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Robust open-loop Nash equilibria in the noncooperative LQ game revisited‡

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SUMMARY

This paper reconsiders existence of worst-case Nash equilibria in noncooperative multi-player differential games, this, within an open-loop information structure. We show that these equilibria can be obtained by determining the open-loop Nash equilibria of an associated differential game with an additional initial state constraint. For the special case of linear-quadratic differential games, we derive both necessary and sufficient conditions for solvability of the finite planning horizon problem. In particular, we demonstrate that, unlike in the standard linear-quadratic differential game setting, uniqueness of equilibria may fail to hold. A both necessary and sufficient condition under which there is a unique equilibrium is provided. A sufficient existence condition for a unique equilibrium is derived in terms of a Riccati differential equation. Consequences for control policies are demonstrated in a simple debt stabilization game. © 2016 The Authors. *Optimal Control Applications and Methods* published by John Wiley & Sons, Ltd.

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1. INTRODUCTION

In the last decades, there is an increased interest in studying diverse problems in economics, optimal control theory, and engineering using dynamic games. Particularly, the framework of linear-quadratic differential games is often used to analyze problems because of its analytic tractability. In environmental economics, marketing, and macroeconomic policy coordination, policy coordination problems are frequently modeled as dynamic games (see, e.g., the books and references in [2–6] and [7]). In optimal control theory, the derivation of robust control strategies (in particular, the H^∞ control problem) can be approached using the theory of (linear-quadratic zero-sum) dynamic games (see the seminal work of [8]). In the area of military operations, pursuit-evasion problems and, more recently, problems of defending assets can also be approached using linear-quadratic modeling techniques (see, e.g., [9–11]). Furthermore, this modeling paradigm has been used in the area of robot formation and communication networks (see, e.g., [12, 13]).

In this note, we consider the open-loop linear-quadratic differential game. This open-loop Nash strategy is often used as one of the benchmarks to evaluate outcomes of the game. Another benchmark that is often used is the state feedback strategy. Recently, Reference [14] compares both strategies to see what the loss in performance of players may be using either one of these strategies. For the scalar game (see the paper for precise details on the game), they find that if there is a large number of players involved in the game, the ratio of losses for an individual player under a feedback

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and open-loop information structure ranges between $\frac{\sqrt{2}}{2}$ and $\sqrt{2}$. This indicates, that the difference in performance using either one of the information structures is not dramatic.

The linear-quadratic differential game problem with an open-loop information structure has been considered by many authors and dates back to the seminal work of Starr and Ho in [15] (see, e.g., [16–23] and [4]). In [24], the (regular indefinite) infinite-planning horizon case for affine systems under the assumption that every player is capable to stabilize the system by his own was studied. These results were generalized in [25] where it is just assumed that the system as a whole is stabilizable.

References [26, 27], and [28] considered for a finite planning horizon the corresponding differential game problem for an open-loop information structure if the system is corrupted by deterministic noise. They introduced the notion of Nash/worst-case equilibrium to model noncooperative behavior in an uncertain environment. And showed that, under some assumptions, the problem has a solution. Implementation of the actions is, however, quite demanding.

In this note, we will relax their equilibrium concept and reconsider the problem from scratch. Following the standard analysis as presented, for example, in [4, Chapter 7], we will derive here both necessary and sufficient conditions for existence of equilibria. In particular, the conditions we obtain here allow for the formulation of sufficient conditions that can be easily implemented using standard methods. Another consequence of our definition is that we can show that situations occur where there exist an infinite number of Nash/worst-case equilibria. Necessary and sufficient conditions are given under which a unique equilibrium occurs.

The outline of the paper is as follows. In Section 2, we introduce the general (nonlinear) problem, formulate the equilibrium concept, and show that the problem can be reformulated as an associated extended differential game with an additional initial state constraint. This result is then used in Section 3 to study the linear-quadratic setting. Furthermore, we present an example where an infinite number of equilibria exist. Section 4 illustrates in a numerical example which consequences the explicit consideration of noise by players may have on policy in a multi-country debt stabilization game. In Section 5, we discuss some obtained results in more detail and raise some issues left for future research. Finally, in the Appendix, we present the proofs of the main results from Section 3.

2. AN EQUIVALENCE RESULT

In this paper, we consider the problem to find Nash equilibria for a differential game that is subject to deterministic noise. With $u(t) := \left[u_1^T(t) \cdots u_N^T\right]^T$, this game is defined by the cost functions

$$J_i(T, x_0, u, w) := \int_0^T g_i(x, u) - w^T(t) R_{wi} w(t) dt + g_{iT}(x(T)). \tag{1}$$

Here, R_{wi} is assumed to be positive definite (> 0), $i \in \bar{N}$[§], and $x(t) \in \mathbb{R}^n$ is the solution of the differential equation:

$$\dot{x}(t) = f(x(t), u(t)) + Dw(t),\ x(0) = x_0. \tag{2}$$

The function $w(.) \in \mathcal{W} := L_2^k[0, T]$ is an unknown disturbance. The controls $u_i(t) \in \mathbb{R}^{m_i}$ considered by player i, $i \in \bar{N}$, are assumed to be such that the control function $u(.)$ belongs to $\mathcal{U} := L_2^m[0, T]$[¶], where $m = m_1 + \cdots + m_N$. Functions f and g_i are such that for all admissible u, w, and x_0, the differential equation and integrals have a well-defined solution.

In this uncertain environment, every player wants to minimize his individual cost function J_i by choosing u_i appropriately. Because all players interact, without making any further specifications, the outcome of the game cannot be predicted. Preferably, every player will base his action on the actions taken by the other players in the game and his expectations concerning the disturbance that will occur. Therefore, depending on the information players have on the game, it is to be expected

[§] $\bar{N} := \{1, \ldots, N\}$

[¶] This set could be chosen in more general. However, this set suffices for most applications.

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that in the end, a set of actions will be chosen from which no individual player has an incentive to deviate. That is, a so-called set of Nash equilibrium actions will be played. We will analyze this problem here under the assumption that the game is played under an open-loop information structure [4, 22]. That is, based on the initial state of the system and system parameters, players will play actions that are just functions of time having the property that if played simultaneously, they constitute a Nash equilibrium. So the reaction of players on each other's action is enforced indirectly.

Inspired by [27], we introduce the next definition of global Nash/worst-case equilibrium.

Definition 2.1
We define the global Nash/worst-case equilibrium in two stages. Consider $u \in \mathcal{U}$, then

1. $\hat{w}_i(u) \in \mathcal{W}$ is the worst-case disturbance from the point of view of the i^{th} player against u if

$$J_i(u, \hat{w}_i(u)) \geqslant J_i(u, w)$$

   holds for each $w \in \mathcal{W}$, $i \in \bar{N}$.
2. Assume for all $u \in \mathcal{U}$, there exists a worst-case disturbance from the point of view of player i. The controls $\left(u_1^*, \ldots, u_N^*\right) \in \mathcal{U}$ form a global Nash/worst-case equilibrium if for all $i \in \bar{N}$,

$$J_i\left(u^*, \hat{w}_i(u^*)\right) \leqslant J_i\left(\left(u_i, u_{-i}^*\right), \hat{w}_i\left(u_i, u_{-i}^*\right)\right)$$

   holds for each admissible control function $\left(u_i, u_{-i}^*\right)$‖ and corresponding worst-case disturbance $\hat{w}_i\left(u_i, u_{-i}^*\right)$.

The aforementioned definition reflects the idea that every player wants to secure against a for him worst-case realization of the disturbance. Matrix R_{wi} models his expectation about the disturbance and can be interpreted as a risk aversion parameter. In case he expects that only a small disturbance $Dw(.)$ might disrupt the system, he can express this by choosing R_{wi} large. A Nash/worst-case equilibrium models then a situation where every player has no incentive to change his policy given his worst-case expectations concerning the disturbance and the actions of his opponents. Clearly, in a situation where players can observe the realization of the disturbance and they can adapt their actions during the game, other solution concepts like the soft-constrained feedback Nash equilibrium (see, e.g., [4, Chapter 9]) are more appropriate.

Remark 2.2
The definition of global Nash/worst-case equilibrium differs from the definition given by Jank *et al.* [27] in that our definition assumes that for all admissible controls u, a worst-case disturbance exists. This assumption is not made in [27]. There it is assumed that only for all controls $\left(u_i, u_{-i}^*\right) \in \mathcal{U}$ from player $i's$ point of view, a worst-case disturbance exists.

Clearly, in case there exists a global Nash/worst-case equilibrium, it is also a Nash/worst-case equilibrium in the sense defined by Jank *et al*. Of course the other way around does not necessarily have to be the case.

Later, we show that global Nash/worst-case equilibria can be identified with open-loop Nash equilibria of an associated extended game. This makes our definition more practicable. In particular, this relationship enhances to formulate sufficient existence conditions for the linear-quadratic differential game, which are numerically more tractable than the conditions given by [27] and [28]. For convenience of notation, we will omit from now on the phrase 'global' in our definition, unless there might be confusion with the definition provided by Jank *et al*.

Next, Theorem 2.3 shows that Nash/worst-case equilibria can be determined as the open-loop Nash (OLN) equilibria of an associated extended differential game.

‖ (v, u_{-i}) equals u where entry u_i is replaced by v. To simplify notation, sometimes, the brackets are dropped.

