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Vertex Weighted Steiner Tree Games

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Abstract

We introduce a variant of the minimum cost spanning tree game in which the players receive a reward if they are connected to a central supplier. The game is called a vertex weighted Steiner tree (VWST) game. The problem is to distribute the total profit in a VWST game (defined as the sum of the rewards minus the contruction costs of the links to the supplier) among the players. We introduce an allocation rule for this type of games which constitutes a core element if and only if it is efficient. We investigate under which conditions this allocation rule is indeed efficient.

1 Introduction

Suppose that a number of customers wish to be connected to a central supplier. We assume that the customers have to pay for the construction costs of the network yet to build. On the other hand, a customer receives a reward if (and only if) he is connected to the central supplier. The total profit of the customers is defined as the sum over the rewards of the connected customers minus the construction costs of the network. The customers are interested in maximizing their total profit. The customers may decide that some of them will not be connected to the supplier if this increases the profit.

Formally, let $N = \{1, 2, \ldots, n\}$ denote the set of customers and let 0 denote the central supplier. The cost of establishing a link between $i \in N \cup \{0\}$ and $j \in N \cup \{0\}$ $(i \neq j)$ is denoted by $d(i,j) \geq 0$. The reward of a customer $i \in N$ is denoted by $r_i \geq 0$. Suppose that the customers decide to

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establish a set E of links. We say that a customer $i \in N$ is connected to the supplier if there exists a path from i to 0 in the graph $G_E = (N \cup \{0\}, E)$. If G_E contains a cycle, then an arbitrary link in this cycle can be deleted, while the connected customers remain connected. This can only increase the total profit. Moreover, suppose that G_E has a component, which contains more than one customer but which does not contain the supplier. Then all links within this component can be deleted, leaving the connected customers connected. Also this can only increase the total profit. Hence, in search for an optimal network, we may restrict ourselves to minimal trees with vertex set $S \cup \{0\}$, where $S \subseteq N$. Let us denote the cost of a minimal tree with vertex set $S \cup \{0\}$ by $c(S)$ and denote $\sum_{i \in S} r_i$ by $r(S)$. Then the maximal profit that the customers can achieve equals

$$v(N) = \max\{r(T) - c(T) \mid T \subseteq N\}.$$

The problem of finding an optimal tree is known as the *vertex weighted Steiner tree problem*, which was first treated by Segev (1987).

If the customers involved in a vertex weighted Steiner tree problem decide to build the optimal network, then together they will obtain a profit $v(N)$. In this paper we address the problem of allocating the total profit among the customers. In order to deal with this allocation problem, we define a transferable utility game associated with the problem. Formally, a game is an ordered pair (N, v), where N is a finite set and v is a real valued function on the subsets of N, assigning 0 to the empty set. The elements of N are called *players*, and the subsets of N are called *coalitions*. The value $v(S)$ is called the *worth* of coalition S. It is usually interpreted as the profit that coalition S can achieve without the cooperation of players outside S. Suppose all players decide to cooperate, and thus receive a total profit of $v(N)$ together. In the negotiations about the division of the profit among the players, the members of an arbitrary coalition S could easily claim that they should receive at least $v(S)$ together, since otherwise S is better off by not cooperating with the other players. The *core* is defined as the set of allocations that satisfy all such possible claims, i.e.

$$\text{Core}(v) := \{x \in \mathbb{R}^N \mid x(N) = v(N) \text{ and } x(S) \geq v(S) \text{ for all } S \subseteq N\}.$$

In case of the vertex weighted Steiner tree problem we assume that a subset $S \subseteq N$ of customers is allowed to build its own network for connecting its members. In doing so, these customers are allowed to use the vertices of customers in $N\backslash S$. However, if the members of S decide to use such a vertex in $N\backslash S$, S will not receive the reward of this customer. Hence,

$$v(S) = \max\{r(S \cap T) - c(T) \mid T \subseteq N\}.$$

We call the game a *vertex weighted Steiner tree* (VWST) game.

