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Graphs with constant $\mu$ and $\bar{\mu}$

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Abstract

A graph $G$ has constant $\mu = \mu(G)$ if any two vertices that are not adjacent have $\mu$ common neighbours. $G$ has constant $\mu$ and $\bar{\mu}$ if $G$ has constant $\mu = \mu(G)$, and its complement $G'$ has constant $\bar{\mu} = \mu(G')$. If such a graph is regular, then it is strongly regular, otherwise precisely two vertex degrees occur. We shall prove that a connected graph has constant $\mu$ and $\bar{\mu}$ if and only if it has two distinct nonzero Laplace eigenvalues. This leads to strong conditions for existence. Several constructions are given and characterized. A list of feasible parameter sets for graphs with at most 40 vertices is generated.

1. Introduction

We say that a noncomplete graph $G$ has constant $\mu = \mu(G)$ if any two vertices that are not adjacent have $\mu$ common neighbours. A graph $G$ has constant $\mu$ and $\bar{\mu}$ if $G$ has constant $\mu = \mu(G)$, and its complement $G'$ has constant $\bar{\mu} = \mu(G')$. It turns out that only two vertex degrees can occur. Moreover, we shall prove that a graph has constant $\mu$ and $\bar{\mu}$ if and only if it has two distinct restricted Laplace eigenvalues. The Laplace eigenvalues of a graph are the eigenvalues of its Laplace matrix. This is a square matrix $Q$ indexed by the vertices, with $Q_{xx} = d_x$, the vertex degree of $x$, $Q_{xy} = -1$ if $x$ and $y$ are adjacent, and $Q_{xy} = 0$ if $x$ and $y$ are not adjacent. Note that if $G$ has $v$ vertices and Laplace matrix $Q$, then its complement $G'$ has Laplace matrix $vI - J - Q$ (where $I$ is an identity matrix and $J$ is an all-one matrix). Since the Laplace matrix has row sums zero, it has an eigenvalue 0 with the all-one vector as eigenvector. The eigenvalues with eigenvectors orthogonal to the all-one vector are called restricted (for a connected graph the restricted Laplace eigenvalues are just the nonzero ones). The restricted multiplicity of an eigenvalue is the dimension of the eigenspace orthogonal to the all-one vector. For more on the Laplace matrix we refer to [5]. Note

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that the graphs with one restricted Laplace eigenvalue are the complete and the empty graphs.

Graphs with constant $\mu$ and $\bar{\mu}$ form a common generalization of two known families of graphs. The regular ones are precisely the strongly regular graphs and for $\mu = 1$ we have the (nontrivial) geodetic graphs of diameter two.

Some similarities with so-called neighbourhood-regular or $\Gamma A$-regular graphs (see [8, 10]) occur. These graphs can be defined as graphs $G$ with constant $\lambda$ and $\bar{\lambda}$, that is, in $G$ any two adjacent vertices have $\lambda$ common neighbours, and in $\overline{G}$ any two adjacent vertices have $\bar{\lambda}$ common neighbours. In such graphs also only two vertex degrees can occur, but there is no easy algebraic characterization.

2. Laplace eigenvalues and vertex degrees

In this section we shall derive some basic properties of graphs with constant $\mu$ and $\bar{\mu}$. We start with an algebraic characterization.

**Theorem 2.1.** Let $G$ be a graph on $v$ vertices. Then $G$ has constant $\mu$ and $\bar{\mu}$ if and only if $G$ has two distinct restricted Laplace eigenvalues $\theta_1$ and $\theta_2$. If so then only two vertex degrees $k_1$ and $k_2$ can occur, and $\theta_1 + \theta_2 = k_1 + k_2 + 1 = \mu + v - \bar{\mu}$ and $\theta_1 \theta_2 = k_1 k_2 + \mu = \mu v$.

**Proof.** Let $G$ have Laplace matrix $Q$. Suppose that $G$ has two distinct restricted Laplace eigenvalues $\theta_1$ and $\theta_2$. Then $(Q - \theta_1 I)(Q - \theta_2 I)$ has spectrum $\{[\theta_1 \theta_2]^{v-1}, [0]^{v-1}\}$ and row sums $\theta_1 \theta_2$, so it follows that $(Q - \theta_1 I)(Q - \theta_2 I) = (\theta_1 \theta_2/v)J$. If $x$ is not adjacent to $y$, so $Q_{xy} = 0$ then $Q_{xy} = \theta_1 \theta_2/v$, and so $\mu = \theta_1 \theta_2/v$ is constant. Since the complement of $G$ has distinct restricted Laplace eigenvalues $v - \theta_1$ and $v - \theta_2$, it follows that $\bar{\mu} = (v - \theta_1)(v - \theta_2)/v$ is also constant.

Now suppose that $\mu$ and $\bar{\mu}$ are constant. If $x$ and $y$ are adjacent then $(vI - J - Q^2)_{xy} = \bar{\mu}$, so $\bar{\mu} = (v^2 I + vJ + Q^2 - 2vJ - 2vQ)_{xy} = Q_{xy}^2 + v$, and if $x$ and $y$ are not adjacent, then $Q_{xy}^2 = \mu$. Furthermore $Q_{xx}^2 = d_x^2 + d_x$, where $d_x$ is the vertex degree of $x$. Now

$$Q^2 = (\bar{\mu} - v)(\text{diag}(d_x) - Q) + \mu(J - I - \text{diag}(d_x) + Q) + \text{diag}(d_x^2 + d_x)$$

$$= (\mu + v - \bar{\mu})Q + \text{diag}(d_x^2 - d_x(\mu + v - \bar{\mu} - 1) - \mu) + \mu J.$$

