frac{1}{|K|} \sum_{r \in K} \frac{1}{|\mathcal{B}^H_r(K)|} \sum_{B \in \mathcal{B}^H_r(K)} \big(m_i^B(N,v,H) - m_j^B(N,v,H)\big) \\
= & \frac{1}{|K|} \sum_{r \in K} \frac{|\mathcal{B}^r_{(i,j)} \cup \mathcal{B}^r_{(j,i)}| + |\mathcal{B}^r_{\{i,j\}}|}{|\mathcal{B}^H_r(K)|} \big(v(\{i\}) - v(\{j\})\big) \\
= & \frac{1}{|K|} \sum_{r \in K} \big(v(\{i\}) - v(\{j\})\big) \\
= & v(\{i\}) - v(\{j\}),
\end{aligned}$$

which shows balanced contributions for interactive players on $\mathcal{G}_N^{\mathcal{H}^{scf}}$. □

The three axioms in Lemma 5.4.3 determine a unique solution on the class of semi-cycle-free hypergraph games.

Lemma 5.4.4. *On the class of semi-cycle-free hypergraph games, component ef-*

ficiency, component fairness, and balanced contributions for interactive players uniquely determine a solution.

Proof. Let ξ be an allocation rule that satisfies component efficiency, component fairness, and balanced contributions for interactive players. For a given $(N,v,H) \in \mathcal{G}_N^{\mathcal{H}^{scf}}$ and $K \in N/H$, let $\{\{i_1\},\dots,\{i_p\},C_{j_1},\dots,C_{j_q}\}$ be the partition of K satisfying the conditions in Lemma 5.4.1. We first show that there are $p+q$ linearly independent equations by applying component efficiency and component fairness. Component efficiency implies that, for each hyperlink $e \in H(K)$,

$$\sum_{i\in K} \xi_i(N,v,H) = v(K) \tag{5.4.3}$$

and

$$\sum_{i\in K'} \xi_i(N,v,H\setminus\{e\}) = v(K'),\ K' \in N/(H\setminus\{e\}). \tag{5.4.4}$$

Then, component fairness implies that, for each $e \in H(K)$,

$$\frac{1}{|K^1|}\Big(\sum_{h\in K^1} \xi_h(N,v,H) - v(K^1)\Big) = \frac{1}{|K^2|}\Big(\sum_{h\in K^2} \xi_h(N,v,H) - v(K^2)\Big),$$

for every $K^1, K^2 \in N/(H\setminus\{e\})$ satisfying $K^1\cap e \neq \emptyset$ and $K^2\cap e \neq \emptyset$. Therefore, similar as in the proof of Lemma 4.3.3, for each $e \in H(K)$ there exists $\alpha_e \in \mathbb{R}$ such that

$$\sum_{h\in K'} \xi_h(N,v,H) = |K'|\alpha_e + v(K'), \tag{5.4.5}$$

for every $K' \in N/(H\setminus\{e\})$ satisfying $K'\cap e \neq \emptyset$.

By (5.4.1) and regarding each C_t, $t \in \{j_1,\dots,j_q\}$, as a team, (5.4.5) and (5.4.3) yield $p+q$ linearly independent equations similar to the proof of Lemma 4.3.3, which implies that the payoffs of each non-connective player and of each team of connective players are fixed. Therefore, we have

$$\xi_i(N,v,H) = a_i, \qquad \text{if } i \in \{i_1,\dots,i_p\}, \tag{5.4.6}$$

$$\sum_{h\in C_t} \xi_h(N,v,H) = b_t, \qquad \text{if } t \in \{j_1,\dots,j_q\}, \tag{5.4.7}$$

where a_i, $i \in \{i_1,\dots,i_p\}$, and b_t, $t \in \{j_1,\dots,j_q\}$, are constants.

Next, we examine the payoffs among C_t for any $t \in \{j_1, \ldots, j_q\}$. If $|C_t| = 1$, then the payoff of the unique player in C_t is fixed in equation (5.4.7). If $|C_t| \geq 2$, then any two players $i, j \in C_t$ are interactive and $H_i = H_j$. Component efficiency implies that $\xi_i(N, v, H_{-j}) = v(\{i\})$ and $\xi_j(N, v, H_{-i}) = v(\{j\})$, and therefore balanced contributions for interactive players reduces to

$$\xi_i(N, v, H) - v(\{i\}) = \xi_j(N, v, H) - v(\{j\}).$$

Obviously, by balanced contributions for interactive players, we obtain $|C_t| - 1$ linearly independent equations. Combined with equation (5.4.7), we can therefore uniquely determine the payoffs of all players in coalition C_t, for every $t \in \{j_1, \ldots, j_q\}$. This implies that $\xi_i(N, v, H)$ is uniquely determined for all $i \in N$. □

From Lemma 5.4.3 and Lemma 5.4.4, we obtain a characterization of the two-step average tree value for semi-cycle-free hypergraph games.

Theorem 5.4.1. *On the class of semi-cycle-free hypergraph games, the two-step average tree value is the unique value that satisfies component efficiency, component fairness, and balanced contributions for interactive players.*

From Theorem 5.4.1 we know that the two-step average tree value satisfies component efficiency and component fairness, as well as balanced contributions for interactive players if the underlying hypergraph is semi-cycle-free. However, if the underlying hypergraph is quasi-cycle-free hypergraph but is not semi-cycle-free, then the two-step average tree value may not satisfy balanced contributions for interactive players.

Example 5.4.1. Consider a quasi-cycle-free hypergraph game $(N, v, H) \in \mathcal{G}_N^{\mathcal{H}}$, where $N = \{1, \ldots, 6\}$ and $H = \{e_1, e_2, e_3\}$, in which $e_1 = \{1, 2\}$, $e_2 = \{2, 3, 4, 5\}$, $e_3 = \{2, 3, 4, 6\}$, as depicted in Figure 5.4.

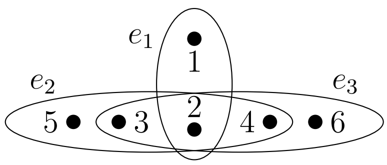


Figure 5.4: The underlying quasi-cycle-free hypergraph in Example 5.4.1.

We notice that player 2 and player 3 are interactive players, since both players belong to e_2 and e_3. Moreover, $H_{-2} = \emptyset$ and $H_{-3} = \{e_1\}$. Therefore, we obtain that

$$\begin{aligned}&ATT_2(N,v,H) - ATT_3(N,v,H)\\&\quad = \frac{1}{6}\big(5(v(\{1,2\}) - v(\{1\})) + v(\{2\})\big) - v(\{3\})\end{aligned}$$

and

$$\begin{aligned}&ATT_2(N,v,H_{-3}) - ATT_3(N,v,H_{-2})\\&\quad = \frac{1}{2}\big((v(\{1,2\}) - v(\{1\})) + v(\{2\})\big) - v(\{3\}).\end{aligned}$$

Hence, it follows that

$$\begin{aligned}&\big(ATT_2(N,v,H) - ATT_3(N,v,H)\big) - \big(ATT_2(N,v,H_{-3}) - ATT_3(N,v,H_{-2})\big)\\= \;&\frac{1}{6}\big(5(v(\{1,2\}) - v(\{1\})) + v(\{2\})\big) - \frac{1}{2}\big((v(\{1,2\}) - v(\{1\})) + v(\{2\})\big)\\= \;&\frac{1}{3}\big(v(\{1,2\}) - v(\{1\}) - v(\{2\})\big).\end{aligned}$$

Therefore, if $v(\{1,2\}) - v(\{1\}) - v(\{2\}) \neq 0$, we have

$$ATT_2(N,v,H) - ATT_2(N,v,H_{-3}) \neq ATT_3(N,v,H) - ATT_3(N,v,H_{-2}),$$

which shows that the two-step average tree value does not satisfy balanced contributions for interactive players. □

The following example illustrates that the three axioms in Theorem 5.4.1 are logically independent.

Example 5.4.2. The following values show the independence of component efficiency, component fairness, and balanced contributions for interactive players.

(1) Let the allocation rule ξ^1 on $\mathcal{G}_N^{\mathcal{H}^{scf}}$ be given by

$$\xi_i^1(N,v,H) = 0, \qquad \text{for all } i \in N.$$

This solution satisfies all axioms except component efficiency.

(2) Let the allocation rule ξ^2 on $\mathcal{G}_N^{\mathcal{H}^{scf}}$ be given by

$$\xi^2(N,v,H) = \mu(N,v,H).$$

This solution satisfies all axioms except component fairness. In van den Nouweland et al. (1992) it is shown that the Myerson value satisfies component efficiency and balanced contributions. Note that the latter axiom is not only for interactive players but also holds for other players. Now, we consider a semi-cycle-free hypergraph game (N, v, H), where $N = \{1, 2, 3, 4, 5\}$, $v = u_{\{1,2,3,4\}}$, and $H = \{e_1, e_2, e_3\}$ with $e_1 = \{1, 2, 3\}$, $e_2 = \{2, 3, 4\}$, and $e_3 = \{4, 5\}$. Then $\mu(N, v, H) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0)$ and $\mu_i(N, v, H \setminus \{e_2\}) = 0$ for all $i \in N$. It follows that the Myerson value fails component fairness, since the average payoff change of component $\{1, 2, 3\}$ is $\frac{1}{4}$, which is not equal to the other component's average payoff change of $\frac{1}{8}$.