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Theorem 2.3
$u^* \in \mathcal{U}$ with corresponding worst-case disturbances $\hat{w}_i(u^*)$ is a Nash/worst-case equilibrium for (1,2) if and only if $(u^*, \hat{w}^e(u^*))$ (with $\hat{w}^e(u^*) := (\hat{w}_1(u^*), \cdots, \hat{w}_N(u^*)))$ is an open-loop Nash equilibrium of the $2N$-player differential game:

$$\begin{bmatrix} \dot{\bar{x}}_1(t) \\ \vdots \\ \dot{\bar{x}}_N(t) \end{bmatrix} = \begin{bmatrix} f(\bar{x}_1(t), u) + Dw_1(t) \\ \vdots \\ f(\bar{x}_N(t), u) + Dw_N(t) \end{bmatrix}, \ \bar{x}^T(0) := \left(\bar{x}_1^T(0), \ldots, \bar{x}_N^T(0)\right) = \left(x_0^T, \ldots, x_0^T\right),$$

where player i likes to minimize his cost function:

$$\bar{J}_i(u, w^e) := \int_0^T g_i(\bar{x}_i(t), u(t)) - w_i^T(t) R_{wi} w_i(t) dt + g_{iT}(\bar{x}_i(T)),$$

w.r.t. u_i, $i \in \bar{N}$, and player i, $i = N+1, \ldots, 2N$, likes to maximize $\bar{J}_i$ w.r.t. w_i.
Moreover, $J_i\,(u^*, \hat{w}(u^*)) = \bar{J}_i\,(u^*, \hat{w}^e(u^*))$, $i \in \bar{N}$.

Proof
$\Rightarrow$ Assume u^* is a Nash/worst-case equilibrium. Then, by definition, $\left(u_i^*, \hat{w}_i(u^*)\right)$ constitutes a saddle-point solution for player i if the other players, j, play u_j^*. Consequently, (see, e.g., [4, Theorem 3.26]),

$$J_i\left(u^*, \hat{w}_i(u^*)\right) = \max_w \min_{u_i} J_i\left(\left(u_i, u_{-i}^*\right), w\right) = \min_{u_i} \max_w J_i\left(\left(u_i, u_{-i}^*\right), w\right). \tag{3}$$

Next, consider the minimization of $\bar{J}_i\left(\left(u_i, u_{-i}^*\right), \hat{w}^e(u^*)\right)$ w.r.t. u_i. Some elementary rewriting shows that this is equivalent to the minimization of

$$\tilde{J}_i := \int_0^T \left\{g_i\left(x_i(t), \left(u_i(t), u_{-i}^*(t)\right)\right) - \hat{w}_i^T(u^*)(t) R_{wi} \hat{w}_i(u^*)(t)\right\} dt + g_{iT}(x_i(T))$$

subject to the system

$$\dot{x}_i(t) = f\left(x_i(t), \left(u_i(t), u_{-i}^*(t)\right)\right) + D\hat{w}_i(u^*)(t), \text{ with } x_i(0) = x_0.$$

By (3), this minimum is attained at $u_i = u_i^*$.
Similarly, we have that the maximization of $\bar{J}_i\left(u^*, \left(w_i, \hat{w}_{-i}^e(u^*)\right)\right)$ w.r.t. w_i is equivalent to the maximization of

$$\tilde{J}_i := \int_0^T g_i(x_i(t), u^*(t)) - w_i^T(t) R_{wi} w_i(t) dt + g_{iT}(x_i(T))$$

subject to the system

$$\dot{x}_i(t) = f(x_i(t), u^*(t)) + Dw_i(t), \ x_i(0) = x_0.$$

From (3) again, it follows that this maximum is attained at $w_i = \hat{w}_i(u^*)$. So $(u^*, \hat{w}^e(u^*))$ is an OLN equilibrium for the $2N$ player differential game.
$\Leftarrow$ Let (u^*, w^{e^*}) be an OLN equilibrium for the $2N$ player differential game. Then

$$\bar{J}_i(u^*, w^{e^*}) \leqslant \bar{J}_i\left(\left(u_i, u_{-i}^*\right), w^{e^*}\right) \text{ for all admissible } u_i, \tag{4}$$

$$\bar{J}_i(u^*, w^{e^*}) \geqslant \bar{J}_i\left(u^*, \left(w_i, w_{-i}^{e^*}\right)\right) \text{ for all admissible } w_i. \tag{5}$$

Now, consider the maximization of $J_i(u^*, w)$ w.r.t. w subject to the system

$$\dot{x}(t) = f(x(t), u^*(t)) + Dw(t), \ x(0) = x_0.$$

A simple elaboration of (5) shows that this maximum is attained at $w = w_i^*$. So $\hat{w}_i(u^*) = w_i^*$. Now, let (u_i, u_{-i}^*) be an arbitrary admissible control and $\hat{w}(u_i, u_{-i}^*))$ a corresponding worst-case control. Then, by definition of worst-case control, $J_i((u_i, u_{-i}^*), \hat{w}(u_i, u_{-i}^*)) \geqslant J_i((u_i, u_{-i}^*), \hat{w}(u^*))$. Therefore, in particular,

$$J_i\left(\left(u_i, u_{-i}^*\right), \hat{w}\left(u_i, u_{-i}^*\right)\right) \geqslant J_i\left(\left(u_i, u_{-i}^*\right), \hat{w}(u^*)\right) \geqslant \min_{u_i} J_i\left(\left(u_i, u_{-i}^*\right), \hat{w}(u^*)\right).$$

From (4), it follows that this last mentioned minimum exists and is attained at u_i^*. That is, $\left(u^*, w_i^*\right)$ is a Nash/worst-case equilibrium.

Finally, that $J_i(u^*, \hat{w}(u^*)) = \bar{J}_i(u^*, \hat{w}^e(u^*))$, $i \in \bar{N}$, follows by a direct comparison of both functions. □

Note that in the aforementioned introduced differential game, the initial condition has a special structure. This usually complicates the analysis.

3. LINEAR QUADRATIC (LQ) NASH/WORST-CASE EQUILIBRIA

In this section, we elaborate the linear-quadratic differential game. This game is defined by the cost functions

$$J_i(T, x_0, u, w) := \frac{1}{2}\int_0^T [x^T(t), u^T(t)]M_i[x^T(t), u^T(t)]^T - w^T(t)R_{wi}w(t)dt + \frac{1}{2}x^T(T)Q_{iT}x(T),$$

$$\text{where } M_i = \begin{bmatrix} Q_i & V_{i11} & \cdots & \cdots & V_{i1N} \\ V_{i11}^T & R_{i1} & V_{i22} & \cdots & V_{i2N} \\ & & \ddots & & \\ V_{i1N}^T & V_{i2N}^T & \cdots & \cdots & R_{iN} \end{bmatrix} \in \mathbb{R}^{(n+m)\times(n+m)}. \tag{6}$$

M_i is assumed to be symmetric, $R_{ii} > 0$ and $R_{wi} > 0$, $i \in \bar{N}$. Note, we do not make definiteness assumptions on matrices Q_i.

Furthermore, $x(t) \in \mathbb{R}^n$ is the solution of the linear differential equation:

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^N B_i u_i(t) + Dw(t),\ x(0) = x_0. \tag{7}$$

From Theorem 2.3, we immediately derive then next Corollary 3.2. In this corollary, we use the next notation.

Notation 3.1
$\hat{E}_{Nn}$ denotes the block-column matrix containing N blocks of $n \times n$ identity matrices and $\bar{E}_{i,j}$ the block-column matrix containing i blocks of $n \times n$ zero matrices with block number matrix j replaced by the identity matrix. Matrices I and 0 (where sometimes we use an index to indicate the size of these matrices) denote the identity matrix and zero matrix of appropriate size, respectively. $\text{diag}(A)_N$ denotes the $N \times N$ block diagonal matrix with diagonal entries matrix A.

$$\bar{A}_N = \text{diag}(A)_N,\ \bar{B}_i = \hat{E}_{Nn}B_i,\ \bar{D}_i = \bar{E}_{N,i}D,\ \bar{c}(t) = \hat{E}_{Nn}c(t),\ \bar{M}_i = \begin{bmatrix} \bar{E}_{N,i}^T & 0 \\ 0 & I_m \end{bmatrix}^T M_i \begin{bmatrix} \bar{E}_{Ni}^T & 0 \\ 0 & I_m \end{bmatrix},$$

$\bar{Q}_{iT} = \bar{E}_{N,i}Q_{iT}\bar{E}_{N,i}^T$ and $\bar{x}_0 = \hat{E}_{Nn}x_0$.

Corollary 3.2
Assume for all $u \in \mathcal{U}$, there exists a worst-case disturbance from the point of view of player i. Then, $u^* \in \mathcal{U}$ with corresponding worst-case disturbances $\hat{w}_i(u^*)$ is a Nash/worst-case equilibrium for (6,7) if and only if $(u^*, \hat{w}(u^*))$ is an open-loop Nash equilibrium of the $2N$-player differential game:

$$\dot{\bar{x}}(t) = \bar{A}_N\bar{x}(t) + \sum_{i=1}^N \bar{B}_i u_i(t) + \sum_{i=1}^N \bar{D}_i w_i(t),\ \bar{x}(0) = \bar{x}_0,$$

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where player i likes to minimize his cost function:

$$\bar{J}_i := \frac{1}{2}\int_0^T [\bar{x}^T(t),\ u^T(t)]\bar{M}_i[\bar{x}^T(t),\ u^T(t)]^T - w_i^T(t)R_{wi}w_i(t)dt + \frac{1}{2}\bar{x}^T(T)\bar{Q}_{iT}\bar{x}(T),$$

w.r.t. u_i, $i \in \bar{N}$, and player i, $i = N+1,\ldots,2N$, likes to maximize $\bar{J}_i$ w.r.t. w_i. Moreover, $J_i(u^*, \hat{w}(u^*)) = \bar{J}_i(u^*, \hat{w}^e(u^*))$, $i \in \bar{N}$.