Core allocations do not necessarily exist for VWST games. Consider the following game with $N = \{a, b, c, A, B, C\}$. For players indicated with lower-case letters we have a cost of 3 to establish a direct link to the supplier. Also, six other links have cost 3, namely $\{A, b\}$, $\{A, c\}$, $\{B, a\}$, $\{B, c\}$, $\{C, a\}$ and $\{C, b\}$. All other links have cost 5. Furthermore, the reward of upper-case players is 5 and the reward of lower-case players is 0. See figure 1 in which only the links with cost 3 are drawn. Let us suppose that x is a core allocation. Observe that $v(N) = 1$, and that also $v(N\backslash\{i\}) = 1$ for all $i \in N$. It follows that $x_i = v(N) - x(N\backslash\{i\}) \leq v(N) - v(N\backslash\{i\}) = 0$ for all $i \in N$. Hence $x \leq 0$, which contradicts the fact that x also satisfies $x(N) = v(N) = 1$. A similar example was given by van Bokhoven (1994).

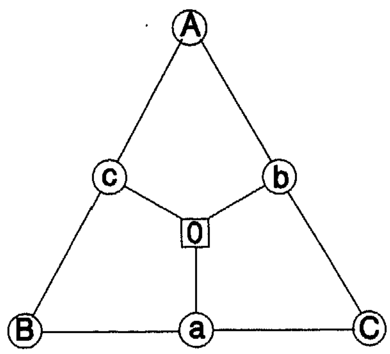

figure 1.

Our game model is closely related to the model of a directed Steiner tree game (see Skorin-Kapov (1992)). For a directed Steiner tree game the worth of a coalition S is determined by a minimum cost Steiner tree directed away from the root, spanning all vertices of S. The vertex weighted Steiner tree problem can be formulated as a directed Steiner tree problem by subtracting the reward of a vertex from all ingoing arcs of this vertex. This does not mean at all that a VWST game is a directed Steiner tree game, since for a VWST game one has to change the data of the network each time that the worth of another coalition is computed, while for a directed Steiner tree game only the set of terminal nodes changes.

2 An Allocation Rule

In this section we shall describe a procedure for allocating the profit of a VWST problem among the customers. The procedure does not necessarily allocate the total profit $v(N)$, but if the procedure does so, we have found a core element. We define the allocation rule in a slightly more general setting. Let (N, c) be a game, and let $r \in \mathbb{R}^N$ with $r \geq 0$. We interpret the game (N, c) as a cost game and the vector r as a reward vector. We define the savings game (N, v) by

$$v(S) = \max\{r(S \cap T) - c(T) \mid T \subseteq N\}.$$

We will refer to the game (N, v) as the *reward game* with respect to r. Observe that the reward game is a VWST game if (N, c) is a minimum cost spanning tree game or if it is a monotone minimum cost spanning tree game (see Granot and Huberman (1981)).

The allocation rule works as follows. We start with a reward vector $r = r^0$. Denote the reward game with respect to r^0 by v^0. Define $z_1 = v^0(N) - v^0(N\backslash\{1\})$ and reduce the reward of player 1 with the amount z_1. Denote the resulting reward vector by r^1 and denote the reward game with respect to the new reward vector by v^1. Now define $z_2 = v^1(N) - v^1(N\backslash\{2\})$ and reduce the reward of player 2 with the amount z_2. Denote the resulting reward vector by r^2 and the associated reward game by v^2. Continue this process and finally allocate $z_n = v^{n-1}(N) - v^{n-1}(N\backslash\{n\})$. Let us denote the i-th unit vector by e_i. Then the allocation-process can be written down as follows.

```
r^0 := r;
for k := 1 to n
do
    z_k := v^{k-1}(N) - v^{k-1}(N\{k});
    r^k := r^{k-1} - z_k e_k;
od.
```

In the following we will refer to the vector z as the z-alloccation.

Lemma 1 *The reward games $v^0, \ldots, v^n$ satisfy the following properties.*

i) $v^k(S) \geq v^{k-1}(S\backslash\{k\})$ *for all* $S \subseteq N$ *with* $k \in S$.

ii) $v^k(N) = v^{k-1}(N\backslash\{k\})$ *for all* $k \in N$.

iii) $v^k(N) = v^k(N\backslash\{l\})$ *for all* $k, l \in N$ *with* $l \leq k$.