(3) Let the allocation rule ξ^3 on $\mathcal{G}_N^{\mathcal{H}^{scf}}$ be given by

$$\xi_i^3(N, v, H) = \begin{cases} ATT_i(N, v, H), & i \in N \setminus R, \\ \frac{\sum_{j \in C_h} ATT_j(N,v,H)}{|C_h|}, & i \in C_h, \text{ for all } h \in \{1, \ldots, m\}, \end{cases}$$

where $\{C_1, \ldots, C_m\}$ is a partition of $R = \{i \in N : |H_i| \geq 2\}$ such that C_h, $h \in \{1, \ldots, m\}$, is a set of interactive players in (N, H) satisfying $H_i = H_j$, for all $i, j \in C_h$. This solution satisfies all axioms except balanced contributions for interactive players. It is clear that this solution satisfies component efficiency and component fairness, since for each C_h, $h \in \{1, \ldots, m\}$, it holds that $\sum_{i \in C_h} \xi_i^3(N, v, H) = \sum_{i \in C_h} ATT_i(N, v, H)$ and the two-step average tree value satisfies the two axioms. For any two interactive players $i, j \in N$, it holds that $i, j \in C_h$ for some $h \in \{1, \ldots, m\}$ such that $|C_h| \geq 2$. Therefore, the contributions of player i to player j is $\xi_i^3(N, v, H) - \xi_i^3(N, v, H_{-j}) = \frac{\sum_{k \in C_h} ATT_k(N,v,H)}{|C_h|} - v(\{i\})$, which in general is not equal to the reverse contributions $\xi_j^3(N, v, H) - \xi_j^3(N, v, H_{-i}) = \frac{\sum_{k \in C_h} ATT_k(N,v,H)}{|C_h|} - v(\{j\})$. Hence, this solution fails balanced contributions for interactive players.

□

Chapter 6

The degree value for hypergraph games

6.1 Introduction

This chapter continues studying games with hypergraph communication structure. We propose a new allocation rule, called the degree value, for these games, and give two characterizations for cycle-free hypergraph games and one characterization for all hypergraph games.

In the literature, the class of hypergraph games is developed from graph games and formally studied in van den Nouweland et al. (1992), and two well-known allocation rules for graph games, the Myerson value and the position value, are extended to allocation rules for hypergraph games. However, the position value is only characterized on the class of cycle-free hypergraph games in van den Nouweland et al. (1992). A characterization of the position value for arbitrary graph games is not completed until Slikker (2005). Together with component efficiency, the axiom of balanced link contributions is proposed to characterize the position value. However, on the class of hypergraph games, the position value fails the natural extension of balanced link contributions on hypergraph games. Instead, Shan and Zhang (2016) propose the property of partial balanced conference contributions and provide a characterization for arbitrary hypergraph games by this axiom combined with component efficiency.

The relationship between the Myerson value and the position value is investigated in Casajus (2007), and later on such kind of approach is named non-axiomatic characterization in Kongo (2010). Moreover, Slikker (2007) discusses for the two values that each of the characterizations involves component efficiency and a property dealing with some kind of balancedness. Following the perspective of Casajus (2007), the Myerson value focuses on the players, while the position value emphasizes the role of the links or hyperlinks. We consider the degree of the players to be of central importance in communicational networks. The degree measures the ability of players to communicate with other players in a communication structure. Therefore, in TU games with communication structure, the degree of a player may reflect the cooperative ability of this player in a sense. In view of this, a value taking into account the degree of the players may have some significance, such as a kind of degree-based solutions introduced in Shan et al. (2016). These values distribute the worth of the grand coalitional proportional to the players' degrees.

This chapter follows the approach that the Myerson value and the position value are defined by applying the Shapley value on a modified game, and proposes an allocation rule by highlighting the role of degree. In a zero-normalized hypergraph game, by splitting the conferences or hyperlinks into separate agents which represent exactly the degrees of the players, we define the so-called degree game associated with the original hypergraph game. In this game the player set comprises all agents generated by splitting all hyperlinks and the characteristic function is defined by applying on this player set the conference game in van den Nouweland et al. (1992). The degree value is proposed by employing the Shapley value on the degree game.

In order to characterize the degree value, we first consider cycle-free hypergraph games. Following the ideas of van den Nouweland et al. (1992), van den Brink et al. (2011), and Algaba et al. (2015), two axioms, called the degree property and the degree measure property, are proposed based on the degree anonymous and the conference unanimous properties, respectively. Together with component efficiency, linearity, and the superfluous conference property, we provide two characterizations of the degree value on the class of cycle-free hypergraph games. On the class of all hypergraph games, by balanced conference contributions and component efficiency, we provide an axiomatic characterization of the degree value. As a natural generalization of balanced link contributions introduced in Slikker (2005), balanced conference contributions states that the total conference contributions of a player towards another player equals the reverse total conference contributions, so for hypergraph games the degree value indicates a kind of fairness similar to the position value for graph games. The degree value and the position value are different in general, but when every hyperlink in a hypergraph contains the same number of players, the two values coincide with each other.

This chapter is partly based on Shan et al. (2018) and the organization of this chapter is as follows. Section 6.2 introduces the degree game and the degree value for hypergraph games. By a numerical example, we show that the degree value is a new allocation rule that does not coincide with the position value or the Myerson value. In Section 6.3, we study the characterizations of the degree value on hypergraph games. Subsection 6.3.1 only considers cycle-free hypergraph games. We provide two new axioms, the degree property and the degree measure property, which are verified to be satisfied by the degree value. These two characterizations are given according to the two new axioms together with component efficiency, linearity, and the superfluous conference property. The characterization on the class of all hypergraph games is studied in Subsection 6.3.2. By defining balanced conference contributions for hypergraph games we provide a characterization of the degree value by this axiom and component efficiency. In Section 6.4 we make a comparison between the degree value and other values.

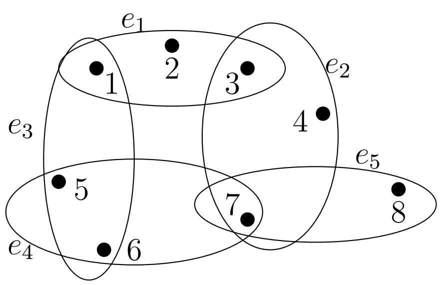
6.2 The degree game and the degree value

In a hypergraph, the degree of a node can be interpreted as its ability to communicate. Therefore, in a hypergraph game, a player's cooperative ability can be represented by his degree on the underlying hypergraph. Motivated by this idea, we introduce a new allocation rule, called the degree value, for hypergraph games. As we have seen before, the Myerson value focuses on the players in terms of the nodes of the underlying communication structure, the position value emphasizes the role of the hyperlinks or conferences, and the (two-step) average tree value takes the connectivity of the hypergraph into account. The degree value proposed below highlights the role of the players' degrees in the underlying hypergraph. Similar to the Myerson value and the position value, the degree value is also defined by applying the Shapley value on a modified game which is induced by the original hypergraph game.

In order to derive the modified game from a given hypergraph game, we first need to introduce the notion of the so-called incidence graph corresponding to a hypergraph.

For a given hypergraph $(N,H) \in \mathcal{H}_N$, the *incidence graph* is a pair $(N \cup H, I(H)) \in \Gamma_{N\cup H}$, where $N\cup H$ is the node set and $I(H) = \{\{i,e\} : i \in e, e \in H\}$ is the link set. Note that in the incidence graph each hyperlink of the original hypergraph is treated as a singleton node. The following example illustrates the concept of incidence graph of a hypergraph.

Example 6.2.1. Consider a hypergraph $(N,H) \in \mathcal{H}_N$, where $N = \{1,\dots,8\}$ and $H = \{e_1,\dots,e_5\}$, in which $e_1 = \{1,2,3\}$, $e_2 = \{3,4,7\}$, $e_3 = \{1,5,6\}$, $e_4 = \{5,6,7\}$, and $e_5 = \{7,8\}$. Its incidence graph is $(N \cup H, I(H)) \in \Gamma_{N\cup H}$, where $N\cup H = \{1,\dots,8,e_1,\dots,e_5\}$ and $I(H) = \{\{1,e_1\}, \{1,e_3\}, \{2,e_1\}, \{3,e_1\}, \{3,e_2\}, \{4,e_2\}, \{5,e_3\}, \{5,e_4\}, \{6,e_3\}, \{6,e_4\}, \{7,e_2\}, \{7,e_4\}, \{7,e_5\}, \{8,e_5\}\}$. Figure 6.1 shows the hypergraph and its incidence graph. □



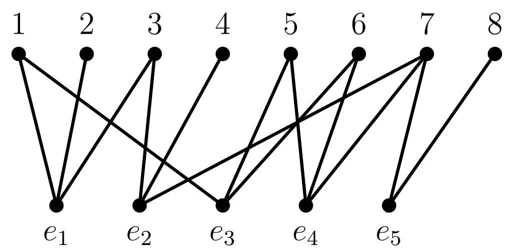


Figure 6.1: The hypergraph and its incidence graph in Example 6.2.1.

