Equilibrium adjustment of disequilibrium prices

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Equilibrium Adjustment of Disequilibrium Prices [1]

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Abstract

We consider an exchange economy in which price rigidities are present. An always converging price and quantity adjustment process for such an economy is presented that is based on a discrete algorithmic procedure rather than more traditional adjustment processes, which are based on difference or differential equations. In the short run all non-numeraire commodities have a flexible price level with respect to the numeraire commodity but their relative prices are mutually fixed. In the long run prices are assumed to be completely flexible. The adjustment process starts with a trivial equilibrium with low enough price level and complete demand rationing on all markets. Along the path followed by the adjustment process, initially all relative prices of the non-numeraire commodities are kept fixed and the price level is increased. Rationing schemes are adjusted to keep markets in equilibrium. Doing so the process reaches a short run equilibrium with only demand rationing and no rationing on the numeraire and at least one of the other commodities. In the long run the process allows for a downward price adjustment of unrationed non-numeraire commodities and reaches a Walrasian equilibrium eventually.

Key words: Exchange economy, Price rigidities, Disequilibrium, Simplicial Algorithm, Adjustment process, Drèze equilibrium, Walrasian equilibrium

1 Introduction

Consider an economy where trade has to take place against a non-Walrasian price system. Then demand is not equal to supply on the markets of some commodities. A Drèze equilibrium can now be obtained by demand and supply rationing on the commodity markets, see Drèze [3]. However, due to the demand and supply rationing, prices have a tendency to change. A well-known price adjustment process is the classical Walrasian tatonnement process. This process adjusts at any point in time the prices of the commodities according to their notional excess demand at that point in time.

The Walrasian tatonnement process has a number of drawbacks. First, the adjustment of the prices according to this process does not guarantee convergence to a Walrasian equilibrium price system. In Scarf [16] examples of economies have been given for which the Walrasian tatonnement process fails to converge to an equilibrium price vector. It has been shown in Saari [15] that any process based on a finite amount of local information fails to converge for a substantial class of economies. This lack of convergence has been solved in Smale [18], van der Laan and Talman [9,10], and Kamiya [7], where several price adjustment processes with much better convergence properties have been presented.

The second drawback of the Walrasian tatonnement process as well as of the other processes mentioned in the previous paragraph is that demand and supply are not in equilibrium as long as the process has not achieved a Walrasian equilibrium price system. So, trade must be excluded until equilibrium is reached. Moreover, as has been noticed in Veendorp [19], the relevant market signals for an adjustment process in an economy are based on the effective demand associated with a Drèze equilibrium instead of the notional demand used in the adjustment processes described above. Therefore, in Veendorp [19] an adjustment process is considered which follows a path of Drèze equilibria and where prices are adjusted as in the Walrasian tatonnement process, with notional excess demand replaced by effective excess demand. In Veendorp [19] (see also the correction in Laroque [11]) a proof of the convergence of this process is given in a model with three commodities and two consumers in case the total excess demand function satisfies a gross substitutability condition. In general, however, such a process does not necessarily converge to a Walrasian equilibrium price system and even chaotic behaviour may be expected (see Day and Pianigiani [2]). The possibility of chaotic behaviour has been confirmed in Böhm [1] in a more complicated model with overlapping generations, producers, and a government.

In this paper a price and quantity adjustment process is presented, which does not suffer from the drawbacks mentioned before. At any point along the path of this adjustment process the economy attains a Drèze equilibrium as in Veendorp [19]. Therefore, trade is possible at each point in time. Furthermore, prices are adjusted according to the market situation at this Drèze equilibrium and not according to the notional excess demand, i.e., the adjustment of prices is based on the relevant market signal. Moreover, the adjustment process does not suffer from lack of convergence, but instead converges to a Walrasian equilibrium in the long run. An artificial function, called the reduced total excess demand function, will be constructed in such a way that the path of zero points of this function yields the path of points followed by the desired adjustment process. The adjustment process will be based on discrete algorithmic procedures as initiated by Scarf [17], called simplicial algorithms, rather than more traditional adjustment processes, which are based on difference or differential equations. A modification of a simplicial algorithm introduced in van der Laan [8] will be presented and will be applied to the reduced total excess demand function. The path of points generated by this simplicial algorithm yields an approximation of the desired adjustment process. The inaccuracy of the approximation can be made arbitrarily small.

The economic interpretation of the adjustment process of this paper is inspired by recent experiences in Eastern European countries and the former Soviet Republic. For general equilibrium type models of the situation in these countries we refer to Polterovich [14]. In these models markets are cleared by means of demand rationing. Thus far this type of equilibrium has not been used in an adjustment process to obtain a Walrasian equilibrium. We assume that one of the commodities is the numeraire having fixed price equal to one. The other commodities, called real commodities, have in the short term a flexible price level with respect to this numeraire commodity, but have mutually fixed relative prices. When the price level is so low that no consumer wants to sell any amount of the real commodities, a trivial equilibrium is obtained by complete demand rationing on all the non-numeraire commodities. We introduce an adjustment process that starts with such a trivial equilibrium and subsequently adjusts prices and rationing schemes in such a way that at any moment during the adjustment process it holds that the markets are kept in equilibrium by rationing the demand for the non-numeraire commodities, while there is no rationing on the supply side of the markets. In the beginning of the process only the price level of the real commodities and the rationing schemes are adjusted simultaneously until at least one of the non-numeraire commodities is no longer rationed in its demand. This part of the process can be seen as the short term adjustment of the rationing scheme given the fixed relative prices of the non-numeraire commodities. Starting from the trivial equilibrium with complete demand rationing, the rationing on demand is reduced until an equilibrium is found in which the rationing on the demand can not be reduced any further without allowing for more price flexibility or supply rationing. From this short term equilibrium at given fixed relative prices, the prices of the unrationed non-numeraire commodities are allowed to decrease relatively with respect to the price level of the real commodities. The process therefore continues by simultaneously changing the price level, the prices of the unrationed commodities at levels below their relative maxima, and the demand constraints of the rationed commodities, in order to keep all markets in equilibrium. It will be shown that this long term process continues until none of the commodities is rationed and a Walrasian equilibrium has been obtained.

This paper is organized as follows. In Section 2 we introduce the model and define the concept of a real demand-constrained equilibrium with given price level. In such an equilibrium the numeraire commodity is not rationed, there may be demand rationing on the other markets, and the price level equals a given value. We show the existence of a trivial equilibrium with complete demand rationing on the markets of all non-numeraire commodities for price levels low enough. In Section 3 we construct the reduced total excess demand function by relating to any element of an $(n+1)$-dimensional set the total excess demand at some price vector and some rationing scheme and we discuss the behaviour of this function. In Section 4 we discuss and illustrate the adjustment process. Starting from the trivial equilibrium this process follows a path of real demand-constrained equilibria until a Walrasian equilibrium is found. In Section 5 we prove by means of simplicial approximation that there indeed exists a path of prices and rationing schemes yielding approximate real demand-constrained equilibria and show that this path connects a trivial real demand-constrained equilibrium with complete demand rationing on the markets of all non-numeraire commodities with an approximate Walrasian equilibrium, whereas the inaccuracy of the approximation can be made arbitrarily small. To relate this path to the set of exact real demand-constrained equilibria, in Section 6 the latter set is considered. It is shown that there exists a connected set of real demand-constrained equilibria containing both a trivial real demand-constrained equilibrium and a Walrasian equilibrium.

2 The model

We consider an exchange economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \tilde{r})$. In this economy there are m consumers, indexed $i = 1, \ldots, m$, and $n+1$ commodities, indexed $j = 1, \ldots, n+1$. For ease of notation, in the sequel we denote the set of indices $\{1, \ldots, k\}$ by I_k. Each consumer $i \in I_m$ is characterized by a consumption set X^i, a preference preordering $\succeq^i$ on X^i, and a vector of initial endowments w^i. We take one of the commodities, say commodity $n+1$, as the numeraire commodity, having a price equal to 1. In this paper we assume that the economy $\mathcal{E}$ is initially faced with completely fixed relative prices for the non-numeraire or real commodities, determined by the vector $\tilde{r} \in \mathbb{R}^n_{++}$. For a given price level $\alpha > 0$, the short term vector of prices is given by $\tilde{p}(\alpha)$ defined by $\tilde{p}_j(\alpha) = \alpha\tilde{r}_j$, for $j \in I_n$, and $\tilde{p}_{n+1}(\alpha) = 1$. By varying the price level, in the short run all the prices of the non-numeraire commodities can simultaneously be adjusted upwards or downwards with respect to the price of the numeraire commodity. In the long term the vector $\widetilde{p}(\alpha)$ serves as an upper bound on the prices of the real commodities and for given price level $\alpha > 0$ the set of admissible prices is given by the set $P(\alpha)$ defined by

$$P(\alpha) = \{p \in \mathbb{R}^{n+1}_{++} \mid p_j \leq \alpha \widetilde{r}_j, \ \forall j \in I_n, \ \text{and } p_{n+1} = 1\}.$$

The following assumptions with respect to the economy $\mathcal{E}$ are made:

Ass 2.1 *For every consumer $i \in I_m$ the consumption set X^i belongs to $\mathbb{R}^{n+1}_{+}$, is closed and convex, and $X^i + \mathbb{R}^{n+1}_{+} \subset X^i$.*

Ass 2.2 *For every consumer $i \in I_m$ the preference preordering $\succeq^i$ on X^i is complete, continuous, strongly monotonic, and strongly convex.*

Ass 2.3 *For every consumer $i \in I_m$ the vector of initial endowments w^i belongs to the interior of X^i.*

Notice that the assumption of strong convexity in 2.2 allows us to work with demand functions instead of demand correspondences.