Proof: Let $S \subseteq N$. Choose $T \subseteq N$ such that

$$v^{k-1}(S\backslash\{k\}) = r^{k-1}(S\backslash\{k\} \cap T) - c(T).$$

Then

$$\begin{aligned} v^{k-1}(S\backslash\{k\}) &= r^{k-1}(S\backslash\{k\} \cap T) - c(T) \\ &\leq r^k(S \cap T) - c(T) \\ &\leq v^k(S). \end{aligned}$$

This proves i).

It follows directly from i) that $v^k(N) \geq v^{k-1}(N\backslash\{k\})$. Choose $T \subseteq N$ such that $v^k(N) = r^k(T) - c(T)$. If $k \notin T$, then we have

$$v^k(N) = r^k(T) - c(T) = r^{k-1}(T) - c(T) \leq v^{k-1}(N\backslash\{k\})$$

and ii) follows. If $k \in T$, then ii) follows from

$$v^k(N) = r^k(T) - c(T) = r^{k-1}(T) - c(T) - z_k \leq v^{k-1}(N) - z_k = v^{k-1}(N\backslash\{k\}).$$

We prove iii) by induction on the number k. The games v^k and v^{k-1} differ only in the reward of player k. Hence, we have $v^k(S) = v^{k-1}(S)$ for all $S \subseteq N$ with $k \notin S$. It follows that $v^k(N) = v^{k-1}(N\backslash\{k\}) = v^k(N\backslash\{k\})$ for all $k \in N$, where the first equality follows from ii). This proves iii) in case $l = k$. Thus we have completely handled the case $k = 1$. Now assume that iii) holds for all pairs l, k with $l \leq k < p$ where $p > 1$. The case $l = k = p$ is already handled, so assume $l < k = p$. We have

$$v^k(N\backslash\{l\}) \geq v^{k-1}(N\backslash\{l\}) - z_k = v^{k-1}(N) - z_k = v^k(N).$$

Now iii) follows, since it trivially holds that $v^k(N\backslash\{l\}) \leq v^k(N)$. □

Theorem 2 *The z-allocation satisfies* $z(S) \leq v(N) - v(N\backslash S)$ *for all* $S \subseteq N$ *and* $z(N) = v(N) - v^n(N)$.

Proof: We shall prove by induction that $z(S) \leq v^k(N) - v^k(N\backslash S)$ for all $S \subseteq \{k+1, \ldots, n\}$ and all $k \in N$. For $k = n - 1$ this is trivially true.

Suppose it is true for all $k > p$. Let $S \subseteq \{p+1, \ldots, n\}$. If $p+1 \notin S$ then according to the induction hypothesis we have

$$z(S) \leq v^{p+1}(N) - v^{p+1}(N \backslash S) \leq v^{p}(N) - v^{p}(N \backslash S),$$

where the second inequality follows from the relations $v^{p+1}(N) = v^{p}(N) - z_p$ and $v^{p+1}(N \backslash S) \geq v^{p}(N \backslash S) - z_p$. If $p+1 \in S$ then

$$\begin{array}{rcl} z(S) & = & z_{p+1} + z(S \backslash \{p+1\}) \\ & \leq & v^{p}(N) - v^{p}(N \backslash \{p+1\}) + v^{p+1}(N) - v^{p+1}(N \backslash S \cup \{p+1\}) \\ & = & v^{p}(N) - v^{p+1}(N \backslash S \cup \{p+1\}) \\ & \leq & v^{p}(N) - v^{p}(N \backslash S), \end{array}$$

where the last inequality follows from lemma 1. By lemma 1 we have $z_k = v^{k-1}(N) - v^{k}(N)$ for all $k \in N$. Hence, $z(N) = \sum_{k \in N}(v^{k-1}(N) - v^{k}(N)) = v^{0}(N) - v^{n}(N)$. □

Corollary 3 *The z-allocation is a core element of the reward game (N, v) if and only if it is efficient.*

Proof : That efficiency of z is a necessary condition follows from the definition of a core element. Let us prove that it also sufficient. Let $S \subseteq N$. According to theorem 2 we have $z(N \backslash S) \leq v(N) - v(S)$. Substituting $v(N) = z(N)$, we find $z(N \backslash S) \leq z(N) - v(S)$, and after rearranging we obtain $z(S) \geq v(S)$. □