3.1. Necessary and sufficient conditions

Before we present results for the general case, for didactic reasons, we first consider the next simplified two-player linear-quadratic differential game.

Consider the case that the system state is described as the outcome of the linear differential equation:

$$\dot{x}(t) = Ax(t) + B_1u_1(t) + B_2u_2(t) + Dw(t), \text{ with } x(0) = x_0, \tag{8}$$

whereas the quadratic cost functional of the two players is given by the following:

$$\begin{aligned} J_i(u_1,u_2,w) := &\frac{1}{2}\int_0^T x^T(t)Q_ix(t) + \sum_{j=1}^{2} u_j^T(t)R_{ij}u_j(t) - w^T(t)R_{wi}w(t)dt \\ &+ \frac{1}{2}x^T(T)Q_{iT}x(T),\ i = 1,2. \end{aligned} \tag{9}$$

Using the shorthand notation $S_i := B_iR_{ii}^{-1}B_i^T$, $S_{D_i} := DR_{wi}^{-1}D^T$, $S := \hat{E}_{2n}[S_1\ S_2] - \text{diag}(S_{D_i})$, $Q := \text{diag}(Q_i)$, $Q_T := \text{diag}(Q_{iT})$, $\breve{Q}_T^T := [I_{2n}\ Q_T]$ and H^+ for the Moore–Penrose inverse of matrix H (see, e.g., [29]), the following theorem is proved in the Appendix.

Theorem 3.3
Consider matrix

$$M := \begin{bmatrix} A & 0 & -(S_1 - S_{D_1}) & -S_2 \\ 0 & A & -S_1 & -(S_2 - S_{D_2}) \\ -Q_1 & 0 & -A^T & 0 \\ 0 & -Q_2 & 0 & -A^T \end{bmatrix}. \tag{10}$$

Assume next four Riccati differential equations:

$$\dot{K}_i(t) = -A^TK_i(t) - K_i(t)A + K_i(t)S_iK_i(t) - Q_i,\ K_i(T) = Q_{iT},\ i = 1,2 \tag{11}$$

$$\dot{L}_i(t) = -A^TL_i(t) - L_i(t)A + L_i(t)S_{D_i}L_i(t) + Q_i,\ L_i(T) = -Q_{iT},\ i = 1,2 \tag{12}$$

have a symmetric solution $K_i(.)$, $L_i(.)$, respectively, on $[0,T]$.

Then, the two-player linear-quadratic differential game (8,9) has a Nash/worst-case equilibrium for every initial state x_0 if and only if

$$\text{with matrix } H(T) := [I_{2n}\ 0_{2n}]e^{-MT}\breve{Q}_T,\ H(T)H^+(T)\begin{bmatrix} I \\ I \end{bmatrix} = \begin{bmatrix} I \\ I \end{bmatrix}. \tag{13}$$

Moreover, if the aforementioned condition (13) applies, with $v(t) := e^{M(t-T)}\breve{Q}_Tz_1$, where $z_1 := H^+\begin{bmatrix} x_0 \\ x_0 \end{bmatrix} + (I - H^+H)q$, $q \in \mathbb{R}^{2n}$ is an arbitrary vector, the set of equilibrium actions/worst-case disturbances are given by the following:

$$u_i(t) = -R_{ii}^{-1}B_i^T\bar{E}_{4,i+2}^Tv(t) \text{ and } w_i(t) = R_{wi}^{-1}D^T\bar{E}_{4,i+2}^Tv(t),\ i = 1,2, \text{respectively.}$$

The set of equilibrium actions is unique if matrix $H(T)$ is invertible. In such case, the unique equilibrium actions can be calculated either from the aforementioned equations or from the linear two-point boundary value problem $\dot{y}(t) = My(t)$, with

$$\bar{P}y(0) + \bar{Q}y(T) = \left[x_0^T \; x_0^T \; 0 \; 0\right]^T. \tag{14}$$

Here,

$$\bar{P} := \begin{bmatrix} I_{2n} & 0_{2n} \\ 0_{2n} & 0_{2n} \end{bmatrix} \text{ and } \bar{Q} := \begin{bmatrix} 0_{2n} & 0_{2n} \\ -Q_T & I_{2n} \end{bmatrix}.$$

Denoting $\left[y_0^T(t),\; v_1^T(t),\; v_2^T(t)\right]^T := y(t)$, with $y_0 \in \mathbb{R}^{2n}$, and $v_i \in \mathbb{R}^n,\; i = 1, 2$, the equilibrium actions are the following:

$$u_i(t) = -R_{ii}^{-1}B_i^T v_i(t) \text{ and } w_i(t) = R_{wi}^{-1}D^T v_i(t),\; i = 1, 2, \text{ respectively.}$$

It is well known that one can associate Riccati differential equations with boundary value problems of the type (14). Furthermore, solutions of these Riccati equations can be used to implement the open-loop control as a feedback control. Below, in Theorem 3.4, we present a sufficient condition in terms of existence of a solution of a Riccati differential equation for the aforementioned boundary value problem. In fact, this condition is both necessary and sufficient for existence of the more general problem where the term of $\left[x_0^T \; x_0^T\right]$ in the initial condition of the boundary value problem (14) is replaced by $\left[x_0^T \; z_0^T\right]$, where both x_0 and z_0 are arbitrary vectors. For the readers' convenience, we included a proof of this result.

Theorem 3.4
Assume the four Riccati differential equations ((11) and (12)) have a solution, and the next nonsymmetric Riccati differential equation has a solution on $[0, T]$:

$$\dot{P}(t) = -\bar{A}_2^T P - P\bar{A}_2 + PSP - Q;\; P(T) = Q_T. \tag{15}$$

Then, the two-player linear-quadratic differential game ((8) and (9)) has a Nash/worst-case equilibrium for every initial state x_0. Moreover, if the aforementioned condition applies, the unique equilibrium actions are given by the following:

$$u_i(t) = -R_{ii}^{-1}B_i^T P_i(t)\bar{x}(t) \text{ and } w_i(t) = R_{wi}^{-1}D^T P_i(t)\bar{x}(t),\; i = 1, 2 \text{ respectively,}$$

where $\left[P_1^T(t) \; P_2^T(t)\right]^T := P(t),\; P_i(t) \in \mathbb{R}^{n\times 2n}$, solve (15), and $\bar{x}(t)$ is the solution of

$$\dot{\bar{x}}(t) = \left(\bar{A}_2 - SP(t)\right)\bar{x}(t), \text{ with } \bar{x}(0) = \bar{x}_0.$$

Proof
First, note that, with $P(.)$ the solution of (15), matrix $U(t)$ defined as the solution of the linear differential equation below exists and is, moreover, invertible on $[0, T]$.

$$\dot{U}(t) = (\bar{A}_2 - SP(t))U(t),\; U(0) = I.$$

Furthermore, with $V(t) := P(t)U(t)$, $W(t) := [\,U^T(t) \;\; V^T(t)\,]^T$ satisfies

$$\begin{aligned}
\dot{W}(t) &= \begin{bmatrix} \dot{U}(t) \\ \dot{P}(t)U(t) + P(t)\dot{U}(t) \end{bmatrix} \\
&= \begin{bmatrix} \bar{A}_2U(t) - SV(t) \\ \left(-\bar{A}_2^T P(t) - P(t)\bar{A}_2 + P(t)SP(t) - Q\right)U(t) + P(t)\bar{A}_2U(t) - P(t)SP(t)U(t) \end{bmatrix} \\
&= \begin{bmatrix} \bar{A}_2U(t) - SV(t) \\ -QU(t) - \bar{A}_2^T V(t) \end{bmatrix} = MW(t),
\end{aligned}$$

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whereas $\bar{P}W(0) + \bar{Q}W(T) = [\, I_{2n} \;\; 0_{2n} \,]^T$. With $W(0) = e^{-MT}W(T)$, spelling this last equality yields

$$[\, I_{2n} \;\; 0_{2n} \,]e^{-MT}W(T) = I_{2n} \text{ and } -Q_TU(T) + V(T) = 0_{2n}.$$

Substitution of $V(T)$ from the second into the first equation shows then that $[\, I_{2n} \;\; 0_{2n} \,]e^{-MT}\breve{Q}_TU(T) = I$, from which we conclude that $H(T)$ must be invertible. The advertized result follows then by Theorem 3.3. □

Example 3.5
In this example, we present a game that has for every initial state an infinite number of Nash/worst-case actions. Note that numerical values presented later for Q_{iT} are approximations of values that were obtained after some extensive theoretical calculations.
Consider the scalar game, with $A = -3$, $B_i = D = Q_i = 1$, $R_{11} = R_{22} = 1$, $R_{w1} = R_{w2} = 1/5$, $Q_{1T} = Q_{2T} = 1.014925$, and $T = 1$.

The corresponding Riccati differential equations ((11) and (12)) are

$$\dot{k}_i(t) = 6k_i(t) + k_i^2(t) - 1,\; k_i(1) = 1.014925;\; i = 1, 2$$
$$\dot{l}_i(t) = 6l_i(t) + 5l_i^2(t) + 1,\; l_i(1) = -1.014925;\; i = 1, 2.$$

Elementary analysis shows that all four differential equations have a solution on the time interval [0, 1].

