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Testing in the Restricted Linear Model Using Canonical Partitions

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ABSTRACT

This paper treats the normal linear model $y \sim N_n(X\beta, \sigma^2 I_n)$ under the restriction $C\beta = 0$ for arbitrary C. Considered is the identifiable testing problem $H_0 : D\beta = 0$ against $H_1 : D\beta \neq 0$. A canonical partition of the rows of C admits a simple form of involved linear subspaces. This leads to a numerically tractable form of the usual F-statistic for this testing problem. © 1997 Elsevier Science Inc.

1. INTRODUCTION

Consider the restricted normal linear regression model

$$y \sim N_n\left(X\beta, \sigma^2 I_n \right)$$

with $(\beta, \sigma^2) \in \mathbb{R}^k \times \mathbb{R}_+$, where β satisfies the (known) restrictions $C\beta = 0$. For the expectation $\mu = X\beta$ this implies that $\mu \in L = X(\mathscr{N}(C))$, the image of $\mathscr{N}(C)$ under the transformation X. Within this model we consider the identifiable testing problem

$$H_0 : D\beta = 0 \quad \text{against} \quad H_1 : D\beta \neq 0.$$

Identifiability takes place if, and only if, this problem is equivalent to

$$H_0 : \mu \in L_0 \quad \text{against} \quad H_1 : \mu \in L - L_0,$$

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where $L_0 = X(\mathcal{N}(C) \cap \mathcal{N}(D))$. This in turn is equivalent to the condition that $D\beta$ can be estimated unbiasedly [see Prakaso Rao (1992, 7.2) or Van Der Genugten (1977, Section 4)]. Still another equivalent condition is

$$\mathscr{R}(D') \subseteq \mathscr{R}(X' \quad C') \tag{1}$$

[see e.g. Rao (1973, 4i.2 (iii), p. 297) or Searle (1987, [8.9]-ii (180), p. 308)].

The usual F-statistic for this testing problem is

$$F = \frac{y' P_{L_1} y / \dim L_1}{y' P_R y / \dim R},$$

where $R = L^{\perp}$, L_1 is the orthogonal complement of L_0 with respect to L, and, for instance, P_R denotes the orthogonal projector onto R. Clearly, F follows the F-distribution with dim L_1 and dim R degrees of freedom and with noncentrality parameter $\mu' P_{L_1} \mu / \sigma^2$.

The problem is to find a numerically tractable expression for the F-statistic in terms of X, C, and D. A good reference to earlier work in this field is Rao (1973, 4a, pp. 231–233 and 4i, pp. 294–302). In general, without imposing any rank conditions, Hartung and Werner (1983, 1984) gave nice expressions using restricted Moore-Penrose inversions. In this paper we derive an alternative nice expression using a so-called canonical partition of the rows of C. The key to the result is that such a partition admits a simple form of relevant corresponding linear subspaces in terms of X, C, and D.

2. CANONICAL PARTITIONS

For typographical reasons we denote row partition by $A = [A_0; A_1]$ for $A = [A_0' \quad A_1']'$, where the $'$ denotes the transpose. We say that A^- is a g-inverse of A if $AA^-A = A$.

Let C_0 be any submatrix of rows from C. Then [see e.g. Rao (1973, 1b.6 (iii), p. 28)]

$$\begin{aligned} \dim \mathscr{R}(X) = r(X) \geqslant \dim X(\mathcal{N}(C_0)) &= r(X; C_0) - r(C_0) \\ \geqslant \dim L &= r(X; C) - r(C). \end{aligned} \tag{2}$$

Hence, there exists an (in general not uniquely determined) submatrix C_0 with a *maximum* number of rows from C such that

$$\mathscr{R}(X) = X(\mathcal{N}(C_0)) \tag{3}$$

or, equivalently,

$$r(X) = r(X;C_0) - r(C_0). \tag{4}$$

Given such C_0, we denote the submatrix of remaining rows by C_1. By reordering we may write $C = [C_0; C_1]$ without loss of generality. We call this a *canonical partition* of C. (It is possible that C_0 or C_1 is empty.) The construction of C_0 is quite easy by inspecting the rows of C subsequently and adding a row to the already obtained set of rows if (4) still holds for the augmented set. Since[1] (3) and (4) are equivalent to $\mathscr{R}(X') \cap \mathscr{R}(C_0') = \{0\}$ and since the number of rows of C_0 is maximal, it follows that the rows of C_1 belong to $\mathscr{R}(X' \quad C_0')$, implying

$$\mathscr{R}(X' \quad C') = \mathscr{R}(X' \quad C_0'). \tag{5}$$

So for D satisfying (1) we see that $[C_0; [C_1; D]]$ is a canonical partition of $[C; D]$.

