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Blackjack in Holland Casino's: basic, optimal and winning strategies

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This paper considers the card game Blackjack according to the rules of Holland Casino's in the Netherlands. Expected gains of strategies are derived with simulation and also with analytic tools. New efficiency concepts based on the gains of the basic and the optimal strategy are introduced. A general method for approximating expected gains for strategies based on card counting systems is developed. In particular it is shown how Thorp's Ten Count system and the High-Low system should be used in order to get positive expected gains. This implies that in Holland Casino's it is possible to beat the dealer in practice.

Key Words and Phrases: card games, optimality, card counting, Ten Count system, High-Low system.

1 Introduction

The card game of Blackjack (also known as Twenty One) is still today one of the most popular casino games. It has engendered much interest since by clever play it is possible for players to get an advantage over the house. This discovery was revealed in the sixties with the publication of the paper THORP (1960) and the subsequent famous book THORP (1966) entitled "Beat the Dealer". THORP showed that the player's expectation varies according to the undealt cards, and he indicated how to identify situations with a positive expectation. By raising the bet in such games an overall positive expected result can be obtained. Such winning strategies will beat the dealer in the long run.

However, casino's took their counter measures and changed the rules in order to get the advantage back. Today, these rules vary strongly between and even within casino's. For most of the variations it is still possible to obtain a serious advantage for professional hard working card counters. Although this mere fact seems to disturb casino boards terribly, the game is still attractive to exploit because most players are really amateurs and lose a lot of money. Another reason is that a winning strategy for one version of the game is a losing one for another variation.

There is a tremendous literature available on BJ (Blackjack). A lot of books are filled with strategy tables to use. Some of them are unreliable because they are based on rough approximating probability calculations; even the class of game variations for which they are supposed to be appropriate is not clearly indicated. The serious

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ultimate guide for references is DALTON (1993). We mention here the easily available and reliable mathematical books and papers BALDWIN et al. (1956), EPSTEIN (1977), GOTTLIEB (1985), GRIFFIN (1988), CHAMBLISS and ROGINSKI (1990), YAKOWITZ and KOLLIER (1992), and the appendix of BLACK (1993). All these publications deal exclusively with the American way of playing: with a dealer's hole card. In Europe in most casino's the game is played without a hole card.

The goal of this paper is to give a profound analysis of Blackjack as it is played in Holland Casino's in the Netherlands (Amsterdam, Breda, Eindhoven, Groningen, Nijmegen, Rotterdam, Scheveningen, Valkenburg, Zandvoort and Schiphol Airport). These BJHC-rules are exactly the same in all cities. They are typical Dutch in so far that the precise combination of the variations does not appear elsewhere. (We will describe the rules exactly in Section 2.)

A keystone for professional playing is the so called *basic strategy*. This strategy for BJHC was published first in VAN DER GENUGTEN (1993). Thereafter this strategy was revealed (and derived independently) by two Dutch professional Blackjack players WIND and WIND (1994).

In this paper we will analyse strategies for the BJHC-game and the concepts on which they are based. Utmost care is taken to give a clear definition of them since in literature this is often a source of confusion. Results are obtained with a special purpose computer package developed by the author.

Much of the material in this paper is, with minor changes, applicable to rules in other European casino's. For rules outside Europe differences are somewhat bigger due to the presence of the hole card.

This paper is a shortened version of VAN DER GENUGTEN (1995) obtained by omitting the description of the computer programs. It is organized as follows.

In Section 2 we give a description of the rules of BJHC.

In Section 3 we discuss the two components of strategies: the betfunction and the playing strategy. We formulate precisely the concept of optimality. This leads to a clear definition of the basic strategy. To fix the ideas we have also included its decision table in this section.

In Section 4 we consider the expected gains of strategies. By means of simulation we can give these gains for some naive strategies and the basic strategy. Also some rough estimates are given for the optimal strategy. We conclude this section by introducing efficiency concepts for arbitrary strategies.

The steady-state analysis in Section 5 makes clear which tools are needed for computer calculations for expected gains. These tools are indicated in the following two sections.

Section 6 describes the calculations for the optimal strategy and for given arbitrary strategies under the theoretical assumption that cards are drawn with replacement. In particular it treats the construction of the basic strategy. Section 7 describes the practical situation that cards are drawn without replacement.

In Section 8 a method is given for estimating the expected gains for optimal betting with arbitrary playing strategies, in particular for the basic strategy and the optimal strategy. This estimation method only gives crude estimates.

In Section 9 we follow another approach by means of linear approximations of expected gains by card fractions. Also its relation to card counting systems for betting is described. Analytic results can be obtained by approximating the distribution of the running count by that of the Brownian bridge. More in detail TTC (Thorp's Ten Count) and HiLo (High-Low) are discussed.

Finally, in Section 10 we describe how the card counting systems of Section 9 can be used for playing decisions. Since optimal betting often involves maximal bets, high budgets are needed. Therefore we consider also some other betting concepts more suitable for low budget players. For readers only interested in practical strategies which beat the dealer this is the most interesting section.

2 BJHC-rules

In this section we will give a description of the BJHC-rules together with some notation to be used in the following. Game constants for which we will consider alternatives are presented as variables together with their standard values.

BJHC is a card game that is played with 2–7 players; mostly the number of players is $a = 7$. The dealer, who is a member of the house, deals the cards out of a device called a *shoe*. A *complete* shoe consists of $n = 6$ decks of playing cards of size 52 (therefore in total $k = 52n = 312$ cards).

Cards are always dealt face up. So, at least in theory, every player can know the composition of the shoe at any stage of the game by observing the dealt cards.

Face cards have the value 10 (T); non-face cards have their indicated value. An A (ace) is counted as 1 or 11 depending on the other cards in the hand. If the sum of a hand with at least one ace counted as 11 would exceed 21, then all aces are counted as 1, otherwise one ace is counted as 11. A hand or sum is called *soft* if it contains an ace counted as 11; otherwise it is called *hard*. The main goal of players it to get hands with a sum as close as possible to but never exceeding 21 by *drawing* (asking the dealer for cards one after another) or *standing* (requesting no more cards) at the right moment. He *busts* (loses) if his (hard) sum exceeds 21. After all players the dealer draws cards too. He has no choice at all: he draws on sums ≤ 16, stands on sums ≥ 17 and ≤ 21 (hard or soft), and busts (loses) on a (hard) sum > 21. If a player and the dealer both stand, then the game is lost for the one holding the smallest sum. The combination (A, T) is called "*Blackjack*" and beats any other sum of 21. Equals sums give a *draw*.

We code cards by their value and the ace by 1. In general the card distribution in the shoe at a certain stage of the game is random and will be denoted by $C = (C(1), \ldots, C(10))$. Realizations will be denoted correspondingly with $c = (c(1), \ldots, c(10))$.

The playing stock C_1 for the first game is the (non-random) complete shoe $c_0 = (kp_1, kp_2, \ldots, kp_{10}) = (4n, 4n, \ldots, 16n) = (24, 24, \ldots, 24, 96)$, where $p_1 = \cdots = p_9 = 1/13$, $p_{10} = 4/13$ are the cards fractions in one deck. The remaining cards in the shoe after the first game become the (random) playing stock C_2 of the second game and so on. Used cards are placed into a discard rack. If during (or at the end of) a game the size $\Sigma C(i)$ of the current stock C in the shoe decreases to a level equal to or less than $k(1-\lambda)$, then after this game the cards are reshuffled and the next game starts again with a complete shoe. In practice the fraction is marked by positioning a *cut card* in the shoe at about a played fraction $\lambda = 2/3$ corresponding to a level of 104 remaining cards. However, in BJHC dealers are allowed to lower the cut card position to $\lambda = 1/2$. This appears to be a disadvantage for the players and is only done when professional card counters join the game. We call a *rowgame* a whole sequence of games, from a complete shoe up to the game in which cut card falls or is reached.

At this moment the HC's in Amsterdam and Zandvoort are experimenting with card shuffling machines. After each game cards are automatically reshuffled. In this case a rowgame consists of exactly one game. If this reshuffling would be completely random, this would correspond to BJ with a fraction $\lambda = 0$. (In practice there is a slight correlation between successive drawings.)

Outside the Netherlands there are still casino's which offer Blackjack without a cut card. This corresponds to a fraction $\lambda = 1$. For that case, and also for other high values of λ, the shoe will get empty during a game. Then the cards in the discard rack are reshuffled and placed into the shoe for playing the remaining part of the game. In this paper we assume that then the next game is started with a reshuffled complete shoe. In BJHC the discard rack is never used for this purpose because the cut card position λ is too small. However, for a general description and analysis it is worthful to consider the whole range $\lambda \in [0, 1]$.

We describe in detail one game together with the decision points of the players.

The game starts with the *betting* of the players. The minimum and maximum bet can vary with the table. Today in BJHC the possible combinations (in Dutch guilders) are (10, 500), (20, 1000), (40, 1500) and in the "cercle privé" (100, 2500) (the combination (5, 500) in Scheveningen no longer exists). Fixing the minimum bet at the unit amount $B_{min} = 1$, the possible values of the maximum bets are $B_{max} = 50$, 37.5 and 25. Bets must be in the range $[1, B_{max}]$.

After the players' betting round the dealer gives one card to each of the players and to himself (the *dealercard*). Then a second card is dealt to each of the players to make it a pair (not yet the dealer: no hole card). So at this stage all hands of players contain two cards.

If the dealercard is an A, every player may ask for *insurance* (IS) against a possible dealer's "Blackjack" later on. This is a side bet with an amount $\frac{1}{2}\times$ his original bet.

A player with the card combination "Blackjack" has to stand.

Next, players without "Blackjack", continue playing their hands, one after another, from player 1 to a.

If both cards of a hand have the same value, a player may *split* (SP) those cards and continue separately with two hands containing one card. To the additional hand a new bet equal to the original bet must be added. The first step in playing a split hand is that the dealer adds one new card to make it a pair. Repeated splitting is allowed without any restriction. However, with a no further split hand of two aces standing is obligatory. Pairs obtained with splitting cannot count as "Blackjack".

If a pair (split or not) has a hard sum 9, 10 or 11 or a soft sum 19, 20 or 21 (not Blackjack), *doubling down* (DD) is permitted. Then the player doubles his original bet, draws exactly one card and has to stand thereafter. A soft sum becomes hard because every ace in this hand gets automatically the value 1.