In general the short term fixed relative prices will not be equal to the relative prices in any Walrasian equilibrium of the economy $\mathcal{E}$, being a price vector $p^* \in \mathbb{R}^{n+1}_{++}$ and consumption vectors $x^{*i} \in X^i$, $\forall i \in I_m$, such that both $\sum_{i=1}^m x^{*i} = \sum_{i=1}^m w^i$ and x^{*i} is a best element for $\succeq^i$ in the budget set $\{x^i \in X^i \mid p^{*\top} x^i \leq p^{*\top} w^i\}$ for every $i \in I_m$. Hence, there may not exist an $\alpha^* > 0$ such that the price vector $p^* = \widetilde{p}(\alpha^*)$ supports a Walrasian equilibrium. To equilibrate the demand and the supply under price restrictions one may introduce an equilibrium concept involving vectors of quantity constraints on the net demand. Given a price vector $p \in \mathbb{R}^{n+1}_{+}$ and a rationing scheme on demand $L \in \mathbb{R}^{n+1}_{+}$, the demand-constrained budget set of consumer $i \in I_m$ is given by

$$B^i(p, L) = \{x^i \in X^i \mid p^\top x^i \leq p^\top w^i, \ x^i - w^i \leq L\}.$$

The corresponding constrained demand $d^i(p, L)$ of consumer i is defined as a best element for $\succeq^i$ in $B^i(p, L)$. Because of the strong convexity and strong monotonicity assumptions this element is unique and lies on the budget hyperplane, i.e., $p^\top d^i(p, L) = p^\top w^i$. A real demand-constrained equilibrium with respect to a given price level $\alpha > 0$ is defined as follows.

Definition 2.4 Real demand-constrained equilibrium

For given $\alpha > 0$, a **real demand-constrained equilibrium** *with price level α* **(RDE$_\alpha$)** *for the economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \widetilde{r})$ is a price system $p^* \in \mathbb{R}^{n+1}_{++}$, a rationing scheme on demand $L^* \in \mathbb{R}^{n+1}_{+}$, and, for every consumer $i \in I_m$, a consumption bundle $x^{*i} \in X^i$ such that*

- *for all* $i \in I_m$, $x^{*i} = d^i(p^*, L^*)$;
- $\sum_{i=1}^m x^{*i} = \sum_{i=1}^m w^i$;
- $p_j^* \leq \alpha \widetilde{r}_j$, $\forall j \in I_n$, *and* $p_{n+1}^* = 1$;
- *for all* $j \in I_n$, $p_j^* < \alpha \widetilde{r}_j$ *implies* $L_j^* > x_j^{*i} - w_j^i$ *for all* $i \in I_m$;
- *for all* $i \in I_m$, $L_{n+1}^* > x_{n+1}^{*i} - w_{n+1}^i$.

A real demand-constrained equilibrium with price level α coincides with the definition of a constrained equilibrium given in Drèze [3] for the set $P(\alpha)$ of admissible prices for given price level α. The rationing scheme on demand is assumed to be uniform, i.e., the same for each consumer. This assumption can easily be relaxed. Condition 2.4 requires that the consumption of each consumer equals his constrained demand while condition 2.4 is the market clearing condition. Condition 2.4 requires that the price vector p^* lies in the set $P(\alpha)$, i.e., for every commodity $j \in I_n$ the price is relatively equal to or smaller than the price level α, whereas the price of the numeraire commodity equals one. Condition 2.4 reflects the natural property that demand rationing on the market of a commodity will only occur if its price is maximal. Condition 2.4 implies that there is no rationing on the market of the numeraire commodity.

A real demand-constrained equilibrium without rationing yields a Walrasian equilibrium. For given price level α, there will indeed exist such an equilibrium for α large enough. On the other hand, it will be shown that for small enough α there exists a uniquely determined trivial RDE_α at which all real commodities are completely rationed and that for α chosen large enough any RDE_α yields a Walrasian equilibrium. In this paper it is shown by means of simplicial approximation that a Walrasian equilibrium can be reached by generating a path of RDE_α's, i.e., at any point on the path all markets are in equilibrium and hence trade is possible, in the following way. The path starts with a trivial RDE_α for α chosen small enough. In the short term, by increasing α and adjusting simultaneously the rationing schemes a path of RDE_α's is generated where all real commodities are demand-rationed and prices are relatively fixed, i.e., in each equilibrium all the price constraints in condition (iii) of Definition 2.4 are binding. In the long term the price constraints will become non-binding and a path of RDE_α's is generated, where the price of a real commodity becomes lower than the maximum price as soon as there is no longer demand rationing on the market of this commodity. To keep all markets in equilibrium, along the path the price level, prices and rationing schemes are adjusted simultaneously. Once the price of a commodity without demand rationing reaches again its maximum price, this commodity becomes demand rationed again and the price is kept equal to its maximum. As soon as no commodity is rationed, a Walrasian equilibrium has been reached. It will be shown that this will indeed be the case eventually.

Define the vector w by $w = \sum_{i \in I_m} w^i$. Since at an RDE$_\alpha$ $(p^*, L^*, x^{*1}, \ldots, x^{*m})$ it holds that $x_j^{*i} - w_j^i < w_j$, $\forall i \in I_m$, $\forall j \in I_{n+1}$, and since there is no rationing on the market of the numeraire commodity, it is useful to consider only demand rationing schemes L satisfying $L_{n+1} = w_{n+1}$ and $0 \leq L_j \leq w_j$, $\forall j \in I_n$. The set of these rationing schemes is denoted by $\mathcal{L}$, so

$$\mathcal{L} = \{L \in \mathbb{R}_+^{n+1} \mid L_j \leq w_j, \forall j \in I_n, \text{ and } L_{n+1} = w_{n+1}\}.$$

In order to show the existence of an RDE$_\alpha$ for any $\alpha > 0$ the following lemma gives a result about the values of the demand if the price of some commodity is relatively low. The lemma states that for every consumer $i \in I_m$ it holds that if the price ratio $\frac{p_j}{p_k}$ for any two commodities $j, k \in I_{n+1}$ is sufficiently small, then his constrained demand for commodity j exceeds the total initial endowments of this commodity if the demand constraint for it is equal to these total initial endowments. Define the set $\overline{\mathcal{L}}$ by $\overline{\mathcal{L}} = \{L \in \mathbb{R}_+^{n+1} \mid L_j \leq w_j,\ \forall j \in I_{n+1}\}$.

Lemma 2.5

Let the economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \tilde{r})$ satisfy the Assumptions 2.1-2.3. Then for every $i \in I_m$ there exists a number $\beta^i > 0$ such that for all $j \in I_{n+1}$ it holds that $d_j^i(p, L) > w_j$ for every $(p, L) \in \mathbb{R}_+^{n+1} \times \overline{\mathcal{L}}$ satisfying both $\frac{p_j}{p_k} \leq \beta^i$ for some $k \in I_{n+1}$ and $L_j = w_j$.

Proof

Suppose that there exists a consumer $i \in I_m$ for which the lemma does not hold. Then without loss of generality there exists a commodity $j \in I_{n+1}$ and a sequence $(p^r, L^r)_{r \in \mathbb{N}}$ of prices and rationing schemes in $\mathbb{R}_+^{n+1} \times \overline{\mathcal{L}}$, satisfying for all $r \in \mathbb{N}$ that $L_j^r = w_j$, $\frac{p_j^r}{p_{k^r}^r} \leq \frac{1}{r}$ for some $k^r \in I_{n+1}$, and $d_j^i(p^r, L^r) \leq w_j$. Because of the homogeneity of degree zero of the demand function we may assume without loss of generality that for any $r \in \mathbb{N}$, $\sum_{h=1}^{n+1} p_h^r = 1$. Hence, there exist a subsequence $(p^{r_s}, L^{r_s}, d^i(p^{r_s}, L^{r_s}))_{s \in \mathbb{N}}$ converging to some $(\overline{p}, \overline{L}, \overline{d}^i) \in \mathbb{R}_+^{n+1} \times \overline{\mathcal{L}} \times \mathbb{R}_+^{n+1}$ satisfying $\overline{p}_j = 0$, $\overline{L}_j = w_j$, and $\overline{d}_j^i \leq w_j$. Since $\sum_{h=1}^{n+1} \overline{p}_h = 1$ and there is no supply rationing, the demand function is continuous at $(\overline{p}, \overline{L})$ according to the lemma on page 304 in Drèze [3]. Consequently, $\overline{d}_j^i = d_j^i(\overline{p}, \overline{L})$. Since $\overline{p}_j = 0$ and $\overline{L}_j = w_j$, it follows from the monotonicity of the preferences that $\overline{d}_j^i = w_j^i + w_j$, which contradicts $\overline{d}_j^i \leq w_j$. $\square$

Given an economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \tilde{r})$, let the numbers β^i, $i \in I_m$, be so small that Lemma 2.5 holds and define $\underline{\alpha}$ by

$$\underline{\alpha} = \frac{\min_{i \in I_m} \beta^i}{\max_{j \in I_n} \tilde{r}_j}.$$

Then $\underline{\alpha}$ corresponds to a price level in the economy which is so low that under the conditions of Lemma 2.5 all consumers are demanding net amounts of all real commodities. This gives us the next theorem.

Theorem 2.6
Let the economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \tilde{r})$ *satisfy the Assumptions 2.1-2.3. Then for any* $\alpha \in (0, \underline{\alpha}]$ *there exists an* RDE_α. *Moreover, for any* RDE_α $(p^*, L^*, x^{*1}, \ldots, x^{*m})$ *with* $0 < \alpha \leq \underline{\alpha}$ *it holds that* $p^* = \tilde{p}(\alpha)$, $L_j^* = 0$, $\forall j \in I_n$, *and* $x^{*i} = w^i$, $\forall i \in I_m$.

Proof
In Herings [5] it is shown that for every $\alpha > 0$ there exists $(p^*, L^*, x^{*1}, \ldots, x^{*m}) \in \mathbb{R}_{++}^{n+1} \times \overline{\mathcal{L}} \times \prod_{i \in I_m} X^i$ such that the first four conditions of Definition 2.4 are satisfied and, moreover, there exists $j \in I_{n+1}$ such that $L_j^* > x_j^{*i} - w_j^i$, $\forall i \in I_m$. Let any $\alpha \in (0, \underline{\alpha}]$ and any $(p^*, L^*, x^{*1}, \ldots, x^{*m})$ with these properties be given.
Suppose $i \in I_m$ and $k \in I_n$ satisfy $L_k^* > x_k^{*i} - w_k^i$. Then $x^{*i} = d^i(p^*, L^*) = d^i(p^*, \tilde{L}^*)$ with $\tilde{L}_j^* = L_j^*$, $\forall j \in I_{n+1} \setminus \{k\}$, and $\tilde{L}_k^* = w_k$. Since $\frac{p_k^*}{p_{n+1}^*} \leq \alpha \tilde{r}_k \leq \underline{\alpha} \tilde{r}_k \leq \beta^i$ and $\tilde{L}_k^* = w_k$ it follows from Lemma 2.5 that $x_k^{*i} > w_k$, which contradicts equilibrium condition 2.4. Consequently, $L_j^* = x_j^{*i} - w_j^i$, $\forall i \in I_m$, $\forall j \in I_n$, and $p^* = \tilde{p}(\alpha)$ due to equilibrium condition 2.4. By equilibrium condition 2.4 it follows for every $j \in I_n$ that $0 = \sum_{i=1}^m (x_j^{*i} - w_j^i) = mL_j^*$ and consequently $L_j^* = 0$ and $x_j^{*i} = w_j^i$, $\forall i \in I_m$. Since there exists $j \in I_{n+1}$ such that $L_j^* > x_j^{*i} - w_j^i$, $\forall i \in I_m$, it holds that $L_{n+1}^* > x_{n+1}^{*i} - w_{n+1}^i$, $\forall i \in I_m$. Consequently, $(p^*, L^*, x^{*1}, \ldots, x^{*m})$ is an RDE_α. $\square$

For $\alpha \in (0, \underline{\alpha}]$, Theorem 2.6 shows the existence of a trivial RDE_α in the sense that the price ratio between the numeraire and any other commodity becomes so high that nobody supplies a non-numeraire commodity and therefore equilibrium is sustained by complete demand rationing on the markets of the non-numeraire commodities.