Since $Q$ and $Q^2$ have row sums zero, it follows that $d_x^2 - d_x(\mu + v - \bar{\mu} - 1) - \mu + \mu v = 0$ for every vertex $x$. So $Q^2 - (\mu + v - \bar{\mu})Q + \mu v I = \mu J$. Now let $\theta_1$ and $\theta_2$ be such that $\theta_1 + \theta_2 = \mu + v - \bar{\mu}$ and $\theta_1 \theta_2 = \mu v$, then $(Q - \theta_1 I)(Q - \theta_2 I) = (\theta_1 \theta_2/v)J$, so $G$ has distinct restricted Laplace eigenvalues $\theta_1$ and $\theta_2$. As a side result we obtained that all vertex degrees $d_x$ satisfy the same quadratic equation, thus $d_x$ can only take two values $k_1$ and $k_2$, and the formulas readily follow. □
If the restricted Laplace eigenvalues are not integral, then they must have the same multiplicities $m_1 = m_2 = \frac{1}{2}(v - 1)$. If the Laplace eigenvalues are integral, then their multiplicities are not necessarily fixed by $v$, $\mu$ and $\bar{\mu}$. For example, there are graphs on 16 vertices with constant $\mu = 2$ and $\bar{\mu} = 6$ with Laplace spectrum $\{[8]^m, [4]^{15-m}, [0]^1\}$ for $m = 5, 6, 7, 8$ and 9.

The following lemma implies that the numbers of vertices of the respective degrees follow from the Laplace spectrum.

**Lemma 2.2.** Let $G$ be a graph on $v$ vertices with two distinct restricted Laplace eigenvalues $\theta_1$ and $\theta_2$ with restricted multiplicities $m_1$ and $m_2$, respectively. Suppose there are $n_1$ vertices of degree $k_1$ and $n_2$ vertices of degree $k_2$. Then $m_1 + m_2 + 1 = n_1 + n_2 = v$ and $m_1 \theta_1 + m_2 \theta_2 = n_1 k_1 + n_2 k_2$.

**Proof.** The first equation is trivial, the second follows from the trace of the Laplace matrix.

The number of common neighbours of two adjacent vertices is in general not constant, but depends on the degrees of the vertices.

**Lemma 2.3.** Let $G$ be a graph with constant $\mu$ and $\bar{\mu}$, and vertex degrees $k_1$ and $k_2$. Suppose $x$ and $y$ are two adjacent vertices. Then the number of common neighbours $\lambda_{xy}$ of $x$ and $y$ equals

\[
\lambda_{xy} = \begin{cases} 
\lambda_{11} = \mu - 1 + k_1 - k_2 & \text{if } x \text{ and } y \text{ both have degree } k_1, \\
\lambda_{12} = \mu - 1 & \text{if } x \text{ and } y \text{ have different degrees,} \\
\lambda_{22} = \mu - 1 + k_2 - k_1 & \text{if } x \text{ and } y \text{ both have degree } k_2.
\end{cases}
\]

**Proof.** Suppose $x$ and $y$ have vertex degrees $d_x$ and $d_y$, respectively. The number of vertices that are not adjacent to both $x$ and $y$ equals $\bar{\mu}$. The number of vertices adjacent to $x$ but not to $y$ equals $d_x - 1 - \lambda_{xy}$, and the number of vertices adjacent to $y$ but not to $x$ equals $d_y - 1 - \lambda_{xy}$. Now we have that $v = 2 + \lambda_{xy} + \bar{\mu} + d_x - 1 - \lambda_{xy} + d_y - 1 - \lambda_{xy}$. Thus $\lambda_{xy} = \bar{\mu} - v + d_x + d_y$. By using that $k_1 + k_2 = \mu + v - \bar{\mu} - 1$, the result follows.

Both Theorem 2.1 and Lemma 2.3 imply the following.

**Corollary 2.4.** A graph with constant $\mu$ and $\bar{\mu}$ is regular if and only if it is strongly regular.
follows that it can be recognized from the Laplace spectrum whether a graph is strongly regular or not. This also follows from the fact that regularity of a graph follows from its Laplace spectrum.

Before proving the next lemma we first look at the disconnected graphs. Since the number of components of a graph equals the multiplicity of its Laplace eigenvalue 0, a graph with constant $\mu$ and $\bar{\mu}$ is disconnected if and only if one of its restricted Laplace eigenvalues equals 0. Consequently this is the case if and only if $\mu = 0$. So in a disconnected graph $G$ with constant $\mu$ and $\bar{\mu}$ any two vertices that are not adjacent have no common neighbours. This implies that two vertices that are not adjacent are in distinct components of $G$. So $G$ is a disjoint union of cliques. Since the only two vertex degrees that can occur are $v - \bar{\mu} - 1$ and 0, $G$ is a disjoint union of $(v - \bar{\mu})$-cliques and isolated vertices.

**Lemma 2.5.** Let $G$ be a graph with two restricted Laplace eigenvalues $\theta_1 > \theta_2$ and vertex degrees $k_1 \geq k_2$. Then $\theta_1 - 1 \geq k_1 \geq k_2 \geq \theta_2$, with $k_2 = \theta_2$ if and only if $G$ or $\overline{G}$ is disconnected.