Matrix $H(T)$, introduced in (13), equals $2.8647\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Consequently, $H^+ = 0.0873\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. It is easily verified that $H(T)H^+(T)\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. So, by Theorem 3.3, for every x_0, this game has an infinite number of Nash/worst-case actions.

With $z_1 := H^+\begin{bmatrix} x_0 \\ x_0 \end{bmatrix} + (I - H^+H)\mu = 0.1745\begin{bmatrix} 1 \\ 1 \end{bmatrix}x_0 + \begin{bmatrix} 1 \\ -1 \end{bmatrix}\mu,\; \mu \in \mathbb{R}$, we obtain

$$v(t) = e^{M(t-T)}\breve{Q}_Tz_1 = S\,\text{diag}\left(e^{2(t-1)}, e^{-2(t-1)}, e^{\sqrt{6}(t-1)}, e^{-\sqrt{6}(t-1)}\right)z_2,$$

where $S = \begin{bmatrix} -1 & -5 & 3-\sqrt{6} & 3+\sqrt{6} \\ 1 & 5 & 3-\sqrt{6} & 3+\sqrt{6} \\ -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ and $z_2 = \begin{bmatrix} (1-5q)\mu_2 \\ (-1+q)\mu_2 \\ 0.1614x_0 \\ 0.0157x_0 \end{bmatrix}$, with $q = Q_{1T}$ and $\mu_2 \in \mathbb{R}$.

So with $v_3(t) := [-e^{2(t-1)},\; -e^{-2(t-1)},\; e^{\sqrt{6}(t-1)},\; e^{-\sqrt{6}(t-1)}]$ and $v_4(t) := [e^{2(t-1)}, e^{-2(t-1)}, e^{\sqrt{6}(t-1)},\; e^{-\sqrt{6}(t-1)}]$, equilibrium/worst-case actions are

$$u_i^*(t) = -v_{i+2}(t)z_2 \text{ and } w_i^*(t) = -5u_i^*(t),\; \text{respectively},\; i = 1, 2.$$

Notice that using these equilibrium actions, the dynamics of the system are described by the following:

$$\dot{x}(t) = -3x(t) - 0.3228e^{\sqrt{6}(t-1)}x_0 - 0.0314e^{-\sqrt{6}(t-1)}x_0 + w(t),\; x(0) = x_0.$$

That is, all equilibrium actions yield the same closed-loop system.

Remark 3.6
It is easily seen that for the N-player case, Theorem 3.3 applies with

$$M = \begin{bmatrix} \bar{A}_N & -S \\ -Q & -A^T{}_N \end{bmatrix} \text{ and } H(T) = [\, I_{Nn} \;\; 0_{Nn} \,]e^{-MT}\breve{Q}_T.$$

Here, $S := \hat{E}_{Nn}[S_1, \ldots, S_N] - \text{diag}(S_{D_i})$ and $\breve{Q}_T^T := [\, I_{Nn} \;\; Q_T \,]$.

3.2. The general case

In this section, we consider the general linear-quadratic differential game (6) and (7).

To obtain the analogs of the previous sections, first, we rewrite the game from Corollary 3.2 into its standard form. We consider the $2N$-player differential game

$$\dot{\bar{x}}(t) = \bar{A}_N \bar{x}(t) + \sum_{i=1}^{N} \bar{B}_i u_i(t) + \sum_{i=1}^{N} \bar{D}_i w_i(t),\ \bar{x}(0) = \bar{x}_0,$$

where player i likes to minimize (w.r.t. u_i) and player $N+i$ to maximize (w.r.t. w_i) the cost function

$$\bar{J}_i := \frac{1}{2}\int_0^T \left[\bar{x}^T(t),\ u^T(t)\ w^T(t)\right]\ \bar{M}_i^e\ \left[\bar{x}^T(t),\ u^T(t)\ w^T(t)\right]^T dt + \frac{1}{2}\bar{x}^T(T)\bar{Q}_{iT}\bar{x}(T),\ i \in \mathbf{N}.$$

Here,

$$\bar{M}_i^e := \begin{bmatrix} \bar{M}_i & 0 \\ 0 & -\bar{R}_{wi} \end{bmatrix},\ i \in \mathbf{N},$$

where $\bar{R}_{wi}$ is the block diagonal matrix where only block i, which equals R_{wi}, differs from zero.

As a follow-up on the notation introduced in Notation 3.1, following [4][Exercise 7.5], consider the next shorthand notation. Standard conventions concerning block-matrix addition and multiplication rules are assumed.

Notation 3.7

$$G := \begin{bmatrix} [0\ I\ 0\cdots 0]\, M_1 \\ \vdots \\ [0\, 0\cdots\ 0\ I]\, M_N \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ I & 0 & \cdots & 0 \\ 0 & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & I \end{bmatrix} = \begin{bmatrix} R_{11} & V_{122} & \cdots & \cdots & V_{12N} \\ V_{222}^T & R_{22} & V_{233} & \cdots & V_{23N} \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & V_{(N-1)NN} \\ V_{N2N}^T & \cdots & \cdots & V_{NNN}^T & R_{NN} \end{bmatrix}.$$

We assume throughout that this matrix G is invertible.

$$\begin{aligned} &B := [\bar{B}_1, \ldots, \bar{B}_N];\ \tilde{B}^T := \operatorname{diag}\left(B_1^T, \ldots, B_N^T\right);\ Z := \operatorname{diag}\left(V_{111}^T, \ldots, V_{N1N}^T\right); \\ &Z_i := [V_{i11}, \ldots, V_{i1N}],\ i \in \mathbf{N};\ \tilde{A}_1 := \bar{A}_N - BG^{-1}Z;\ \tilde{S} := BG^{-1}\tilde{B}^T - \operatorname{diag}(S_{D_i}); \\ &\tilde{Q} := Q - \begin{bmatrix} Z_1 \\ \vdots \\ Z_N \end{bmatrix} G^{-1}Z;\ \tilde{A}_2^T := \bar{A}_N^T - \begin{bmatrix} Z_1 \\ \vdots \\ Z_N \end{bmatrix} G^{-1}\tilde{B}^T;\ \text{and } \tilde{M} := \begin{bmatrix} \tilde{A}_1 & -\tilde{S} \\ -\tilde{Q} & -\tilde{A}_2^T \end{bmatrix}. \end{aligned}$$

In the Appendix, we show how one arrives then at corresponding results of the previous section. Because from a computational point of view, Theorem 3.4 is the most important result from the previous section; we state this generalization here separately.

Theorem 3.8

Assume next Riccati differential equations

$$\begin{aligned} &\dot{K}_i(t) = -A^T K_i(t) - K_i(t)A + (K_i(t)B_i + V_{i1i})R_{ii}^{-1}(B_i^T K_i(t) + V_{i1i}^T) - Q_i,\ K_i(T) = Q_{iT},\ i \in \mathbf{N}, \\ &\dot{L}_i(t) = -A^T L_i(t) - L_i(t)A + L_i(t)S_{D_i}L_i(t) + Q_i,\ L_i(T) = -Q_{iT},\ i \in \mathbf{N}, \end{aligned}$$

have a symmetric solution $K_i(.),\ L_i(.)$, respectively, on $[0, T]$.

If the next nonsymmetric Riccati differential equation

$$\dot{P}(t) = -\tilde{A}_2^T P - P\tilde{A}_1 + P\tilde{S}P - \tilde{Q};\ P(T) = Q_T, \tag{16}$$

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has a solution on $[0, T]$, the linear-quadratic differential game ((6) and (7)) has a unique Nash/worst-case equilibrium for every initial state x_0.

If $P^T(.) := \left[P_1^T(.), \ldots, P_N^T(.)\right],\ P_i(.) \in \mathbb{R}^{n \times nN}$, is the solution of (16), the worst-case controls and corresponding worst-case disturbances are given by the following:

$$u^*(t) = -G^{-1}(Z + \tilde{B}^T P(t))\bar{x}(t) \text{ and } \hat{w}_i(t) = R_{wi}^{-1} D^T P_i(t)\bar{x}(t),\ i \in \mathbf{N}.$$

Here, $\bar{x}(t)$ satisfies, with $A_{cl}(t) := \bar{A}_N - BG^{-1}(Z + \tilde{B}^T P(t)) + \text{diag}(S_{D_i})P(t)$, the differential equation $\dot{\bar{x}}(t) = A_{cl}(t)\bar{x}(t);\ \bar{x}(0) = \bar{x}_0$.

Moreover, worst-case expected costs by player i are

$$J_i = \frac{1}{2}\bar{x}_0^T \bar{C}_i(0)\bar{x}_0,\ i \in \mathbf{N},$$

where $\bar{C}_i(.),\ i \in \mathbf{N}$, is the solution of the linear matrix differential equation:

$$\begin{aligned}\dot{\bar{C}}_i(t) = & -A_{cl}^T(t)\bar{C}_i(t) - \bar{C}_i(t)A_{cl}(t) - \left[I, -(Z + \tilde{B}^T P(t))^T G^{-T}\right]\bar{M}_i\left[I, -(Z + \tilde{B}^T P(t))^T G^{-T}\right]^T \\ & - P_i^T(t) S_{D_i} P_i(t), \\ \bar{C}_i(T) = & \ \bar{Q}_{iT}.\end{aligned}$$

Notice that the aforementioned expected worst-case costs will almost never realize for every player, as only one disturbance signal $w(.)$ will occur.