Finally, if a hand is not doubled, the player can *draw* or *stand* (D/S) as long as he did not stand or bust. Standing on a (hard or soft) 21 is obligatory. A non-split hand of three sevens gets a bonus of $1\times$ the original bet.

After all players have played their hands the dealer draws cards for himself according to the fixed rule already indicated.

A winning player gains an amount $1\times$ his original bet and even $1\frac{1}{2}\times$ if he wins with "Blackjack". A losing player loses his bet. In case of a draw a player neither gains nor loses: his bet is returned.

If at least one player has taken insurance against a dealer's ace then, even in the case that no player stands, the dealer must draw at least one card to see if he gets "Blackjack". If he has "Blackjack" then the player gains $2\times$ his insurance, otherwise he loses this insurance. Therefore, the dealer gives him immediately this gain and removes the player's cards from the table. This particular form of insurance is called *evenmoney*. (Of course, just for evenmoney the dealer would not draw a card.)

In the following we consider the number of decks n, the cut card position λ, the number of players a and the maximum bet $B_{\max}$ as parameters. For the standard values $n = 6$, $\lambda = 2/3$, $a = 7$ the playing time needed for one game is about 1 minute. Reshuffling takes 2 minutes. Since one rowgame contains approximately 10 games, this gives 12 minutes per rowgame or 5 rowgames per hour. So a professional player can play 10,000 rowgames (or 100,000 games) yearly if he works hard for 2000 hours per year. This should be kept in mind in judging expected gains per (row) game of strategies. For theoretical purposes concerning approximations we will also consider games in which every card is drawn with replacement. We refer to these games by the parameter values $n = \infty$ and $\lambda = 0$. This implies that rowgames coincide with games.

3 Strategies and optimality

Consider a game with fixed parameters n, a λ and $B_{\max}$. A strategy (H_v, S_v) for a player v consists of two parts: a *betting* strategy H_v which prescribes the betsize at the start of each new game, and a *playing* strategy S_v which prescribes the playing decisions IS, SP, DD, D/S at any stage of the game.

We restrict the class of all possible strategies of a player v in the following way. His betsize at the start of a game shall only depend on the stock at that moment; therefore it can be characterized by a *betfunction* $H_v(c) \in [1, B_{\max}]$, $c \in \mathcal{C}$, with $\mathcal{C} = \{c_0\} \cup \{c : \Sigma c(i) > k(1-\lambda)\}$ the class of possible stocks which can be encountered with betting.

The playing decisions of the player v at a certain stage of the game shall only depend on the current or past stocks in that game and the exposed cards on the table at that stage. So a playing strategy S_v is a function which specifies the playing decisions for every possible stock c at the start of a game. More precisely, let $d_0(c)$ denote the sequence of the $2a+1$ cards dealt by the dealer ($v = 0$) before the playing round starts, $d_v(c)$ (for $v = 0, \ldots, a$) the whole sequence of cards used by the players $0, \ldots, v$ and, more specific, $d_{vj} = (d_{v-1}(c), x_{1v}, \ldots, x_{jv})$ the sequence up to the stage in which player v already got additionally j cards $x_{1v}, \ldots, x_{jv}$. Then S_v contains the relevant playing decisions at any stage d_{vj} during this game. This constitutes a class $\mathcal{S}_v$ of playing strategies. The stocks during successive games only depend on the playing strategies $S_v \in \mathcal{S}_v$ for $v = 1, \ldots, a$ of the players and not on their betfunctions. The restriction to such playing strategies gives no loss of generality at all.

Denote by $G_1(c)$ the (random) gain of a player v for a game with starting stock $c \in \mathcal{C}$ and minimum bet $B_{\min} = 1$. Then the (random) gain $G(c)$ of this player using the betfunction $H(c)$ is given by $G(c) = H(c)G_1(c),\ c \in C$.

For given playing strategies $S_1, \ldots, S_a$ the probability distribution $\mathcal{L}(G_1$ instead of $(Gc))$ is fixed. Given these strategies we call the betfunction H_v of player v optimal if it maximizes $E(G(c))$ for all $c \in \mathcal{C}$. Clearly, H_v is optimal for

$$H_v(c) = \begin{cases} 1 & \text{if } E(G_1(c)) \leq 0 \\ B_{\max} & \text{if } E(G_1(c)) > 0 \end{cases}$$

For fixed S_j, $j \neq v$, the distribution $\mathcal{L}(G_1(c))$ only depends on the choice $S_v \in \mathcal{S}_v$. Given the S_j with $j \neq v$ we call the playing strategy S_v optimal if $S_v(d_{vj})$ maximizes $E(G_1(c) \mid d_{vj})$ for every stage d_{vj} of the game that can be reached by player v and for every stock $c \in \mathcal{C}$.

Optimality for player v depends on the playing strategies S_j of other players as well. In analyzing strategies for player v we must make a specific choice for the playing strategies of the other players. A reasonable and pragmatic approach is to consider possible improvements of player v amid other players of moderate skill playing independently of each other and following a simple so called basic strategy. Although in practice moderate players do not quite reach the level of this strategy, we choose it as a well defined reference point (see e.g. Bond, 1974, Keren and Wagenaar, 1985, Wagenaar, 1988, Chau and Phillips, 1995).

We define the basic strategy S_{bas} as the playing strategy which would be optimal under the theoretical assumption that all cards are drawn with replacement (i.e. the game with $n = \infty$ and $\lambda = 0$). Clearly, under this assumption $E(G_1(c_0) \mid d_{vj})$ will only depend on d_{vj} through the dealercard and the cards in the hand(s) of player v and not on the playing strategies S_j of the other players $j \neq v$. Therefore S_{bas} is the same for all players and can be tabulated as a function of the dealercard and characteristics of the player's hand. We describe its construction in Section 6. Table 1 gives the result.

So from now on while evaluating numerically the quality of the strategy of a particular player we assume that the other players follow the basic strategy S_{bas}. Therefore the optimal playing strategy S_{opt} will only depend on the number of decks n, the number of players a, the cut card position λ and the particular player v. We denote by H_{bas}, H_{opt} the optimal betfunctions belonging to S_{bas}, S_{opt}, respectively. These functions depend on B_{max} too.

4 Expected gains and efficiency

Consider fixed parameters n, a, λ and B_{max}. For a fixed choice of playing strategies for each player, we consider the expected gain of a particular player with strategy (H, S).

The random sequence of all successive stocks by dealing one card after another during the mth game starting with stock C_m and ending with C_{m+1} determines the

Table 1. Basic Strategy S_{bas} of BJHC

INSURANCE: never

SPLITTING (Split = X; No Split = —)

Dealercard Pair	A	2	3	4	5	6	7	8	9	T
AA	—	X	X	X	X	X	X	X	X	X
22	—	X	X	X	X	X	X	—	—	—
33	—	X	X	X	X	X	X	—	—	—
44	—	—	—	—	X	X	—	—	—	—
55	—	—	—	—	—	—	—	—	—	—
66	—	X	X	X	X	X	—	—	—	—
77	—	X	X	X	X	X	X	—	—	—
88	—	X	X	X	X	X	X	X	X	—
99	—	X	X	X	X	X	—	X	X	—
TT	—	—	—	—	—	—	—	—	—	—

DOUBLE DOWN (DDown = X; No DDown = —)

Dealercard Sum	A	2	3	4	5	6	7	8	9	T
Hard 9	—	—	X	X	X	X	—	—	—	—
Hard 10	—	X	X	X	X	X	X	X	X	—
Hard 11	—	X	X	X	X	X	X	X	X	—
Soft 19–21	—	—	—	—	—	—	—	—	—	—

DRAW/STAND (Draw = X; Stand = —)

Dealercard Sum	A	2	3	4	5	6	7	8	9	T
Hard $\leq$ 11	X	X	X	X	X	X	X	X	X	X
Hard 12	X	X	X	—	—	—	X	X	X	X
Hard 13	X	—	—	—	—	—	X	X	X	X
Hard 14	X	—	—	—	—	—	X	X	X	X
Hard 15	X	—	—	—	—	—	X	X	X	X
Hard 16	X	—	—	—	—	—	X	X	X	X
Hard $\geq$ 17	—	—	—	—	—	—	—	—	—	—
Soft $\leq$ 17	X	X	X	X	X	X	X	X	X	X
Soft 18	X	—	—	—	—	—	—	—	X	X
Soft $\geq$ 19	—	—	—	—	—	—	—	—	—	—

gain G_m of the mth game. Then the average gain $\mu_G = \mu_G(H, S)$ per game in the long run is given by

$$\mu_G = \lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^{m} G_i, \quad \text{a.s.} \tag{1}$$

(The average bet $\mu_B = \mu_B(H, S)$ per game in the long run is defined similarly.) Let GR_m be the sum of all gains in the mth rowgame and N_m the number of games in this rowgame. Then the average gain μ_{GR} and number of games μ_N per rowgame is given by

$$(\mu_{GR}, \mu_N) = \lim_{M\to\infty} \frac{1}{M}\sum_{m=1}^{M}(GR_m, N_m), \quad \text{a.s.} \tag{2}$$

Clearly,

$$\mu_G = \mu_{GR}/\mu_N \tag{3}$$

For a given set of playing strategies the values of μ_{GR}, μ_N and μ_G can be determined with simulation. For a reasonable accuracy a simulation length of about $M = 50{,}000{,}000$ rowgames is needed. On a PC-Pentium 90 such a simulation run requires 4 days.

In order to given an idea about the losses that are suffered with simple naive strategies we performed a simulation for BJHC with $a = 7$ players, giving player v the naive strategy "stand if sum $\geq v + 11$ and draw otherwise". The strategy "stand for sum ≥ 12" means "never bust" and "stand for sum ≥ 17" is called "mimic the dealer". The betsize is 1.