**Proof.** Assume that $G$ is not regular, otherwise $G$ is strongly regular and the result easily follows. First, suppose that the induced graph on the vertices of degree $k_1$ is not a coclique. So there are two vertices of degree $k_1$ that are adjacent. Then the $2 \times 2$ submatrix of the Laplace matrix $Q$ of $G$ induced by these two vertices has eigenvalues $k_1 \pm 1$, and since these interlace (cf. [9]) the eigenvalues of $Q$, we have that $k_1 + 1 \leq \theta_1$. Since $k_1 + k_2 + 1 = \theta_1 + \theta_2$, then also $k_2 \geq \theta_2$.

Next, suppose that the induced graph on the vertices of degree $k_2$ is not a clique. So there are two vertices of degree $k_2$ that are not adjacent. Now the $2 \times 2$ submatrix of $Q$ induced by these two vertices has two eigenvalues $k_2$, and since these also interlace the eigenvalues of $Q$, we have that $k_2 \geq \theta_2$, and then also $\theta_1 - 1 \geq k_1$.

The remaining case is that the induced graph on the vertices of degree $k_1$ is a coclique and the induced graph on the vertices of degree $k_2$ is a clique. Suppose we have such a graph. Since a vertex of degree $k_1$ only has neighbours of degree $k_2$, and $\lambda_{12} = \mu - 1$, we find that $k_1 = \mu$. Since any two vertices of degree $k_1$ have $\mu$ common neighbours, it follows that every vertex of degree $k_1$ is adjacent to every vertex of degree $k_2$, and we find that $k_2 \geq k_1$, which is a contradiction. So the remaining case cannot occur, and we have proven the inequalities.

Now suppose that $G$ or $\overline{G}$ is disconnected. Then it follows from the observations before the lemma or looking at the complement that $k_2 = \theta_2$. On the other hand, suppose that $k_2 = \theta_2$. Then it follows that $k_1 = \theta_1 - 1$ and from the equation $\theta_1 \theta_2 = k_1 k_2 + \mu$ it then follows that $k_2 = \mu$. Now take a vertex $x_2$ of degree $k_2$ that is adjacent to a vertex $x_1$ of degree $k_1$. If there are no such vertices then $G$ is disconnected and we are done. It follows that every vertex that is not adjacent to $x_2$, is adjacent to all neighbours of $x_2$, so also to $x_1$. Since $x_1$ and $x_2$ have $\mu - 1$ common neighbours, $x_1$ is also adjacent to all neighbours of $x_2$. So $x_1$ is adjacent to all other vertices, and so $\overline{G}$ is disconnected. □
We conclude this section with so-called Bruck-Ryser conditions.

**Proposition 2.6.** Let $G$ be a graph with constant $\mu$ and $\bar{\mu}$ on $v$ vertices, with $v$ odd, and with restricted Laplace eigenvalues $\theta_1$ and $\theta_2$. Then the Diophantine equation

$$x^2 = (\theta_1 - \theta_2)^2 y^2 + (-1)^{(v-1)/2} \mu z^2$$

has a nontrivial integral solution $(x, y, z)$.  

**Proof.** Let $Q$ be the Laplace matrix of $G$, then

$$Q = (Q - \frac{1}{2}(\theta_1 + \theta_2)I)(Q - \frac{1}{2}(\theta_1 + \theta_2)I)^T$$

$$= Q^2 - (\theta_1 + \theta_2)Q + \frac{1}{4}(\theta_1 + \theta_2)^2 I$$

$$= \mu J + (\frac{1}{4}(\theta_1 + \theta_2)^2 - \theta_1 \theta_2)I = \frac{1}{4}(\theta_1 - \theta_2)^2 I + \mu J.$$  

Since $Q - \frac{1}{2}(\theta_1 + \theta_2)I$ is a rational matrix, it follows from a lemma by Bruck and Ryser (cf. [2]) that the Diophantine equation

$$x^2 = \frac{1}{4}(\theta_1 - \theta_2)^2 y^2 + (-1)^{(v-1)/2} \mu z^2$$

has a nontrivial integral solution, which is equivalent to stating that the Diophantine equation above has a nontrivial integral solution. $\square$

**3. Cocliques**

If $k_1 - k_2 > \mu - 1$, then the induced graph on the set of vertices of degree $k_2$ is a coclique, since two adjacent vertices of degree $k_2$ would have a negative number of common neighbours. It turns out (see the table in Section 9) that this is the case in many examples. Therefore we shall have a closer look at cocliques. If $G$ is a graph, then we denote by $\alpha(G)$ the maximal size of a coclique in $G$.

**Lemma 3.1.** Let $G$ be a graph on $v$ vertices with largest Laplace eigenvalue $\theta_1$ and smallest vertex degree $k_2$. Then $\alpha(G) \leq v(\theta_1 - k_2)/\theta_1$.

**Proof.** Let $C$ be a coclique of size $\alpha(G)$. Partition the vertices of $G$ into $C$ and the set of vertices not in $C$, and partition the Laplace matrix $Q$ of $G$ according to this partition of the vertices. Let $B$ be the matrix of average row sums of the blocks of $Q$, then

$$B = \begin{pmatrix} k & -k \\ -k & \alpha(G) \frac{v - \alpha(G)}{v} \end{pmatrix}.$$  

where $k$ is the average degree of the vertices in $C$. Since $B$ has eigenvalues 0 and $kv/(v - \alpha(G))$, and since these interlace the eigenvalues of $Q$ (cf. [9]), we have that $kv/(v - \alpha(G)) \leq \theta_1$. The result now follows from the fact that $k_2 \leq k$. $\square$
Another bound is given by the multiplicities of the eigenvalues.

**Lemma 3.2.** Let \( G \) be a connected graph with Laplace spectrum \( \{[\theta_1]^m, [\theta_2]^m, [0]^1\} \), where \( \theta_1 > \theta_2 > 0 \), such that \( \bar{G} \) is also connected. Then \( \alpha(G) \leq \min\{m_1, m_2 + 1\} \).