In fact, the incidence graph $(N \cup H, I(H))$ of a hypergraph $(N,H) \in \mathcal{H}_N$ is a bipartite graph (or bigraph), since the node set of the incidence graph can be divided into two parts, N and H, and every link in $I(H)$ connects a node in N to a node in H. Note that any hypergraph has a unique incidence graph and any incidence graph corresponds to exactly one hypergraph.

For the incidence graph $(N \cup H, I(H))$ of a hypergraph $(N,H) \in \mathcal{H}_N$, let $I(H)_i = \{\{i,e\} \in I(H) : e \in H\}$ for $i \in N$ and $I(H)_e = \{\{i,e\} \in I(H) : i \in N\}$ for $e \in H$, then $|I(H)_i|$ and $|I(H)_e|$ are the degrees of $i \in N$ and $e \in H$, respectively, in $(N\cup H, I(H))$. It is easy to check that for any hypergraph $(N,H) \in \mathcal{H}_N$ it holds that $|I(H)_i| = |H_i|$ for all $i \in N$, $|I(H)_e| = |e|$ for all $e \in H$, and

$$\sum_{i\in N} |I(H)_i| = \sum_{e\in H} |I(H)_e| = |I(H)| = \sum_{i\in N} |H_i|. \tag{6.2.1}$$

These equalities show that in the incidence graph the number of links equals the sum of the degrees of the nodes in any of the two parts, N and H, and it also equals the total degree of the underlying hypergraph.

Now, we consider hypergraph games. For the sake of convenience and to avoid that additional notation may distract from the main result, throughout this chapter we focus attention on zero-normalized hypergraph games, but all results also hold for arbitrary hypergraph games. Under the interpretation that the degree of a player in a hypergraph game represents the player's ability to cooperate, we first consider the worth of the cooperative abilities among the players. For a hypergraph game $(N, v, H) \in \widehat{\mathcal{G}}_N^{\mathcal{H}}$, assuming that each element $\{i, e\} \in I(H)$ represents a degree[1] of player $i \in N$, we have the following definition.

Definition 6.2.1. For any $(N, v, H) \in \widehat{\mathcal{G}}_N^{\mathcal{H}}$, the *degree game* $(I(H), \bar{v}) \in \mathcal{G}_{I(H)}$ is given as

$$\bar{v}(S) = v^N(H[S]) = \sum_{K \in N/H[S]} v(K), \text{ for all } S \subseteq I(H), \tag{6.2.2}$$

where $H[S] = \{e \in H : I(H)_e \subseteq S\}$, $S \subseteq I(H)$.

For the degree game of a hypergraph game, the worth of a set of degrees is the worth of the corresponding hyperlinks in the hyperlink game. Note that a hyperlink $e \in H$ is divided into the link set $I(H)_e$ in $I(H)$. Therefore, the worth of coalition $S \subseteq I(H)$ in the degree game is produced by the hyperlinks that are induced by S on the hyperlink game.

Next we provide the definition of the degree value for hypergraph games.

Definition 6.2.2. On the class of hypergraph games, the *degree value* assigns to every $(N, v, H) \in \widehat{\mathcal{G}}_N^{\mathcal{H}}$, a payoff vector $\mathcal{D}(N, v, H)$ given by

$$\mathcal{D}_i(N, v, H) = \sum_{l \in I(H)_i} Sh_l(I(H), \bar{v}), \quad \text{for all } i \in N. \tag{6.2.3}$$

The degree value for hypergraph games first assigns to each degree a Shapley payoff on the degree game deduced by a hypergraph game, and then it distributes to each player the sum of the Shapley payoffs of all his degrees. The allocation rule is called the degree value due to the fact that it is obtained from determining the

[1] Here we abuse the notion of the degree to represent the ability of a player to cooperate, i.e., the links of the incidence graph.

payoffs of the players' degrees. Note that if $(N, v, H) \in \mathcal{G}_N^{\mathcal{H}}$ is not zero-normalized, then the degree value is given by

$$\mathcal{D}_i(N, v, H) = v(\{i\}) + \sum_{l \in I(H)_i} Sh_l(I(H), \overline{v^0}), \qquad \text{for all } i \in N,$$

where v^0 is the zero-normalized characteristic function of v, defined by, $v^0(S) = v(S) - \sum_{i \in S} v(\{i\})$, for all $S \subseteq N$.

From the definition, we can see that the degree value highlights the role of degree, in which a player's degree reflects a kind of cooperative ability of this player. Therefore, this value possesses a new perspective which differs from the Myerson value, the position value, and the (two-step) average tree value, which values emphasize the role of the players, the hyperlinks, and connectivity, respectively.

In order to illustrate the degree value, we consider the following example.

Example 6.2.2. Consider the hypergraph game $(N, v, H) \in \widehat{\mathcal{G}}_N^{\mathcal{H}}$, where $N = \{1, 2, \ldots, 6\}$ is the player set, $v = u_{\{1,2,3\}}$, and $H = \{e_1, e_2, e_3, e_4\}$ is the set of hyperlinks, where $e_1 = \{1, 4\}, e_2 = \{2, 5\}, e_3 = \{3, 6\}, e_4 = \{4, 5, 6\}$. The hypergraph is depicted in Figure 6.2.

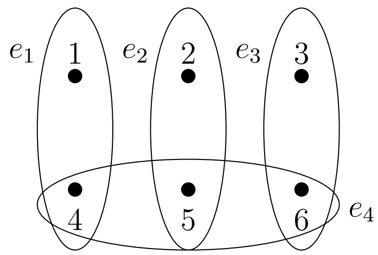


Figure 6.2: The underlying hypergraph in Example 6.2.2.

By Definition 6.2.1, we have the deduced degree game $(I(H), \bar{v})$ in which the player set equals

$$I(H) = \{\{1, e_1\}, \{2, e_2\}, \{3, e_3\}, \{4, e_1\}, \{4, e_4\}, \{5, e_2\}, \{5, e_4\}, \{6, e_3\}, \{6, e_4\}\},$$

and the characteristic function $\bar{v}$ is given by

$$\bar{v}(S) = \begin{cases} 1, & \text{if } S = I(H), \\ 0, & \text{otherwise.} \end{cases}$$

Hence, we have $Sh_l(I(H), \bar{v}) = \frac{1}{9}$ for all $l \in I(H)$, and the degree value is given by

$$\mathcal{D}_1(N,v,H) = \mathcal{D}_2(N,v,H) = \mathcal{D}_3(N,v,H) = \frac{1}{9},$$
$$\mathcal{D}_4(N,v,H) = \mathcal{D}_5(N,v,H) = \mathcal{D}_6(N,v,H) = \frac{1}{9} + \frac{1}{9} = \frac{2}{9}.$$

Note that $\mu_i(N,v,H) = AT_i(N,v,H) = ATT_i(N,v,H) = \frac{1}{6}$ for all $i \in N$ and the position value is $\pi(N,v,H) = (\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{5}{24}, \frac{5}{24}, \frac{5}{24})$. It is clear that the degree value is a new allocation rule for hypergraph games that differs from both the Myerson value and the position value. Additionally, it can be seen that in this example the degree value distributes a higher payoff to the players who have a larger degree than the Myerson and position value payoffs, e.g., $\mathcal{D}_6(N,v,H) > \pi_6(N,v,H) > \mu_6(N,v,H)$. Even though the position value also has somehow the feature that benefits the players with a large degree, the degree value may strengthen this effect because $\mathcal{D}_6(N,v,H) > \pi_6(N,v,H)$. □

6.3 Axiomatizations of the degree value

In this section we give axiomatic characterizations of the degree value, first for the subclass of cycle-free hypergraph games and subsequently for arbitrary hypergraph games.

6.3.1 Cycle-free hypergraph games

This subsection provides two characterizations of the degree value on the class of cycle-free hypergraph games. Motivated by the studies in van den Nouweland et al. (1992), van den Brink et al. (2011), and Algaba et al. (2015), two new axioms are introduced to characterize the degree value. Both characterizations follow the research line of the position value as in van den Nouweland et al. (1992). In fact, except for the influence property (see Section 2.3), the degree value satisfies all the properties in the characterization of the position value provided in van den Nouweland et al. (1992). The following example shows that the degree value fails the influence property.

Example 6.3.1. Consider the hypergraph game as described in Example 6.2.2.