Table 2. Sim. gains of naive playing strategies ($n = 6$, $\lambda = 2/3$, $a = 7$, $H \equiv 1$)
playing strategy: never IS, SP or DD; S if sum $\geq v + 11$ and D otherwise
($M = 50{,}200{,}000$ rowgames $-$ $\mu_N = 10.13$ games)

Pv	μ_{GR}	$\pm 95\% CI$	$\mu_G(1, S_{v+11})$
D	4.622	0.004	0.4562
P5	−0.524	0.001	−0.0517
P4	−0.527	0.001	−0.0521
P6	−0.571	0.001	−0.0564
P3	−0.586	0.001	−0.0579
P2	−0.688	0.001	−0.0679
P1	−0.814	0.001	−0.0804
P7	−0.911	0.001	−0.0899

In the first column the player D refers to the dealer and Pv to player v. The third column contains the half length of a 95% confidence interval for μ_{GR}. We see that these simple strategies lead to a disaster. Even the relatively best player P5 standing on 16 suffers a loss of more than 5%. This is much more than a pure chance game such as Roulette would cost! Certainly such players are welcome at the Blackjack tables in HC.

All players can do much better with a little bit more effort by following the basic strategy. Since the use of pencil and paper is strictly forbidden in BJHC, they just have to learn table 1 by heart. A simulation results is given in Table 3.

Table 3. Sim. gains of S_{bas} ($n = 6$, $\lambda = 2/3$, $a = 7$, $H \equiv 1$)
playing strategy: see Table 1
($M = 50{,}000{,}000$ rowgames $-\mu_N = 9.86$ games)

P	μ_{GR}	+95%CI	$\mu_G(1, S_{bas})$
D	0.3726	0.005	0.0378
P5	−0.053	0.001	−0.0054
P2	−0.053	0.001	−0.0054
P6	−0.053	0.001	−0.0054
P4	−0.053	0.001	−0.0054
P7	−0.054	0.001	−0.0054
P3	−0.054	0.001	−0.0054
P1	−0.054	0.001	−0.0055

We see that $\mu_G(1, S_{bas}) = -0.0054$ is almost the same for all players and therefore independent of the position at the table. Although the value is still negative it is much higher than the values of μ_G for the naive strategies in Table 2.

The gain $\mu_G(1, S_{bas})$ for the basic strategy does hardly depend on the number of players. Table 4 gives the simulation result for 1 instead of 7 players.

Table 4. Sim. gains of S_{bas} ($n = 6$, $\lambda = 2/3$, $a = 1$, $H \equiv 1$)
($M = 1{,}000{,}000{,}000$ rowgames $-\mu_N = 39.5$ games)

	μ_{GR}	$\pm$95%CI	$\mu_G(1, S_{bas})$
P1	−0.217	0.001	−0.0050

The value −0.0050 differs slightly from $\mu_G(1, S_{bas}) = -0.0054$ for $n = 6$, $\lambda = 2/3$, $a = 7$. Roughly speaking, the basic strategy with bet 1 gives a loss of 0.0050 to 0.0055 for all players and is independent of the number of players a.

Rather crude estimates of μ_G can be given for the optimal betfunctions H_{bas}, H_{opt} in combination with the playing strategies S_{bas}, S_{opt}. Table 5 gives some results for a particular player, thereby assuming that the other players play the basic strategy. (For details see Van der Genugten, 1995, sections 4 and 8.)

Table 5. Estimated gains ($n = 6$, $\lambda = 2/3$, $B_{max} = 50$)

Strategy	μ_G
$(1, S_{bas})$	−0.005
$(1, S_{opt})$	−0.004
(H_{bas}, S_{bas})	+0.08
(H_{opt}, S_{opt})	+0.11

The table shows that there exist strategies (H, S) with positive expected gains. Using such strategies will beat the dealer in the long run.

Consider for fixed playing strategies of the other players the strategy (H, S) of a particular player. We define the *total efficiency* $\mathrm{TE}(H, S)$, the *betting efficiency* $\mathrm{BE}(H, S)$ and the (playing) *strategy efficiency* $\mathrm{SE}(H, S) = \mathrm{SE}(S)$ by, respectively,

$$\mathrm{TE}(H, S) = \frac{\mu_G(H, S) - \mu_G(1, S_{bas})}{\mu_G(H_{opt}, S_{opt}) - \mu_G(1, S_{bas})}$$

$$\mathrm{BE}(H, S) = \frac{\mu_G(H, S) - \mu_G(1, S)}{\mu_G(H_{opt}, S_{opt}) - \mu_G(1, S)}$$

$$\mathrm{SE}(S) = \frac{\mu_G(1, S) - \mu_G(1, S_{bas})}{\mu_G(1, S_{opt}) - \mu_G(1, S_{bas})}$$

Clearly,

$$\mathrm{TE}(H, S) = \mathrm{BE}(H, S) + TM.\mathrm{SE}(S).(1 - \mathrm{BE}(H, S))$$

where TM is the *table multiplier* (not depending on S) defined by

$$\mathrm{TM} = \frac{\mu_G(1, S_{opt}) - \mu_G(1, S_{bas})}{\mu_G(H_{opt}, S_{opt}) - \mu_G(1, S_{bas})}$$

For obtaining a high betting efficiency of the strategy (H, S) we see that much effort should be put into the approximation H of the optimal betfunction H_{bas} in a simple playable way; the improvement of the playing strategy S from S_{bas} towards S_{opt} is less important. This is even more true when the table multiplier TM is small. Then the total efficiency TE of (H, S) is almost completely determined by its betting efficiency BE. So the improvement of S towards S_{opt} for influencing SE is of minor importance.

For $n = 6$ and $B_{max} \in [25, 50]$, $\lambda \in [1/2, 2/3]$ the table multipliers TM of BJHC are in the range $0.01 < \mathrm{TM} < 0.03$ and therefore very small. (The figures in Table 5 are in agreement with this.) In fact the large number of decks $n = 6$ has for a great deal reduced the effect of skill to betting.

Figure 6 shows the BE (betting efficiency) of (H_{bas}, S_{bas}) for $1/6 \leq \lambda \leq 5/6$ and $B_{max} = 1$, 2, 25, 37.5 and 50. For $B_{max} = 1$ the efficiency is 0 by definition. For $B_{max} = 2$ the betting efficiency is very low (between 0.3 and 0.4 for varying values of λ. For the usual BJHC-values $B_{max} = 25$, 37.5 and 50 the efficiency is more or less constant and only depends on λ. It decreases from 0.7 for $\lambda = 0.3$ to 0.6 for $\lambda = 0.8$. Clearly, for S_{bas} it is much more important to use a good betfunction than improving S_{bas} itself. (For details see Van der Genugten, 1995, section 8.)

5 Steady-state analysis

Consider a fixed choice of playing strategies. The random sequence $C_1, C_2, \ldots$ of starting stocks form an ergodic Markov chain with state space $\mathcal{C}$ and initial value $C_1 = c_0$. Denote by

$$\pi(c) = \lim_{m \to \infty} P\{C_m = c\}, \quad c \in \mathcal{C} \tag{4}$$

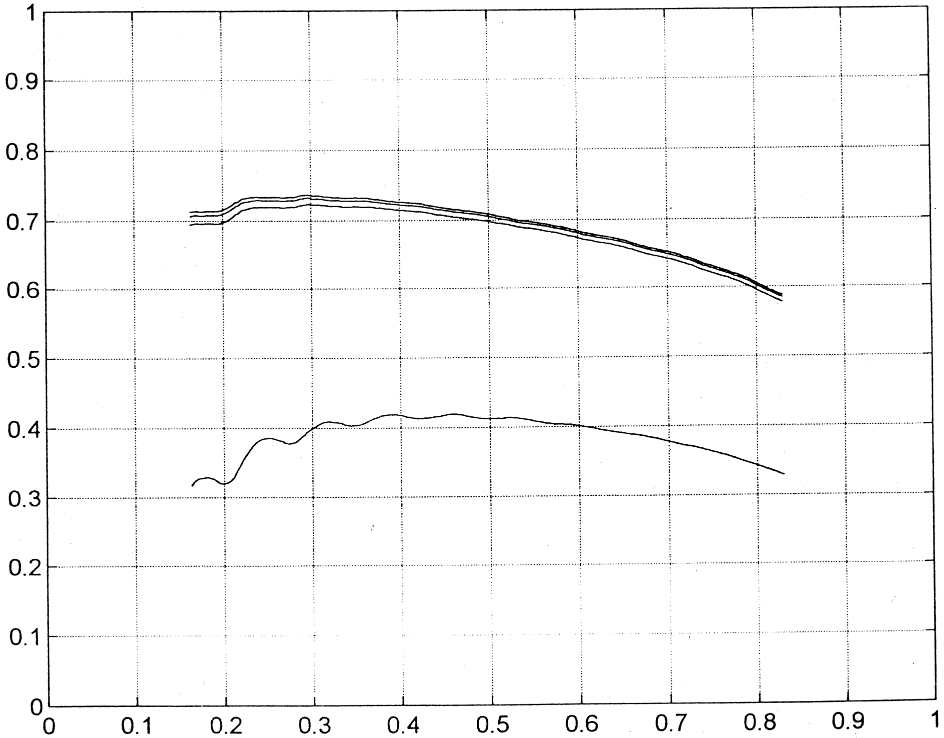

Fig. 6. BE(H_{bas}, S_{bas}) for $B_{\max} = 1, 2, 25, 37.5, 50$ and for $1/6 < \lambda < 5/6$ ($n = 6$, $a = 7$).

its limit distribution (independent of c_0). We can express the average gains in the long run as expectations of gains in only one game if we start this game with a random stock $C_1 = C$ with $\mathcal{L}(C) = \pi$ (the steady state). Then for $G_1 = G_1(C)$ and $G = G(C) = H(C)G_1(C)$ we have according to the LLN for Markov chains:

$$\mu_{G_1} = E(G_1) = E(G_1(C)) = \sum_{c \in C} \pi(c)E(G_1(c)) \tag{5}$$

and more general,

$$\mu_G = E(G) = E(H(C)G_1(C)) = \sum_{c \in C} \pi(c)H(c)E(G_1(c)) \tag{6}$$

So, at least in theory, we can use (6) for calculating the expected gain μ_G of any betfunction H by determining $\pi(c)$ and $E(G_1(c))$, $c \in \mathcal{C}$.

In evaluating numerically the strategy of a particular player we taken for π the limit probabilities for the assumed standard case that all players follows S_{bas}. So we neglect the effect that π will change when the particular player deviates from S_{bas}. In practice this effect is small and good approximations will be obtained. Neglecting this effect, we see from (5) that the playing strategy S_{opt} of a player as defined in Section 3 maximizes his μ_{G_1}. The corresponding betfunction H_{opt} (depending on S_{opt})

maximizes his μ_G in (6). The same holds for the optimal betfunction H_{bas} corresponding to S_{bas}.