**Proof.** Suppose \( C \) is a coclique with size greater than \( m_1 \). Consider the submatrix of the Laplace matrix \( Q \) induced by the vertices of \( C \). This matrix only has eigenvalues \( k_1 \) and \( k_2 \), and since these interlace the eigenvalues of \( Q \), we find that \( k_2 \leq \theta_2 \). This is in contradiction with Lemma 2.5, since \( G \) and \( \bar{G} \) are connected. If \( C \) is a coclique of size greater than \( m_2 + 1 \), we find by interlacing that \( k_1 \geq \theta_1 \), which is again a contradiction. \( \square \)

As remarked before, if \( G \) is a graph with constant \( \mu \) and \( \bar{\mu} \) with \( \lambda_{22} < 0 \), then the vertices of degree \( k_2 \) form a coclique. If this is the case, then \( n_2 \leq m_2 \), and we know the adjacency spectrum of the induced graph on the vertices of degree \( k_1 \).

**Proposition 3.3.** Let \( G \) be a connected graph with Laplace spectrum \( \{[\theta_1]^m, [\theta_2]^m, [0]^1\} \), where \( \theta_1 > \theta_2 > 0 \), such that \( \bar{G} \) is also connected. Suppose that the \( n_2 \) vertices of degree \( k_2 \) induce a coclique, then \( n_2 \leq m_2 \), and the \( n_1 \) vertices of degree \( k_1 \) induce a graph with adjacency spectrum \( \{[\lambda_1]^1, [k_1 - \theta_2]^m - n_1, [\lambda_2]^1, [-1]^n - 1, [k_1 - \theta_1]^m - n_2\} \), where \( \lambda_1 \) and \( \lambda_2 \) are such that \( \lambda_1^2 + \lambda_2^2 = n_1k_1 - n_2k_2 - (m_2 - n_2)(k_1 - \theta_2)^2 - (n_2 - 1) - (m_1 - n_2)(k_1 - \theta_1)^2 \) and \( \lambda_1 + \lambda_2 = k_1 - 1 \).

**Proof.** The adjacency matrix \( A_1 \) of the graph induced by the vertices of degree \( k_1 \) is a submatrix of the matrix \( k_1I - Q \), where \( Q \) is the Laplace matrix of \( G \). From interlacing it follows that \( A_1 \) has second largest eigenvalue \( k_1 - \theta_2 \) with multiplicity at least \( m_2 - n_2 \) and smallest eigenvalue \( k_1 - \theta_1 \) with multiplicity at least \( m_1 - n_2 \). Note that we did not use here that the vertices of degree \( k_2 \) induce a coclique. Now let

\[
A = \begin{pmatrix}
A_1 & N^T \\
N & O
\end{pmatrix}
\]

be the adjacency matrix of \( G \), where the partition is induced by the degrees of the vertices. Two vertices of degrees \( k_2 \) have \( \mu \) common neighbours, so \( NN^T = k_2I + \mu(J - I) \). A vertex of degree \( k_2 \) and a vertex of degree \( k_1 \) have \( \mu - 1 \) or \( \mu \) common neighbours, depending on whether they are adjacent or not, so \( NA_1 = \mu J - N \). Let \( \{v_i|i = 1, \ldots, n_2\} \) be an orthonormal set of eigenvectors of \( NN^T \), with \( v_1 \) the constant vector, then \( NN^Tv_i = (k_2 - \mu)v_i \), \( i = 2, \ldots, n_2 \). Now

\[
A_1(N^Tv_i) = (NA_1)^Tv_i = (\mu - N)^Tv_i = -N^Tv_i, \quad i = 2, \ldots, n_2.
\]

Since \( k_2 > \mu \) (otherwise \( G \) or \( \bar{G} \) is disconnected), it follows that \( A_1 \) has \(-1\) as an eigenvalue with multiplicity at least \( n_2 - 1 \).
By Lemma 3.2 we have \( n_2 \leq m_2 + 1 \). Suppose that \( n_2 = m_2 + 1 \). Then \( n_1 = m_1 \) and it follows that \( A_1 \) has spectrum \([\lambda_1]^1, [-1]^{n_1-1}, [k_1 - \theta_1]^{m_1-n_1}\) for some \( \lambda_1 \). Since \( A_1 \) has zero trace, and using Lemma 2.5, we have \( \lambda_1 = n_2 - 1 + (m_1 - n_2) \) \((\theta_1 - k_1) > n_1 - 1\), which is a contradiction. Hence \( n_2 \leq m_2 \). Now let \( \lambda_1 \geq \lambda_2 \) be the remaining two eigenvalues of \( A_1 \). These eigenvalues (i.e., the equations in the statement) follow from the trace of \( A_1 \) and the trace of \( A_1^T \). Since \( \lambda_1 \leq k_1 \) (interlacing), it follows that \( \lambda_2 \geq -1 \).

If the vertices of degree \( k_2 \) form a coclique, then Lemma 3.1 implies that

\[ n_2 \leq v(\theta_1 - k_2)/\theta_1. \]

If this bound is tight, then it follows from tight interlacing that the partition of the vertices into vertices of degree \( k_1 \) and vertices of degree \( k_2 \) is regular, that is, every block in the partitioned matrix in the proofs of Lemmas 3.1 and 3.3 has constant row sums. So \( N \) is the incidence matrix of a \( 2-(n_2, \kappa, \mu) \) design, where \( \kappa = n_2k_2/n_1 \). Furthermore, the adjacency matrix of the induced graph \( G_1 \) on the vertices of degree \( k_1 \) has spectrum

\[ \{[k_1 - \kappa]^1, [k_1 - \theta_2]^{m_1+1-n_1}, [-1]^{n_1-1}, [k_1 - \theta_1]^{m_1-n_2}\} \]

so \( G_1 \) is a regular graph with at most four eigenvalues. It follows from the multiplicities that \( \theta_1 \) and \( \theta_2 \) must be integral.