Then the influence of the players is given by

$$I_i(N,H) = \sum_{e \in H_i} |e|^{-1} = \frac{1}{2} \text{ for } i = 1,2,3$$

and

$$I_i(N,H) = \sum_{e \in H_i} |e|^{-1} = \frac{5}{6} \text{ for } i = 4,5,6.$$

It can be verified that (N, v, H) is conference anonymous, since, for any $A \subseteq H$, $v^N(A) = f(|A|)$, where

$$f(k) = \begin{cases} 0, & \text{if } k \in \{0,1,2,3\}; \\ 1, & \text{if } k = 4. \end{cases}$$

By the influence of the players, it holds that $\pi_i(N,v,H) = \frac{1}{4} I_i(N,H)$, for all $i \in N$. But, there does not exist a real number α such that $\mathcal{D}_i(N,v,H) = \alpha I_i(N,H)$ for all $i \in N$. Therefore, the degree value does not satisfy the influence property. $\square$

In order to give a characterization of the degree value, we first introduce an axiom called the degree property. This property is different from the degree property for graph games in Borm et al. (1992) and is also not an extension. Similar to conference anonymity, we introduce so-called degree anonymity. A hypergraph game $(N, v, H) \in \widehat{\mathcal{G}}_N^{\mathcal{H}}$ is *degree anonymous* if there exists a function $g : \{0, 1, \ldots, |I(H)|\} \to \mathbb{R}$ such that

$$\bar{v}(S) = g(|S|), \text{ for all } S \subseteq I(H). \tag{6.3.1}$$

The degree property: For any degree anonymous $(N, v, H) \in \widehat{\mathcal{G}}_N^{\mathcal{H}}$, there exists $\alpha \in \mathbb{R}$ such that

$$\xi(N,v,H) = \alpha d(N,H),$$

where $d(N,H) \in \mathbb{R}^n$ is the *degree vector* in hypergraph $(N,H) \in \mathcal{H}_N$ given by $d_i(N,H) = |H_i|$ for all $i \in N$.

The degree property says that if a hypergraph game is degree anonymous then the payoff of a player can be expressed as a ratio of the player's degree. Note that in the influence property the payoff of a player is a ratio of the player's influence which is different from the degree property.

The following result exhibits some of the properties that are satisfied by the degree value.

Lemma 6.3.1. *On the class of hypergraph games, the degree value satisfies component efficiency, linearity, the superfluous conference property, and the degree property.*

Proof. Let $(N, v, H) \in \widehat{\mathcal{G}}_N^{\mathcal{H}}$ be a hypergraph game and $K \in N/H$. Then we have

$$\begin{aligned}
\sum_{i \in K} \mathcal{D}_i(N, v, H) &= \sum_{i \in K} \sum_{l \in I(H)_i} Sh_l(I(H), \bar{v}) \\
&= \sum_{l \in I(H(K))} Sh_l(I(H), \bar{v}) \\
&= \sum_{l \in I(H(K))} Sh_l(I(H(K)), \bar{v}_{I(H(K))}) \\
&= \bar{v}_{I(H(K))}\big(I(H(K))\big) = \bar{v}\big(I(H(K))\big) = v(K),
\end{aligned}$$

where the second equality follows from the fact that

$$\bigcup_{i \in K} I(H)_i = \{\{i, e\} \in I(H) : i \in K\} = \{\{i, e\} \in I(H) : e \in H(K)\} = I(H(K)),$$

and the third equality follows from the Shapley value in (2.1.4), from (2.1.3), and from the fact that

$$\begin{aligned}
\bar{v}(S \cup \{l\}) - \bar{v}(S) &= \bar{v}\big((S \cap I(H(K))) \cup \{l\}\big) - \bar{v}\big(S \cap I(H(K))\big) \\
&= \bar{v}_{I(H(K))}\big((S \cap I(H(K))) \cup \{l\}\big) - \bar{v}_{I(H(K))}\big(S \cap I(H(K))\big),
\end{aligned}$$

for any $S \subseteq I(H) \setminus \{l\}$ such that $l \in I(H(K))$, where $\bar{v}_{I(H(K))}$ is the subgame of $\bar{v}$ on player set $I(H(K))$. This shows component efficiency.

For any $(N, v, H), (N, w, H) \in \widehat{\mathcal{G}}_N^{\mathcal{H}}$, $a, b \in \mathbb{R}$, and $S \subseteq I(H)$, we have the following fact about their degree games

$$\begin{aligned}
(\overline{av + bw})(S) &= (av + bw)^N(H[S]) = \sum_{K \in N/H[S]} (av + bw)(K) \\
&= \sum_{K \in N/H[S]} \big(av(K) + bw(K)\big) \\
&= av^N(H[S]) + bw^N(H[S]) = a\bar{v}(S) + b\bar{w}(S).
\end{aligned}$$

Hence, it turns out that, for every $i \in N$,

$$\begin{aligned}\mathcal{D}_i(N, av+bw, H) &= \sum_{l \in I(H)_i} Sh_l(I(H), \overline{av+bw}) \\ &= a \sum_{l \in I(H)_i} Sh_l(I(H), \bar{v}) + b \sum_{l \in I(H)_i} Sh_l(I(H), \bar{w}) \\ &= a\mathcal{D}_i(N, v, H) + b\mathcal{D}_i(N, w, H),\end{aligned}$$

which shows that $\mathcal{D}$ satisfies linearity.

Now suppose $e \in H$ is a superfluous hyperlink for $(N, v, H) \in \widehat{\mathcal{G}}_N^{\mathcal{H}}$, i.e., it holds that $v^N(A \setminus \{e\}) = v^N(A)$ for all $A \subseteq H$. Then, for any $S \subseteq I(H)$, we see that $e \notin H[S \setminus \{l\}]$ for all $l \in I(H)_e$, and

$$\begin{aligned}\bar{v}(S \setminus \{l\}) &= v^N(H[S \setminus \{l\}]) = v^N(H[S] \setminus \{e\}) \\ &= v^N(H[S]) = \bar{v}(S), \quad \text{for all } l \in I(H)_e,\end{aligned}$$

where the third equality follows because $e \in H$ is superfluous. Hence, each element of $I(H)_e$ is a null-player in $(I(H), \bar{v})$, and therefore $Sh_l(I(H), \bar{v}) = 0$ for all $l \in I(H)_e$. From this we obtain

$$\begin{aligned}\mathcal{D}_i(N, v, H) &= \sum_{l \in I(H)_i} Sh_l(I(H), \bar{v}) = \sum_{l \in I(H)_i \setminus I(H)_e} Sh_l(I(H), \bar{v}) \\ &= \sum_{l \in I(H \setminus \{e\})_i} Sh_l(I(H \setminus \{e\}), \bar{v}_{I(H \setminus \{e\})}) \\ &= \mathcal{D}_i(N, v, H \setminus \{e\}), \quad \text{for all } i \in N,\end{aligned}$$

where the second equality follows from the null-player property of the Shapley value.

Suppose $(N, v, H) \in \widehat{\mathcal{G}}_N^{\mathcal{H}}$ is degree anonymous and let g be as in (6.3.1). Since all $l \in I(H)$ are symmetric in the degree game $(I(H), \bar{v})$, we obtain

$$Sh_l(I(H), \bar{v}) = \bar{v}(I(N)) \cdot |I(H)|^{-1} = v^N(H) \cdot |I(H)|^{-1}, \text{ for all } l \in I(H).$$

Therefore,

$$\begin{aligned}\mathcal{D}_i(N,v,H) &= \sum_{l\in I(H)_i} Sh_l(I(H),\bar{v}) \\ &= \sum_{l\in I(H)_i} v^N(H)\cdot |I(H)|^{-1} \\ &= |I(H)_i|\cdot v^N(H)\cdot |I(H)|^{-1} \\ &= |H_i|\cdot v^N(H)\cdot |I(H)|^{-1}, \qquad \text{for all } i\in N.\end{aligned}$$

Let $\alpha = v^N(H)\cdot |I(H)|^{-1}$, then the degree property is proved. □

Let $\widehat{\mathcal{G}}_N^{\mathcal{H}^{cf}}$. denote the subclass of cycle-free zero-normalized hypergraph games on N. The following result shows that the four axioms in Lemma 6.3.1 exactly determine one solution if the underlying hypergraph is cycle-free. The proof is similar as the uniqueness proof for the position value in van den Nouweland et al. (1992).

Lemma 6.3.2. *On the class of cycle-free hypergraph games, component efficiency, linearity, the superfluous conference property, and the degree property determine a unique solution.*

Proof. Let ξ be an allocation rule that satisfies the four axioms, and let $(N,v,H)\in \widehat{\mathcal{G}}_N^{\mathcal{H}^{cf}}$ be a cycle-free hypergraph game such that $v=u_T$, for some $T\subseteq N$ with $|T|\geq 2$. Since ξ satisfies linearity, it is sufficient to prove that ξ can be uniquely determined on (N,v,H) by the other axioms. Now we consider the following two cases:

Case 1. If $T\nsubseteq S$ for any $S\in C^H(N)$, i.e., T is not contained in any feasible coalition, then $v^N(A)=0$ for all $A\subseteq H$, since v is zero-normalized. Hence, by component efficiency and the degree property, we have $\xi_i(N,v,H)=0$ for all $i\in N$.