For BJ with $a = 1$ player we can calculate $E(G_1(c)|\, d_{1j})$ for every $c \in \mathcal{C}$ and for every card sequence d_{1j} of the player. This can be done not only for a given playing strategy but also for the optimal strategy. We distinguish the cases $n = \infty$ (drawing with replacement) and $n < \infty$ (drawing without replacement).

For $n = \infty$ the calculations are relatively simple because the card fractions in the stock remain unchanged. The computer programs solve the problems for a given stock c in about 0.5 sec on a PC-Pentium 90. We discuss the results in Section 6.

For $n < \infty$ the calculations are very complicated since all possible stock developments from a given stock c have to be taken into account. Yet, by a special coding system for such developments we were able to solve the problem within a finite time. However, the needed computer time for a given stock c of moderate size with $n = 6$ decks is about 5 days on the PC-Pentium 90 (and on a VAX mainframe still 19 hours). About 80% of the needed time is due to the calculations for repeated splitting. For many $c \in \mathcal{C}$ the differences between the values of $E(G_1(c))$ for $n = \infty$ and moderate finite n are small. This will be discussed in Section 7. Therefore in applying (6) we can take $n = \infty$ for approximations with values of λ not too close to 1.

In BJ with a number of players $a > 1$, for a particular player v there is also information contained in $d_{v-1}(c)$ and conditioning has to be performed for a whole sequence d_{vj}. This is simply impossible to do. However, the differences with $a = 1$ player seem to be small. Therefore we will use the obtained results for one player also as approximations for the general case of a particular player among the other players.

With these approximations a straightforward computation of μ_G by (6) is still impossible. The problem is the large number of stocks in $\mathcal{C}$ (about $(4n + 1)^9(16n + 1)$; for $n = 6$ resulting in 3.7×10^{14}). Therefore we follow an approach which mixes simulation and analysis. Results are described in Section 8. They are based on conditioning to the fraction t of played cards. This method is also used for the analysis of the card counting methods in Sections 9 and 10.

6 Expected gains for infinite decks

In this section we assume that cards are drawn with replacement. Given a stock $c \in \mathcal{C}$ we can maximize $E(G_1(c)|\, d_{1j})$ for any sequence $d_{1j} = (d_0(c), x_{11}, \ldots, x_{1j})$, where $d_0(c)$ contains the dealercard and the hands of two cards of all players and where $x_{11}, \ldots, x_{1j}$ denotes the cards of the player thereafter. (Since all cards are drawn with replacement the stock at stage d_{1j} is still c.)

Table 7 gives the result for the starting stock c_0 for $n = 6$. The unconditional mean becomes $E(G_1(c_0)) = -0.00614$. It is in fact an extension of Table 1 containing S_{bas} since it optimizes decisions for the starting stock c_0. The main part has an entry for each dealercard 1, ..., 10. Each hand has three columns:

Dec = coded optimal decision,
Opt = expected gains for the optimal decision,
Dif = difference with the expected gain of the second best decision.

Table 7. Optimal decisions and expected gains for the starting stock (with replacement)
Stock: 312 24 24 24 24 24 24 24 24 24 96

GAME VALUE: −0.006144
(Decisions: 0 = Stand 1 = Draw 2 = Double Down 3 = Split)

INSURANCE
Decision: No − Opt: 0.000 − Dif: 0.038

Dealer:		1	2	3	4	5	6	7	8	9	T
SPLITTING											
A A	Dec	1	3	3	3	3	3	3	3	3	3
	Opt	−0.322	0.609	0.658	0.707	0.757	0.817	0.633	0.507	0.368	0.119
	Dif	0.176	0.528	0.554	0.581	0.600	0.631	0.468	0.412	0.368	0.260
2 2	Dec	1	3	3	3	3	3	3	1	1	1
	Opt	−0.483	−0.084	−0.015	0.060	0.153	0.225	0.007	−0.159	−0.241	−0.344
	Dif	0.414	0.031	0.067	0.109	0.165	0.214	0.096	0.015	0.124	0.257
3 3	Dec	1	3	3	3	3	3	3	1	1	1
	Opt	−0.518	−0.138	−0.056	0.031	0.125	0.195	−0.052	−0.217	−0.293	−0.389
	Dif	0.413	0.003	0.051	0.103	0.160	0.208	0.099	0.012	0.123	0.255
4 4	Dec	1	1	1	1	3	3	1	1	1	1
	Opt	−0.444	−0.022	0.008	0.039	0.076	0.140	0.082	−0.060	−0.210	−0.307
	Dif	0.522	0.145	0.099	0.050	0.005	0.025	0.212	0.227	0.256	0.381
5 5	Dec	1	2	2	2	2	2	2	2	2	1
	Opt	−0.251	0.359	0.409	0.461	0.513	0.576	0.392	0.287	0.144	−0.054
	Dif	0.750	0.552	0.526	0.497	0.461	0.464	0.584	0.631	0.663	0.679
6 6	Dec	1	3	3	3	3	3	1	1	1	1
	Opt	−0.550	−0.212	−0.124	−0.031	0.066	0.132	−0.213	−0.272	−0.340	−0.429
	Dif	0.486	0.041	0.110	0.180	0.233	0.286	0.044	0.131	0.230	0.349
7 7	Dec	1	3	3	3	3	3	3	1	1	1
	Opt	−0.612	−0.131	−0.048	0.040	0.131	0.232	−0.049	−0.372	−0.431	−0.507
	Dif	0.432	0.162	0.204	0.251	0.298	0.386	0.273	0.017	0.125	0.236
777	Dec	1	3	3	3	3	3	3	1	1	1
	Opt	−0.535	−0.131	−0.048	0.040	0.131	0.232	−0.049	−0.295	−0.354	−0.430
	Dif	0.509	0.085	0.127	0.174	0.221	0.309	0.196	0.094	0.202	0.312
8 8	Dec	1	3	3	3	3	3	3	3	3	1
	Opt	−0.666	0.076	0.149	0.223	0.300	0.413	0.325	−0.020	−0.387	−0.575
	Dif	0.222	0.369	0.401	0.434	0.467	0.566	0.740	0.438	0.123	0.039
9 9	Dec	0	3	3	3	3	3	3	3	3	0
	Opt	−0.377	0.196	0.259	0.324	0.393	0.472	0.400	0.235	−0.077	−0.242
	Dif	0.329	0.074	0.111	0.148	0.194	0.189	0.030	0.129	0.106	0.195
T T	Dec	0	0	0	0	0	0	0	0	0	0
	Opt	0.146	0.640	0.650	0.661	0.670	0.704	0.773	0.792	0.758	0.435
	Dif	0.649	0.275	0.238	0.200	0.158	0.128	0.259	0.396	0.525	0.542
DOUBLE DOWN											
H 9	Dec	1	1	2	2	2	2	1	1	1	1
	Opt	−0.353	0.074	0.121	0.182	0.243	0.317	0.172	0.098	−0.052	−0.218
	Dif	0.562	0.013	0.020	0.053	0.085	0.121	0.068	0.125	0.249	0.367
H10	Dec	1	2	2	2	2	2	2	2	2	1
	Opt	−0.251	0.359	0.409	0.461	0.513	0.576	0.392	0.287	0.144	−0.054
	Dif	0.374	0.176	0.203	0.230	0.256	0.288	0.136	0.089	0.028	0.108
H11	Dec	1	2	2	2	2	2	2	2	2	1
	Opt	−0.209	0.471	0.518	0.566	0.615	0.667	0.463	0.351	0.228	0.033
	Dif	0.331	0.232	0.257	0.283	0.307	0.334	0.171	0.121	0.070	0.021

continued

Table 7—*continued*

Dealer:		1	2	3	4	5	6	7	8	9	T
S19	Dec	0	0	0	0	0	0	0	0	0	0
	Opt	−0.115	0.386	0.404	0.423	0.440	0.496	0.616	0.594	0.288	−0.019
	Dif	0.800	0.325	0.284	0.241	0.196	0.179	0.512	0.620	0.589	0.566
S20	Dec	0	0	0	0	0	0	0	0	0	0
	Opt	0.146	0.640	0.650	0.661	0.670	0.704	0.773	0.792	0.758	0.435
	Dif	0.771	0.281	0.241	0.200	0.158	0.128	0.381	0.505	0.614	0.597
S21	Dec	0	0	0	0	0	0	0	0	0	0
	Opt	0.331	0.882	0.885	0.889	0.892	0.903	0.926	0.931	0.939	0.812
	Dif	0.871	0.411	0.368	0.323	0.277	0.235	0.463	0.580	0.711	0.800
DRAW/STAND											
H 3	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	−0.465	−0.101	−0.069	−0.036	0.000	0.024	−0.057	−0.131	−0.215	−0.322
	Dif	0.304	0.192	0.183	0.175	0.167	0.178	0.418	0.380	0.328	0.254
H 4	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	−0.483	−0.115	−0.083	−0.049	−0.012	0.011	−0.088	−0.159	−0.241	−0.344
	Dif	0.287	0.178	0.170	0.162	0.155	0.165	0.387	0.351	0.302	0.232
H 5	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	−0.501	−0.128	−0.095	−0.061	−0.024	−0.001	−0.119	−0.188	−0.267	−0.366
	Dif	0.269	0.165	0.157	0.150	0.143	0.153	0.356	0.322	0.277	0.210
H 6	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	−0.518	−0.141	−0.107	−0.073	−0.035	−0.013	−0.152	−0.217	−0.293	−0.389
	Dif	0.251	0.152	0.145	0.138	0.132	0.141	0.323	0.293	0.251	0.187
H 7	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	−0.522	−0.109	−0.077	−0.043	−0.007	0.029	−0.069	−0.211	−0.285	−0.371
	Dif	0.247	0.184	0.176	0.168	0.160	0.183	0.407	0.300	0.258	0.204
H 8	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	−0.444	−0.022	0.008	0.039	0.071	0.115	0.082	−0.060	−0.210	−0.307
	Dif	0.325	0.271	0.260	0.250	0.238	0.269	0.558	0.451	0.333	0.269
H 9	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	−0.353	0.074	0.101	0.129	0.158	0.196	0.172	0.098	−0.052	−0.218
	Dif	0.416	0.367	0.354	0.340	0.325	0.350	0.647	0.609	0.491	0.358
H10	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	−0.251	0.182	0.206	0.230	0.256	0.288	0.257	0.198	0.117	−0.054
	Dif	0.518	0.475	0.458	0.442	0.423	0.441	0.732	0.708	0.660	0.522
H11	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	−0.209	0.238	0.260	0.283	0.307	0.334	0.292	0.230	0.158	0.033
	Dif	0.561	0.531	0.513	0.494	0.475	0.487	0.768	0.740	0.701	0.609
H12	Dec	1	1	1	0	0	0	1	1	1	1
	Opt	−0.550	−0.253	−0.234	−0.211	−0.167	−0.154	−0.213	−0.272	−0.340	−0.429
	Dif	0.219	0.039	0.019	0.002	0.026	0.017	0.263	0.239	0.203	0.147
H13	Dec	1	0	0	0	0	0	1	1	1	1
	Opt	−0.582	−0.293	−0.252	−0.211	−0.167	−0.154	−0.269	−0.324	−0.387	−0.469
	Dif	0.187	0.015	0.039	0.063	0.090	0.082	0.206	0.187	0.156	0.106
H14	Dec	1	0	0	0	0	0	1	1	1	1
	Opt	−0.612	−0.293	−0.252	−0.211	−0.167	−0.154	−0.321	−0.372	−0.431	−0.507
	Dif	0.157	0.069	0.096	0.124	0.154	0.147	0.154	0.139	0.112	0.068
777	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	−0.535	−0.216	−0.175	−0.134	−0.090	−0.077	−0.244	−0.295	−0.354	−0.430
	Dif	0.234	0.077	0.077	0.077	0.077	0.077	0.231	0.216	0.189	0.145