In this way it can be proved that there is no graph on 25 vertices with constant \( \mu = 2 \) and \( \bar{\mu} = 12 \), with 10 vertices of degree 6. These 10 vertices induce a coclique for which the bound is tight. The induced graph on the remaining 15 vertices has spectrum \([4]^1, [3]^3, [-1]^9, [-2]^2\] but such a graph cannot exist (cf. [7]).

Examples for which the bound is tight are obtained by taking an affine plane for the design and a disjoint union of cliques for \( G_1 \). This is family \( b \) of Section 4. Another example is constructed from a polarity with \( q\sqrt{q} + 1 \) absolute points in \( \text{PG}(2, q) \) where \( q \) is the square of a prime power (cf. Section 5).

In Section 6 we find a large family of graphs for which the bound of Lemma 3.2 is tight.

Also if \( \lambda_1 = 0 \), so that the vertices of degree \( k_2 \) do not necessarily form a coclique, we find a bound on the number of vertices \( n_2 \) of degree \( k_2 \).

**Lemma 3.4.** If \( k_1 - k_2 \geq \mu - 1 \), then \( n_2 \leq v - \mu \).

**Proof.** Fix a vertex \( x_1 \) of degree \( k_1 \). If \( x_1 \) has no neighbours of degree \( k_2 \) then \( n_1 \geq k_1 + 1 \geq \mu + k_2 \geq \mu \), and so \( n_2 \leq v - \mu \). If \( x_1 \) has a neighbour \( x_2 \) of degree \( k_2 \), then \( x_1 \) and \( x_2 \) cannot have a common neighbour \( y_2 \) of degree \( k_2 \), since otherwise \( x_2 \) and \( y_2 \) have a common neighbour \( x_1 \), so that \( 0 \geq \mu - 1 + k_2 - k_1 = \lambda_2 \geq 0 \), which is a contradiction. So all common neighbours of \( x_1 \) and \( x_2 \) have degree \( k_1 \), so \( n_1 \geq \lambda_1 + 1 = \mu \), and so \( n_2 \leq v - \mu \).
4. Geodetic graphs of diameter two

A geodetic graph is a graph in which any two vertices are connected by a unique shortest path. Thus a geodetic graph of diameter two is a graph with constant $\mu = 1$. It is proved (see [3, Theorem 1.17.1]) that if $G$ is a geodetic graph of diameter two, then either

(i) $G$ contains a vertex adjacent to all other vertices, or
(ii) $G$ is strongly regular, or
(iii) precisely two vertex degrees $k_1 > k_2$ occur. If $X_1$ and $X_2$ denote the sets of vertices with degrees $k_1$ and $k_2$, respectively, then $X_2$ induces a coclique, maximal cliques meeting both $X_1$ and $X_2$ have size two, and maximal cliques contained in $X_1$ have size $k_1 - k_2 + 2$. Moreover, $v = k_1 k_2 + 1$.

If $G$ is of type (i), then $G$ need not have constant $\mu$. Note that its complement is disconnected, so see Section 2. If $G$ is of type (ii), then clearly it has constant $\mu$. Now suppose that $G$ is of type (iii). Since $\mu = 1$, every edge is in a unique maximal clique. Let $x$ and $y$ be two adjacent vertices, then $x$ and $y$ cannot both be in $X_2$. If one is in $X_1$, and the other in $X_2$, then they have no common neighbour, since maximal cliques meeting both $X_1$ and $X_2$ have size 2. So $\lambda_{12} = 0$ and then $\mu_{12} = v - k_1 - k_2$. If both $x$ and $y$ are in $X_1$, then by the previous argument they have no common neighbours in $X_2$, and since every maximal clique contained in $X_1$ has size $k_1 - k_2 + 2$, they have $k_1 - k_2$ common neighbours in $X_1$. So $\lambda_{11} = k_1 - k_2$, and then also $\mu_{11} = v - k_1 - k_2$. So $G$ has constant $\mu$.

The following four families of graphs are all known examples of type (iii).

(a) Take a clique and a coclique of size $k_1$, and an extra vertex. Join the vertices of the clique and the coclique by a matching, and join the extra vertex to every vertex of the coclique (see also Section 6).

(b) Take an affine plane. Take as vertices the points and lines of the plane. A point is adjacent to a line if it is on the line, and two lines are adjacent if they are parallel.

(c) Take the previous example and add the parallel classes to the vertices. Join each line to the parallel class it is in, and join all parallel classes mutually.

(d) Take a projective plane with a polarity $\sigma$. Take as vertices the points of the plane, and join two points $x$ and $y$ if $x$ is on the line $y^\sigma$ (cf. Section 5).

5. Symmetric designs with a polarity

Let $D$ be a symmetric design. A polarity of $D$ is a one-one correspondence $\sigma$ between its points and blocks such that for any point $p$ and any block $b$ we have that $p \in b$ if and only if $b^\sigma \in p^\sigma$. A point is called absolute (with respect to $\sigma$) if $p \in p^\sigma$. Now $D$ has a polarity if and only if it has a symmetric incidence matrix $A$. An absolute point corresponds to a one on the diagonal of $A$.