Case 2. If there exists a feasible coalition containing T, then there exists a unique feasible coalition $\bar{T}\in C^H(N)$ satisfying $T\subseteq \bar{T}$ and $\bar{T}\subseteq S$ for any $S\in C^H(N)$ such that $T\subseteq S$, i.e., $\bar{T}$ is the unique minimal feasible coalition containing T. Note that only cycle-free hypergraphs can guarantee that $\bar{T}$ is unique, see van den Nouweland et al. (1992), which is a generalization of the results in Owen (1986).

Therefore, for any $A \subseteq H$, we have

$$v^N(A) = \begin{cases} 1, & \text{if } H(\bar{T}) \subseteq A, \\ 0, & \text{otherwise.} \end{cases}$$

This implies that all hyperlinks in $H \setminus H(\bar{T})$ are superfluous in (N, v, H). By the superfluous conference property, it holds that

$$\xi(N, v, H) = \xi(N, v, H(\bar{T})).$$

Furthermore, $(N, v, H(\bar{T}))$ is degree anonymous because there exists a function g such that $g(0) = \cdots = g(|H(\bar{T})| - 1) = 0$, while $g(|H(\bar{T})|) = 1$. According to component efficiency and the degree property, we have

$$\xi_i(N, v, H(\bar{T})) = \frac{|H(\bar{T})_i|}{\sum_{j \in N} |H(\bar{T})_j|} = \frac{|H(\bar{T})_i|}{|I(H(\bar{T}))|}, \quad \text{for all } i \in N.$$

□

From Lemma 6.3.1 and Lemma 6.3.2, we obtain the following result.

Theorem 6.3.1. *On the class of cycle-free hypergraph games, the degree value is the unique allocation rule that satisfies component efficiency, linearity, the superfluous conference property, and the degree property.*

The second characterization follows along a similar approach as the characterization of the position value in van den Brink et al. (2011) as well as in Algaba et al. (2015). In van den Brink et al. (2011), an axiom called the degree measure property is provided to replace the degree property for graph games. By adopting a similar method, we propose the following axiom.

A hypergraph game $(N, v, H) \in \widehat{\mathcal{G}}_N^{\mathcal{H}}$ is *conference unanimous* if (N, v, H) is conference anonymous with $f(|H|) = v^N(H)$ and $f(|A|) = 0$ for all $A \subsetneq H$, in other words, (N, v, H) is conference unanimous if $v^N(A) = 0$ for all $A \subsetneq H$.

The degree measure property: For any conference unanimous $(N, v, H) \in \widehat{\mathcal{G}}_N^{\mathcal{H}}$, there exists $\alpha \in \mathbb{R}$ such that

$$\xi(N, v, H) = \alpha d(N, H).$$

The degree measure property states that if a hypergraph game is conference unanimous then the payoff of a player can be expressed as a ratio of the player's degree. Note that the degree measure property differs from the degree property due to the different preconditions. However, the next lemma shows that the degree measure property and the degree property are equivalent.

Lemma 6.3.3. *Let $(N, v, H) \in \widehat{\mathcal{G}}_N^{\mathcal{H}}$ be a hypergraph game. Then (N, v, H) is degree anonymous if and only if (N, v, H) is conference unanimous.*

Proof. It is clear that if (N, v, H) is conference unanimous, then (N, v, H) is degree anonymous with $g(|S|) = 0$ for all $S \subsetneq I(H)$. Next, we prove that if $(N, v, H) \in \widehat{\mathcal{G}}_N^{\mathcal{H}}$ is degree anonymous, then the function g defined in (6.3.1) satisfies $g(k) = 0$ for $k = 0, \ldots, |I(H)| - 1$. Since $\bar{v}(I(H)) = v^N(H)$, it is sufficient to prove that $\bar{v}(S) = v^N(H[S]) = 0$ for all $S \subsetneq I(H)$.

It is trivial for $|H| = 1$ since in that case $H[S] = \emptyset$ for all $S \subsetneq I(H)$. Now we consider $|H| > 1$. For any $S \subsetneq I(H)$ such that $|H[S]| \geq 1$ and $|S| \leq |I(H)| - 2$, there exists $S' \subseteq I(H)$ satisfying a) $|S'| = |S|$ and $|H[S']| < |H[S]|$, or b) $|S'| < |S|$ and $H[S'] = H[S]$. In case a), we obtain that

$$v^N(H[S']) = g(|S'|) = g(|S|) = v^N(H[S]),$$

which implies

$$v^N(\emptyset) = v^N(A) = 0,$$

for every $A \subsetneq H$. That is, $\bar{v}(S) = v^N(H[S]) = 0$ for all $S \subsetneq I(H)$. In case b), let S' be the minimal subset satisfying $H[S'] = H[S]$, then we have

$$g(|S'|) = v^N(H[S']) = v^N(H[S]) = g(|S|).$$

Replacing S by S' and repeating the step of case a), this completes the proof. □

From the lemma, we can deduce that the degree property is equivalent to the degree measure property. Therefore, the next result follows immediately from Theorem 6.3.1 and Lemma 6.3.3.

Corollary 6.3.1. *On the class of cycle-free hypergraph games, the degree value is the unique allocation rule that satisfies component efficiency, linearity, the superfluous conference property, and the degree measure property.*

The following example shows the logical independence of the axioms in Theorem 6.3.1 and Corollary 6.3.1. By providing four different allocation rules that each of them does not satisfy a specific axiom but does satisfy the remaining axioms, we prove that all of the axioms are non-redundant.

Example 6.3.2. The following four allocation rules illustrate the independence of component efficiency, linearity, the superfluous conference property, and the degree property, or the degree measure property.

- Let the allocation rule ξ^1 on $\widehat{\mathcal{G}}_N^{\mathcal{H}^{cf}}$ be given by

$$\xi_i^1(N, v, H) = 0, \text{ for all } i \in N.$$

  This solution satisfies all properties except component efficiency.

- Let the allocation rule ξ^2 on $\widehat{\mathcal{G}}_N^{\mathcal{H}^{cf}}$ be given by

$$\xi_i^2(N, v, H) = \begin{cases} \frac{|H_i| \cdot v(K)}{|I(H(K))|}, & \text{if } i \in K, K \in N/H \text{ with } |K| \geq 2, \\ 0, & \text{if } \{i\} \in N/H. \end{cases}$$

  Linearity and component efficiency follow immediately from the definition of this solution. Take any conference unanimous $(N, v, H) \in \widehat{\mathcal{G}}_N^{\mathcal{H}^{cf}}$. If $|N/H| \geq 2$ and $1 \leq |H(K)| < |H|$ for some $K \in N/H$, then it implies $v(K) = 0$ for all $K \in N/H$ due to $v^N(A) = 0$ for all $A \subsetneq H$. So, $\alpha = 0$ and $\xi_i^2(N, v, H) = 0$ for all $i \in N$. If $|N/H| \geq 2$ and $H(K) = H$ for some $K \in N/H$, then let $\alpha = \frac{v(K)}{|I(H(K))|}$ for $K \in N/H$ such that $H(K) = H$. Note that $|H_i| = 0$ for all $i \in N \setminus K$. If $|N/H| = 1$, then there exists $\alpha = \frac{v(N)}{|I(H)|}$ such that $\xi^2(N, v, H) = \alpha d(N, H)$. Therefore, this solution satisfies the degree measure property, and from Lemma 6.3.3, it also satisfies the degree property. Take any $(N, v, H) \in \widehat{\mathcal{G}}_N^{\mathcal{H}^{cf}}$ and superfluous $e \in H$. Let $H' = H \setminus \{e\}$, then $\xi_i^2(N, v, H') = \frac{|H_i'| \cdot v(K)}{|I(H'(K))|}$, if $i \in e \cap K$ and $K \in N/H'$ with $|K| \geq 2$, otherwise, $\xi_i^2(N, v, H') = 0$ if $i \in e$ and $|H_i'| = 0$. Therefore, in general $\xi^2(N, v, H') \neq \xi^2(N, v, H)$, which implies that this solution fails the superfluous conference property.