continued

Table 7—*continued*

Dealer:		1	2	3	4	5	6	7	8	9	T
H15	Dec	1	0	0	0	0	0	1	1	1	1
	Opt	−0.640	−0.293	−0.252	−0.211	−0.167	−0.154	−0.370	−0.417	−0.472	−0.543
	Dif	0.129	0.124	0.154	0.185	0.218	0.212	0.106	0.094	0.072	0.033
H16	Dec	1	0	0	0	0	0	1	1	1	1
	Opt	−0.666	−0.293	−0.252	−0.211	−0.167	−0.154	−0.415	−0.458	−0.509	−0.575
	Dif	0.104	0.178	0.212	0.245	0.282	0.277	0.061	0.052	0.034	0.001
H17	Dec	0	0	0	0	0	0	0	0	0	0
	Opt	−0.639	−0.153	−0.117	−0.081	−0.045	0.012	−0.107	−0.382	−0.423	−0.464
	Dif	0.055	0.383	0.414	0.446	0.478	0.520	0.377	0.124	0.131	0.152
H18	Dec	0	0	0	0	0	0	0	0	0	0
	Opt	−0.377	0.122	0.148	0.176	0.200	0.283	0.400	0.106	−0.183	−0.242
	Dif	0.364	0.744	0.768	0.793	0.815	0.891	0.991	0.697	0.433	0.433
H19	Dec	0	0	0	0	0	0	0	0	0	0
	Opt	−0.155	0.386	0.404	0.423	0.440	0.496	0.616	0.594	0.288	−0.019
	Dif	0.694	1.115	1.132	1.150	1.166	1.219	1.331	1.308	1.003	0.732
H20	Dec	0	0	0	0	0	0	0	0	0	0
	Opt	0.146	0.640	0.650	0.661	0.670	0.704	0.773	0.792	0.758	0.435
	Dif	1.044	1.495	1.505	1.516	1.525	1.558	1.625	1.643	1.609	1.296
S12	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	−0.322	0.082	0.104	0.127	0.156	0.186	0.165	0.095	0.000	−0.142
	Dif	0.448	0.375	0.356	0.338	0.324	0.340	0.641	0.606	0.543	0.434
S13	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	−0.347	0.047	0.074	0.102	0.133	0.162	0.122	0.054	−0.038	−0.174
	Dif	0.422	0.339	0.326	0.314	0.301	0.315	0.598	0.565	0.505	0.402
S14	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	−0.373	0.022	0.051	0.080	0.112	0.139	0.080	0.013	−0.075	−0.206
	Dif	0.397	0.315	0.303	0.291	0.279	0.293	0.555	0.524	0.468	0.370
S15	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	−0.398	−0.000	0.029	0.059	0.092	0.118	0.037	−0.027	−0.112	−0.237
	Dif	0.372	0.293	0.281	0.270	0.259	0.272	0.512	0.483	0.431	0.339
S16	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	−0.422	−0.021	0.009	0.040	0.073	0.099	−0.005	−0.067	−0.149	−0.268
	Dif	0.347	0.272	0.261	0.251	0.241	0.253	0.470	0.444	0.395	0.307
S17	Dec	1	1	1	1	1	1	1	1	1	1
	Opt	−0.432	−0.000	0.029	0.059	0.091	0.128	0.054	−0.073	−0.150	−0.259
	Dif	0.207	0.152	0.146	0.140	0.136	0.116	0.161	0.309	0.273	0.206
S18	Dec	1	0	0	0	0	0	0	0	1	1
	Opt	−0.372	0.122	0.148	0.176	0.200	0.283	0.400	0.106	−0.101	−0.210
	Dif	0.005	0.059	0.058	0.057	0.052	0.093	0.229	0.066	0.082	0.032
S19	Dec	0	0	0	0	0	0	0	0	0	0
	Opt	−0.115	0.386	0.404	0.423	0.440	0.496	0.616	0.594	0.288	−0.019
	Dif	0.196	0.262	0.255	0.248	0.237	0.256	0.395	0.442	0.280	0.140
S20	Dec	0	0	0	0	0	0	0	0	0	0
	Opt	0.146	0.640	0.650	0.661	0.670	0.704	0.773	0.792	0.758	0.435
	Dif	0.397	0.457	0.444	0.431	0.414	0.416	0.516	0.594	0.642	0.489
GAIN FOR SUM = 21											
	BJ	1.038	1.500	1.500	1.500	1.500	1.500	1.500	1.500	1.500	1.385
	NoBJ	0.331	0.882	0.885	0.889	0.892	0.903	0.926	0.931	0.939	0.812

(Note that during the game "splitting" comes before "double down", and "double down" before "draw/stand".) For example, with a hand (5, 5) against a dealercard 8 we see under "splitting" Dec = 2. So we should not split but double down. In this case the expected gain is Opt = 0.287. Splitting would give a difference Dif = 0.631 compared with the optimal decision, leading to an expected gain of 0.287 − 0.631 = −0.344. The sum of (5, 5) is H(ard)10. Under "double down" for H10 we get the same value Opt = 0.287. The second best decision (draw or stand) has a difference Dif = 0.089, leading to an expected gain 0.287 − 0.089 = 0.198 for "not double down". Under "draw/stand" we find that this value corresponds to Dec = 1 (drawing). The difference is 0.708 leading to an expected gain of 0.198 − 0.708 = −0.510 for standing.

Under "splitting" and "draw/stand" the code 777 refers to the situation that the extra bonus for three sevens can be obtained and 77 or H14 to the situation that this is not the case.

For $c \neq c_0$ the optimal decisions can be quite different from those of S_{bas}.

We will not describe the algorithms leading to the results above. Only the algorithm for insurance is very easy. Let $f_{10} = c(10)/\Sigma c(i)$ be the fraction of tens in the current stock c. This equals the probability that the dealer gets BJ. Therefore the expected gain with insurance is $-\frac{1}{2} + \frac{3}{2} f_{10}$. So we should insure if $f_{10} > \frac{1}{3}$. For the starting stock c_0 we have $f_{10} = p_{10} < \frac{1}{3}$. Therefore the basic strategy prescribes "never insure".

For a discussion of the other algorithms we refer to VAN DER GENUGTEN (1995), section 6.

7 Expected gains for finite decks

In this section we assume that cards are drawn without replacement. We have to maximize $E(G_1(c)|\, d_{1j})$ for any sequence $d_{1j} = (d_0(c),\ x_{11}, \ldots, x_{1j})$ for $a = 1$ player.

For the unconditional mean with $n = 6$ decks we find $E(G_1(c_0)) = -0.0052$. For the infinite case $n = \infty$ we found the value −0.0061 (see Section 6). The difference is mainly due to the different procedures in drawing cards.

Intermediate results are the expected gains $E(G_1(c_0)|\, d_0(c_0))$ given the dealercard and the hand of two cards of the player. So these three cards are removed from the starting stock before the calculations for splitting, double down and draw/stand are made. It is interesting to compare these results with the corresponding infinite case. It appears that the optimal decisions for $n = 6$ and $n = \infty$ coincide for almost all players' hands. There are some exceptions, e.g. for $n = 6$ we should draw for (10, 2) and stand for all other hands against a dealercard 4. However, the effect on the expected gain is very small (For details we refer to VAN DER GENUGTEN, 1995, section 7.)

There is a systematic pattern in the unconditional mean for varying n. Table 8 contains maximal expected gains for varying n. The table shows that S_{opt} (or S_{bas}) gives a higher expected gain when the number of decks n decreases, although for $n \geq 1$ the value is still negative.

Table 8. Expected gains $E(G_1(c_0))$ for S_{opt} $(a = 1)$

n	$E(G_1(c_0))$	n	$E(G_1(c_0))$	n	$E(G_1(c_0))$	n	$E(G_1(c_0))$
∞	-0.0061	20	-0.0059	4	-0.0047	1	-0.0029
100	-0.0061	6	-0.0052	3	-0.0043	$\frac{1}{2}$	$+0.0071$
50	-0.0060	5	-0.0050	2	-0.0033		

8 Estimation of expected gains

Consider a fixed choice of playing strategies and one particular player. We continue the steady-state analysis in Section 5 by conditioning to the fraction t of played cards. Let $\mathcal{C}_t = \{c : \Sigma c(j) = k - kt\}$ denote the set of stocks containing $k - kt$ cards $(t = 0,\ 1/k, \ldots, \lambda - 1/k)$. Then from (6) we get

$$\mu_G = \sum_{t=0}^{\lambda - 1/k} p(t) E_t(G) \tag{7}$$

with

$$p(t) = P\{C \in \mathcal{C}_t\} \tag{8}$$

$$E_t(G) = E_t(H(C) G_1(C)) \tag{9}$$

Here E_t denotes the conditional expectation given $\{C \in \mathcal{C}_t\}$. In particular, for the optimal betfunction H corresponding to the chosen playing strategy we get

$$E_t(G) = E_t(G_1(C)) + (B_{\max} - 1) E_t\{G_1(C)) I(G_1(C) > 0)\} \tag{10}$$

Note that substitution of $B_{\max} = 1$ gives the result for the unit betfunction $H = 1$.