Suppose that $D$ is a symmetric 2-(v, k, $\lambda$) design with a polarity $\sigma$. Let $G = P(D)$ be the graph on the points of $D$, where two distinct points $x$ and $y$ are adjacent if
Then the only vertex degrees that can occur are \( k \) and \( k - 1 \). The number of vertices with degree \( k - 1 \) is the number of absolute points of \( \sigma \). Let \( A \) be the corresponding symmetric incidence matrix, then \( Q = kI - A \) is the Laplace matrix of \( G \). Since \( A \) is a symmetric incidence matrix of \( D \), we find that \((kI - Q)^2 = A^2 = AA^T = (k - \lambda)I + \lambda J \), so \( Q^2 - 2kQ + (k^2 - k - \lambda)I = \lambda J \). Thus \( Q \) has two distinct restricted eigenvalues \( k \pm \sqrt{k - \lambda} \). The converse is also true.

**Theorem 5.1.** Let \( G \) be a graph with constant \( \mu \) and \( \bar{u} \) on \( v \) vertices, with vertex degrees \( k \) and \( k - 1 \). Then \( G \) comes from a symmetric \( 2-(v, k, \mu) \) design with a polarity.

**Proof.** Let \( G \) have restricted Laplace eigenvalues \( \theta_1 \) and \( \theta_2 \), then \( \theta_1 + \theta_2 = 2k \) and \( \mu = \theta_1 \theta_2 / v = k(k - 1)/(v - 1) \). Hence we have that \( Q^2 - 2kQ + \mu I = \mu J \). Now let \( A = kI - Q \), then \( A \) is a symmetric \((0,1)\)-matrix with row sums \( k \), and \( AA^T = A^2 = k^2I - 2kQ + Q^2 = (k^2 - \mu)I + \mu J = (k - \mu)I + \mu J \), so \( A \) is the incidence matrix of a symmetric \( 2-(v, k, \mu) \) design with a polarity. \( \Box \)

Since the polarities in the unique \( 2-(7, 3, 1) \), \( 2-(11, 5, 2) \) and \( 2-(13, 4, 1) \) designs are unique, the graphs we obtain from these designs are also uniquely determined by their parameters.

In a projective plane of order \( n \), where \( n \) is not a square, any polarity has \( n + 1 \) absolute points. If \( n \) is a square, then the number of absolute points in a polarity lies between \( n + 1 \) and \( n \sqrt{n} + 1 \). The projective plane \( PG(2, q) \) admits a polarity with \( q + 1 \) absolute points for every prime power \( q \) and a polarity with \( q \sqrt{q} + 1 \) absolute points whenever \( q \) is the square of a prime power. If a polarity in a projective plane of order \( n \) has \( n + 1 \) absolute points then the set of absolute points forms a line if \( n \) is even, and an oval if \( n \) is odd, that is, no three points are on one line (cf. [2, Section VIII.9]). Using this, we find that there is precisely one graph from a polarity with \( 5 \) absolute points in the projective plane of order \( 4 \), and precisely one graph from a polarity with \( 6 \) absolute points in the projective plane of order \( 5 \). Using the remarks in Section 3 we also find precisely one graph from a polarity with \( 9 \) absolute points in the projective plane of order \( 4 \).

By Paley's construction of Hadamard matrices (cf. [2, Theorem I.9.11]) we obtain symmetric \( 2-(2e(q + 1) - 1, 2e-1(q + 1) - 1, 2e-2(q + 1) - 1) \) designs with a polarity with \( 2e-1(q + 1) - 1 \) absolute points, for every odd prime power \( q \) and every \( e > 0 \).

Furthermore, we found polarities with \( 0, 4, 8, 12 \) and \( 16 \) absolute points in a \( 2-(16, 6, 2) \) design, a polarity in the \( 2-(37, 9, 2) \) design from the difference set (cf. [2, Example VI.4.3]) and a polarity with \( 16 \) absolute points in the \( 2-(40, 13, 4) \) design \( PG(2, 3) \). Spence (private communication) found polarities with \( 3, 7, 11, 15 \) and \( 19 \) absolute points in \( 2-(15, 7, 3) \) designs, polarities in \( 2-(25, 9, 3) \) and \( 2-(30, 13, 3) \) designs, polarities with \( 5, 11, 17, 23, 29 \) and \( 35 \) absolute points in \( 2-(35, 17, 8) \) designs, polarities with \( 0, 6, 12, 18, 24, 30 \) and \( 36 \) absolute points in \( 2-(36, 15, 6) \) designs and polarities with \( 10, 16, 22, 28, 34 \) and \( 40 \) absolute points in \( 2-(40, 13, 4) \) designs.
6. Other graphs from symmetric designs

Let $D$ be a symmetric 2-$(w, k, \lambda)$ design. Fix a point $x$. We shall construct a graph $G = G(D)$ that has constant $\mu$ and $\bar{\mu}$. The vertices of $G$ are the points and the blocks of $D$, except for the point $x$. Between the points there are no edges. A point $y$ and a block $b$ will be adjacent if and only if precisely one of $x$ and $y$ is incident with $b$. Two blocks will be adjacent if and only if both blocks are incident with $x$ or both blocks are not incident with $x$. It is not hard to show that the resulting graph $G$ has constant $\mu = k - \lambda$ and constant $\bar{\mu} = w - k - 1 + \lambda$. In $G$ the $n_1 = w$ blocks have degrees $k_1 = w - 1$, and the $n_2 = w - 1$ points have degrees $k_2 = 2(k - \lambda)$. Note that $D$ and the complement of $D$ give rise to the same graph $G$. We have the following characterization of $G(D)$.