The z-allocation is not necessarily a core element of (N, v), which follows in a trivial way from the fact that VWST games with an empty core exist. For instance, if we compute the z-allocation for the earlier given example of a 6-player VWST game with an empty core, we obtain $z = 0$ and $v^{n}(N) = v(N) = 1$. Even when the core of a reward game is non-empty, the z-allocation is not necessarily an element of it. Define the 4-player cost game (N, c) by

$$c(S) = \begin{cases} 8 & \text{if } |S| = 1 \\ 12 & \text{if } |S| = 2 \\ 20 & \text{if } |S| = 3 \\ 24 & \text{if } |S| = 4, \end{cases}$$

and let $r = (7,7,7,7)$. The reward game (N,v) is then given by

$$v(S) = \begin{cases} 0 & \text{if } |S| = 1 \\ 2 & \text{if } |S| = 2 \\ 2 & \text{if } |S| = 3 \\ 4 & \text{if } |S| = 4. \end{cases}$$

Observe that the game (N,v) has exactly one core element, namely $(1,1,1,1)$. For the z-allocation we have $z_1 = v(N) - v(234) = 2$. Hence, the z-allocation is not in the core of the reward game.

Let us formalize the situation of this example. Since the game (N,c) is interpreted as a cost game it makes little sense to apply the usual core concept to this game. The anti-core of the game (N,c) is defined by

$$\text{ACore}(c) := \{x \in \mathbb{R}^N \mid x(N) = c(N) \text{ and } x(S) \leq c(S) \text{ for all } S \subseteq N\}.$$

In the example the vector $x := (6,6,6,6)$ lies in the anti-core of (N,c), and the vector $r - x = (1,1,1,1)$ lies in the core of the reward game. This is no coincidence as is shown in the following theorem.

Theorem 4 *Let (N,c) be a (cost) game and let $r \geq 0$ be a reward vector. If a vector $x \in ACore(c)$ exists with $0 \leq x \leq r$, then $r - x \in Core(v)$, where (N,v) is the reward game with respect to r.*

Proof: Let $S \subseteq N$. We have

$$\begin{aligned} v(S) &= \max\{r(S \cap T) - c(T) \mid T \subseteq N\} \\ &\leq \max\{r(S \cap T) - x(T) \mid T \subseteq N\} \\ &= r(S) - x(S). \end{aligned}$$

Efficiency follows from $v(N) \geq r(N) - c(N) = r(N) - x(N)$. □

Clearly, the existence of a core element for the reward game may depend on the reward vector. Let us call a cost game (N,c) *strongly balanced* if the reward game (N,v) has a non-empty core for every non-negative reward vector r.

Theorem 5 *Let (N,c) be a strongly balanced cost game, let $r \geq 0$ be a reward vector, and let (N,v) be the corresponding reward game. Then the z-allocation lies in $Core(v)$.*

Proof: It is sufficient to prove that $v^n(N) = 0$. It follows from lemma 1 that $v^n(N) = v^n(N\backslash\{i\})$ for all $i \in N$. Hence, the only possible core allocation for (N, v^n) is the 0-allocation. Since (N, c) is strongly balanced this must indeed be a core allocation. It follows that $v^n(N) = 0$. □

A game (N, v) is called *convex* if $v(S) + v(T) \leq v(S \cap T) + v(S \cup T)$ for all $S, T \subseteq N$. It is called *concave* if $v(S) + v(T) \geq v(S \cap T) + v(S \cup T)$ for all $S, T \subseteq N$.

Theorem 6 *Concave cost games are strongly balanced.*

Proof: Let (N, c) be a concave game, let $r \geq 0$ be a reward vector, and let the game (N, v) be the corresponding reward game. We will prove that (N, v) is convex. Let $S, T \subseteq N$. Choose $U, V \subseteq N$ such that $v(S) = r(S \cap U) - c(U)$ and $v(T) = r(T \cap V) - c(V)$. Convexity of the game (N, v) follows from

$$\begin{aligned}
v(S) + v(T) &= r(S \cap U) + r(T \cap V) - c(U) - c(V) \\
&\leq r(S \cap U) + r(T \cap V) - c(U \cap V) - c(U \cup V) \\
&= r(S \cap T \cap U \cap V) - c(U \cap V) + \\
&\quad r((S \cap U) \cup (T \cap V)) - c(U \cup V) \\
&\leq r(S \cap T \cap U \cap V) - c(U \cap V) + \\
&\quad r((S \cup T) \cap (U \cup V)) - c(U \cup V) \\
&\leq v(S \cap T) + v(S \cup T).
\end{aligned}$$