By giving all players the basic strategy S_{bas}, the probabilities $p(t)$ can be determined by simulation. Figure 9 gives a graphical presentation of $p(t)/p(0)$ for $0 < t < 1$ for $n = 6$, $\lambda = 1$, $a = 7$ based on a simulation of $M = 40{,}000{,}000$ rowgames.

Note the oscillating pattern due to the fact that every rowgame starts anew with the same starting stock c_0. From the $p(t)$ for $\lambda = 1$ we easily get the $p(t)$ for arbitrary λ by truncation and rescaling. As mentioned at the end of Section 5 we cannot calculate (7) from (9) due to the fact that the calculation of $E(G_1(c))$ for all c is too time consuming. For optimal betting we used simulation of a restricted size to get for all t an approximation of the conditional expectations $E_t(G_1(C))$ and $E_t(G_1(C) I(G_1(C) > 0))$ appearing in (10). We did this for the basic strategy S_{bas} as well as the optimal playing strategy S_{opt}. With (7) this leads to the corresponding expected gains μ_G for (H_{bas}, S_{bas}) and (H_{opt}, S_{opt}). The results have already been given in Table 5. (For details see Van der Genugten, 1995, section 8.)

9 Card counting systems for betting

A card counting system is a vector $\varphi \in \mathbb{R}^{10}$ with the interpretation that card j gets the score φ_j, $j = 1, \ldots, 10$. During a rowgame a player cumulates the scores of all

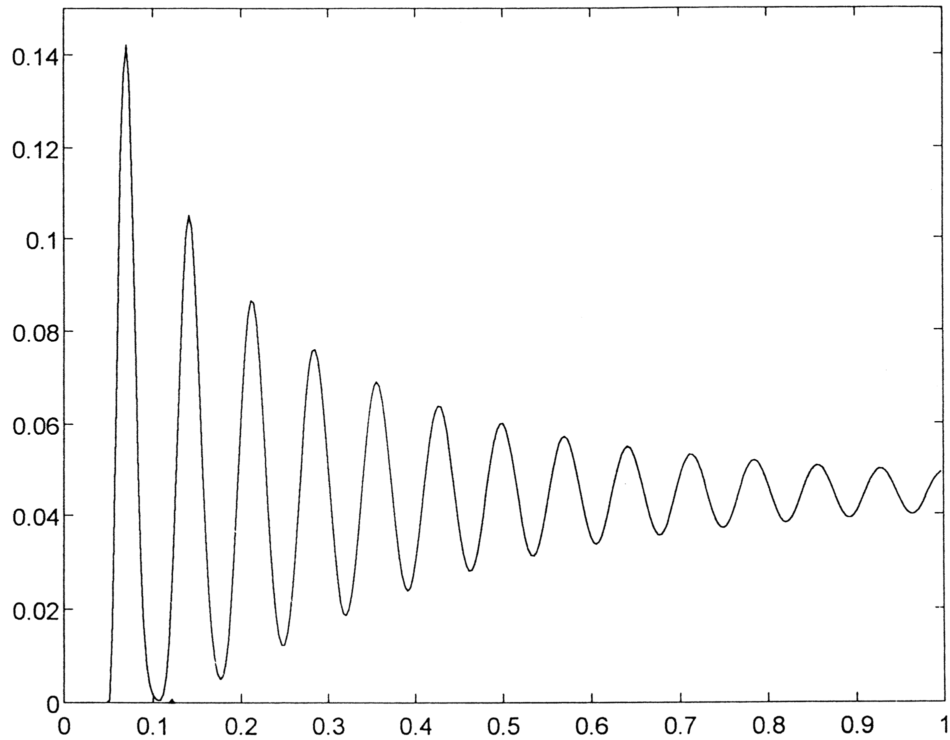

Fig. 9. Graphical representation of $p(t)/p(0)$ for $0 < t < 1$ ($n = 6$, $\lambda = 1$, $a = 7$, S_{bas}; $p(0) = 0.0690$), ($M = 40{,}000{,}000$ rowgames).

successive dealt cards $X_1, X_2, \ldots$ using his counting system φ. This sum of scores is called the *running count*. So

$$V(t) = \sum_{i=1}^{kt} \varphi_{X_i} \tag{11}$$

is the running count up to a played card fraction t. The player makes his betting and playing decisions according to the *true count* $U(t)$, by definition the running count $V(t)$ divided by the number of remaining cards $k - kt$ in the shoe:

$$U(t) = \frac{V(t)}{k(1-t)} = \sum_{j=1}^{10} \varphi_j F_j(t) \tag{12}$$

where

$$F_j(t) = \frac{c_0(j) - C(j)}{k - kt} \tag{13}$$

denotes the played card fraction of card j in a stock $C \in \mathcal{C}_t$.

The basic idea behind the introduction of counting systems is that for large n the conditional gain distribution $\mathcal{L}_t(G_1(C))$ depends approximately only on the card fractions $F_j(t)$. In particular we can try to find a linear combination $U(t)$ such that

$$\mathcal{L}_t(G_1(C)) \approx \mathcal{L}(g_0 + U(t)) \tag{14}$$

where g_0 is a norming constant, say

$$g_0 = E(G_1(c_0)) \tag{15}$$

This is done by calculating $E(G_1(c))$ for c in a sufficiently large and widely spread-out set of stocks $\mathcal{C}_{\text{fit}}$. Then we take for φ the LS-approximation in $\mathcal{C}_{\text{fit}}$. Often the restriction $\mu_\varphi = 0$ for a *centered* system is added. Here $\mu_\varphi = \Sigma p_j \varphi_j$.

For the basic strategy S_{bas} the LS-solution under the restriction $\mu_\varphi = 0$ for a certain set $\mathcal{C}_{\text{fit}}$ is given by Table 10.

Table 10. φ-values of S_{bas} ($g_0 = -0.0061$)

card j	φ_j	card j	φ_j
1	−0.3411	6	+0.2253
2	+0.1861	7	+0.1154
3	+0.2153	8	−0.0254
4	+0.2708	9	−0.1006
5	+0.3451	10	−0.2227

The *low* cards 2–6 exhibit an increasing pattern of $E(G_1(c))$ with $F_j(t)$. So playing stocks containing few low cards are a disadvantage for players. For the *high* cards 7–10 and $A = 1$ we see that card 7 gives a slightly positive pattern, card 8 is neutral, card 9 is slightly negative and the cards $T = 10$ and $A = 1$ are strongly negative. So stocks rich with tens or aces are advantageous for players.

Now suppose that for a particular playing strategy a betfunction H based on φ is used, say

$$H(c) = \bar{H}(U(t)), \quad c \in \mathcal{C}_t \tag{16}$$

Then with (14):

$$\mathcal{L}_t(G) = \mathcal{L}\{\bar{H}(U(t))G_1(C)\} \approx \mathcal{L}\{\bar{H}(U(t))(g_0 + U(t))\} \tag{17}$$

and, in particular,with (7)–(9):

$$\mu_G \approx \sum_{t=0}^{\lambda - 1/k} p(t)E\{\bar{H}(U(t))(g_0 + U(t))\} \tag{18}$$

It can be proved that $(U(t),\ 0 < t < 1)$ converges for $n \to \infty$ after standardization to the Brownian bridge. This leads to the following approximation of $U(t)$ for large n:

$$U(t) \overset{\mathcal{L}}{\approx} \frac{t}{1-t}\mu_\varphi + \sqrt{\frac{t}{1-t}}\frac{1}{\sqrt{k-1}}\sigma_\varphi Z \tag{19}$$

with $\sigma_\varphi^2 = \Sigma p_j(\varphi_j - \mu_\varphi)^2$ and $Z \sim N(0, 1)$. The approximations (17)–(19) are very useful for analytic purposes. In particular, the corresponding optimal betfunction can be found approximately by maximizing the expectation on the right hand side of (18). This leads to $\bar{H}$ given by

$$\bar{H}(U(t)) = \begin{cases} 1 & \text{if } U(t) \le -g_0 \\ B_{\max} & \text{if } U(t) > g_0 \end{cases} \tag{20}$$

For the corresponding maximal expected gain given $\{C \in \mathcal{C}_t\}$ we get:

$$E_t(G) \approx E\{\bar{H}(U(t))(g_0 + U(t))\} \tag{21}$$

For a centered counting system ($\mu_\varphi = 0$) substitution of (19) into (21) leads after some calculations to

$$\mu_G \approx g_0 + (B_{\max} - 1)\sum_{t=0}^{\lambda - 1/k} p(t)\{g_0 Z_0(-g_0/g_1(t)) + g_1(t) Z_1(-g_0/g_1(t))\} \tag{22}$$

with

$$g_1(t) = \sqrt{\frac{t}{1-t}}\frac{1}{\sqrt{k-1}}\sigma_\varphi \tag{23}$$

$$Z_k(t) = \int_t^\infty \frac{1}{\sqrt{2\pi}} u^k \,\mathrm{e}^{-1/2u^2}\, \mathrm{d}u, \quad k = 0, 1 \tag{24}$$

For the basic strategy S_{bas} the corresponding approximating optimal strategy $\bar{H}_{\text{bas}}$ follows immediately from (20) with φ in Table 22. The corresponding approximations for the expected gain follow from (22). Table 11 gives the results for the interesting values of λ and $B_{\max}$. The table indicates clearly the influence of the cut card position λ and the maximum bet $B_{\max}$.