4. EXAMPLE

Consider the problem to find equilibrium strategies for the next noncooperative three-player game. The game might be interpreted as a debt stabilization problem within a two-country setting, where countries engaged in a monetary union. The model can be viewed as the first extension of the well-known Tabellini model [30] to a two-country setting, including uncertainty. Within that setting, the variables $x_i(t)$, introduced later, can be interpreted as the government debt, scaled to the level of national output, of country i. u_i as the primary fiscal deficit, also scaled to output, whereas the monetary financing undertaken by the central bank, measured as a fraction of aggregate output, will be denoted by u_E. It is assumed that uncertainty is caused by some outside factor that will hit the economies of both countries in the same way. This is modeled by incorporating into the system a disturbance variable $w(t)$. All parameters, below, are assumed to be positive. The welfare loss-function of country $i,\ i = 1, 2$, and central bank respectively are modeled by the following:

$$J_1(u_1(.)) = \frac{1}{2}\int_0^T u_1^2(t) + \beta_1 x_1^2(t) - r_{w1}w^2(t)dt, \tag{17}$$

$$J_2(u_2(.)) = \frac{1}{2}\int_0^T u_2^2(t) + \beta_2 x_2^2(t) - r_{w2}w^2(t)dt, \tag{18}$$

$$J_E(u_E(.)) = \frac{1}{2}\int_0^T u_E^2(t) + \beta_E(\omega x_1(t) + (1-\omega)x_2(t))^2 - r_{wE}w^2(t)dt. \tag{19}$$

Here, r_{wi} models the expectation by different players how strong the disturbance will hit the economies. The evolution of debt in both countries over time is assumed to be described by the differential equations:

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix}u_1(t) + \begin{bmatrix}0\\1\end{bmatrix}u_2(t) + \begin{bmatrix}-\gamma_1\\ -\gamma_2\end{bmatrix}u_E(t) + \begin{bmatrix}1\\1\end{bmatrix}w(t). \tag{20}$$

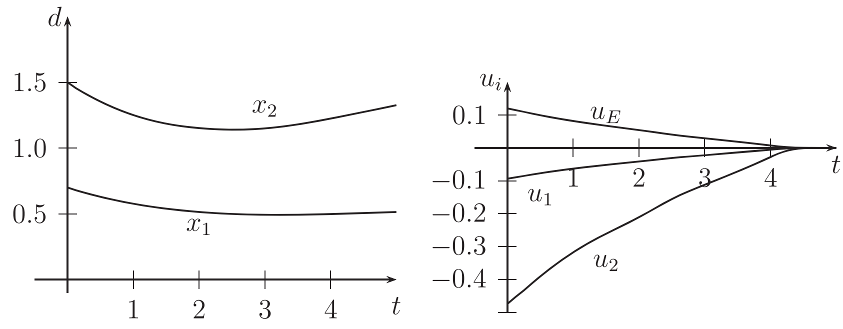
With

$$A := \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix};\ B_1 := \begin{bmatrix} 1 \\ 0 \end{bmatrix};\ B_2 := \begin{bmatrix} 0 \\ 1 \end{bmatrix};\ B_E := -\begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix};\ B := [B_1\ B_2\ B_E];\ D := \begin{bmatrix} 1 \\ 1 \end{bmatrix};$$
$$Q_1 := \begin{bmatrix} \beta_1 & 0 \\ 0 & 0 \end{bmatrix};\ Q_2 := \begin{bmatrix} 0 & 0 \\ 0 & \beta_2 \end{bmatrix};\ \text{and } Q_E := \beta_E \begin{bmatrix} \omega^2 & \omega(1-\omega) \\ \omega(1-\omega) & (1-\omega)^2 \end{bmatrix},$$

this model fits then into our framework.

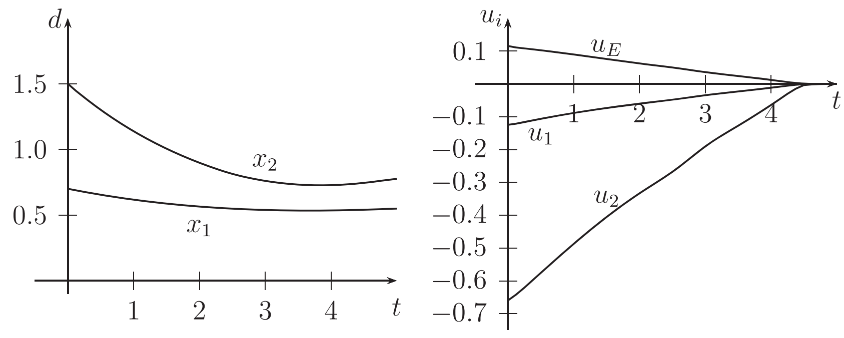
Now, choose $x_1(0) = 0.7$, $x_2(0) = 1.5$, $\alpha_1 = 0.03$, $\alpha_2 = 0.08$, $\gamma_1 = 1$, $\gamma_2 = 0.5$, $\beta_1 = 0.04$, $\beta_2 = 0.08$, $\beta_E = 0.04$, $\omega = 0.3$ and a time horizon of $T = 5$. This models a case where country 2 has initially an approximately twice as higher debt than country 1. Because financial markets are less sure whether country 2 will be able to pay its future debts than in the case of country 1, country 2 has to pay much higher interest payments on debt than country 1. The model specifies, moreover, that country 1 is less willing to use its fiscal instrument in reducing debt than country 2. The central bank is assumed to be concerned approximately twice as much about debt in country 2 than debt in country 1. Furthermore, monetary instruments are twice as much effective when used in country 1 than in country 2.

To see the effect of taking into account disturbance expectations in this model, we simulated two cases. The first case, visualized in Figure 1, models the case that players expect no serious disturbances will affect the economies in the nearby future. This is modeled by choosing the risk aversion parameters r_{wi} large. In this experiment, we choose $r_{w1} = r_{wE} = 10$ and $r_{w2} = 15$. The second simulation, visualized in Figure 2, considers a case where fiscal players do expect disturbances will affect their economy in the nearby future, and the central bank shares this view. The risk aversion



(a) Debt, x_i, country i. (b) Fiscal and monetary policies.

Figure 1. Debt and policies for the benchmark parameters with $r_{w1} = r_{wE} = 10$and$r_{w2} = 15$. That is, almost no disturbance expected. Actual $w(.) = 0$.



(a) Debt, x_i, country i. (b) Fiscal and monetary policies.

Figure 2. Debt and policies for the benchmark parameters with expected disturbance: $r_{w1} = r_{wE} = 1$and$r_{w2} = 1.5$. Actual $w(.) = 0$.

parameters r_{wi} are chosen in these experiments $r_{w1} = r_{wE} = 1$ and $r_{w2} = 1.5$. To determine the equilibrium strategies, we use Theorem 3.4.

The first thing we verified is whether (11) and (12) have a solution. To that end, notice that, because Q_i are semi-positive definite, (11) will always have a solution, whatever T is. So only existence of a solution of (12) had to be checked. Next, the Nash/worst-case open-loop equilibrium strategies for fiscal players and the central bank were calculated using Theorem 3.4. These strategies are plotted both for the 'almost noise-free expectations' and the 'noisy expectations' scenario in Figure 1(b) and (b), respectively. Notice that these strategies will be played, because of the assumed open-loop information structure of the game, whatever the realization of the noise $w(.)$ will be. In Figures 1(a) and 2(a), we plotted the realization of debt if actually no noise occurs during the planning horizon $[0, T]$ in both scenarios, that is, under the assumption that $w(.) = 0$.

As one might expect, fiscal instruments are much more aggressively used by country 2 than in country 1 in order to reduce its debt. Including disturbance expectations shows that in particular, both countries react by a more aggressive fiscal policy. Although fiscal player 2 expects less disturbances will hit the economy than the other players do, we observe that its response in policy terms is much higher. This leads to a significant reduction of its final debt compared with the 'noise-free expectations' case.

So including model uncertainty has a stabilizing effect in this model.

5. CONCLUDING REMARKS

In this paper, we considered the finite planning horizon open-loop differential game that is disrupted by deterministic noise. We showed that Nash/worst-case equilibria for this game can be found by determining the open-loop Nash equilibria of an associated extended differential game that has an additional restriction on its initial state. More specifically, we showed that Nash/worst-case equilibria can be calculated from a 'virtual' differential game where with every player, a 'nature' player is introduced that tries to maximize the performance criterion with respect to the disturbance.

Based on this equivalence result we derived for the linear-quadratic differential game both necessary and sufficient conditions for Nash/worst-case equilibria along the lines, these conditions are obtained for the noise-free case. The only difficulty arises in the extra condition that is imposed on the initial state of the extended system. In an example, we showed that, different from the noise-free case, in the linear-quadratic case, multiple equilibria may occur. As a result of the analysis, we could also easily establish both necessary and sufficient conditions for existence of a unique equilibrium.

Implementation of the equilibrium actions seems to be computationally less demanding than equilibrium actions derived in a similar setting by Jank *et al.* [26–28]. In an example, we illustrated its use. The example shows that the consideration of model uncertainty in a debt stabilization game leads to a more active control pursued by players in the short run, which has a stabilizing effect on closed-loop response. It supports the intuition that if players incorporate a worst-case realization of disturbances into their policy making, they will respond by implementing a policy that is, particularly in the short run, more directed to the realization of the goals expressed in the cost functional with respect to the state variables, this, because they expect 'nature' will try to manipulate the state variables as worse as possible from their cost functional point of view. So if debt stabilization is the major issue, policy is directed more towards achieving this goal in order to mitigate the attempts by 'nature' to destabilize debt.