Since convex games have a non-empty core, the theorem follows. □

3 VWST games defined on graphs

Up to here we assumed that it was possible to establish any link between two customers or between a customer and the supplier. Let $G = (N \cup \{0\}, E)$ be a connected undirected graph. In this section we assume that only the edges in E can be established and that the cost of establishing such a link is given by a non-negative cost function d on the edges in E. Let us call the graph G *Steiner balanced* if every VWST game defined on G has a non-empty core, regardless of the cost function d, the rewards of the players and the location of the supplier. It follows from theorem 5 that the z-allocation is always a core element for a VWST game if it is defined on a graph which is Steiner balanced.

Suppose we are given a graph G which is not Steiner balanced. Then it is possible to define a VWST game (N, v) on G such that it has an empty core. Hence, the z-allocation cannot be efficient for this game, and it follows that $v^n(N) > 0$. According to lemma 1 the game (N, v^n) also satisfies $v^n(N) = v^n(N \backslash \{i\})$ for all $i \in N$. This shows that there are several optimal trees for the grand coalition, which are all embedded in G. Furthermore, the collection of optimal trees cannot be just any collection of trees. For instance, if two trees have only the root-vertex in common, then they cannot both be optimal, since the superposition of two such trees gives a strictly better tree. In the following we will give a refinement of this observation, and this will lead to necessary conditions on a graph such that it is not Steiner balanced.

Let us first introduce some terminology for trees. Let $\Gamma = (V, E)$ be a tree-graph, and let $0 \in V$ be a special vertex of this tree, called the root-vertex. A vertex $i \in V$ is called a *leaf* of Γ if $i \neq 0$ and if the degree of i is 1. Vertex $i \in V$ is called a descendant of $j \in V$ if the (unique) path from 0 to i contains j. Two vertices $i, j \in V$ are called *adjacent* if $\{i, j\} \in E$. Vertex i is called a *child* of $j \in V$ if i and j are adjacent and if i is a descendant of j. Vertex j is called the *father* of i. Let $D(i) \subseteq V$ denote the set of descendants of i. Note that $i \in D(i)$. The subgraph of Γ with vertex-set $D(i)$ is again a tree, and this tree is called the *subtree of* Γ *rooted at* i.

Let $G = (V, E)$ be a graph, and let $\mathcal{T}$ be a collection of trees embedded in G. The collection $\mathcal{T}$ is said to have the *merger property* if it satisfies the following conditions.

ROOT There is a special vertex which is a vertex of every tree in $\mathcal{T}$. This vertex is designated as the root-vertex.

NE The collection $\mathcal{T}$ is non-empty and the tree consisting of the root alone is not in $\mathcal{T}$.

MERGE If i is a vertex of both $\Gamma \in \mathcal{T}$ and $\Omega \in \mathcal{T}$, and if j is a child of i in the tree Γ, then the subtree of Γ rooted at j contains another vertex which is also in Ω.

LEAF For every vertex $i \in V$ that is a leaf in some tree of $\mathcal{T}$, there is another tree in $\mathcal{T}$ that does not contain i at all.

We say that the graph G has the merger property if it contains a collection of trees with the merger property.

Theorem 7 *If a graph $G = (V, E)$ does not have the merger property, then it is Steiner balanced.*

Proof: Suppose that G is not Steiner balanced. We will prove that G has the merger property. We have observed earlier that it is possible to define a VWST game (N, v) on this graph such that $v(N) > 0$ and $v(N) = v(N \backslash \{i\})$ for all $i \in N$. Here the set N consists of all vertices of V except one, and the remaining vertex is designated as the root. Call an optimal tree for coalition N minimal if there is no other optimal tree whose vertex-set is a proper subset of the first tree. Let $\mathcal{T}$ denote the collection of all optimal trees for coalition N that are minimal. Trivially, the conditions **ROOT** and **NE** are satisfied by $\mathcal{T}$.