Table 11. Expected gains μ_G for $(\bar{H}_{\text{bas}}, S_{\text{bas}})$ $(n = 6)$

$B_{\max}$	1	2	25	37.5	50
$\lambda = \frac{1}{2}$	−0.0061	−0.0052	0.016	0.027	0.038
$\lambda = \frac{2}{3}$	−0.0061	−0.0046	0.031	0.050	0.069

Counting systems φ based on LS are called *theoretical* because the scores φ_j are not nice figures and therefore too complicated to use in practice. Therefore the next step is to replace φ by a counting system ψ, obtained from φ by rounding off. This leads to a *practical* running count

$$S(t) = \sum_{i=1}^{kt} \psi_{X_i} \tag{25}$$

and a practical true count

$$T(t) = \frac{S(t)}{k - kt} = \sum_{j=1}^{10} \psi_j F_j(t) \tag{26}$$

A betfunction based on the practical system ψ is constructed from a betfunction based on the theoretical true count φ through the LS-estimate $\hat{\varphi}$ of φ based on ψ. We restrict ourselves to centered systems ($\mu_\varphi = \mu_\psi = 0$). Then

$$\hat{\varphi} = (\rho_{\psi\varphi}\sigma_\varphi/\sigma_\psi)\psi \tag{27}$$

where $\rho_{\psi\varphi}$ is the correlation coefficient between ψ and φ (i.e., $\rho_{\psi\varphi} = \sigma_{\psi\varphi}/(\sigma_\varphi\sigma_\psi)$ with $\sigma_{\psi\varphi} = \Sigma p_j(\varphi_j - \mu_\varphi)(\psi_j - \mu_\psi)$). With (12) and (16) this leads to the corresponding estimates

$$\hat{U}(t) = \sum_{j=1}^{10} \hat{\varphi}_j F_j(t) = \rho_{\psi\varphi}\frac{\sigma_\varphi}{\sigma_\psi} T(t) \tag{28}$$

$$H(c) = \bar{H}(\hat{U}(t)) = \hat{H}(T(t)), \quad c \in \mathcal{C}_t \tag{29}$$

The corresponding optimal betfunction $\hat{H}$ based on $T(t)$ follows from (20) and (28):

$$\hat{H}(T(t)) = \begin{cases} 1 & \text{if } T(t) \leq t_0 \\ B_{\max} & \text{if } T(t) > t_0 \end{cases} \tag{30}$$

where

$$t_0 = -g_0 \frac{\sigma_\psi}{\rho_{\varphi\psi}\sigma_\varphi} \tag{31}$$

The approximation of the corresponding expected gain is given by (22), where in the definition (23) of $g_1(t)$ we replace σ_φ by $\rho_{\psi\varphi}\sigma_\varphi$ (in (22) the factor σ_ψ cancels out). Therefore the rounding off procedure to get ψ from φ can be judged just on the base of the correlation coefficient! In literature $\rho_{\psi\varphi}$ is called the *betting correlation*.

Figure 12 gives the estimated μ_G as a function of $\rho_{\phi\psi} \in (0.5, \ 1)$ for $\lambda = 2/3$, $n = 6$ and interesting values of $B_{\max}$. The figure shows clearly the strong increase of the expected gain by using ψ-approximations of φ with increasing correlation coefficient $\rho_{\varphi\psi}$.

The results of this section up till here can be generalized in several ways (higher dimensional count systems or side counting, non-linear approximations). We refer to Van der Genugten (1995), sections 9 and 10 for a general setup.

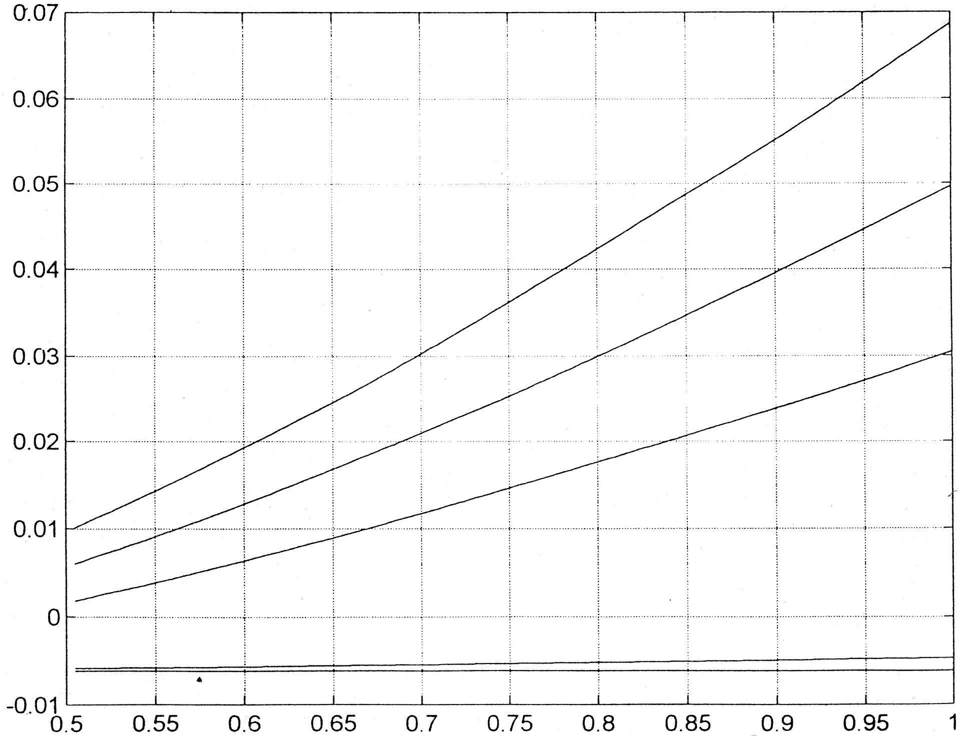

Fig. 12. Est. μ_G for S_{bas} as function of $\rho_{\varphi\psi}$ ($B_{max} = 1, 2, 25, 37.5$ and 50; $\lambda = 2/3$, $n = 6$).

We consider hereafter only two famous practical (centered) card counting systems: TTC (Thorp's Ten Count) and HiLo (High-Low). Table 13 gives the definition of both systems.

Table 13. ψ-values of TTC and HiLo

card	1	2	3	4	5	6	7	8	9	10
TTC	4	4	4	4	4	4	4	4	4	−9
HiLo	−1	1	1	1	1	1	0	0	0	−1

The TTC-system is the most simple card counting system. Its true count $T = T(t)$ can be easily expressed in terms of the so-called T-ratio $TR = TR(t)$ of a stock: the number of non-tens divided by the number of tens:

$$T = (9 - 4TR)/(1 + TR)$$

For HiLo we use HL $= 52 \times$ True Count.

In relation to the φ-values of the basic strategy, Table 14 follows from Tables 10 and 13.

Table 14. $(\varphi,\ \psi)$-values of counting systems with respect to S_{bas}

	$\rho_{\varphi\psi}$	t_0	
TTC	0.66	+0.248	$TR_0 = 2.06$
HiLo	0.96	+0.0248	$HL_0 = 1.3$

This leads to the following expected gains in Table 15 for the betting strategies $\hat{H}_{\text{TTC}}$ and $\hat{H}_{\text{HiLo}}$ based on (30) (the values of Table 11 are included for comparison).

Table 15. μ_G of counting systems for S_{bas} $(n = 6)$

	$B_{\max}$	1	2	25	37.5	50
$\lambda = \frac{1}{2}$	$\hat{H}_{\text{TTC}}$	−0.0061	−0.0058	0.0015	0.0055	0.010
	$\hat{H}_{\text{HiLo}}$	−0.0061	−0.0053	0.014	0.024	0.034
	$\bar{H}_{\text{bas}}$	−0.0061	−0.0052	0.016	0.027	0.038
$\lambda = \frac{2}{3}$	$\hat{H}_{\text{TTC}}$	−0.0061	−0.0055	0.0092	0.017	0.025
	$\hat{H}_{\text{HiLo}}$	−0.0061	−0.0047	0.028	0.046	0.063
	$\bar{H}_{\text{bas}}$	−0.0061	−0.0046	0.031	0.050	0.069

Clearly, the performance of the HiLo-system is much better than that of the TTC-system. However, TCC is much easier to use in practice.

As a final check on the estimations in Table 15 we performed a simulation for $\lambda = 2/3$, $B_{\max} = 50$ with the ψ-values of TTC and HiLo and the underlying φ for S_{bas}. Table 16 gives the results. Comparing Table 16 with $B_{\max} = 50$, $\lambda = 2/3$ in Table 15, we see a slight bias in the estimates. This bias follows a systematic pattern for varying n (compare Section 7).

Table 16. Sim. gains of counting systems for S_{bas} $(n = 6,\ \lambda = 2/3,\ B_{\max} = 50)$ $(M = 50{,}000{,}000$ rowgames—$\mu_N = 9.86)$

betfunction	μ_{GR}	±95%	μ_G
$\hat{H}_{\text{TTC}}$ $(TR_0 = 2.06)$	0.294	±0.02	0.0299
$\hat{H}_{\text{HiLo}}$ $(HL_0 = 1.3)$	0.687	±0.02	0.0698
$\bar{H}_{\text{bas}}$ $(t_0 = 0.0061)$	0.733	±0.02	0.0744

10 Card counting systems for playing

Card counting systems are also used for playing decisions. For BJHC the number of decks $n = 6$ is large and therefore playing decisions different from the basic strategy can only increase the expected gain by a small amount. We only consider playing decisions for TTC and HiLo.

Table 17 gives the TTC-playing strategy S_{TTC} and Table 18 the HiLo-playing strategy S_{HiLo}.