3 The reduced total excess demand function

To describe the price and quantity adjustment process, we relate to any element of the $(n+1)$-dimensional set C^{n+1} given by

$$C^{n+1} = \{q \in \mathbb{R}^{n+1} \mid 0 \leq q_{n+1} < 1, \ 0 \leq q_j \leq 2, \ \forall j \in I_n, \ \text{and} \ \exists k \in I_n, \ q_k \leq 1\},$$

a price and a rationing vector. The set C^{n+1} is illustrated in Figure 1 for $n = 2$. Observe that the boundary $q_{n+1} = 1$ does not belong to the set.

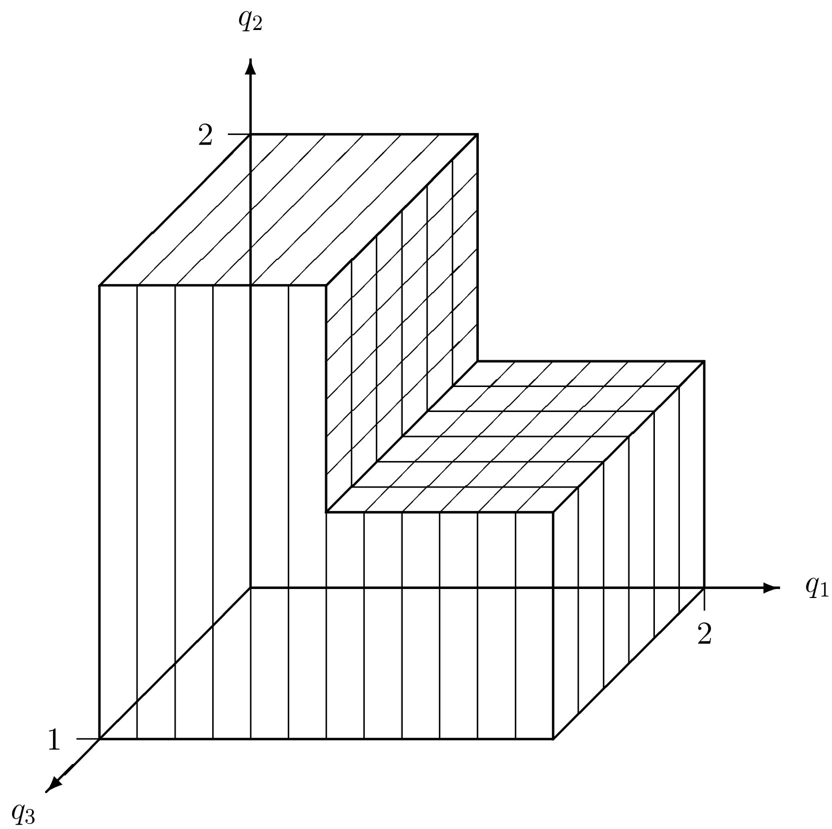

Figure 1: The set C^3.

For $q \in C^{n+1}$, the price level $\widehat{\alpha}(q) > 0$, the price system $\widehat{p}(q) \in \mathbb{R}^{n+1}_{++}$, and the rationing scheme $\widehat{L}(q) \in \mathcal{L}$ are defined by

$$\begin{aligned}
\widehat{\alpha}(q) &= \frac{\underline{\alpha}}{1 - q_{n+1}}, && (3.1)\\
\widehat{p}_j(q) &= \min\{1, 2 - q_j\}\widehat{\alpha}(q)\widetilde{r}_j,\ \forall j \in I_n, && (3.2)\\
\widehat{p}_{n+1}(q) &= 1, && (3.3)\\
\widehat{L}_j(q) &= \min\{1, q_j\}w_j,\ \forall j \in I_n, && (3.4)\\
\widehat{L}_{n+1}(q) &= w_{n+1}. && (3.5)
\end{aligned}$$

Notice that $\widehat{\alpha}(q)$ is well-defined because $q_{n+1} < 1$ for all $q \in C^{n+1}$. Furthermore, for every $j \in I_n$, $\widehat{L}_j(q) = 0$ if $q_j = 0$, and $\widehat{L}_j(q) = w_j$ if $q_j \geq 1$. Moreover, for every $j \in I_n$, $\widehat{p}_j(q) = \widehat{\alpha}(q)\widetilde{r}_j$ if $q_j \leq 1$, and $\widehat{p}_j(q) < \widehat{\alpha}(q)\widetilde{r}_j$ if $q_j > 1$. If $q_{n+1} = 0$ then $\widehat{\alpha}(q) = \underline{\alpha}$, and if $q_{n+1} > 0$ then $\widehat{\alpha}(q) > \underline{\alpha}$.

For $q \in C^{n+1}$ we call $\widehat{B}^i(q) = B^i(\widehat{p}(q), \widehat{L}(q))$ the constrained budget set of consumer $i \in I_m$ at q, i.e.,

$$\widehat{B}^i(q) = \{x^i \in X^i \mid \widehat{p}(q)^\top x^i \leq \widehat{p}(q)^\top w^i,\ x^i_j - w^i_j \leq \widehat{L}_j(q), \forall j \in I_{n+1}\}.$$

Let $\widehat{d}^i(q)$ denote the best element for $\succeq^i$ in the constrained budget set $\widehat{B}^i(q)$ of consumer $i \in I_m$, so $\widehat{d}^i(q) = d^i(\widehat{p}(q), \widehat{L}(q))$, and define the total excess demand at q by

$$\widehat{z}(q) = \sum_{i=1}^{m} \widehat{d}^i(q) - \sum_{i=1}^{m} w^i.$$

The function $\widehat{z} : C^{n+1} \to \mathbb{R}^{n+1}$ is called the reduced total excess demand function.

For $q^* \in C^{n+1}$, it holds that $(\widehat{p}(q^*), \widehat{L}(q^*), \widehat{d}^1(q^*), \ldots, \widehat{d}^m(q^*))$ is an $\mathrm{RDE}_{\widehat{\alpha}(q^*)}$ if and only if $\widehat{z}(q^*) = \underline{0}$. Clearly, $q^* = \underline{0}$ corresponds to the trivial $\mathrm{RDE}_{\underline{\alpha}}$ given by $(\widetilde{p}(\underline{\alpha}), (\underline{0}^\top, w_{n+1})^\top, w^1, \ldots, w^m)$. Finally, define the set $\widehat{C}^{n+1}$ by $\widehat{C}^{n+1} = \{q \in C^{n+1} \mid \min_{j \in I_n} q_j = 1\}$. The set $\widehat{C}^{n+1}$ corresponds to the crossed area in Figure 1 for $n = 2$. If $q \in \widehat{C}^{n+1}$ then $\widehat{L}(q) = w$ and hence the rationing constraints are non-binding. So, we have that for every $q \in \widehat{C}^{n+1}$ it holds that any $\mathrm{RDE}_{\widehat{\alpha}(q)}$ $(\widehat{p}(q), \widehat{L}(q), \widehat{d}^1(q), \ldots, \widehat{d}^m(q))$ is a WE.

In Section 5 a constructive proof is given of the existence of a path of approximate zeros of $\widehat{z}$ in C^{n+1} corresponding to approximate RDE_α's. This path connects $q = \underline{0}$, corresponding to the trivial RDE_α for $\alpha = \underline{\alpha}$, with an approximate zero point q^* of $\widehat{z}$ on the boundary $\widehat{C}^{n+1}$ of C^{n+1}, inducing an approximate Walrasian equilibrium. In Section 6 we will show the existence of a connected set of RDE_α's connecting the trivial $\mathrm{RDE}_{\underline{\alpha}}$ and a Walrasian equilibrium by considering the limit of a sequence of paths of approximate RDE_α's. The price and quantity adjustment process follows the path of zero points of $\widehat{z}$. Starting from the trivial $\mathrm{RDE}_{\underline{\alpha}}$ at the point $q = \underline{0}$, it proceeds with increasing the price level by increasing the variable q_{n+1}. Since, initially, all consumers are net demanders of all real commodities, the only way to maintain equilibrium is to ration all demands completely, i.e., $q_1, \ldots, q_n$ are initially all kept equal to zero. At some point some consumer will start to supply some commodity $k \in I_n$. Then it is possible to weaken the demand rationing on the market of commodity k, i.e., to increase q_k. Continuing the adjustment of the price level, there will also become supply by some consumers on other markets, making it possible to decrease the amount of demand rationing on these markets too. Continuing this adjustment of the price level by adjusting q_{n+1} and keeping the markets of all commodities in equilibrium by adjusting q_j, $\forall j \in I_n$, it will be shown that at some point a short term equilibrium is reached at which there is no longer demand rationing on the market of at least one real commodity, say, commodity $j^1 \in I_n$. From this point in time it is allowed that the price of commodity j^1 decreases relative to the price level and is adjusted in such a way that the market of this commodity is kept in equilibrium. Next, at some point in time it will either happen that there is no demand rationing on the market of another commodity, say, commodity $j^2 \in I_n$, or in order to keep the market of commodity j^1 in equilibrium, its price has to be increased above the maximum price on the market of commodity j^1 indicated by the price level. In the former case the price of commodity j^2 is also allowed to decrease from the maximum price and together with the price level and the rationing schemes of the other commodities the prices of the commodities j^1 and j^2 are adjusted simultaneously such that the markets of these commodities are kept in equilibrium. In the latter case the price of commodity j^1 is kept equal again to the maximum price, whereas the market of commodity j^1 is equilibrated by introducing demand rationing again. More generally, at some point in time let $J \subset I_n$ be the subset of commodities with no demand rationing. Then the process proceeds by adjusting simultaneously the price level, the prices of the commodities in J, and the demand rationing schemes of the real commodities in the set $I_n \setminus J$, such that all markets are kept in equilibrium. As soon as for some $j \in J$ the price reaches its maximum, the price of this commodity is kept equal again to the maximum price and the process proceeds as above with $J \setminus \{j\}$ as the set of unrationed real commodities, whereas the process proceeds with the set $J \cup \{k\}$ as the set of unrationed commodities as soon as there is no longer demand rationing for some commodity $k \in I_n \setminus J$. It will be shown that eventually there will be no demand rationing on any market and therefore a Walrasian equilibrium will be reached. Before we give more details of the process, in the following lemmas we describe some properties of the reduced total excess demand function $\widehat{z}$.