Let Γ and Ω be two trees in $\mathcal{T}$, and let i be a vertex of both trees. Assume that i is not a leaf in Γ and let j be a child of i in the tree Γ. Denote the subtree of Γ rooted at j by Γ_j. Note that the sum of the rewards of the vertices in Γ_j minus the costs of the edges in this tree is strictly greater than $d(i, j)$, since otherwise the subtree could be deleted from Γ without reducing the value of Γ, contradicting the minimality of Γ. Now, suppose that Ω does not contain any vertex of Γ_j. Then the complete subtree Γ_j can be attached to vertex i, thereby increasing the value of the tree Ω, a contradiction. This proves that $\mathcal{T}$ satisfies condition **MERGE**.

Finally, suppose that $i \in N$ is a leaf of $\Gamma \in \mathcal{T}$. Note that the reward of i is strictly positive, since otherwise i could be deleted from Γ without reducing the value of the tree, which contradicts the minimality of Γ. Let Ω be a minimal optimal tree for coalition $N \backslash \{i\}$. The tree Ω cannot contain vertex i, since otherwise

$$v(N) \geq v(N \backslash \{i\}) + r_i > v(N \backslash \{i\} = v(N),$$

a contradiction. This proves that $\mathcal{T}$ satisfies condition **LEAF**. □

Theorem 8 *Every 5-persons VWST game has a non-empty core.*

Proof: We show that a graph which is not Steiner balanced must have at least 7 vertices. Let G be a graph which is not Steiner balanced. Let $\mathcal{T}$ be a collection of trees in G with the merger property. Denote the root by 0. Choose $\Gamma_1 \in \mathcal{T}$, and let i be adjacent to 0 in this tree. The subtree rooted at i contains at least two leaves. This is seen as follows. Suppose the subtree has only one leaf, i.e. the subtree is a path, with say l as an endpoint. Let Ω

be an arbitrary tree of $\mathcal{T}$. By applying condition **MERGE** repeatedly, we see that Ω contains l. Since Ω was chosen arbitrarily, we conclude that all trees in $\mathcal{T}$ contain l. This contradicts condition **LEAF**. Hence, the subtree of Γ_1 rooted at i has at least two leaves, and it follows that Γ_1 consists of at least 4 vertices, namely 0, i, and at least two leaves, say l_1 and l_2. Denote the vertex set of Γ_1 by V_1.

By condition **LEAF**, there exists $\Gamma_2 \in \mathcal{T}$ that does not contain l_1. By condition **MERGE**, the tree Γ_2 cannot contain the predecessor of l_1 in Γ_1 either. Denote the vertex set of Γ_2 by V_2. Of course, also Γ_2 contains at least 4 vertices. Since V_1 contains at least two vertices which are not in V_2, it follows already that

$$|V| \geq |V_1 \cup V_2| \geq |V_2| + 2 \geq 6.$$

We want to show however that $|V| \geq 7$.

If Γ_2 has at least 5 vertices, we are finished. So assume that $|V_2| = 4$. We know that Γ_2 has at least two vertices which are not in Γ_1. Hence, it has at most two vertices in common with Γ_1. Consequently, Γ_1 contains at least two vertices which are not in Γ_2, and it follows that

$$|V| \geq |V_1 \cup V_2| \geq |V_1| + 2 \geq 6.$$

Again, if Γ_1 has at least 5 vertices, we are finished. So assume that also $|V_1| = 4$. Now we conclude that Γ_1 and Γ_2 have exactly two vertices in common. One of these vertices is the root 0. The other vertex in common must be a leaf in both trees, since otherwise a contradiction with condition **MERGE** would occur. Denote this common leaf by l. By condition **LEAF** it follows that there must be a third tree in $\mathcal{T}$, say Γ_3, that does not contain l. Let f_1 denote the father of l in the tree Γ_1, and let f_2 denote the father of l in Γ_2. These fathers must be different vertices, and by condition **MERGE** it follows that Γ_3 cannot contain f_1 and f_2 either. It follows that Γ_3 can have at most 3 vertices in common with $V_1 \cup V_2$. Since Γ_3 has at least 4 vertices, we conclude that Γ_3 has at least 1 vertex which is neither in Γ_1 nor in Γ_2. This means that the graph G has at least 7 vertices. □