**Theorem 6.1.** Let $G$ be a graph with constant $\mu$ and $\bar{\mu}$ on $2w - 1$ vertices, such that both $G$ and $\bar{G}$ are connected. Suppose $G$ has $w$ vertices of degree $k_1$, and $w - 1$ vertices of degree $k_2$, and suppose that the vertices of degree $k_2$ induce a coclique. Then $k_1 = w - 1$, $k_2 = 2\mu$, and $G = G(D)$, where $D$ is a symmetric 2-$(w, k, k - \mu)$ design.

**Proof.** Let

$$A = \begin{pmatrix} A_1 & N^T \\ N & O \end{pmatrix}$$

be the adjacency matrix of $G$, where the partition is induced by the degrees of the vertices. It follows from Lemmas 3.2 and 3.3 that $m_1 = m_2 = n_2$, and that $A_1$ has spectrum $\{[\lambda_1]^1, [\lambda_2]^1, [-1]^{w-2}\}$, with $\lambda_1 + \lambda_2 = k_1 - 1$, and $\lambda_1 \geq \lambda_2 \geq -1$. On the other hand, it follows from the trace of $A_1$ that $\lambda_1 + \lambda_2 = w - 2$, so that $k_1 = w - 1$. Since $k_1k_2 = \mu(w - 1)$, we then find that $k_2 = 2\mu$.

Suppose that $\lambda_2 = -1$, then $\lambda_1 = w - 2 - \lambda_2 = w - 1$, so $A_1 = J - I$. But then $G$ is disconnected, which is a contradiction. Now $A_1 + I$ is positive semidefinite of rank two with diagonal $1$, and so it is the Gram matrix of a set of vectors of length $1$ in $\mathbb{R}^2$, with mutual inner products $0$ or $1$. It follows that there can only be two distinct vectors, and $A_1$ is the adjacency matrix of a disjoint union of two cliques, say of sizes $k$ and $w - k$.

Let $N = (N_1N_2)$ be partitioned according to the partition of $A_1$ into two cliques, where $N_1$ has $k$ columns and $N_2$ has $w - k$ columns. From the equation $NA_1 = \mu J - N$ we derive that $N_1J = N_2J = \mu J$, so both $N_1$ as $N_2$ have row sums $\mu$.

Now let

$$M = \begin{pmatrix} j^T & 0^T \\ J - N_1 & N_2 \end{pmatrix}$$

then $M$ is square of size $w$, with row sums $k$. Furthermore, we find that $(J - N_1)(J - N_1)^T + N_2N_2^T = (k - 2\mu)J + NN^T = (k - 2\mu)J + (k_2 - \mu)I + \mu J = \ldots$
\[ \mu I + (k - \mu)J, \] and so we have that \[ MM^T = \mu I + (k - \mu)J, \] so \( M \) is the incidence matrix of a symmetric \((w, k, k - \mu)\) design \( D \), and \( G = G(D) \). \( \square \)

The matrix \( N \) that appears in the proof above is the incidence matrix of a structure, that is called a pseudo design by Marrero and Butson [11] and a 'near-square' \( \lambda \)-linked design by Woodall [15]. An alternative proof of Theorem 6.1 uses Theorem 3.4 of [11] that states that a pseudo \((w \neq 4\mu, k_2 = 2\mu, \mu)\)-design comes from a symmetric design in the way described above. The problem then is to prove the case \( w = 4\mu \), however.

For every orbit of the action of the automorphism group of the design \( D \) on its points, we get a different graph \( G(D) \) by taking the fixed point \( x \) from that orbit. Since the trivial \( 2-(k_1 + 1, 1, 0) \) (here we get family \( a \) of geodetic graphs given in Section 5), the \( 2-(7, 3, 1) \), the \( 2-(11, 5, 2) \) and the \( 2-(13, 4, 1) \) designs are unique and have an automorphism group that acts transitively on the points, the graphs we obtain are uniquely determined by their parameters. According to Spence (private communication), the five \( 2-(15, 7, 3) \) designs have respectively 1, 2, 3, 2 and 2 orbits, the three \( 2-(16, 6, 2) \) designs all have a transitive automorphism group, and the six \( 2-(19, 9, 4) \) designs have respectively 7, 5, 3, 3, 3 and 1 orbits. Thus we get precisely ten graphs on 29 vertices with constant \( \mu = 4 \) and \( \bar{\mu} = 10 \), three graphs on 31 vertices with constant \( \mu = 4 \) and \( \bar{\mu} = 11 \), and 22 graphs on 37 vertices with constant \( \mu = 5 \) and \( \bar{\mu} = 13 \).