Lemma 3.1
Let the economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \widetilde{r})$ satisfy the Assumptions 2.1-2.3. Then the reduced total excess demand function $\widehat{z}$ is continuous on C^{n+1} and $\widehat{p}(q)^\top \widehat{z}(q) = 0$, $\forall q \in C^{n+1}$.

Proof
By the lemma in Drèze [3] (p. 304) it follows that, for every $i \in I_m$, B^i is continuous on $\mathbb{R}^n_+ \times \{1\} \times \mathcal{L}$, using that $p_{n+1} = 1$ and there is no supply rationing on the market of the numeraire commodity. Using the continuity and the strong convexity of the preferences, and the maximum theorem it follows that, for every $i \in I_m$, d^i is continuous on $\mathbb{R}^n_+ \times \{1\} \times \mathcal{L}$. By the continuity of the functions $\widehat{p}$ and $\widehat{L}$ in q it follows that $\widehat{z}$ is continuous on C^{n+1}. The strong monotonicity of the preferences yields that $\widehat{p}(q)^\top \widehat{z}(q) = 0$, $\forall q \in C^{n+1}$. $\square$

When $q = \underline{0}$, each consumer wants to sell the numeraire commodity in exchange for any of the other commodities. However, as long as $q_j = 0$ for all $j \in I_n$, none of the non-numeraire commodities can be bought. So, the consumers must keep their initial endowments of the numeraire commodity and we have an equilibrium. When $q_k > 0$ for just one commodity $k \in I_n$, there is no longer complete demand rationing and the consumers want to buy good k against the numeraire. We then have that $\widehat{z}_{n+1}(q) < 0$ and $\widehat{z}_k(q) > 0$ and therefore the economy is out of equilibrium. In the following lemma this reasoning is generalized to the case that $q_j > 0$ for at least one $j \in I_n$.

Lemma 3.2
Let the economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \widetilde{r})$ satisfy the Assumptions 2.1-2.3. Then, for every $j \in I_n$, if $q \in C^{n+1}$ and $q_j = 0$, then $\widehat{z}_j(q) \leq 0$. Furthermore, if $q \in C^{n+1}$ and $q_{n+1} = 0$, then $\widehat{z}_j(q) \geq 0$, $\forall j \in I_n$. Moreover, for every $k \in I_n$, if $q \in C^{n+1}$, $q_{n+1} = 0$, and $q_k > 0$, then $\widehat{z}_k(q) > 0$.

Proof
If $q_j = 0$ for some $j \in I_n$, then $\widehat{L}_j(q) = 0$ and hence $\widehat{z}_j(q) = \sum_{i=1}^m (\widehat{d}^i_j(q) - w^i_j) \leq m\widehat{L}_j(q) = 0$.
Let $q \in C^{n+1}$ with $q_{n+1} = 0$ be given and suppose that $\widehat{z}_k(q) \leq 0$ for some $k \in I_n$ with $q_k > 0$. Then, for some $i \in I_m$, $\widehat{d}^i_k(q) \leq w^i_k$. Since $q_k > 0$ and hence $\widehat{L}_k(q) > 0$ we have that $\widehat{L}_k(q)$ is non-binding for this consumer. Therefore, $\widehat{d}^i(q) = d^i(\widehat{p}(q), \widetilde{L})$ with $\widetilde{L} \in \mathcal{L}$ defined by $\widetilde{L}_j = \widehat{L}_j(q)$, for all $j \in I_{n+1} \setminus \{k\}$, and $\widetilde{L}_k = w_k$. Moreover, $\frac{\widehat{p}_k(q)}{\widehat{p}_{n+1}(q)} \leq \underline{\alpha}\widetilde{r}_k \leq \beta^i$. By Lemma 2.5, $d^i_k(\widehat{p}(q), \widetilde{L}) > w_k$, a contradiction. Consequently, for every $k \in I_n$, if $q \in C^{n+1}$, $q_{n+1} = 0$, and $q_k > 0$, then $\widehat{z}_k(q) > 0$. From the continuity of $\widehat{z}$ it follows that $q \in C^{n+1}$ and $q_{n+1} = 0$ implies $\widehat{z}_j(q) \geq 0$, $\forall j \in I_n$. □

We now want to consider the behaviour of $\widehat{z}$ near the boundary of C^{n+1} where $q_j = 2$ for some $j \in I_n$ or where $q_{n+1} = 1$, i.e., when the numeraire commodity is relatively very cheap. To do so, define the positive number δ by

$$\delta = \min\{\tfrac{1}{2}, (\underline{\alpha} \min_{j \in I_n} \widetilde{r}_j)^2\}. \tag{3.6}$$

Lemma 3.3
Let the economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \widetilde{r})$ satisfy the Assumptions 2.1-2.3. Then, for every $j \in I_n$, if $q \in C^{n+1}$ and $q_j = 2$, then $\widehat{z}_j(q) > 0$. If $q \in C^{n+1}$ and $q_{n+1} \geq 1 - \delta$, then $\widehat{z}_{n+1}(q) > 0$.

Proof
When $q_j = 2$ then $\widehat{p}_j(q) = 0$ and $\widehat{L}_j(q) = w_j > 0$. From the monotonicity of preferences it follows that $\widehat{z}_j(q) > 0$.
By definition of C^{n+1} there exists a commodity $k \in I_n$ such that $q_k \leq 1$. If $q_{n+1} \geq 1 - \delta$, then

$$\frac{\widehat{p}_{n+1}(q)}{\widehat{p}_k(q)} = \frac{1 - q_{n+1}}{\underline{\alpha}\widetilde{r}_k} \leq \frac{\delta}{\underline{\alpha}\widetilde{r}_k} \leq \underline{\alpha} \min_{j \in I_n} \widetilde{r}_j \leq \min_{i \in I_m} \beta^i.$$

Hence, by Lemma 2.5, $d^i_{n+1}(\widehat{p}(q), \widehat{L}(q)) > w_{n+1}$, for all $i \in I_m$, and so $\widehat{z}_{n+1}(q) > mw_{n+1} - w_{n+1} \geq 0$. □

4 An illustration of the price and quantity adjustment process

In this section we consider the adjustment process induced by following the path of real demand-constrained equilibria. The path first proceeds from the trivial $\mathrm{RDE}_{\underline{\alpha}}$ to an RDE_{α}, where at least one real commodity is not being rationed. At this point $q_j = 1$ holds for at least one $j \in I_n$. Then the process continues by keeping the relative prices of the rationed commodities maximal and by allowing a decrement of the relative price of the unrationed commodity by increasing the corresponding value of the variable q_j. Continuing we have that in order to keep total excess demand equal to zero the process adjusts simultaneously the prices of the unrationed commodities (corresponding to the indices j with $q_j > 1$) below their relative upper bound, the price level $\widehat{\alpha}(q)$, and the rationing schemes of the commodities with prices still on their relative upper bound (corresponding to the indices j with $q_j < 1$). As soon as for some $j \in I_n$ the value of q_j increases to one, the corresponding regime switches from rationing adjustment under fixed relative price to price adjustment without rationing, while the reverse happens if the value of q_j becomes equal to one from above. Eventually, the process reaches a point in which all values of q_j, $j \in I_n$, are equal to or greater than one and hence a Walrasian equilibrium has been obtained.

A typical example of the process is illustrated in Figure 2 for $n = 2$ by drawing the projection of the path in the (q_1, q_2)-space. From Theorem 2.6 it follows that the point $q = \underline{0}$ induces the trivial $\mathrm{RDE}_{\underline{\alpha}}$. Moreover, it follows from Lemma 3.2 that any other point q with $q_{n+1} = 0$ can not induce an equilibrium. Therefore, to keep all markets in equilibrium, initially we have to increase the value of q_3 inducing an increase of the price level. This means that the projection does not change and remains equal to the point $\underline{0}$ in the (q_1, q_2)-space. Suppose next that a consumer starts to supply commodity 1. Then also the value of q_1 starts to increase. So, the projection goes from $\underline{0}$ in the direction of the point A, generating $\mathrm{RDE}_{\widehat{\alpha}(q)}$'s by relaxing the constraint on the demand of commodity 1 according to the value of q_1 and changing the price level according to q_3. At point A also the value of q_2 becomes positive, inducing a non-zero demand constraint on the market of commodity 2. At point B the path reaches an $\mathrm{RDE}_{\widehat{\alpha}(q)}$ with no rationing on the market of commodity 1. Then the path continues with values of q_1 above one. This part of the path induces $\mathrm{RDE}_{\widehat{\alpha}(q)}$'s in which for commodity 1 a situation corresponding to condition 2.4 of Definition 2.4 occurs, i.e., no rationing on the demand of commodity 1, while the price of this commodity is relatively below the price level $\widehat{\alpha}(q)$, which is determined by the value of q_3. From Lemma 3.3 we know that $\widehat{z}_1(q) > 0$ if $q_1 = 2$ and that $\widehat{z}_3(q) > 0$ if $q_3 \geq 1 - \delta$. Since at the path all markets are in equilibrium the path can neither reach values of q_3 above $1 - \delta$ nor the boundary of C^3 where $q_1 = 2$. Moreover, if q_2 becomes

Figure 2: Illustration of the adjustment path; $n = 2$.

equal to zero, the path is continued along this boundary of C^3 by adjusting q_1 and q_3 and complete demand rationing on the market of commodity 2. Therefore, by continuing the process in the region where $q_1 > 1$, the path has to reach either a point on the boundary of C^3 where $q_2 = 1$ or q_1 must become equal to one again. The former case is similar to point W and is discussed below. The latter case is illustrated in the figure where the path reaches point C. At this point a second $\text{RDE}_{\widehat{\alpha}(q)}$ with $q_1 = 1$ is reached. From this point on the path induces again $\text{RDE}_{\widehat{\alpha}(q)}$'s with rationing on both commodities. By continuing the process the path must reach again a point q with $q_j = 1$ for either $j = 1$ or $j = 2$. This happens at point D where $q_2 = 1$. From this point on the path induces $\text{RDE}_{\widehat{\alpha}(q)}$'s with no rationing on the market of commodity 2. Similarly to the reasoning given before, the path must reach either a point where $q_2 = 1$ or a point where $q_1 = 1$. In the former case the path continues as in point C. The latter case is illustrated in the figure by point W, where the process reaches a WE. Notice that along the path initially the value of q_3 increases. However, in general it is not guaranteed that this value increases monotonically. Along some parts of the path it is possible that the value of the variable q_3 determining the price level will decrease and hence the price level $\alpha(q)$ will decrease in order to keep the total excess demand equal to zero.