On a theoretical level, open issues that remain to be settled are how these results can be used to arrive at both necessary and sufficient conditions for existence of Nash/worst-case equilibria for an arbitrary planning horizon $[0, t_f]$, where t_f ranges between 0 and T. In the noise-free case, necessary and sufficient conditions for solvability of this problem can be formulated in terms of existence to a set of coupled Riccati differential equations. Unfortunately, due to the initial state restrictions for the extended system, such a generalization for the disturbed game is less obvious. Clearly, as we did here, by, for example, ignoring this restriction and assuming that the corresponding 'noise-free' Riccati differential equations have a solution, one easily obtains a set of sufficient conditions for the existence to the problem. However, the question remains how far these conditions are necessary too. Probably, the presented conditions will also help to solve the corresponding infinite-planning horizon problem.

From a practical applications point of view, the assumption that for all admissible controls there exists for all players a worst-case disturbance signal is in most cases not too unrealistic. For, if such a signal does not exist for some players for some admissible control, this probably implies in most cases that the corresponding worst-case cost for that player is infinite. Therefore, likely, this control will not be a Nash/worst-case equilibrium. But, one should be cautious, as maybe no equilibrium might exist under such conditions. It seems that cases like this can be easily incorporated within the current framework as long as no equilibrium occurs where some players have infinite cost. Concerning the technical assumptions imposed in the theorems in Chapter 3, we recall from [4][p.269] that the assumption that Riccati equations ((11) and (12)) have a solution is usually satisfied if the game has an equilibrium. Moreover, in case the nonsymmetric Riccati equation (15) has no solution, one can relapse to Theorem 3.4 to decide whether a solution exists and, if so, find its solution. Similar remarks apply for the general case. Finally, the results presented here for the linear-quadratic case may be useful too to analyze the same problem setting under the additional assumption that the underlying system is described by a set of linear differential-algebraic equations. Settings that often naturally occur when considering linearized models of systems.

APPENDIX

Proof of Theorem 3.3

First, notice that for a fixed $u \in \mathcal{U}$, the maximization of J_i w.r.t. w is an affine linear-quadratic control problem. As we assumed (12) has a solution, this control problem has a unique solution (see, e.g., [4][Theorem 5.11]). This implies that for all $u \in \mathcal{U}$, there exists a unique worst-case disturbance from the point of view of player i.

By Corollary 3.2, the two-player linear-quadratic differential game ((8)–(9)) has a Nash/worst-case equilibrium for every initial state x_0 if and only if with $\bar{x}(t) := \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$; $\bar{A}_2 := \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$; $\bar{B}_i := \begin{bmatrix} B_i \\ B_i \end{bmatrix}$; $\bar{D}_1 := \begin{bmatrix} D \\ 0 \end{bmatrix}$; $\bar{D}_2 := \begin{bmatrix} 0 \\ D \end{bmatrix}$; $\bar{Q}_1 := \begin{bmatrix} Q_1 & 0 \\ 0 & 0 \end{bmatrix}$; $\bar{Q}_2 := \begin{bmatrix} 0 & 0 \\ 0 & Q_2 \end{bmatrix}$; $\bar{Q}_{1T} := \begin{bmatrix} Q_{1T} & 0 \\ 0 & 0 \end{bmatrix}$; and $\bar{Q}_{2T} := \begin{bmatrix} 0 & 0 \\ 0 & Q_{2T} \end{bmatrix}$, the next four-player game has a Nash equilibrium for every initial state x_0.

$$\dot{\bar{x}}(t) = \bar{A}_2\bar{x}(t) + \bar{B}_1u_1(t) + \bar{B}_2u_2(t) + \bar{D}_1w_1(t) + \bar{D}_2w_2(t), \text{ with } \bar{x}(0) = \bar{x}_0 := \begin{bmatrix} x_0 \\ x_0 \end{bmatrix} \quad (21)$$

and cost functional for the players given by the following:

$$\begin{aligned} J_i(u_1, u_2, w_i) = \frac{1}{2}\int_0^T \Bigg\{ & \bar{x}^T(t)\bar{Q}_i\bar{x}(t) + \sum_{j=1}^{2} u_j^T(t)R_{ij}u_j(t) \\ & - w_i^T(t)R_{wi}w_i(t)\Bigg\} dt + \frac{1}{2}\bar{x}^T(T)\bar{Q}_{iT}\bar{x}(T), \end{aligned} \quad (22)$$

and $J_{i+2}(u_1, u_2, w_i) = -J_i(u_1, u_2, w_i),\ i = 1, 2.$

Unfortunately, the initial state of this extended system cannot be arbitrarily chosen. Therefore, we cannot use directly existing results on open-loop LQ games to derive both necessary and sufficient existence conditions for a Nash equilibrium. However, we can follow the lines of the proof for the standard case (see, e.g., proof of [4, Theorem 7.1]) to obtain these conditions.

Suppose that $(u_i^*(.), w_i^*(.))$ is a Nash equilibrium. Then, by the maximum principle, the Hamiltonian

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$$H_i = \frac{1}{2}\left(\bar{x}^T\bar{Q}_i\bar{x} + u_1^TR_{i1}u_1 + u_2^TR_{i2}u_2 - w_i^T(t)R_{wi}w_i(t)\right) + \psi_i^T\left(\bar{A}_2\bar{x} + \bar{B}_1u_1 + \bar{B}_2u_2 + \bar{D}_1w_1 + \bar{D}_2w_2\right)$$

is minimized by player i w.r.t. u_i, $i = 1, 2$, and

$$H_{i+2} = \frac{1}{2}\left(-\bar{x}^T\bar{Q}_i\bar{x} - u_1^TR_{i1}u_1 - u_2^TR_{i2}u_2 + w_i^T(t)R_{wi}w_i(t)\right) + \psi_{i+2}^T\left(\bar{A}_2\bar{x} + \bar{B}_1u_1 + \bar{B}_2u_2 + \bar{D}_1w_1 + \bar{D}_2w_2\right)$$

is minimized by player $i+2$ w.r.t. w_i, $i = 1, 2$. This yields the necessary conditions

$$u_i^*(t) = -R_{ii}^{-1}\bar{B}_i^T\psi_i(t) \text{ and } w_i^*(t) = -R_{wi}^{-1}\bar{D}_i^T\psi_{i+2}(t),\ i = 1, 2,$$

where the $2n$-dimensional vectors $\psi_i(t)$ satisfy

$$\dot{\psi}_i(t) = -\bar{Q}_i\bar{x}(t) - \bar{A}_2^T\psi_i(t), \psi_i(T) = \bar{Q}_{iT}\bar{x}(T),$$
$$\dot{\psi}_{i+2}(t) = \bar{Q}_i\bar{x}(t) - \bar{A}_2^T\psi_{i+2}(t), \psi_{i+2}(T) = -\bar{Q}_{iT}\bar{x}(T),\ i = 1, 2,$$

and

$$\dot{\bar{x}}(t) = \bar{A}_2\bar{x}(t) - \bar{S}_1\psi_1(t) - \bar{S}_2\psi_2(t) - \bar{S}_{D_1}\psi_3(t) - \bar{S}_{D_2}\psi_4(t),$$

with $\bar{x}^T(0) = \begin{bmatrix} x_0^T & x_0^T \end{bmatrix}$. Here, $\bar{S}_i := \begin{bmatrix} S_i & S_i \\ S_i & S_i \end{bmatrix}$, $\bar{S}_{D_1} := \begin{bmatrix} S_{D_1} & 0 \\ 0 & 0 \end{bmatrix}$, and $\bar{S}_{D_2} := \begin{bmatrix} 0 & 0 \\ 0 & S_{D_2} \end{bmatrix}$. In other words, if the problem has an open-loop Nash equilibrium, then, with $\tilde{y}(t) := \left[\bar{x}^T(t), \psi_1^T(t), \ldots, \psi_4^T(t)\right]^T$, the differential equation

$$\dot{\tilde{y}}(t) = \begin{bmatrix} \bar{A}_2 & -\bar{S}_1 & -\bar{S}_2 & -\bar{S}_{D_1} & -\bar{S}_{D_2} \\ -\bar{Q}_1 & -\bar{A}_2^T & 0 & 0 & 0 \\ -\bar{Q}_2 & 0 & -\bar{A}_2^T & 0 & 0 \\ \bar{Q}_1 & 0 & 0 & -\bar{A}_2^T & 0 \\ \bar{Q}_2 & 0 & 0 & 0 & -\bar{A}_2^T \end{bmatrix}\tilde{y}(t) \tag{23}$$

with boundary conditions $\bar{x}^T(0) = \begin{bmatrix} x_0^T & x_0^T \end{bmatrix}$, $\psi_1(T) - \bar{Q}_{1T}\bar{x}(T) = 0$, $\psi_2(T) - \bar{Q}_{2T}\bar{x}(T) = 0$, $\psi_3(T) + \bar{Q}_{1T}\bar{x}(T) = 0$, and $\psi_4(T) + \bar{Q}_{2T}\bar{x}(T) = 0$ has a solution. Next, split $\psi_i^T =: \begin{bmatrix} \psi_{i1}^T & \psi_{i2}^T \end{bmatrix}$. Some detailed inspection of (23) shows that $\psi_{12} = \psi_{21} = \psi_{32} = \psi_{41} = 0$, $\psi_{31} = -\psi_{11}$ and $\psi_{42} = -\psi_{22}$. So we conclude that if the problem has an open-loop Nash equilibrium, then with $y^T(t) := \begin{bmatrix} x_1^T(t) & x_2^T(t) & \psi_{11}^T(t) & \psi_{22}^T(t) \end{bmatrix}$, the next linear two-point boundary value problem has a solution for every x_0.

$$\dot{y}(t) = My(t), \text{ with } \bar{P}y(0) + \bar{Q}y(T) = \begin{bmatrix} \bar{x}_0^T & 0 & 0 \end{bmatrix}^T. \tag{24}$$

Some elementary rewriting shows that the aforementioned two-point boundary value problem (24) has a solution for every initial state x_0 if and only if

$$(\bar{P} + \bar{Q}e^{MT})y(0) = \begin{bmatrix} \bar{x}_0^T & 0 & 0 \end{bmatrix}^T,$$

or, equivalently,

$$(\bar{P}e^{-MT} + \bar{Q})e^{MT}y(0) = [x_0^T\ x_0^T\ 0\ 0]^T, \tag{25}$$

is solvable for every x_0.