- Let the allocation rule ξ^3 on $\widehat{\mathcal{G}}_N^{\mathcal{H}^{cf}}$ be given by

$$\xi^3(N, v, H) = \begin{cases} \xi^2(N, v, H), & \text{if } (N, v, H) \text{ is conference unanimous}, \\ \xi^2(N, v, \bar{H}), & \text{else}, \end{cases}$$

  where $\bar{H} \subseteq H$ is the set of hyperlinks that are not superfluous in (N, v, H). The degree property and the degree measure property follow directly from the definitions of ξ^3 and ξ^2. To prove component efficiency, take any $(N, v, H) \in \widehat{\mathcal{G}}_N^{\mathcal{H}^{cf}}$. If (N, v, H) is conference unanimous, then component efficiency follows from ξ^2. If (N, v, H) is not conference unanimous, then we consider $(N, v, \bar{H})$ and any $K \in N/H$ containing some superfluous hyperlinks in $(N, v, \bar{H})$. Note that $|N/\bar{H}| \geq |N/H|$. Let $K_1, \ldots, K_k$ be the components of K in $(N, \bar{H})$, then we have

$$\begin{aligned}\sum_{i \in K} \xi_i^3(N, v, H) &= \sum_{j=1}^{k} \sum_{i \in K_j} \xi_i^2(N, v, \bar{H}) = \sum_{j=1}^{k} v(K_j) = \sum_{T \in N/\bar{H}(K)} v(T) \\ &= v^N(\bar{H}(K)) = v^N(H(K)) = v(K),\end{aligned}$$

where the fifth equality follows from the fact that all hyperlinks in $H(K) \setminus \bar{H}(K)$ are superfluous. Therefore, this solution satisfies component efficiency. To prove the superfluous conference property, let $e \in H$ be superfluous for a given $(N, v, H) \in \widehat{\mathcal{G}}_N^{\mathcal{H}^{cf}}$. If (N, v, H) is conference unanimous, then $v^N(H) = v^N(H \setminus \{e\}) = 0$, which implies $v^N(A) = 0$, for all $A \subseteq H$, and $\xi_i^3(N, v, H \setminus \{e\}) = 0 = \xi_i^3(N, v, H)$, for all $i \in N$. Suppose (N, v, H) is not conference unanimous, we consider two cases. If $(N, v, H \setminus \{e\})$ is not conference unanimous, then it follows that $\xi^3(N, v, H \setminus \{e\}) = \xi^2(N, v, \overline{H \setminus \{e\}}) = \xi^2(N, v, \bar{H})$. If $(N, v, H \setminus \{e\})$ is conference unanimous, then $(N, v, \bar{H})$ is conference unanimous due to $v^N(A) = 0$ for all $A \subseteq \bar{H} \subsetneq H$. Therefore,

$$\xi^3(N, v, H \setminus \{e\}) = \xi^2(N, v, H \setminus \{e\}) = \xi^2(N, v, \bar{H}) = \xi^3(N, v, H).$$

Hence, ξ^3 satisfies the superfluous conference property. The next example shows that ξ^3 fails linearity. Consider $(N, v, H) \in \widehat{\mathcal{G}}_N^{\mathcal{H}^{cf}}$, where $N = \{1, \ldots, 4\}$, $v = u_{\{1,2\}} + u_{\{2,3\}}$, and $H = \{\{1, 4\}, \{2, 4\}, \{3, 4\}\}$. It is clear that (N, v, H) is not conference unanimous and has no superfluous conference. Hence, $\xi^3(N, v, H) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1)$. However, $\xi^3(N, u_{\{1,2\}}, H) = (\frac{1}{4}, \frac{1}{4}, 0, \frac{1}{2})$ and $\xi^3(N, u_{\{2,3\}}, H) = (0, \frac{1}{4}, \frac{1}{4}, \frac{1}{2})$. So, we have $\xi^3(N, v, H) \neq \xi^3(N, u_{\{1,2\}}, H) + \xi^3(N, u_{\{2,3\}}, H)$.

- Let the allocation rule ξ^4 on $\widehat{\mathcal{G}}_N^{\mathcal{H}^{cf}}$ be given by $\xi^4(N, v, H) = \pi(N, v, H)$. The position value satisfies efficiency, linearity, and the superfluous conference property, see van den Nouweland et al. (1992). The hypergraph game as in Example 6.2.2 is degree anonymous and conference unanimous. It shows that there does not exist a real number $\alpha \in \mathbb{R}$ such that $\pi(N, v, H) =$

$\alpha d(N, H)$, where $\pi(N, v, H) = (\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{5}{24}, \frac{5}{24}, \frac{5}{24})$ and $d(N, H) = (1, 1, 1, 2, 2, 2)$. Therefore, ξ^4 fails the degree property and the degree measure property.

□

6.3.2 Arbitrary hypergraph games

From Lemma 6.3.1 and Lemma 6.3.3 we see that on the class of all hypergraph games the degree value satisfies component efficiency, linearity, the superfluous property, and both the degree property and the degree measure property. However, in the proof of Theorem 6.3.2, only cycle-free hypergraph games can be characterized by these axioms to characterize the degree value. In this subsection we provide a characterization for arbitrary hypergraph games by employing a deletion hyperlink property.

Before presenting the characterization, we first recall an axiom proposed in Slikker (2005) for graph games, called balanced link contributions (see, Section 2.3). This axiom says that the total link contributions of two players towards each other are equal, where each link contribution of a player, say $i \in N$, towards another player, say $j \in N$, is the sum of the payoff differences of player j by breaking any one of the incident links of i. This axiom in Slikker (2005) is used to characterize the position value for graph games. Now we extend this axiom to hypergraph games as follows.

Balanced conference contributions: For any $(N, v, H) \in \mathcal{G}, \mathcal{G} \subseteq \widehat{\mathcal{G}}_N^{\mathcal{H}}$, and $i, j \in N$, it holds that

$$\sum_{e \in H_j} \big(\xi_i(N, v, H) - \xi_i(N, v, H \setminus \{e\})\big) = \sum_{e \in H_i} \big(\xi_j(N, v, H) - \xi_j(N, v, H \setminus \{e\})\big).$$

Balanced conference contributions follows the idea of balanced link contributions and also deals with the gains of players contributing to each other. Formally, the total conference contributions of a player towards another player equals the reverse total conference contributions, where the total conference contributions of a player towards another player is the sum of the payoff changes a player can influence another player by breaking one of his conferences.

It is clear that balanced conference contributions is a natural extension of balanced link contributions from graph games to hypergraph games. Note that if the underlying structure is a graph then balanced conference contributions coincides with balanced link contributions. However, the position value for hypergraph games does not satisfy balanced conference contributions in general (see Example 6.3.3). Instead, Shan and Zhang (2016) show that the position value satisfies partial balanced conference contributions (see Section 2.3). In the sequel, we show that the degree value satisfies the property of balanced conference contributions and it can be uniquely characterized by this property together with component efficiency.

First, we examine a property of the degree value regarding the deletion of hyperlinks. The following lemma says that the effect of deleting a hyperlink is the same as removing any element induced by this hyperlink in the degree game.

Lemma 6.3.4. *For any* $(N, v, H) \in \widehat{\mathcal{G}}_N^{\mathcal{H}}$, $i \in N$, $e \in H$, *and* $l' \in I(H)_e$, *it holds that*

$$\mathcal{D}_i(N, v, H \setminus \{e\}) = \sum_{l \in (I(H)\setminus\{l'\})_i} Sh_l\big(I(H) \setminus \{l'\}, \bar{v}_{I(H)\setminus\{l'\}}\big), \text{ for all } i \in N.$$

Proof. Take any $i \in N$. By equation (6.2.3), we have

$$\mathcal{D}_i(N, v, H \setminus \{e\}) = \sum_{l \in I(H\setminus\{e\})_i} Sh_l(I(H \setminus \{e\}), \bar{v}_{I(H\setminus\{e\})}). \tag{6.3.2}$$

Hence, it is sufficient to prove that

$$\sum_{l \in I(H\setminus\{e\})_i} Sh_l(I(H \setminus \{e\}), \bar{v}_{I(H\setminus\{e\})}) = \sum_{l \in (I(H)\setminus\{l'\})_i} Sh_l(I(H) \setminus \{l'\}, \bar{v}_{I(H)\setminus\{l'\}}). \tag{6.3.3}$$

Since $l' \in I(H)_e$ and $I(H \setminus \{e\}) = I(H) \setminus I(H)_e$, it holds that $I(H \setminus \{e\}) \subseteq I(H) \setminus \{l'\}$. Therefore, for any $S \subseteq I(H) \setminus \{l'\}$, by the definition of the degree game $\bar{v}$, we have

$$\begin{aligned} \bar{v}_{I(H)\setminus\{l'\}}(S) &= \bar{v}(S \cap (I(H) \setminus \{l'\})) \\ &= \bar{v}(S \cap I(H \setminus \{e\})) \\ &= \bar{v}_{I(H\setminus\{e\})}(S \cap I(H \setminus \{e\})). \end{aligned}$$

The second equation holds since the players in $(I(H) \setminus \{l'\}) \setminus (I(H \setminus \{e\}))$ are null-players in $(I(H) \setminus \{l'\}, \bar{v}_{I(H)\setminus\{l'\}})$.

Hence, it turns out that

$$Sh_l(I(H) \setminus \{l'\}, \bar{v}_{I(H)\setminus\{l'\}}) = Sh_l(I(H \setminus \{e\}), \bar{v}_{I(H\setminus\{e\})}), \text{ for all } l \in I(H \setminus \{e\}),$$

and

$$Sh_l(I(H) \setminus \{l'\}, \bar{v}_{I(H)\setminus\{l'\}}) = 0, \text{ for all } l \in (I(H) \setminus \{l'\}) \setminus (I(H \setminus \{e\})).$$

This implies that equation (6.3.3) holds. □

The next lemma gives an alternative expression of the degree value. This expression is obtained by applying the Harsanyi dividends, which is similar to the formulation of the Shapley value in (2.1.5).