Using the definition of $\widehat{p}(q)$ we can translate the picture of Figure 2 in the (q_1, q_2)-

Figure 3: The partition of the price space in disequilibrium regimes; $n = 2$.

space to a picture in the (p_1, p_2)-space. Recall that $p_3 = 1$ is fixed. To do so, we first consider Figure 3. Assuming that there is no rationing on the market of the numeraire commodity, in Figure 3 we have drawn the different rationing regimes according to the values of p_1 and p_2. The point W' denotes the Walrasian equilibrium values of the prices. The curves going through this point separate the different regimes of rationing. At a point in Region IV the values of p_1 and p_2 are rather high and supply rationing on both markets is needed in order to equilibrate the markets. In Region II (III) the value of p_2 (p_1) is rather low and the value of p_1 (p_2) high and therefore demand rationing on market 2 (market 1) and supply rationing on market 1 (market 2) is needed. At a point in Region I demand rationing on both markets is necessary. At the intersection of two regions we need only rationing on one of the markets, for instance there is no rationing on the market of commodity 1 where the Regions I and II meet. At such a point market 1 switches from demand rationing in Region I to supply rationing in Region II. Of course, at point W' the markets are equilibrated without rationing. The regions are drawn again in Figure 4. In this figure the straight line leaving the origin represents the initially fixed relative prices of the non-numeraire commodities. At any point on this line we have that $p = \widetilde{p}(\alpha)$ for some price level $\alpha > 0$. Point O reflects the price level $\underline{\alpha}$. At this point the trivial equilibrium is obtained with complete demand rationing on both commodities.

Figure 4: Illustration of the adjustment path in the price space; $n = 2$.

Translating Figure 2 to Figure 4 the path starts at the point O. Increasing the value of q_3 corresponds to an increase of the price level and hence in Figure 4 the path goes upwards along the ray of fixed relative prices, until at the point O' some consumer starts to supply commodity 1. This point still corresponds with the point $\underline{0}$ in Figure 2, because this latter point is the projection of the part of the path along which only q_3 increases. At the point O' the complete demand rationing is relaxed and q_1 becomes positive. Going from $\underline{0}$ to A in Figure 2 corresponds to going from O' to A' in Figure 4. The path from $\underline{0}$ to A shows that the demand rationing on commodity 1 is relaxed from zero, while the path from O' to A' shows that the price level increases simultaneously. At point A also q_2 becomes positive. Continuing along the path in Figure 2 from A to B, Figure 4 shows that simultaneously the price level (i.e. $\widehat{\alpha}(q)$) increases until at point B' corresponding to point B in Figure 2 the boundary between Region I and Region II is reached, at which the market regime for commodity 1 switches from demand rationing into supply rationing. At this point the path in Figure 2 continues with values of q_1 above 1 and hence with price p_1 below the maximum according to the price level, while the markets are kept in equilibrium without rationing on the market of commodity 1. In Figure 4 this is illustrated by the fact that the path leaves the ray through O in upward direction, inducing a price ratio $\frac{p_1}{p_2} < \frac{\widetilde{r}_1}{\widetilde{r}_2}$, by following the curve between Region I and Region II. At point C' corresponding to point C in Figure

2 this curve again meets the ray of fixed relative prices. Observe that going along this curve from B' to C' the absolute value of p_2 first is increasing and afterwards decreasing, showing that the price level and hence q_3 does not increase monotonically. Continuing at point C the path in Figure 2 again induces an equilibrium with fixed relative prices and demand rationing on both markets, and hence the corresponding path in Figure 4 continues along the ray through O going further upwards in Region I. At this part of the path the price level increases again. At point D' corresponding to the point D in Figure 2 the border between Region I and Region III is reached. Now the path continues along the curve between these regions, keeping the markets in equilibrium by allowing the price of commodity 2 to vary below the allowed maximum value ($q_2 > 1$) and imposing a demand constraint on the market of commodity 1 ($q_1 < 1$), until at point W' corresponding to W in Figure 2 the Walrasian equilibrium values of the prices are reached.

5 Approximate real demand-constrained equilibria

In this section attention is focused on approximate real demand-constrained equilibria. By using simplicial techniques we give a constructive proof of the existence of a path of points in C^{n+1} inducing a path of approximate equilibria connecting the trivial $\mathrm{RDE}_{\underline{\alpha}}$ induced by $q = \underline{0}$ with an approximate WE induced by some point $q \in C^{n+1}$ such that $\min_{j \in I_n} q_j = 1$. Applying the simplicial technique provides us with the possibility to follow the path described in the previous section. By taking the mesh size of the underlying triangulation small enough the excess demand at the approximate equilibria being generated can be made arbitrarily close to zero. In the following definition an approximate RDE_α, for any price level $\alpha > 0$, is introduced.

Definition 5.1 ε-RDE_α and ε-WE
For a given price level $\alpha > 0$ and a real number $\varepsilon \geq 0$, an **ε-RDE_α (ε-WE)** *for the economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \tilde{r})$ is a price system p, a rationing scheme L, and consumption bundles $x^1, \ldots, x^m$ such that all conditions of an RDE_α (WE) are satisfied, except that the condition of equality of demand and supply is replaced by $\|\sum_{i=1}^m x^i - w\|_\infty \leq \varepsilon$.*

Clearly, an 0-RDE_α is an RDE_α and an 0-WE is a WE. In order to show the existence of a path of ε-RDE_α's connecting the trivial $\mathrm{RDE}_{\underline{\alpha}}$ and an ε-WE for arbitrary $\varepsilon > 0$, we will use some techniques of simplicial approximation of functions. This approach is also used in van der Laan [8] and Herings [6]. For given $t \in \mathbb{N}$, $0 \leq t \leq k$, a t-dimensional simplex or t-simplex is defined as the convex hull of $t+1$ affinely independent vectors in $\mathbb{R}^k$, $q^1, \ldots, q^{t+1}$, and is denoted by $\sigma(q^1, \ldots, q^{t+1})$ or shortly by σ. The vectors $q^1, \ldots, q^{t+1}$ are called the vertices of σ. A $(t-1)$-simplex τ being the convex hull of t vertices of

$\sigma(q^1, \ldots, q^{t+1})$ is called a facet of σ. For $h \in I_{n+1}$ the facet $\tau(q^1, \ldots, q^{h-1}, q^{h+1}, \ldots, q^{t+1})$ is called the facet of σ opposite to the vertex q^h. For $0 \leq j \leq t$, a j-simplex being the convex hull of $j+1$ vertices of a t-simplex σ is called a face of σ. A finite collection $\mathcal{T}$ of k-simplices is a triangulation of a compact subset S of some Euclidean space if:

1. S is the union of all simplices in $\mathcal{T}$;

2. the intersection of two simplices in $\mathcal{T}$ is either empty or a common face of both.

It can be shown that if S is homeomorphic to a convex set, then each facet τ of a k-simplex $\sigma \in \mathcal{T}$ either lies in the relative boundary of S and is only a facet of σ or it is a facet of exactly one other k-simplex in $\mathcal{T}$. The mesh of a triangulation $\mathcal{T}$ is defined by $\mathrm{mesh}(\mathcal{T}) = \max_{\sigma \in \mathcal{T}} \max\{\|\tilde{q} - \hat{q}\|_\infty \mid \tilde{q}, \hat{q} \in \sigma\}$.

In this section the set $C_\delta^{n+1} = \{q \in C^{n+1} \mid q_{n+1} \leq 1 - \delta\}$ will be triangulated, where δ is as defined in equation (3.6) of Section 3. It is also useful to define the set $\widehat{C}_\delta^{n+1} = \{q \in C_\delta^{n+1} \mid \min_{j \in I_n} q_j = 1\}$. An example of a triangulation of C_δ^{n+1} with arbitrarily small mesh size is obtained by using the K-triangulation described in Freudenthal [4]. The K-triangulation of C_δ^{n+1} is obtained as follows. For $k \in I_n$, let e^k denote the vector in $\mathbb{R}^{n+1}$ with $e_k^k = 1$ and $e_j^k = 0$, for all $j \in I_{n+1} \setminus \{k\}$, and let e^{n+1} denote the vector in $\mathbb{R}^{n+1}$ with $e_{n+1}^{n+1} = 1 - \delta$ and $e_j^{n+1} = 0$, for all $j \in I_n$. Let $r \in \mathbb{N}$ be given, then the K-triangulation of C_δ^{n+1} with grid size r^{-1} is the collection of all simplices $\sigma_{(q^1,\pi)}$ with vertices $q^1, \ldots, q^{n+2}$ in C_δ^{n+1} such that q_j^1 is a multiple of r^{-1} if $j \in I_n$, q_{n+1}^1 is a multiple of $(1-\delta)r^{-1}$, $\pi = (\pi_1, \ldots, \pi_{n+1})$ is a permutation of the elements of I_{n+1}, and for every $h \in I_{n+1}$, $q^{h+1} = q^h + r^{-1}e^{\pi_h}$. The mesh size of the K-triangulation of C_δ^{n+1} with grid size r^{-1} is r^{-1}.