Theorem 9 *Let $G = (V, E)$ be a tree-graph. Then G is Steiner balanced.*

Proof : Suppose G is not Steiner balanced. Then let $\mathcal{T}$ be a collection of trees embedded in G with the merger property. Choose $\Gamma \in \mathcal{T}$ with a minimal number of vertices, and let l be a leaf of Γ. By condition **LEAF** there is another tree, say $\Omega \in \mathcal{T}$, that does not contain l. We will show that there are vertices i and j such that i is the father of j in Γ, while j is also a vertex of Ω and i is not. Assume that this is not true, i.e. if a vertex k of Γ is not in Ω, then also none of its children is in Ω. By repeated application of this assumption it follows that if a vertex k of Γ is not in Ω, then none of the vertices in the subtree of Γ rooted at k is in Ω. We know that l is not in Ω. Let k denote the first vertex on the path from 0 to l in Γ which is not in Ω. Then the whole subtree of Γ rooted at k contains no vertex of Ω. By condition **MERGE** it follows that the father of k cannot be in Ω either, and we have derived a contradiction.

Now choose father i and child j in Γ such that j is in Ω and i is not. The paths from 0 to j in Γ and Ω are not the same, since the path in Γ contains vertex i and the path in Ω does not. Then G cannot be a tree-graph, since paths between vertices in a tree are unique. □

An alternative proof of theorem 9 is obtained if we use a result by Granot et al. (1996) which states that a monotone minimum cost spanning tree game is concave if it is defined on a tree-graph. According to theorem 6 such a game is strongly balanced, and hence the corresponding VWST game has a non-empty core. It even follows from the proof of theorem 6 that such a VWST game is convex.

We conlude with an open problem. We were unable to prove (nor disprove) that a VWST game has a non-empty core in case the underlying graph is series-parallel. We tried to prove directly that a graph having the merger property cannot be series-parallel. We also tried to formulate a VWST game defined on a series-parallel graph as a linear production game (Owen (1975)) or as a generalized linear production game (Granot (1986)) by using a description of the arborescence polytope for series-parallel graphs (Goemans (1994)). Such an approach works fine for the directed Steiner tree game (see Skorin-Kapov (1992)), but it failed for the VWST game. Finally, we tried to derive the result by inspection of the linear time algorithm (Wald and Colbourn 1982) that solves the directed Steiner tree problem in case the underlying graph is series-parallel. Also this approach was unsuccessful. Note that if a series-parallel graph is Steiner balanced, then a core element for the associated VWST game can be computed in $\mathcal{O}(n^2)$ elementary operations, since computation of the z-allocation requires the determination of $v^k(N)$ for $k = 0, 1, \ldots, n$, and $v^k(N)$ can be computed in linear time.

We conjecture that a VWST game has a non-empty core if it is defined on a series-parallel graph. This conjecture is based mainly on our fruitless efforts to construct a series-parallel graph with the merger property. In fact, these efforts have led us to the stronger conjecture that a graph has the merger property if and only if it contains the graph of figure 1 as a minor. If this conjecture is true, then this completely settles the question of which graphs are Steiner balanced and which are not.

References

Van Bokhoven M (1994) Voluntary connection to a source. Bachelor Thesis, University of Tilburg.

Goemans MX (1994) Arborescence polytopes for series-parallel graphs, *Discrete Applied Mathematics*, 51:277-289.

Granot D, and Huberman G (1981) Minimum cost spanning tree games, *Mathematical Programming*, 21:1–18.

Granot D (1986) A generalized linear production model: a unifying model, *Mathematical Programming*, 34:212-222.

Granot D, Maschler M, Owen G, and Zhu WR (1996) The kernel/nucleolus of a standard tree game, *International Journal of Game Theory*, 25:219–244.

Owen G (1975) On the core of linear production games, *Mathematical Programming*, 9:358–370.

Segev A (1987) The node-weighted Steiner tree problem, *Networks*, 17:1–18.

Skorin-Kapov D (1992) A fixed-cost spanning forest problem on series-parallel graphs. *Proceedings of the second Conference on Operational Research*, KOI'92, Rovinj (1992), 85–94.

Wald JA, and Colbourn CJ (1983) Steiner trees, partial 2-trees, and minimum IFI networks. *Networks*, 13:159–167.