Lemma 6.3.5. *For any $(N, v, H) \in \widehat{\mathcal{G}}_N^{\mathcal{H}}$, it holds that*

$$\mathcal{D}_i(N, v, H) = \sum_{S \subseteq I(H)} \Delta_{\bar{v}}(S) \frac{|S \cap I(H)_i|}{|S|}, \text{ for all } i \in N.$$

Proof. The degree game $\bar{v}$ on player set $I(H)$ can be represented by a unique linear combination of unanimity games,

$$\bar{v} = \sum_{S \subseteq I(H)} \Delta_{\bar{v}}(S) u_S.$$

According to equations (6.2.3) and (2.1.5), for any $i \in N$, we have

$$\begin{aligned} \mathcal{D}_i(N, v, H) &= \sum_{l \in I(H)_i} Sh_l(I(H), \bar{v}) \\ &= \sum_{l \in I(H)_i} \sum_{S \subseteq I(H): S \ni l} \frac{\Delta_{\bar{v}}(S)}{|S|} \\ &= \sum_{S \subseteq I(H)} \Delta_{\bar{v}}(S) \frac{|S \cap I(H)_i|}{|S|}, \end{aligned}$$

which completes the proof. □

Using Lemma 6.3.4 and Lemma 6.3.5, we achieve a result about the degree value for hypergraph games, as is stated in the following lemma.

Lemma 6.3.6. *On the class of hypergraph games, the degree value satisfies balanced conference contributions.*

Proof. Let $(N, v, H) \in \widehat{\mathcal{G}}_N^{\mathcal{H}}$ be a hypergraph game and $i, j \in N$ with $i \neq j$. Then, by Lemma 6.3.4 and Lemma 6.3.5, we have

$$\begin{aligned}
&\sum_{e \in H_j} [\mathcal{D}_i(N, v, H) - \mathcal{D}_i(N, v, H \setminus \{e\})] \\
= &\sum_{l \in I(H)_j} \Big[\sum_{S \subseteq I(H)} \Delta_{\bar{v}}(S) \frac{|S \cap I(H)_i|}{|S|} - \sum_{S \subseteq I(H) \setminus \{l\}} \Delta_{\bar{v}}(S) \frac{|S \cap I(H)_i|}{|S|} \Big],
\end{aligned}$$

where $l = \{j, e\}$ for some $e \in H_j$. Note that for every $l \in I(H)_j$ it holds that $\Delta_{\bar{v}_{I(H)\setminus\{l\}}}(S) = \Delta_{\bar{v}}(S)$, for all $S \subseteq I(H) \setminus \{l\}$, and $I(H)_i \setminus \{l\} = I(H)_i$ due to $i \neq j$. As a consequence,

$$\begin{aligned}
&\sum_{e \in H_j} [\mathcal{D}_i(N, v, H) - \mathcal{D}_i(N, v, H \setminus \{e\})] \\
= &\sum_{l \in I(H)_j} \sum_{S \subseteq I(H): S \ni l} \Delta_{\bar{v}}(S) \frac{|S \cap I(H)_i|}{|S|} \\
= &\sum_{S \subseteq I(H)} |S \cap I(H)_j| \Delta_{\bar{v}}(S) \frac{|S \cap I(H)_i|}{|S|} \\
= &\sum_{S \subseteq I(H)} \Delta_{\bar{v}}(S) \frac{|S \cap I(H)_i| \cdot |S \cap I(H)_j|}{|S|} \\
= &\sum_{e \in H_i} [\mathcal{D}_j(N, v, H) - \mathcal{D}_j(N, v, H \setminus \{e\})],
\end{aligned}$$

where the third equality follows by the symmetry of players i and j. □

The following example illustrates Lemma 6.3.6. In addition, it also shows that the Myerson value and position value do not satisfy balanced conference contributions.

Example 6.3.3. Consider the hypergraph game as described in Example 6.2.2. For the degree value, the Myerson value, and the position value, the payoffs are exhibited in Table 6.1. For the degree value, the total hyperlink contributions of player 6 to player 1 equals $(\frac{1}{9} - 0) + (\frac{1}{9} - 0) = \frac{2}{9}$, by breaking successively hyperlink e_3 and hyperlink e_4. The reverse contributions of player 1 to player 6 equals $\frac{2}{9} - 0 = \frac{2}{9}$ as well. Hence, it shows that the degree value indeed satisfies balanced conference contributions.

However, the total hyperlink contributions towards players may be not balanced when applying the Myerson value or the position value. Indeed, according

Hyperlinks A	$\mathcal{D}(N,v,A)$	$\mu(N,v,A)$	$\pi(N,v,A)$
$A=H$	$(\frac{1}{9},\frac{1}{9},\frac{1}{9},\frac{2}{9},\frac{2}{9},\frac{2}{9})$	$(\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6})$	$(\frac{1}{8},\frac{1}{8},\frac{1}{8},\frac{5}{24},\frac{5}{24},\frac{5}{24})$
$A\subsetneq H$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$	$(0,0,0,0,0,0)$

Table 6.1: Degree values, Myerson values, and position values for Example 6.3.3.

to the Myerson value (μ), the total hyperlink contributions of player 1 to player 6 equals $\frac{1}{6}-0=\frac{1}{6}$, while the reverse contributions equals $(\frac{1}{6}-0)+(\frac{1}{6}-0)=\frac{1}{3}$. According to the position value (π), these contributions are $\frac{5}{24}-0=\frac{5}{24}$ and $(\frac{1}{8}-0)+(\frac{1}{8}-0)=\frac{1}{4}$, respectively. □

Now we provide a characterization of the degree value by using the result of Lemma 6.3.6. The proof strategy is similar to the proof of the uniqueness of the position value for graph games in Slikker (2005).

Theorem 6.3.2. *On the class of hypergraph games, the degree value is the unique allocation rule that satisfies component efficiency and balanced conference contributions.*

Proof. As Lemma 6.3.1 and Lemma 6.3.6 state, the degree value for hypergraph games satisfies component efficiency and balanced conference contributions. Conversely, by assuming that an allocation rule ξ on $\widehat{\mathcal{G}}_N^{\mathcal{H}}$ satisfies the two axioms, we show that ξ coincides with $\mathcal{D}$. The proof will be executed by induction on the cardinality of hyperlinks.

For any $(N,v,H)\in\widehat{\mathcal{G}}_N^{\mathcal{H}}$ with $|H|=0$, it follows directly that $\xi(N,v,H)=\mathcal{D}(N,v,H)$, since the two allocation rules satisfy component efficiency. Assume that $t\geq 1$ and ξ and $\mathcal{D}$ coincide with each other for all $(N,v,H)\in\widehat{\mathcal{G}}_N^{\mathcal{H}}$ with $|H|\leq t-1$.

Let $(N,v,H)\in\widehat{\mathcal{G}}_N^{\mathcal{H}}$ be a hypergraph game with $|H|=t$. It suffices to show that $\xi_i(N,v,H)=\mathcal{D}_i(N,v,H)$ for each component $K\in N/H$ and $i\in K$. This equality follows directly from component efficiency if $|K|=1$. Therefore, we may assume that $|K|\geq 2$. Without loss of generality, let $K=\{1,\ldots,k\}$. For notational convenience, we write $\xi(H)$ and $\xi(H\setminus\{e\})$ instead of $\xi(N,v,H)$ and $\xi(N,v,H\setminus\{e\})$, respectively. By applying balanced conference contributions and component efficiency to ξ, we have the following k equations,

$$|H_2|\xi_1(H) - |H_1|\xi_2(H) = \sum_{e\in H_2} \xi_1(H\setminus\{e\}) - \sum_{e\in H_1} \xi_2(H\setminus\{e\}),$$
$$\dots$$
$$|H_k|\xi_1(H) - |H_1|\xi_k(H) = \sum_{e\in H_k} \xi_1(H\setminus\{e\}) - \sum_{e\in H_1} \xi_k(H\setminus\{e\}),$$
$$\sum_{i\in K} \xi_i(H) = v(K),$$

where the first $k-1$ equations are implied by balanced conference contributions and the last equation follows from component efficiency.

By the induction hypothesis, we obtain the following $k-1$ linearly independent equations,

$$|H_2|\xi_1(H) - |H_1|\xi_2(H) = \sum_{e\in H_2} \mathcal{D}_1(H\setminus\{e\}) - \sum_{e\in H_1} \mathcal{D}_2(H\setminus\{e\}),$$
$$\dots$$
$$|H_k|\xi_1(H) - |H_1|\xi_k(H) = \sum_{e\in H_k} \mathcal{D}_1(H\setminus\{e\}) - \sum_{e\in H_1} \mathcal{D}_k(H\setminus\{e\}),$$

combined with

$$\sum_{i\in K} \xi_i(H) = v(K).$$

Hence, there are k linearly independent equations with k variables in component K. Therefore, for each $K \in N/H$, all payoffs $\xi_i(N,v,H)$, $i \in K$, are uniquely determined. Since the degree value $\mathcal{D}(N,v,H)$ also satisfies component efficiency and balanced conference contributions, $(\mathcal{D}_i(N,v,H))_{i\in K}$ is a solution of the system of equations, and so it is the unique solution. Therefore, $\xi(N,v,H) = \mathcal{D}(N,v,H)$. This completes the proof. □

6.4 Comparison to other values

As we already mentioned in the beginning of Section 6.2, the Myerson value, the position value, and the (two-step) average tree value are proposed in different perspectives. The Myerson focuses on the players, the position value emphasizes the role of the hyperlinks, and the (two-step) average tree value takes the connectivity of the hypergraph into account, while the degree value highlights the role of the players' degrees. Example 6.2.2 also shows the differences between the degree value and the other mentioned values for hypergraph games. In this section, we first discuss the relationship between the degree value and the position value, and then summarize the properties of the different values for hypergraph games mentioned in this thesis.