Let the labelling function $\phi : C_\delta^{n+1} \to I_{n+1}$ be defined by $\phi(q) = \max[\arg\min\{\hat{z}_j(q) \mid j \in I_{n+1}\}]$, i.e., the last component for which the total excess demand at q is minimal. Let some triangulation $\mathcal{T}$ of C_δ^{n+1} be given. Now a procedure is used which starts at $q = \underline{0}$ and generates a sequence of simplices of varying dimension being faces of simplices in $\mathcal{T}$. For a simplex $\sigma(q^1, \ldots, q^{t+1})$ in this sequence it holds for every $j \in I_{n+1}$ that $q_j = 0$ for every $q \in \sigma$ or $j \in \phi(\{q^1, \ldots, q^{t+1}\})$. In the first case $\hat{z}_j(q) \leq 0$ for every $q \in \sigma$ by Lemma 3.1 and Lemma 3.2, and in the second case $\hat{z}_j(q^i) \leq 0$ for a vertex q^i of σ with $\phi(q^i) = j$ by the definition of the labelling function ϕ and the fact that $\hat{z}$ satisfies Walras' law. It will be shown below that these properties guarantee that for a point q in such a simplex $\hat{z}(q)$ is approximately zero. Two subsequent simplices in the sequence either share a common facet, or one simplex is a facet of the other. Such simplices are said to be adjacent. The procedure used is closely related to the one given in van der Laan [8] and is described below. For $J \subset I_{n+1}$, define the sets

$$A(J) = \{q \in C_\delta^{n+1} \mid q_j = 0, \ \forall j \in I_{n+1} \setminus J\},$$

$$\mathcal{T}(J) \;\; = \;\; \{\sigma \cap A(J) \mid \sigma \in \mathcal{T} \text{ and } \dim(\sigma \cap A(J)) = |J|\},$$

with $|J|$ denoting the number of elements of the set J. It can be shown that $\mathcal{T}(J)$ is a triangulation of $A(J)$. Denote by $\overline{C}_\delta^{n+1}$ the part of the boundary of C_δ^{n+1} where some component of q is maximal, so

$$\overline{C}_\delta^{n+1} = \{q \in C_\delta^{n+1} \mid \exists k \in I_n,\; q_k = 2, \text{ or } \min_{j \in I_n} q_j = 1, \text{ or } q_{n+1} = 1 - \delta\}.$$

The set $\overline{C}_\delta^{n+1} \setminus \widehat{C}_\delta^{n+1}$ corresponds to the striped area in Figure 1. In the description of the procedure given below, σ^j will denote a simplex and q^i a vertex generated by the procedure. J^k is a subset of labels of I_{n+1} generated by the procedure and induces a set $A(J^k)$ and a triangulation $\mathcal{T}(J^k)$ in which the procedure generates simplices. Given a set $S \subset \mathbb{R}^k$, $\mathrm{co}(S)$ denotes the convex hull of the set S. The procedure operates as follows.

<u>Procedure</u>

Step 0. Set $t = 0$, $q^1 = \underline{0}$, $\sigma^1 = \sigma(q^1)$, $J^1 = \emptyset$, $i = j = k = 1$. Go to Step 1.

Step 1. If $\phi(q^i) \notin J^k$, then go to Step 3. Otherwise there is a unique vertex $\overline{q}$ of σ^j such that $\overline{q} \neq q^i$ and $\phi(\overline{q}) = \phi(q^i)$. Go to Step 2.

Step 2. Let $\overline{\tau}$ be the facet of σ^j opposite $\overline{q}$. If there exists $h \in J^k$ such that $\overline{\tau} \subset A(J^k \setminus \{h\})$, then go to Step 4. If $\overline{\tau} \subset \overline{C}_\delta^{n+1}$, then stop. Otherwise there is a unique point $q^{i+1} \in A(J^k)$ such that $\sigma^{j+1} = \mathrm{co}(\overline{\tau} \cup \{q^{i+1}\})$ is a t-simplex of $\mathcal{T}(J^k)$ and $\sigma^{j+1} \neq \sigma^j$. Increase the values of i and j by 1. Go to Step 1.

Step 3. Define $J^{k+1} = J^k \cup \{\phi(q^i)\}$. There is a unique point $q^{i+1} \in A(J^{k+1})$ such that $\sigma^{j+1} = \mathrm{co}(\sigma^j \cup \{q^{i+1}\})$ is a $(t+1)$-simplex of $\mathcal{T}(J^{k+1})$. Increase the values of i, j, k, t by 1. Go to Step 1.

Step 4. Let $\overline{\overline{q}}$ be the unique vertex of σ^j such that $\phi(\overline{\overline{q}}) = h$ and $\overline{\overline{q}} \neq \overline{q}$. Define $J^{k+1} = J^k \setminus \{h\}$. Define $\sigma^{j+1} = \overline{\tau}$. Increase the values of j and k by 1 and decrease the value of t by 1. Let $\overline{q}$ be the element $\overline{\overline{q}}$. Go to Step 2.

The procedure is illustrated in Figure 5 for $n = 1$ and $r = 3$. The procedure starts with the 0-dimensional simplex $\sigma^1 = \{\underline{0}\}$ in $A(\emptyset)$ and terminates with a simplex having a facet in $\overline{C}_\delta^{n+1} \cap A(\{2\})$. After the starting simplex $\{\underline{0}\}$ the procedure generates a 1-simplex in $A(\{2\})$. Then two adjacent 2-simplices in $A(\{1,2\})$ are generated. Subsequently, two adjacent 1-simplices in $A(\{1\})$ are obtained, followed by eight adjacent 2-simplices in $A(\{1,2\})$. Finally two adjacent 1-simplices in $A(\{2\})$ are generated, with the last simplex having the facet determined in Step 2 in the set $\overline{C}_\delta^{n+1}$. It is easily verified that the properties of a triangulation guarantee that each step in the procedure is feasible and unique.

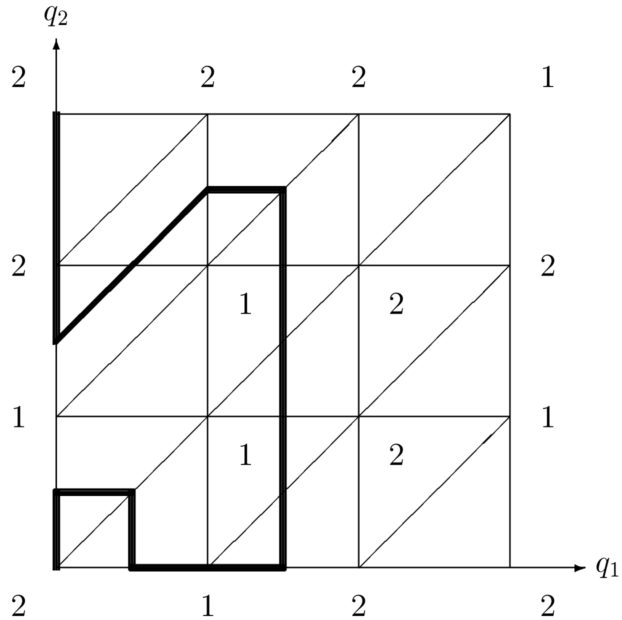

Figure 5: Illustration of the procedure; $n = 1, r = 3$.

Definition 5.2 *J*-completeness
Let $J \subset I_{n+1}$ be given with $|J| = t$. A $(t-1)$-simplex $\tau(q^1, \ldots, q^t)$ in C_δ^{n+1} is ***J*-complete** *if $\phi(\{q^1, \ldots, q^t\}) = J$.*

A J-complete simplex τ in $A(J)$ and a $\overline{J}$-complete simplex $\overline{\tau}$ in $A(\overline{J})$ are said to be adjacent complete simplices if either $J = \overline{J}$ and τ and $\overline{\tau}$ are both facets of a same simplex σ in $\mathcal{T}(J)$, or if τ is a facet of $\overline{\tau}$ and $\overline{\tau}$ is a simplex in $A(J)$, or if $\overline{\tau}$ is a facet of τ and τ is a simplex in $A(\overline{J})$. It is easily verified that if two simplices $\sigma^j \in \mathcal{T}(J)$ and $\sigma^{j+1} \in \mathcal{T}(\overline{J})$ are subsequently generated by the procedure then $\tau^j = \sigma^j \cap \sigma^{j+1}$ is a $(J \cup \overline{J})$-complete simplex in $A(J \cap \overline{J})$. Let $\sigma^1, \sigma^2, \ldots$ be the sequence of simplices generated by the procedure and consider the sequence $\tau^1, \tau^2, \ldots$ given by $\tau^j = \sigma^j \cap \sigma^{j+1}$, for $j \geq 1$. The subsequent simplices in the latter sequence are adjacent complete simplices. It will be shown that by generating a finite number of simplices in $\cup_{J \subset I_{n+1}} \mathcal{T}(J)$ the procedure terminates in Step 2 with a simplex having, for some $J \subset I_{n+1}$, a J-complete facet in $\widehat{C}_\delta^{n+1}$. To prove this, we first give the next lemma.

Lemma 5.3
Let a triangulation $\mathcal{T}$ of C_δ^{n+1} and a labelling function $\phi : C_\delta^{n+1} \rightarrow I_{n+1}$ be given. Let τ be a J-complete simplex in $A(J)$ for some $J \subset I_{n+1}$. Then τ has exactly one adjacent complete simplex if $\tau = \{\underline{0}\}$ or if τ lies in $\overline{C}_\delta^{n+1}$. Otherwise, τ has two adjacent complete simplices.