Denoting $z := e^{MT}y(0)$ and $[\,W_1\ W_2\,] := \begin{bmatrix} I_{2n} & 0_{2n} \end{bmatrix}e^{-MT}$, the question whether (25) is solvable for every x_0 is equivalent with the question whether

$$\begin{bmatrix} W_1 & W_2 \\ -Q_T & I_{2n} \end{bmatrix}z = \begin{bmatrix} \bar{x}_0 \\ 0 \end{bmatrix} \tag{26}$$

has a solution for every x_0. An elementary analysis shows that (26) has a solution for every x_0 if and only if, with $H(T) := \left[\, I_{2n} \;\; 0_{2n} \,\right] e^{-MT} \left[\, I_{2n} \;\; Q_T \,\right]^T$,

$$H(T)z_1 = \bar{x}_0 \tag{27}$$

has a solution for every x_0. This last problem is equivalent with the problem under which conditions the matrix equation $H(T)X = \begin{bmatrix} I \\ I \end{bmatrix}$ has a solution X. From this, one directly obtains condition (13) (see, e.g., [29, Exercise 10.49]).

If (13) holds, all solutions of (27) are

$$z_1 = H^+\bar{x}_0 + (I - H^+H)q, \text{ where } q \text{ is an arbitrary vector.}$$

From (26), it follows that the set of points z yielding a consistent initial boundary value problem are

$$z = [\, I_{2n},\; Q_T \,]^T z_1.$$

The z corresponding consistent initial state of the boundary value problem is $y_0 = e^{-MT}z$. The solution of the initial boundary value problem is then

$$y(t) = e^{Mt}y_0 = e^{M(t-T)}z = e^{M(t-T)} \begin{bmatrix} I_{2n} \\ Q_T \end{bmatrix} z_1.$$

Using the definition of $y(t)$ in terms of the state and co-state variables yields then the corresponding equilibrium controls:

$$u_i(t) = -R_{ii}^{-1}\bar{B}_i^T \begin{bmatrix} \psi_{i1}(t) \\ \psi_{i2}(t) \end{bmatrix} = -R_{ii}^{-1}B_i^T\psi_{ii}(t) = -R_{ii}^{-1}B_i^T\bar{E}_{4,i+2}^T e^{M(t-T)} \begin{bmatrix} I_{2n} \\ Q_T \end{bmatrix} z_1(t) \text{and} \tag{28}$$

$$w_i(t) = -R_{wi}^{-1}\bar{D}_i^T \begin{bmatrix} \psi_{i+21}(t) \\ \psi_{i+22}(t) \end{bmatrix} = R_{wi}^{-1}D^T\psi_{ii}(t) = R_{wi}^{-1}D^T\bar{E}_{4,i+2}^T e^{M(t-T)} \begin{bmatrix} I_{2n} \\ Q_T \end{bmatrix} z_1(t),\; i = 1, 2. \tag{29}$$

"$\Leftarrow$ **part**" By assumption, the Riccati differential equations ((11) and (12)) have a solution on $[0, T]$. Because $H(T)$ satisfies (13), it is clear from the '$\Rightarrow$ part' of the proof that the two-point boundary value problem (14) has for every x_0 a consistent initial value y_0 yielding a unique solution for the boundary value problem. Assume y_0 is with a x_0 consistent initial value of the boundary value problem. Denote the solution $y(t)$ of this two-point boundary value problem (14) by $\left[\, x_1^T(t),\; x_2^T(t),\; \psi_{11}(t),\; \psi_{22}(t) \,\right]^T$, where the dimension of x_i and ψ_i is n. Now, consider

$$m_i(t) := \psi_{ii}(t) - K_i(t)x_i(t) \text{ and } m_{i+2}(t) := -\psi_{ii}(t) - L_i(t)x_i(t),\; i = 1, 2.$$

Then, $m_i(T) = 0$. Furthermore, differentiation of $m_1(t)$ gives

$$\begin{aligned} \dot{m}_1(t) &= \dot{\psi}_{11}(t) - \dot{K}_1(t)x_1(t) - K_1(t)\dot{x}_1(t) = -Q_1x_1(t) - A^T\psi_{11}(t) - [-A^TK_1(t) - K_1(t)A \\ &\quad + K_1(t)S_1K_1(t) - Q_1]x_1(t) - K_1(t)[Ax_1(t) - S_1\psi_{11}(t) - S_2\psi_{22}(t) + S_{D_1}\psi_{11}(t)] \\ &= -A^Tm_1(t) - K_1(t)S_1K_1(t)x_1(t) + K_1(t)S_1[m_1(t) + K_1(t)x_1(t)] \\ &\quad + K_1(t)S_2[m_2(t) + K_2(t)x_2(t)] + K_1(t)S_{D_1}[m_3(t) + L_1(t)x_1(t)] \\ &= -A^Tm_1(t) + K_1(t)[S_1m_1(t) + S_2m_2(t) + S_{D_1}m_3(t)] + K_1(t)[S_2K_2(t)x_2(t) + S_{D_1}L_1(t)x_1(t)]. \end{aligned}$$

Next, consider

$$u_i^*(t) = -R_{ii}^{-1}B_i^T(K_i(t)x_i(t) + m_i(t)) \text{ and } w_i^*(t) = -R_{wi}^{-1}D^T(L_i(t)x_i(t) + m_{i+2}(t))\; i = 1, 2.$$

By [4, Theorem 5.11], the minimization w.r.t. u_1 of $J_1\left(u_1, u_2^*, w_1^*, w_2^*\right) :=$

$$\frac{1}{2}\int_0^T \left\{\bar{x}^T(t)\bar{Q}_i\bar{x}(t) + u_1^T(t)R_{11}u_1(t) + u_2^{*T}(t)R_{12}u_2^*(t) - w_1^{*T}(t)R_{w1}w_1^*(t)\right\}dt + \frac{1}{2}\bar{x}^T(T)\bar{Q}_{iT}\bar{x}(T),$$

where $\dot{\bar{x}}(t) = \bar{A}_2\bar{x}(t) + \bar{B}_1 u_1(t) + \bar{B}_2 u_2^*(t) + \sum_{i=1}^2 \bar{D}_i w_i^*(t)$, with $\bar{x}(0) = \bar{x}_0 = [x_0^T, \ x_0^T]^T$, has a unique solution. This solution is $\bar{u}_1(t) = -R_{11}^{-1}\bar{B}_1^T(\bar{K}_1(t)\bar{x}(t) + \bar{m}_1(t))$, where $\bar{K}_1(t)$ solves the Riccati differential equation:

$$\dot{\bar{K}}_1(t) = -\bar{A}_2^T\bar{K}_1(t) - \bar{K}_1(t)\bar{A}_2 + \bar{K}_1(t)\bar{S}_1\bar{K}_1(t) - \bar{Q}_1, \text{ with } \bar{K}_1(T) = \bar{Q}_{1T}$$

and $\bar{m}_1(t)$ solves the linear differential equation:

$$\dot{\bar{m}}_1(t) = (\bar{K}_1(t)\bar{S}_1 - \bar{A}_2^T)\bar{m}_1(t) - \bar{K}_1(t)(\bar{B}_2 u_2^*(t) + \sum_{i=1}^{2}\bar{D}_i w_i^*(t)), \ \bar{m}_1(T) = 0. \tag{30}$$

It is easily verified that $\bar{K}_1(t) := \begin{bmatrix} K_1(t) & 0 \\ 0 & 0 \end{bmatrix}$ solves the Riccati differential equation, where $K_1(t)$ solves (11). Using this in (30) shows that $\bar{m}_1(t) = \begin{bmatrix} \tilde{m}_1(t) \\ 0 \end{bmatrix}$, where $\tilde{m}_1(t)$ solves the differential equation

$$\dot{\tilde{m}}_1(t) = (K_1(t)S_1 - A^T)\tilde{m}_1(t) - K_1(t)(B_2 u_2^*(t) + Dw_1^*(t)), \ \tilde{m}_1(T) = 0. \tag{31}$$

Consequently, $\bar{u}_1(t) = -R_{11}^{-1}B_1^T(K_1(t)\bar{x}_1(t) + \tilde{m}_1(t))$, where $\bar{x}_1(t)$ is the optimal control implied solution of the differential equation:

$$\dot{\bar{x}}_1(t) = (A - S_1K_1)\bar{x}_1(t) - S_1\tilde{m}_1(t) + B_2u_2^*(t) + Dw_1^*(t) \text{ with } \bar{x}_1(0) = x_0. \tag{32}$$