(For the construction we refer to Van der Genugten, 1995, section 10.) We studied the effect of S_{bas}, S_{TTC} and S_{HiLo} in combination with various betfunctions. Based on

Table 17. Playing strategy S_{TTC} (TR = Ten Ratio)

INSURANCE: insure if $TR \leq 2$

SPLITTING (Split if $TR \leq$, if underlined then split it $TR \geq$)

Dealercard Pair	A	2	3	4	5	6	7	8	9	T
AA	1.4	4.2	4.5	4.7	5.0	5.3	4.3	3.9	3.7	3.2
22	0	3.0	3.7	∞	∞	∞	∞	$\underline{2.6}$	$\underline{8.0}$	0
33	0	2.3	∞	∞	∞	∞	∞	$\underline{2.4}$	$\underline{5.5}$	0
44	0	1.4	1.6	1.9	2.3	2.4	$\underline{6.2}$	$\underline{7.0}$	0	0
55	0	0	0	0	0	0	0	0	0	0
66	0	2.4	2.7	3.0	3.3	3.7	0	0	0	0
77	0	3.6	4.1	4.8	5.8	∞	∞	2.1	0	0
777	0	3.0	3.3	3.7	4.2	5.3	4.9	0	0	0
88	0	∞	∞	∞	∞	∞	∞	∞	∞	$\underline{2.8}$
99	0	2.6	2.9	3.1	3.5	3.4	1.8	4.1	4.1	0
TT	0	1.4	1.5	1.6	1.7	1.8	0	0	0	0

DOUBLE DOWN (DDown if $TR \leq$)

Dealercard Sum	A	2	3	4	5	6	7	8	9	T
H9	0	2.2	2.4	2.6	2.9	3.2	1.8	1.5	0	0
H10	0	3.6	3.9	4.2	4.8	5.1	3.5	3.0	2.5	0
H11	0	3.9	4.1	4.5	5.0	5.4	3.9	3.3	2.8	2.1
S19	0	1.2	1.3	1.4	1.5	1.6	0	0	0	0
S20	0	1.4	1.5	1.6	1.7	1.8	0	0	0	0
S21	0	0	0	0	1.0	1.0	0	0	0	0

DRAW/STAND (Draw if $TR \geq$)

Dealercard Sum	A	2	3	4	5	6	7	8	9	T
H12	0	2.0	2.2	2.3	2.4	2.3	0	0	0	1.1
H13	1.2	2.3	2.5	2.7	2.9	2.8	0	0	0	1.3
H14	1.3	2.7	3.0	3.2	3.5	3.3	0	0	1.2	1.6
777	0	2.2	2.4	2.5	2.7	2.7	0	0	0	0
H15	1.4	3.2	3.5	3.9	4.3	4.1	0	0	1.4	1.9
H16	1.5	3.8	4.2	4.7	5.4	5.0	0	1.4	1.8	2.2
H17	3.0	∞	∞	∞	∞	∞	∞	∞	∞	∞
H18	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
S17	0	0	0	0	0	0	0	0	0	0
S18	2.1	5.1	5.5	6.2	6.6	∞	∞	5.1	0	0
S19	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞

Table 14 and some further simulations with S_{TTC} and S_{HiLo} we took the (modified) simple bounds of Table 19.

The betfunctions always take the maximum bet $B_{\max}$ for appropriate values of TR and HL respectively. This leads to a large variance in the gain. So these strategies can only be played in practice by high budget players with a large starting capital. For low budget players with a low (or moderate) capital other betfunctions come into view. Therefore we consider also two low budget betfunctions $\tilde{H}_{\text{TTC}}$ and $\tilde{H}_{\text{HiLo}}$ specified in

Table 18. Playing strategy S_{HiLo} ($HL = 52 \times$ true count)

INSURANCE: insure if $HL \geq 3$

SPLITTING (Split if $HL >$)

Dealercard Pair	A	2	3	4	5	6	7	8	9	T
AA	$+\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	-9
22	$+\infty$	$-3\frac{1}{2}$	-6	-8	$-9\frac{1}{2}$	$-\infty$	$-\infty$	$+4\frac{1}{2}$	$+\infty$	$+\infty$
33	$+\infty$	0	-5	-8	$-9\frac{1}{2}$	$-\infty$	$-\infty$	$+5\frac{1}{2}$	$+\infty$	$+\infty$
44	$+\infty$	$+\infty$	$+8$	$+3$	$-\frac{1}{2}$	$-1\frac{1}{2}$	$+\infty$	$+\infty$	$+\infty$	$+\infty$
55	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$
66	$+\infty$	-2	$-4\frac{1}{2}$	$-6\frac{1}{2}$	$-8\frac{1}{2}$	$-\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$
7(7)7	$+\infty$	-7	$-8\frac{1}{2}$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$+\infty$	$+\infty$	$+\infty$
88	$+\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$+\infty$
99	$+\infty$	-3	$-4\frac{1}{2}$	$-5\frac{1}{2}$	-7	$-7\frac{1}{2}$	$+3$	-9	$-\infty$	$+\infty$
TT	$+\infty$	$+\infty$	$+8\frac{1}{2}$	$+6\frac{1}{2}$	$+5$	$+4\frac{1}{2}$	$+\infty$	$+\infty$	$+\infty$	$+\infty$

DOUBLE DOWN (DDown if $HL >$)

Dealercard Sum	A	2	3	4	5	6	7	8	9	T
H9	$+\infty$	$+1$	-1	-3	$-4\frac{1}{2}$	$-\frac{1}{2}$	$+3\frac{1}{2}$	$+7\frac{1}{2}$	$+\infty$	$+\infty$
H10	$+\infty$	-9	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-6\frac{1}{2}$	$-4\frac{1}{2}$	$-1\frac{1}{2}$	$+\infty$
H11	$+\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-7\frac{1}{2}$	-5	$+3\frac{1}{2}$
S19	$+\infty$	$+\infty$	$+\infty$	$+9$	$+6\frac{1}{2}$	$+7$	$+\infty$	$+\infty$	$+\infty$	$+\infty$
S20	$+\infty$	$+\infty$	$+8\frac{1}{2}$	$+6\frac{1}{2}$	$+5$	$+4\frac{1}{2}$	$+\infty$	$+\infty$	$+\infty$	$+\infty$
S21	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+8$	$+8$	$+\infty$	$+\infty$	$+\infty$	$+\infty$

DRAW/STAND (Draw if $HL \leq$)

Dealercard Sum	A	2	3	4	5	6	7	8	9	T
H12	$+\infty$	$+3$	$+1\frac{1}{2}$	0	$-1\frac{1}{2}$	-1	$+\infty$	$+\infty$	$+\infty$	$+\infty$
H13	$+\infty$	-1	$-2\frac{1}{2}$	-4	$-5\frac{1}{2}$	$-5\frac{1}{2}$	$+\infty$	$+\infty$	$+\infty$	$+\infty$
H14(777)	$+\infty$	-4	-5	$-6\frac{1}{2}$	-8	$-8\frac{1}{2}$	$+\infty$	$+\infty$	$+\infty$	$+\infty$
H15	$+9\frac{1}{2}$	-6	-7	$-8\frac{1}{2}$	$-9\frac{1}{2}$	$-\infty$	$+\infty$	$+\infty$	$+8$	$+4\frac{1}{2}$
H16	$+8\frac{1}{2}$	$-9\frac{1}{2}$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$+8$	$+7$	$+4\frac{1}{2}$	0
H17	$-6\frac{1}{2}$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
H18	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
S17	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$+\infty$
S18	$+1$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$+\infty$	$+\infty$
S19	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$

Table 20. The betting strategy $\tilde{H}_{\mathrm{TTC}}$ aims to minimize the probability of ruin starting with a moderate capital.

According to WIND and WIND (1994) the betting strategy $\tilde{H}_{\mathrm{HiLo}}$ follows from a conservative interpretation of the Kelly-principle to choose the bet in such a way that

Table 19. Modified (ψ, φ)-values w.r.t. S_{bas}

	t_0	
TTC	$+0.194$	$TR_0 = 2.1$
HiLo	$+0.0288$	$HL_0 = 1.3$

Table 20. Betting of low and high budget players ($B_{max} = 50$)

	TTC			HiLo	
Class	$\hat{H}_{TTC}$ (High)	$\tilde{H}_{TTC}$ (Low)	Class	$\hat{H}_{HiLo}$ (High)	$\tilde{H}_{HiLo}$ (Low)
$TR > 2.3$	1	1	$HL < 1/2$	1	1
$2.1 < TR \leq 2.3$	1	1	$1/2 \leq HL < 1$	1	1
$2.0 < TR \leq 2.1$	50	2	$1 \leq HL < 2$	50	5
$1.9 < TR \leq 2.0$	50	6	$2 \leq HL < 3$	50	10
$1.8 < TR \leq 1.9$	50	9	$3 \leq HL < 4$	50	15
$1.7 < TR \leq 1.8$	50	13	$4 \leq HL < 5$	50	20
$TR \leq 1.7$	50	21	$HL \geq 5$	50	25

Table 21. Simulated gains ($n = 6$, $a = 7$, $\lambda = 2/3$, $B_{max} = 50$)
($M = 50{,}000{,}000$ rowgames—$\mu_N = 9.98$ games)

Strategy	μ_{GR}	σ_{GR}	$\pm 95\% CI$	μ_G	μ_B
$(\hat{H}_{HiLo}, S_{HiLo})$	+0.918	87.7	0.027	+0.0931	14.7
$(\hat{H}_{HiLo}, S_{bas})$	+0.702	86.5	0.027	+0.0711	14.4
$(\hat{H}_{TTC}, S_{TTC})$	+0.569	75.5	0.023	+0.0576	11.2
$(\hat{H}_{TTC}, S_{bas})$	+0.321	73.8	0.023	+0.0326	10.8
$(\tilde{H}_{HiLo}, S_{HiLo})$	+0.240	20.9	0.006	+0.0243	3.60
$(\tilde{H}_{HiLo}, S_{bas})$	+0.171	20.1	0.006	+0.0174	3.51
$(\tilde{H}_{TTC}, S_{TTC})$	+0.062	11.2	0.003	+0.0063	2.02
$(\tilde{H}_{TTC}, S_{bas})$	+0.009	10.7	0.003	+0.0009	1.97
$(H \equiv 1, S_{TTC})$	−0.0425	3.51	0.0011	−0.0043	1.11
$(H \equiv 1, S_{HiLo})$	−0.0456	3.50	0.0011	−0.0046	1.10
$(H \equiv 1, S_{bas})$	−0.0524	3.50	0.0011	−0.0053	1.10

the expected growth of one's capital is maximized. For details we refer to VAN DER GENUGTEN (1995), section 10.

We made several simulation runs to obtain the performance of these strategies. Table 21 gives an overview. This table contains also the standard deviation σ_{GR} of one rowgame and the mean bet μ_B of one game. We see that the HiLo-system is better (but also more complicated) than the TTC-system. There is a large difference between the high-budget systems $\hat{H}$ and the low-budget systems $\tilde{H}$. However, even low budget players can beat the dealer in the long run!

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