Proof
First, consider the simplex $\sigma^1 = \tau^1 = \{\underline{0}\}$. This is a J-complete simplex in $A(J)$ if and

only if $J = \{\phi(\underline{0})\}$. Since $\mathcal{T}(\{\phi(\underline{0})\})$ is a triangulation of $A(\{\phi(\underline{0})\})$ and τ^1 is a facet in the relative boundary of $A(\{\phi(\underline{0})\})$, there is a unique 1-simplex $\sigma^2 = \sigma(\underline{0}, q)$ in $A(\{\phi(\underline{0})\})$ such that τ^1 is a facet of σ^2. Either $\phi(q) = \phi(\underline{0})$ and $\tau^2 = \{q\}$ is a $\{\phi(\underline{0})\}$-complete simplex in $A(\{\phi(\underline{0})\})$, or $\phi(q) \neq \phi(\underline{0})$ and $\tau^2 = \sigma^2$ is a $\{\phi(\underline{0}), \phi(q)\}$-complete simplex in $A(\{\phi(\underline{0}), \phi(q)\})$. Hence, τ^1 has exactly one adjacent complete simplex.
Secondly, let $\tau^* = \tau(q^1, \ldots, q^t)$ be J-complete in $A(J)$ with $|J| = t$, while τ^* is a subset of $\overline{C}_\delta^{n+1}$, so τ^* lies in the relative boundary of $A(J)$. It is easily shown that τ^* cannot lie in $A(J')$ for a proper subset J' of J. Since $\mathcal{T}(J)$ is a triangulation of $A(J)$ there is a unique simplex $\sigma^* = \sigma(q^1, \ldots, q^{t+1})$ in $\mathcal{T}(J)$ containing τ^* as a facet. Either $\phi(q^{t+1}) \in J$ and σ^* has a unique J-complete facet in $A(J)$ not equal to τ^*, or $\phi(q^{t+1}) \notin J$ and σ^* is a $J \cup \{\phi(q^{t+1})\}$-complete simplex in $A(J \cup \{\phi(q^{t+1})\})$. Since τ^* does not lie in $A(J')$ for any proper subset J' of J this shows that τ^* has exactly one adjacent complete simplex.
Now let $\tau(q^1, \ldots, q^t)$ be a J-complete simplex in $A(J)$ with $|J| = t$, $\tau \neq \{\underline{0}\}$, and τ not being a subset of $\overline{C}_\delta^{n+1}$. There are two possibilities, either τ lies in $A(J')$ for some uniquely determined proper subset J' of J or τ does not lie in the relative boundary of $A(J)$. In the first case, by the properties of a triangulation, there is a unique t-simplex $\sigma(q^1, \ldots, q^{t+1})$ in $\mathcal{T}(J)$ having τ as a facet. As in the previous paragraph, either σ is $J \cup \{\phi(q^{t+1})\}$-complete in $A(J \cup \{\phi(q^{t+1})\})$ or σ has a J-complete facet $\tau' \neq \tau$ in $A(J)$. This yields exactly one adjacent complete simplex to τ. The other adjacent complete simplex is given by the unique J'-complete facet of τ. Hence, τ has exactly two adjacent complete simplices. In case τ does not lie in the relative boundary of $A(J)$, then by the properties of a triangulation there are exactly two different simplices in $\mathcal{T}(J)$ containing τ as a common facet, and as before this yields exactly two adjacent complete simplices to τ. It is easily verified that there can not be any other adjacent complete simplex to τ. $\square$

Theorem 5.4
Let a triangulation $\mathcal{T}$ of C_δ^{n+1} and a labelling function $\phi : C_\delta^{n+1} \to I_{n+1}$ be given. Then the procedure terminates, after generating a finite number of simplices in $\cup_{J \subset I_{n+1}} \mathcal{T}(J)$, in Step 2 of the procedure with a simplex having a J-complete facet in $A(J) \cap \overline{C}_\delta^{n+1}$ for some $J \subset I_{n+1}$.

Proof
Let $\sigma^1, \sigma^2, \ldots$ be the sequence of adjacent simplices generated by the procedure. Either the procedure terminates, after generating a finite number of simplices, in Step 2 with a t-simplex in $A(J)$ having a J-complete facet in $A(J) \cap \overline{C}_\delta^{n+1}$, or due to the finiteness of the number of simplices in $\cup_{J \subset I_{n+1}} \mathcal{T}(J)$, after a finite number of steps a J-complete simplex in $A(J)$ is generated which already has been generated before. However, applying the well-known door-in-door-out principle of Lemke and Howson [12] (see also Scarf [17])

it follows from Lemma 5.3 that each J-complete simplex in $A(J)$ can be visited at most once. Hence, the procedure must terminate. $\square$

So given any triangulation of C_δ^{n+1} the procedure generates a finite number, say M, of simplices $\sigma^1, \ldots, \sigma^M$ and a corresponding sequence of adjacent complete facets $\tau^1, \ldots, \tau^M$ with $\sigma^1 = \tau^1 = \{\underline{0}\}$, $\tau^j = \sigma^j \cap \sigma^{j+1}$, $\forall j \in I_{M-1}$, and $\tau^M = \sigma^M \cap \overline{C}_\delta^{n+1}$. The simplex σ^1 induces the trivial RDE$_{\underline{\alpha}}$ with complete demand rationing on all non-numeraire commodities. In the following theorem it is shown that the maximal absolute value of the total excess demand, $\|\widehat{z}(q)\|_\infty$, at any point q in any simplex generated by the procedure can be made arbitrarily small by taking the mesh size of the triangulation small enough.

Theorem 5.5

Let the economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \widetilde{r})$ satisfy the Assumptions 2.1-2.3. Then for every $\varepsilon > 0$, there exists $\gamma > 0$ such that for every triangulation $\mathcal{T}$ with $\mathrm{mesh}(\mathcal{T}) \leq \gamma$, for every point q in any simplex generated by the procedure it holds that $\|\widehat{z}(q)\|_\infty < \varepsilon$.

Proof

Let σ be any simplex generated by the procedure and take any point q' in σ. For some $J \subset I_{n+1}$, σ contains a J-complete simplex τ in $A(J)$ with vertices $q^1, \ldots, q^{|J|}$. It will be shown that $n+1 \in J$. Suppose not, then $q_{n+1} = 0$, for all $q \in \tau$. By Lemma 3.2 it holds then for any vertex q^h of τ that $\widehat{z}_j(q^h) \geq 0$, for all $j \in I_n$. By Lemma 3.1, $\widehat{p}(q^h)^\top \widehat{z}(q^h) = 0$ and hence $\widehat{z}_{n+1}(q^h) \leq 0$. So $\phi(q^h) = n+1$, a contradiction with $n+1 \notin J$. Moreover, for every $k \in J$ there exists some vertex q^h of τ such that $\widehat{z}_k(q^h) \leq 0$. If $k \in I_{n+1} \setminus J = I_n \setminus J$, then for every $q \in \tau$, $q_k = 0$, and by Lemma 3.1, $\widehat{z}_k(q) \leq 0$. Consequently, for every $j \in I_{n+1}$ there exists a point $q \in \tau$ with $\widehat{z}_j(q) \leq 0$. Define $\overline{\varepsilon} = \frac{\min_{j \in I_{n+1}} \widetilde{p}_j(\underline{\alpha})}{\sum_{j=1}^{n+1} \widetilde{p}_j(\underline{\alpha}/\delta)}\varepsilon$. Since $\widehat{z}$ is a continuous function on a compact set C_δ^{n+1} there exists $\gamma > 0$ such that for every $\widetilde{q}, \widehat{q} \in C_\delta^{n+1}$ it holds that $\|\widetilde{q} - \widehat{q}\|_\infty \leq \gamma$ implies $\|\widehat{z}(\widetilde{q}) - \widehat{z}(\widehat{q})\|_\infty < \overline{\varepsilon}$. Hence, $\mathrm{mesh}(\mathcal{T}) \leq \gamma$ implies $\widehat{z}_k(q') < \overline{\varepsilon} \leq \varepsilon$, for all $k \in I_{n+1}$. Since by Lemma 3.1, $\widehat{p}(q')^\top \widehat{z}(q') = 0$ it holds for every $k \in I_{n+1}$ that

$$\widehat{z}_k(q') = -\frac{\sum_{j \in I_{n+1} \setminus \{k\}} \widehat{p}_j(q') \widehat{z}_j(q')}{\widehat{p}_k(q')} > -\overline{\varepsilon} \frac{\sum_{j \in I_{n+1} \setminus \{k\}} \widehat{p}_j(q')}{\widehat{p}_k(q')} > -\varepsilon.$$

Hence, $\|\widehat{z}(q')\|_\infty < \varepsilon$. $\square$

The next corollary follows immediately from the fact that $\widehat{z}(\underline{0}) = \underline{0}$. The corollary implies that initially only the price level is increased.

Corollary 5.6
Let the economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \tilde{r})$ *satisfy the Assumptions 2.1-2.3. Then* $\phi(\underline{0}) = n+1$.

If $\|\widehat{z}(q)\|_\infty < \varepsilon$, then it is easily verified that $(\widehat{p}(q), \widehat{L}(q), \widehat{d}^1(q), \ldots, \widehat{d}^m(q))$ satisfies all properties of an ε-RDE$_{\widehat{\alpha}(q)}$, except possibly the requirement that demand rationing on commodities with a price below the maximum price or demand rationing on the numeraire commodity is non-binding. However, recall that we defined $\widehat{L}_j(q) = w_j$ if $q \in C^{n+1}$ and $q_j \geq 1$, and $\widehat{L}_{n+1}(q) = w_{n+1}$, for every $q \in C^{n+1}$. So, if $\varepsilon < \min_{i \in I_m} \min_{j \in I_{n+1}} w^i_j$, then for every consumer $i \in I_m$, for every commodity $j \in I_n$, $\widehat{d}^i_j(q) - w^i_j \leq \widehat{z}_j(q) + w_j - w^i_j \leq \varepsilon + w_j - w^i_j < w_j$, and an ε-RDE$_{\widehat{\alpha}(q)}$ is obtained. Define $\widehat{\varepsilon} = \min_{i \in I_m} \min_{j \in I_{n+1}} w^i_j$. Since $q_j \geq 1$ implies that $\widehat{L}_j(q) = w_j$, we now have that for every $\varepsilon < \widehat{\varepsilon}$, $\|\widehat{z}(q)\|_\infty < \varepsilon$ and $q \in \widehat{C}^{n+1}_\delta$ implies $(\widehat{p}(q), \widehat{L}(q), \widehat{d}^1(q), \ldots, \widehat{d}^m(q))$ is an ε-WE. We are now able to prove the next theorem, saying that there indeed exists a path of approximate equilibria.

Theorem 5.7
Let the economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \tilde{r})$ *satisfy the Assumptions 2.1-2.3. Then for every* $\varepsilon > 0$ *there exists a piecewise linear, continuous function* $\pi : [0,1] \to C^{n+1}_\delta$ *satisfying*
(i) $(\widehat{p}(\pi(0)), \widehat{L}(\pi(0)), \widehat{d}^1(\pi(0)), \ldots, \widehat{d}^m(\pi(0)))$ *is the trivial RDE*$_{\underline{\alpha}}$,
(ii) $(\widehat{p}(\pi(1)), \widehat{L}(\pi(1)), \widehat{d}^1(\pi(1)), \ldots, \widehat{d}^m(\pi(q)))$ *is an* ε*-WE, and*
(iii) $(\widehat{p}(\pi(t)), \widehat{L}(\pi(t)), \widehat{d}^1(\pi(t)), \ldots, \widehat{d}^m(\pi(t)))$ *is an* ε*-RDE*$_{\widehat{\alpha}(\pi(t))}$, *for all* $t \in [0,1]$.