It is clear that the axiom of balanced conference contributions in hypergraph games is a natural extension of balanced link contributions in graph games. Therefore, the degree value for graph games can be characterized by component efficiency and balanced link contributions. On the other hand, the position value for graph games is the unique allocation rule that on the class of graph games satisfies component efficiency and balanced link contributions (Slikker (2005)). This implies that on the class of graph games the degree value coincides with the position value. In fact, this equivalence relation can be traced back to Kongo (2010), who provides a so-called non-axiomatic characterization of the position value for graph games, i.e., an alternative expression of the position value for graph games. By applying the degree value on graph games, it is easy to verify that the degree value for graph games is the alternative expression of the position value in Kongo (2010).

The work of Slikker (2005) does not extend directly to hypergraph games, until Shan and Zhang (2016) propose the axiom of partial balanced conference contributions (see Section 2.3) and characterize the position value on the class of all hypergraph games. Compared with balanced conference contributions, we see that for a hypergraph game if the underlying hypergraph is uniform then balanced conference contributions coincides with partial balanced conference contributions. Therefore, from the characterization of the position value in Shan and Zhang (2016) and Theorem 6.3.2, the degree value may coincide with the position value on the class of uniform hypergraph games. Indeed, the following result shows the equivalence. Let $\widehat{\mathcal{G}}_N^{\mathcal{H}^u}$ be the set of uniform hypergraph games on N.

Theorem 6.4.1. *On the class of uniform hypergraph games, the degree value coincides with the position value.*

Proof. First, for any k-uniform $(N, v, H) \in \widehat{\mathcal{G}}_N^{\mathcal{H}^u}$, we set g as a mapping from $\Pi(I(H))$ to $\Pi(H)$ such that, for every $e_1, e_2 \in H$ and permutation $\sigma \in \Pi(I(H))$, $g(\sigma)(e_1) < g(\sigma)(e_2)$ if and only if $\max\{\sigma(l) : l \in I(H)_{e_1}\} < \max\{\sigma(l') : l' \in I(H)_{e_2}\}$.

By this setting and the definition of $\bar{v}$, we can compare the marginal contributions between $(I(H), \bar{v})$ and (H, v^N). For any $e \in H$ and $l \in I(H)_e$, if $\sigma(l) = \max\{\sigma(l') : l' \in I(H)_e\}$, then we have $m_l^\sigma(I(H), \bar{v}) = m_e^{g(\sigma)}(H, v^N)$, otherwise, $m_l^\sigma(I(H), \bar{v}) = 0$. Therefore, for every $\sigma \in \Pi(I(H))$ and $e \in H$, we have

$$\sum_{l \in I(H)_e} m_l^\sigma(I(H), \bar{v}) = m_e^{g(\sigma)}(H, v^N).$$

Note that $g(\Pi(I(H))) = \Pi(H)$ and $|e| = |I(H)_e| = k$ for all $e \in H$. Some different permutations of $\Pi(I(H))$ can be mapped by g into the same permutation of $\Pi(H)$ and for every $\sigma \in \Pi(H)$ the number of permutations $\sigma \in \Pi(I(H))$ that satisfy $g(\sigma) = \delta$ equals $|\Pi(I(H))|/|\Pi(H)|$. Then, for every $e \in H$, we have

$$\frac{1}{|\Pi(I(H))|} \sum_{\sigma \in \Pi(I(H))} \sum_{l \in I(H)_e} m_l^\sigma(I(H), \bar{v}) = \frac{1}{|\Pi(H)|} \sum_{\delta \in \Pi(H)} m_e^\delta(H, v^N).$$

By the definition of the Shapley value as in (2.1.4), we obtain that

$$\sum_{l \in I(H)_e} Sh_l(I(H)), \bar{v}) = Sh_e(H, v^N), \qquad \text{for all } e \in H.$$

By the definition of $\bar{v}$, for any $l, l' \in I(H)_e$, $e \in H$, and $K \subseteq I(H) \setminus \{l, l'\}$, it holds that

$$\bar{v}(K \cup \{l\}) = \bar{v}(K) = \bar{v}(K \cup \{l'\}).$$

Hence, l and l' are symmetric in $(I(H), \bar{v})$. So, by the symmetry of the Shapley value, it follows that for each $e \in H$ it holds that $Sh_l(I(H), \bar{v}) = Sh_{l'}(I(H), \bar{v})$ for every $l, l' \in I(H)_e$. Thus, for each $e \in H$, we have

$$Sh_l(I(H), \bar{v}) = \frac{1}{|I(H)_e|} Sh_e(H, v^N) = \frac{1}{|e|} Sh_e(H, v^N), \text{ for all } l \in I(H)_e,$$

which implies,

$$\mathcal{D}_i(N, v, H) = \sum_{l \in I(H)_i} Sh_l(I(H), \bar{v}) = \sum_{e \in H_i} \frac{1}{|e|} Sh_e(H, v^N) = \pi_i(N, v, H),$$

for all $i \in N$. □

As a natural consequence motivated by Theorem 6.3.2 and Theorem 6.4.1, a characterization of the position value can be proposed on the class of uniform hypergraph games by employing component efficiency and balanced conference contributions. Since the proof is essentially the same as the characterizations of the degree value in Subsection 6.3.2 and the position value in Slikker (2005), it is omitted. Note that when removing a hyperlink from a uniform hypergraph, the resulting hypergraph is still uniform.

Theorem 6.4.2. *On the class of uniform hypergraph games, the position and degree value is the unique allocation rule that satisfies component efficiency and balanced conference contributions.*

Furthermore, the influence property shows that the position value looks at the sizes of the incident hyperlinks of a player, in the sense that the larger the size of a hyperlink is, the less influence the player has in this hyperlink. The degree (measure) property shows that the degree value looks just at the number of incident hyperlinks of a player. Therefore, the influence property and the degree (measure) property precisely show the differences between the position value and the degree value. This interpretation also makes immediately clear that the two values coincide for uniform hypergraph games.

To conclude, in Table 6.2 the properties of the different values for hypergraph games are summarized. In this table '+' has the meaning that the value satisfies the property and '−' that it does not. Note that the properties of the degree value also hold on any subclass $\mathcal{G}$, $\mathcal{G} \subseteq \mathcal{G}_N^{\mathcal{H}}$, as mentioned above Definition 6.2.1.

	μ	π	AT	ATT	$\mathcal{D}$
Component efficiency on $\mathcal{G}_N^{\mathcal{H}}$	+	+	+	+	+
Linearity on $\mathcal{G}_N^{\mathcal{H}}$	+	+	+	+	+
Fairness on $\mathcal{G}_N^{\mathcal{H}}$	+	−	−	−	−
Component fairness on $\mathcal{G}_N^{\mathcal{H}^{cf}}$	−	−	+	+	−
Component fairness on $\mathcal{G}_N^{\mathcal{H}^{qcf}}$	−	−	−	+	−
Balanced contributions on $\mathcal{G}_N^{\mathcal{H}}$	+	−	−	−	−
Balanced contributions for interactive players on $\mathcal{G}_N^{\mathcal{H}^{qcf}}$	+	−	−	+	−
Balanced conference contributions on $\mathcal{G}_N^{\mathcal{H}}$	−	−	−	−	+
Balanced conference contributions on $\mathcal{G}_N^{\mathcal{H}^u}$	−	+	−	−	+
Partial balanced conference contributions on $\mathcal{G}_N^{\mathcal{H}}$	−	+	−	−	−
Restricted null-player property on $\mathcal{G}_N^{\mathcal{H}}$	+	−	+	+	−
Weak symmetry on $\mathcal{G}_N^{\mathcal{H}^c}$	+	−	−	+	−
Independence in unanimity games on $\mathcal{G}_N^{\mathcal{H}^t}$ or $\mathcal{G}_N^{\mathcal{C}}$	−	−	+	+	−
Symmetry in unanimity games on $\mathcal{G}_N^{\mathcal{C}}$	+	−	+	+	−
The influence property on $\mathcal{G}_N^{\mathcal{H}}$	−	+	−	−	−
The degree (measure) property on $\mathcal{G}_N^{\mathcal{H}}$	−	−	−	−	+
The superfluous conference property on $\mathcal{G}_N^{\mathcal{H}}$	+	+	−	−	+

Table 6.2: The properties of the values for hypergraph games

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