7. Switching in strongly regular graphs

Let \( G \) be a strongly regular graph with parameters \((v = 2k + 1, k, \lambda, \mu^*)\). Fix a vertex \( x \) and 'switch' between the set of neighbours of \( x \) and the set of vertices (distinct from \( x \)) that are not neighbours of \( x \), that is, a vertex that is adjacent to \( x \) and a vertex that is not adjacent to \( x \) are adjacent if and only if they are not adjacent in \( G \). All other adjacencies remain the same. If the adjacency eigenvalues of \( G \) are \( k, r \) and \( s \), then we obtain a graph with restricted Laplace eigenvalues \( 2(\lambda + 1) - s \) and \( 2(\lambda + 1) - r \). The graph has constant \( \mu = k - \mu^* = \lambda + 1 \) and \( \bar{\mu} = \mu^* \), and there is one vertex of degree \( k \) and \( 2k \) vertices of degree \( 2(\lambda + 1) \). Almost all examples have \( k = 2(\lambda + 1) = 2\mu^* \), so that we get a (strongly) regular graph. The only known (to us) examples for which \( k \neq 2(\lambda + 1) \) are the triangular graph \( T(7) \) and its complement. (Note that from one pair of complementary graphs we get another pair of complementary graphs.) \( T(7) \) is the strongly regular graph on the unordered pairs \{i, j\}, \( i, j = 1, \ldots, 7, i \neq j \), where two distinct pairs are adjacent if they intersect. From the complement of \( T(7) \) we get a graph with constant \( \mu = 4 \) and \( \bar{\mu} = 6 \) on 21 vertices with one vertex of degree 10 and 20 vertices of degree 8. The subgraph induced by the neighbours of the vertex \( x \) of degree 10 is the Petersen graph (the complement of \( T(5) \)).

This construction can be reversed, that is, if \( G \) is a graph on \( v \) vertices with constant \( \mu \) and \( \bar{\mu} \), such that there is one vertex of degree \( k = \frac{1}{2}(v - 1) \) and \( 2k \) vertices of degree \( 2\mu \), then it must be constructed from a strongly regular graph in the above way. Since
$T(7)$ is uniquely determined by its parameters, and it has a transitive automorphism group. It follows that there is precisely one graph with constant $\mu = 4$ and $\bar{\mu} = 6$ on 21 vertices with one vertex of degree 10 and 20 vertices of degree 8.

Next, let $G$ be a strongly regular graph with parameters $(v^* = 2k + 1, k, \lambda, \mu^*)$ with a regular partition into two parts, where one part has $k_2$ vertices and the induced graph is regular of degree $k_2 - \mu^* - 1$, and the other part has $v^* - k_2$ vertices and the induced graph is regular of degree $k - \mu^*$. (Then $k_2(k - k_2 + \mu^* + 1) = (v^* - k_2)\mu^*$.)

Add an isolated vertex to the second part and then switch with respect to this partition, that is, two vertices from different parts will be adjacent if and only if they are not adjacent in $G$, and two vertices from the same part will be adjacent if and only if they also are adjacent in $G$. The obtained graph has one vertex of degree $k_2$ and $v^*$ vertices of degree $k_1 = k_2 + k - 2\mu^*$.

If the adjacency eigenvalues of $G$ are $k$, $r$ and $s$, then we obtain a graph with restricted Laplace eigenvalues $k_1 - s$ and $k_1 - r$, and it has constant $\mu = k_2 - \mu^*$ and $\bar{\mu} = k + 1 - k_2 + \mu$. Again, we obtain a (strongly) regular graph if $k = 2\mu^*$.

Also here the construction can be reversed. A graph on $v$ vertices with constant $\mu$ and $\bar{\mu}$, such that $\mu + \bar{\mu} = \frac{1}{2}v$ and there is one vertex of degree $k_2$ must be constructed from a strongly regular graph in the above way.

If we take $T(7)$ and take for one part of the partition a 7-cycle or the disjoint union of a 3-cycle and a 4-cycle, then we find that there are precisely two nonisomorphic graphs on 22 vertices with constant $\mu = 3$ and $\bar{\mu} = 8$, with 21 vertices of degree 9 and one vertex of degree 7. In $T(7)$ there cannot be a regular partition with $k_2 = 12$ (which is the other value satisfying the quadratic equation) since this would give a graph which is the complement of a graph with $\lambda_{22} = 0$ and $n_1 < \mu$, contradicting Lemma 3.4.

8. The number of small strongly regular graphs

For completeness, here we shall give some results on the numbers of nonisomorphic strongly regular graphs with parameters $(v, k, \lambda, \mu)$ on $v \leq 40$ vertices.

The 15 graphs with parameters $(25, 12, 5, 6)$ and 10 graphs with parameters $(26, 10, 3, 4)$ were found by Paulus [12]. An exhaustive computer search by Arlazarov et al. [1] showed that these are all the graphs with these parameters. In the same paper 41 graphs with parameters $(29, 14, 6, 7)$ were found by an incomplete search (see also [6]). Independent exhaustive searches by Bussemaker and Spence (cf. [14]) showed that these are all. Bussemaker et al. [6] also give 82 graphs with parameters $(37, 18, 8, 9)$.

According to Spence [14] there exist at least 3854 graphs with parameters $(35, 16, 6, 8)$, 32548 graphs with parameters $(36, 15, 6, 6)$ and 180 graphs with parameters $(36, 14, 4, 6)$. Spence (private communication) recently classified all graphs with parameters $(40, 12, 2, 4)$; there are 28 such graphs, one more than the 27 that were already mentioned in [13].
For other parameter sets occurring in the table in Section 9 we refer to [4].

9. Feasible parameter sets

By computer we generated all feasible parameter sets for graphs on \( v \) vertices with constant \( \mu \) and \( \bar{\mu} \), having restricted Laplace eigenvalues \( \theta_1 > \theta_2 \) and vertex degrees \( k_1 \geq k_2 \), for \( v \leq 40 \), satisfying \( 0 < \mu \leq \bar{\mu} \). If \( \lambda_{22} < 0 \), then also the condition \( n_2 \leq v(\theta_1 - k_2)/\theta_1 \) is satisfied. The results are given in Table 1.

Table 1

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Note: By # we denote the number of nonregular graphs. If there are any strongly regular graphs, then their number is denoted in between brackets. By Bruck-Ryser(p) we denote that the Bruck-Ryser condition is not satisfied modulo p.

10. References