Substitution of $u_2^*(t)$ and $w_1^*(t)$ into (31) and (32) shows that the variables $\tilde{m}_1$ and $\tilde{x}$ satisfy

$$\begin{aligned}\dot{\tilde{m}}_1(t) &= (K_1(t)S_1 - A^T)\tilde{m}_1(t) + K_1(t)(S_2(K_2(t)x_2(t) + m_2(t)) \\ &\quad + S_{D_1}(L_1(t)x_1(t) + m_3(t))), \ \tilde{m}_1(T) = 0, \\ \dot{\bar{x}}_1(t) &= (A - S_1K_1(t))\bar{x}_1(t) - S_2(K_2(t)x_2(t) + m_2(t)) \\ &\quad - S_{D_1}(L_1(t)x_1(t) + m_3(t)) - S_1\tilde{m}_1(t), \ \bar{x}_1(0) = x_0.\end{aligned}$$

It is easily verified that a solution of this set of differential equations is given by $\tilde{m}_1(t) = m_1(t)$ and $\bar{x}_1(t) = x_1(t)$. Because its solution is unique, this implies that $\bar{u}_1(t) = u_1^*(t)$ or stated differently,

$$J_1\left(u_1^*, u_2^*, w_1^*, w_2^*\right) \leqslant J_1\left(u_1, u_2^*, w_1^*, w_2^*\right), \text{ for all } u_1.$$

Similarly, it can be shown that the corresponding inequalities for the cost functions for the other players apply, which shows that $\left(u_1^*, u_2^*, w_1^*, w_2^*\right)$ is a Nash equilibrium for the extended game. So by Corollary 3.2, these strategies yield a Nash/worst-case equilibrium. Notice, $\psi_{ii}(t) = K_i(t)x_i(t) + m_i(t) = -(L_i(t)x_i(t) + m_{i+2}(t))$. This observation directly yields the stated equilibrium strategies.

Finally, the fact that equilibrium strategies are unique if $H(T)$ is invertible follows directly from (27) as the set of all equilibrium strategies satisfy (28) and (29).

Outline Proof Theorem 3.8

Here, we sketch the main points in arriving at the generalization of Theorems 3.3 and 3.4. Following [4][Exercise 7.5], consider the next shorthand notation. Standard conventions concerning block-matrix addition and multiplication rules are assumed again.

$$G^e := \begin{bmatrix} [0\ I\ 0 \cdots 0]\,\bar{M}_1^e \\ \vdots \\ [0\ 0 \cdots\ 0\ I]\,\bar{M}_{2N}^e \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ I & 0 & \cdots & 0 \\ 0 & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & I \end{bmatrix} = \begin{bmatrix} R_{11} & V_{122} & \cdots & V_{12N} & 0 \\ V_{222}^T & \ddots & \cdots & \vdots & \vdots \\ \vdots & & \ddots & V_{(N-1)(N-1)N} & \vdots \\ V_{N2N}^T & \cdots & V_{N(N-1)N}^T & R_{NN} & 0 \\ 0 & \cdots & \cdots & 0 & R_w \end{bmatrix},$$

where $R_w = \text{diag}(R_{wi})$. We assume that this matrix G^e is invertible.

$B^e := [\bar{B}_1, \ldots, \bar{B}_N, \bar{D}_1, \ldots, \bar{D}_N]$; $\tilde{B}^{e^T} := \text{diag}\left(\bar{B}_1^T, \ldots, \bar{B}_N^T, \bar{D}_1^T, \ldots, \bar{D}_N^T\right)$;
$\tilde{B}_i^{e^T} :=$ block-column i of $\tilde{B}^{e^T}$;

$$Z^e := \begin{bmatrix} [0\ I\ 0 \cdots 0]\ \bar{M}_1^e \\ \vdots \\ [0\ 0 \cdots 0\ I]\ \bar{M}_{2N}^e \end{bmatrix} \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \text{diag}(V_{111}^T, \ldots, V_{N1N}^T) \\ 0_{Nk \times Nn} \end{bmatrix};$$

$$Z_i^e := [I\ 0 \cdots 0]\bar{M}_i^e \begin{bmatrix} 0 \\ I \end{bmatrix} = \begin{bmatrix} & 0_{(i-1)n \times Nn} & & 0_{(i-1)n \times m} \\ V_{i11} & \cdots & V_{i1N} & 0_{n \times m} \\ & 0_{(N-i)n \times Nn} & & 0_{(N-i)n \times m} \end{bmatrix}, i \in \mathbf{N},\ Z_{i+N} := -Z_i,\ i \in \mathbf{N};$$

$\tilde{Q}_i^e := \bar{E}_{N,i} Q_i \bar{E}_{N,i}^T - Z_i^e G^{e^{-1}} Z^e,\ i \in \mathbf{N}$; $\tilde{Q}_{i+N}^e := -\tilde{Q}_i^e,\ i \in \mathbf{N}$; $\tilde{Q}^e := \left[\tilde{Q}_1^{e^T} \cdots \tilde{Q}_{2N}^{e^T}\right]^T$;
$\tilde{S}_i^e := B^e G^{e^{-1}} \tilde{B}_i^{e^T}$; $\tilde{S}^e := \left[\tilde{S}_1^{e^T} \cdots \tilde{S}_{2N}^{e^T}\right]$;

$$\tilde{A}^e := \bar{A}_N - B^e G^{e^{-1}} Z^e; \text{ and } \tilde{A}_{2N^2}^{e^T} := \bar{A}_{2N^2}^T - \begin{bmatrix} Z_1^e \\ \vdots \\ Z_{2N}^e \end{bmatrix} G^{e^{-1}} \tilde{B}^{e^T}.$$

Then, with

$$\tilde{M}^e := \begin{bmatrix} \tilde{A}^e & -\tilde{S}^e \\ -\tilde{Q}^e & -\tilde{A}_{2N^2}^{e^T} \end{bmatrix},$$

along the lines of [4][Exercise 7.5] and the aforementioned proof of Theorem 3.3, we have that the general linear-quadratic differential game ((6) and (7)) has a solution iff the next generalization of the two-point boundary value problem (23) has a solution:

$$\dot{\tilde{y}}^e(t) = \tilde{M}^e \tilde{y}^e(t), \ \text{ where } \tilde{y}^{e^T} = \left[\bar{x}^T\ \tilde{\psi}_1^{e^T} \cdots \tilde{\psi}_{2N}^{e^T}\right]^T,$$

with boundary conditions $\bar{x}(0) = \bar{x}_0,\ \tilde{\psi}_i^e(T) - \bar{Q}_{iT}\bar{x}(T) = 0,\ i \in \mathbf{N}$ and $\tilde{\psi}_i^e(T) + \bar{Q}_{iT}\bar{x}(T) = 0,\ i = N+1, \ldots, 2N$. Moreover, $G^e \left[u_1^{*^T} \cdots u_N^{*^T}\ w_1^{*^T} \cdots w_N^{*^T}\right]^T = -[Z^e\ \tilde{B}^{e^T}]\tilde{y}^e(t)$. A spelling of this two-point boundary value problem shows then, along the lines of the proof of Theorem 3.3, that it has a solution iff the next two-point boundary value problem has a solution:

$$\dot{\tilde{y}}(t) = \tilde{M}\tilde{y}(t), \text{ where } \tilde{y} = [\bar{x}^T\ \tilde{\psi}_1^T \cdots \tilde{\psi}_N^T]^T,$$

with boundary conditions $\bar{x}(0) = \bar{x}_0,\ \tilde{\psi}_i(T) - \bar{Q}_{iT}\bar{x}(T) = 0,\ i \in \mathbf{N}$. Furthermore,

$$G^e \left[u_1^{*^T} \cdots u_N^{*^T}\ w_1^{*^T} \cdots w_N^{*^T}\right]^T = -\begin{bmatrix} Z & \tilde{B}^T \\ 0 & -\text{diag}(D^T) \end{bmatrix} \tilde{y}(t).$$

From this, the generalization of Theorem 3.3 readily follows.

Following the lines of the proof of Theorem 3.4, one arrives then directly at the results presented in Theorem 3.8.

The corresponding worst-case cost advertized in this Theorem results by direct substitution of the worst-case actions into J_i. To illustrate this, we elaborate later the case for J_1. Using the advertized differential equations for $C_1(t)$, it follows that

$$\begin{aligned}2J_1 &= \int_0^T [\bar{x}^T(t)\; u^T(t)]\bar{M}_1[\bar{x}^T(t)\; u^T(t)]^T - w_1^T(t)R_{w1}w_1(t)dt + \bar{x}^T(T)\bar{Q}_{1T}\bar{x}(T)\\ &= \int_0^T \bar{x}^T(t)\left[[I,\, -(Z+\tilde{B}^TP(t))^TG^{-T}]\bar{M}_1[I,\, -(Z+\tilde{B}^TP(t))^TG^{-T}]^T\right.\\ &\qquad \left.-P_1^T(t)S_{D_1}P_1(t)\right]\bar{x}(t)dt + \bar{x}^T(T)\bar{Q}_{1T}\bar{x}(T)\\ &= \int_0^T \bar{x}^T(t)\left[-A_{cl}^T(t)C_1(t) - C_1(t)A_{cl}(t) - \dot{C}_1(t)\right]\bar{x}(t)dt + \bar{x}^T(T)\bar{Q}_{1T}\bar{x}(T)\\ &= -\int_0^T \frac{d}{dt}\left\{\bar{x}^T(t)C_1(t)\bar{x}(t)\right\}dt + \bar{x}^T(T)\bar{Q}_{1T}\bar{x}(T) = \bar{x}^T(0)C_1(0)\bar{x}(0).\end{aligned}$$

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