Proof
Without loss of generality take $\varepsilon < \widehat{\varepsilon}$. Choose γ as in Theorem 5.5 and consider the sequence $\tau^1, \ldots, \tau^M$ of adjacent complete simplices obtained by using the procedure. Each simplex in this sequence is J-complete in $A(J)$ for some $J \subset I_{n+1}$. For $j \in I_M$, let b^j denote the barycentre of τ^j. Clearly, $b^1 = \underline{0}$. Since for every $j \in I_{M-1}$ the convex hull of the union of τ^j and τ^{j+1} equals σ^{j+1} and a simplex is convex, it holds that convex combinations of the barycentres of τ^j and τ^{j+1} are elements of σ^{j+1}. Let $N = M-1$ and define $\pi : [0,1] \to C^{n+1}_\delta$ by

$$\pi(t) = (1 - Nt + \lfloor Nt \rfloor) b^{\lfloor Nt \rfloor + 1} + (Nt - \lfloor Nt \rfloor) b^{\lfloor Nt \rfloor + 2}, \text{ for all } t \in [0,1],$$

where $\lfloor r \rfloor$ denotes for any real number r the greatest integer less than or equal to r. Notice that in case $t = 1$, $b^{N+2} = b^{M+1}$ can be taken equal to an arbitrary vector. Clearly, π is a continuous, piecewise linear function, $\pi(0)$ yields the trivial RDE$_{\underline{\alpha}}$, and for all $t \in [0,1]$, $\pi(t)$ induces an ε-RDE$_{\widehat{\alpha}(\pi(t))}$. It remains to be verified that $\pi(1)$ induces an ε-WE, or equivalently $\pi(1) \in \widehat{C}^{n+1}_\delta$. Clearly $\pi(1) \in \overline{C}^{n+1}_\delta$, so it is sufficient to show that $\max_{j \in I_n} \pi_j(1) < 2$ and $\pi_{n+1}(1) < 1 - \delta$. Let $q^1, \ldots, q^t$ be the vertices of τ^M. Suppose $\pi_{n+1}(1) = 1 - \delta$. Then since $\pi_{n+1}(1) = b^M$, being the barycentre of τ^M, it holds for every

$j \in I_t$ that $q^j_{n+1} = 1 - \delta$ and by Lemma 3.3 that $\hat{z}_{n+1}(q^j) > 0$, so $\phi(q^j) \neq n+1$. But then τ^M is J-complete in $A(J)$ for some J not containing $n+1$, implying $q^j_{n+1} = 0$, for all $j \in I_t$, a contradiction.

Finally, suppose $\pi_k(1) = 2$ for some $k \in I_n$. Then $k \in J$ and $q^h_k = 2$, for all $h \in I_t$. By Lemma 3.3 $\hat{z}_k(q^h) > 0$ and so $\phi(q^h) \neq k$, for all $h \in I_t$, yielding again a contradiction. Consequently, $\pi(1) = b^M \in \widehat{C}^{n+1}_\delta$. □

6 Real demand-constrained equilibria

So far the existence of a continuous piecewise linear path of ε-RDE$_\alpha$'s has been shown for every $\varepsilon > 0$. Moreover, this path has been constructed by applying the technique of simplicial approximation. In this final section the case $\varepsilon = 0$ will be considered. We conjecture that under suitable differentiability conditions on utility functions the path of points $q^* \in C^{n+1}_\delta$ satisfying $\hat{z}(q^*) = \underline{0}$ is generically a piecewise differentiable 1-manifold with boundary. Moreover, one of the components of this 1-manifold is homeomorphic to the unit interval and has two boundary points, $q^* = \underline{0}$ inducing the trivial RDE$_{\underline{\alpha}}$, and a point in $\widehat{C}^{n+1}_\delta$ inducing a WE. To substantiate this conjecture, notice that the first n components of the function $\hat{z}$ map an $(n+1)$-dimensional set into an n-dimensional set, thereby leaving one degree of freedom. Clearly, a zero point of the first n components of the function $\hat{z}$ yields a zero point of $\hat{z}$ by Walras' law.

In this section we will take another approach. We will not make any differentiability assumptions, instead we only make the Assumptions 2.1-2.3. The result, being that the set of points $q^* \in C^{n+1}_\delta$ satisfying $\hat{z}(q^*) = \underline{0}$ contains a component containing both the point $q^* = \underline{0}$ and a point in $\widehat{C}^{n+1}_\delta$, holds for every economy satisfying the previously mentioned assumptions. The proof of the result follows the approach of Herings [6]. Given an economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}^m_{i=1}, \tilde{r})$, define the set Q as

$$Q = \{q^* \in C^{n+1}_\delta \mid \hat{z}(q^*) = \underline{0}\}.$$

A topological space is connected if it is not the union of two non-empty, disjoint, closed sets. A subset of a topological space is connected if it becomes connected when given the induced topology. The component of a point in a topological space equals the union of all connected subsets of the topological space containing the point. It is not difficult to show that the component of a point is the largest connected subset of the topological space containing the point. The collection of components of a set partitions the set. For a non-empty compact set $S \subset \mathbb{R}^k$ define the distance function $g_S : \mathbb{R}^k \to \mathbb{R}$ by

$$g_S(\tilde{s}) = \min_{s \in S} \|s - \tilde{s}\|_\infty, \ \forall \tilde{s} \in \mathbb{R}^k.$$

It is easily shown that the function g_S is continuous. Let S^1 and S^2 be non-empty, compact subsets of $\mathbb{R}^k$. Define $e(S^1, S^2)$ by

$$e(S^1, S^2) = \min_{(s^1,s^2)\in S^1\times S^2} \|s^1 - s^2\|_\infty.$$

Obviously, S^1 and S^2 being disjoint implies $e(S^1, S^2) > 0$.

Theorem 6.1

Let the economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \tilde{r})$ satisfy the Assumptions 2.1-2.3. Then Q has a component containing $\underline{0}$ and an element in $\widehat{C}_\delta^{n+1}$, i.e., there exists a connected set of points in C_δ^{n+1} inducing a set of RDE$_\alpha$'s containing both the trivial RDE$_{\underline{\alpha}}$ and some WE.

Proof

Let π^r, $r \in \mathbb{N}$, denote a function π as defined in Theorem 5.7 satisfying $\|\widehat{z}(\pi^r(t))\|_\infty < \frac{1}{r}$, for all $t \in [0,1]$. Consider an accumulation point of the sequence $\{\pi^r(1)\}_{r\in\mathbb{N}}$, say q^*. Clearly, $q^* \in \widehat{C}_\delta^{n+1}$ and $\widehat{z}(q^*) = \underline{0}$, so q^* induces a WE. So, $\underline{0} \in Q$ and $q^* \in Q$. Exercise 4c of Section 5.1 in Munkres [13] (p. 235) states that the component of a point in a compact Hausdorff space equals the intersection of all sets containing the point which are both open and close in the compact Hausdorff space. Suppose q^* is not an element of the component of $\underline{0}$. From the fact that $\widehat{C}_\delta^{n+1}$ is a compact Hausdorff space when given the induced topology, it follows that there exist compact disjoint sets Q^1 and Q^2 such that $\underline{0} \in Q^1$, $q^* \in Q^2$, and $Q^1 \cup Q^2 = Q$. Hence, there exists $\varepsilon > 0$ such that $e(Q^1, Q^2) > \varepsilon$. Consider a subsequence $(\pi^{r_s})_{s\in\mathbb{N}}$ with $\|\pi^{r_s}(1) - q^*\|_\infty < \frac{\varepsilon}{2}$ for all $s \in \mathbb{N}$. For $s \in \mathbb{N}$ define the function $f^s : [0,1] \to \mathbb{R}$ by

$$f^s(t) = g_{Q^1}(\pi^{r_s}(t)) - g_{Q^2}(\pi^{r_s}(t)),\ \forall t \in [0,1].$$

By the continuity of the functions g_{Q^1}, g_{Q^2}, and π^{r_s} it follows that, for any $s \in \mathbb{N}$, the function f^s is continuous. Moreover, $f^s(0) < -\varepsilon$ and $f^s(1) > 0$. Let $t^s \in [0,1]$ satisfy $f^s(t^s) = 0$. Then $g_{Q^1}(\pi^{r_s}(t^s)) = g_{Q^2}(\pi^{r_s}(t^s)) = g_Q(\pi^{r_s}(t^s)) > \frac{\varepsilon}{2}$. Consider the sequence $(\pi^{r_s}(t^s))_{s\in\mathbb{N}}$ in the compact set C_δ^{n+1}. Without loss of generality $\lim_{s\to\infty} \pi^{r_s}(t^s)$ exists and is equal to, say, $\overline{\pi} \in C_\delta^{n+1}$. It holds that

$$\widehat{z}(\overline{\pi}) = \widehat{z}(\lim_{s\to\infty} \pi^{r_s}(t^s)) = \lim_{s\to\infty} \widehat{z}(\pi^{r_s}(t^s)) = \underline{0}.$$

Hence, $g_Q(\overline{\pi}) = 0$. Since

$$g_Q(\overline{\pi}) = g_Q(\lim_{s\to\infty} \pi^{r_s}(t^s)) = \lim_{s\to\infty} g_Q(\pi^{r_s}(t^s)) \geq \frac{\varepsilon}{2},$$

a contradiction is obtained. $\square$

Corollary 6.2

Let the economy $\mathcal{E} = (\{X^i, \succeq^i, w^i\}_{i=1}^m, \widetilde{r})$ satisfy the Assumptions 2.1-2.3. Then there exists a connected set of RDE$_\alpha$'s of $\mathcal{E}$ containing the trivial RDE$_{\underline{\alpha}}$ and some WE.

Proof

Consider the set of RDE$_{\widehat{\alpha}(q)}$'s

$$\{(\widehat{p}(q), \widehat{L}(q), \widehat{d}^1(q), \ldots, \widehat{d}^m(q)) \in \mathbb{R}_+^{n+1} \times \mathbb{R}_+^{n+1} \times \prod_{i=1}^m \mathbb{R}_+^{n+1} \mid q \in Q^0\},$$

with Q^0 the component of the set Q containing $\underline{0}$. By Theorem 6.1 the set above contains the trivial RDE$_{\underline{\alpha}}$ and a WE, and since the image of a connected set by a continuous function is connected, the corollary follows. □

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