From now on we only consider the general case of nonempty C_0 and C_1. Let $[H_0 \quad G_0] = [X'X; C_0]^-$ be some g-inverse of $[X'X; C_0]$. We have:

THEOREM 1. *Let $\tilde{L}$ be the orthogonal complement of $L = X(\mathscr{N}(C))$ with respect to $\mathscr{R}(X)$. Then*

$$\dim \tilde{L} = r(C) - r(C_0), \tag{6}$$

$$\tilde{L} = \mathscr{R}(XH_0'C_1'). \tag{7}$$

Proof. The relation (6) follows from (2), (4), (5), and $\dim \tilde{L} = r(X) - \dim L$. Let $J_0 = [X'X; C_0]^-[X'X; C_0] = H_0X'X + G_0C_0$. Then from (5) we get $C_1J_0 = C_1$. Together with (3) this gives

$$\begin{aligned} L &= X(\mathscr{N}(C_0; C_1)) = X(\mathscr{N}(C_0; C_1H_0X'X)) \\ &= \{\mu \in X(\mathscr{N}(C_0)) : C_1H_0X'\mu = 0\} \\ &= \{\mu \in \mathscr{R}(X) : C_1H_0X'\mu = 0\} = X(\mathscr{N}(C_1H_0X'X)) \end{aligned}$$

or $\tilde{L} = \mathscr{R}((C_1H_0X')') = \mathscr{R}(XH_0'C_1')$, and this proves (6). ■

[1] I thank the anonymous referee for this remark, leading to a simple description of the properties of a canonical partition.

COROLLARY.

$$P_{\tilde{L}} = XH_0'C_1'(C_1H_0X'XH_0'C_1')^- C_1H_0X'. \tag{8}$$

Expressions for the denominator of the F-statistic follow immediately from the orthogonal splitting $R = \mathscr{R}(X)^\perp + \tilde{L}$:

$$\dim R = n - r(X) + r(C) - r(C_0), \tag{9}$$

$$P_R = \left[I_n - X(X'X)^- X'\right] + P_{\tilde{L}}. \tag{10}$$

For the expressions in the numerator we find:

THEOREM 2.

$$\dim L_1 = r(C; D) - r(C), \tag{11}$$

$$L_1 = \mathscr{R}(XH_{01}'D') \tag{12}$$

with

$$H_{01} = \left[I_k - C_1'(C_1H_0X'XH_0'C_1')^- C_1H_0X'XH_0'\right]'H_0. \tag{13}$$

Proof. Let $\tilde{L}_0$ be the orthogonal complement of L_0 with respect to $\mathscr{R}(X)$. Then we have the orthogonal splitting $\tilde{L}_0 = L_1 + \tilde{L}$.

We apply Theorem 1 to $L_0 = X(\mathscr{N}(C; D))$ and the corresponding canonical partition $[C; D] = [C_0; [C_1; D]]$. Then (6) implies $\dim \tilde{L}_0 = r(C; D) - r(C_0)$, and so (11) follows from $\dim L_1 = \dim \tilde{L}_0 - \dim \tilde{L}$ and (6). Furthermore, (7) implies $\tilde{L}_0 = \mathscr{R}(XH_0'C_1' \; XH_0'D')$. Combining this with $\tilde{L} = \mathscr{R}(XH_0'C_1')$, (8), and (13), we see that L_1 is spanned by the columns of $(I_n - P_{\tilde{L}})XH_0'D' = XH_{01}'D'$. This proves (12). ∎

COROLLARY.

$$P_{L_1} = XH_{01}'D'(DH_{01}X'XH_{01}'D')^- DH_{01}X'. \tag{14}$$

We have only treated testing problems. Identification and estimating under restrictions can be analyzed as well. Formulae can be presented in such a way that the differences with the corresponding full model appear explicitly. For such comparisons the following choice of a g-inverse $[H_0 \; G_0]$ of $[X'X; C_0]$ should be made:

$$G_0 = (I_k - J)(C_0 - C_0 J)^-, \qquad H_0 = (I_k - G_0 C_0)(X'X)^- \quad (15)$$

with $J = (X'X)^- X'X$. Substitution of (15) leads to results generalizing those in Searle (1987, [8, 7]). For details we refer to Van Der Genugten (1993).

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