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Publication date:
2017

Document Version
Early version, also known as pre-print

Link to publication in Tilburg University Research Portal

Citation for published version (APA):
Dietzenbacher, B., Borm, P., & Estevez Fernandez, M. A. (2017). *NTU-Bankruptcy Problems: Consistency and the Relative Adjustment Principle.* (CentER Discussion Paper; Vol. 2017-044). CentER, Center for Economic Research.

CentER

Discussion Paper

No. 2017-044

NTU-BANKRUPTCY PROBLEMS: CONSISTENCY AND THE RELATIVE ADJUSTMENT PRINCIPLE

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6 November 2017

ISSN 0924-7815
ISSN 2213-9532

Tilburg University

NTU-Bankruptcy Problems: Consistency and the Relative Adjustment Principle

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November 6, 2017

Abstract

This paper axiomatically studies bankruptcy problems with nontransferable utility by adequately generalizing and analyzing properties for bankruptcy rules. In particular, we discuss several consistency notions and introduce the class of parametric bankruptcy rules. Moreover, we introduce the class of adjusted bankruptcy rules and study the relative adjustment principle based on relative symmetry, truncation invariance, and minimal rights first.

Keywords: NTU-bankruptcy problem, axiomatic analysis, consistency, parametric bankruptcy rules, adjusted bankruptcy rules, relative adjustment principle
JEL classification: C79, D63, D74

1 Introduction

A bankruptcy problem with nontransferable utility, shortly an NTU-bankruptcy problem, arises when a set of claimants have individual and incompatible claims on a divisible and deficient bundle of resources which generate utility. The corresponding set of induced utility allocations constitutes the estate of the bankruptcy problem and bankruptcy rules assign to each such a bankruptcy problem a feasible utility allocation. NTU-bankruptcy problems form a natural generalization of TU-bankruptcy problems where the assumption of linear and transferable utility is dropped. TU-bankruptcy problems are well-studied in the literature (cf. Thomson (2003), Thomson (2013), and Thomson (2015)) and the question arises whether and how bankruptcy theory can be extended to NTU-bankruptcy problems. However, this passage is in general fraught with difficulties.

Orshan, Valenciano, and Zarzuelo (2003), Estévez-Fernández, Borm, and Fiestras-Janeiro (2014), and Dietzenbacher (2017) studied NTU-bankruptcy problems from a game theoretic perspective by defining an appropriate coalitional bankruptcy game and focusing on the structure of the core. Instead, we continue on the axiomatic approach of Dietzenbacher, Estévez-Fernández, Borm, and Hendrickx (2016) by formulating some appropriate properties for bankruptcy rules and studying their implications.

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Dietzenbacher et al. (2016) explored proportionality, equality, and duality in the context of NTU-bankruptcy problems and introduced the proportional rule, the constrained relative equal awards rule, and the constrained relative equal losses rule. They extended axiomatic characterizations by adequately generalizing the corresponding properties for TU-bankruptcy rules to NTU-bankruptcy rules. In particular, they defined the relative symmetry property which imposes a relatively equal treatment of claimants with relatively equal claims, i.e. equal claims in relation to their utopia values. Moreover, they defined the property of truncation invariance, which imposes invariance of the prescribed allocation under truncation of the claims by the utopia values.

A well-studied rule for bankruptcy problems with transferable utility is the so-called Talmud rule. Aumann and Maschler (1985) showed that the Talmud rule is the unique TU-bankruptcy rule satisfying consistency and the contested garment principle. This paper studies generalizations of these two concepts to bankruptcy problems with nontransferable utility. The question whether there also exists a unique NTU-bankruptcy rule satisfying both generalized properties is not addressed here, but is of interest for future research.

Consistency requires that application of a bankruptcy rule to a subproblem leads to the same payoffs for the involved claimants as within the original bankruptcy problem. We examine the relation of the proportional rule, the constrained relative equal awards rule, and the constrained relative equal losses rule with several consistency notions and introduce the class of parametric bankruptcy rules. However, the design of subproblems of NTU-bankruptcy problems is not straightforward or trivial, and it turns out that different modeling choices have different consequences.

The contested garment principle for TU-bankruptcy rules describes a standard solution for bankruptcy problems with two claimants where they first concede the minimal rights to each other and subsequently divide the remaining estate equally. There, the minimal right of a claimant (cf. Curiel, Maschler, and Tijs (1987)) is defined as the maximal payoff within the estate when all other claimants are allocated their claims. To adequately generalize this two-claimant procedure to the new relative adjustment principle for NTU-bankruptcy rules, we define the minimal rights first property and introduce the class of adjusted bankruptcy rules.

The minimal rights first property requires that application of a bankruptcy rule to the remaining bankruptcy problem when all claimants are first allocated their minimal rights leads to the same payoff allocation as the original bankruptcy problem. It turns out that the proportional rule, the constrained relative equal awards rule, and the constrained relative equal losses rule do not satisfy minimal rights first. Inspired by Thomson and Yeh (2008), we introduce the truncation operator and minimal rights operator which 'force' bankruptcy rules to satisfy truncation invariance and minimal rights first, respectively. All bankruptcy rules that result from applying these operators to relative symmetric bankruptcy rules satisfy the relative adjustment principle.

This paper is organized in the following way. In Section 2, we provide an overview of NTU-bankruptcy theory based on Dietzenbacher et al. (2016). Section 3 discusses several consistency notions and introduces the class of parametric bankruptcy rules. In Section 4, we introduce the class of adjusted bankruptcy rules and studies the relative adjustment principle.

2 Preliminaries

Let N be a nonempty and finite set of *claimants*. The collection of all subsets of N is denoted by $2^N = \{S \mid S \subseteq N\}$. For any set of payoff allocations $E \subseteq \mathbb{R}^N_+$,

- the *comprehensive hull* is given by $\text{comp}(E) = \{x \in \mathbb{R}^N_+ \mid \exists_{y \in E} : y \geq x\}$;
- the *weak upper contour set* is given by $\text{WUC}(E) = \{x \in \mathbb{R}^N_+ \mid \neg\exists_{y \in E} : y > x\}$;
- the *weak Pareto set* is given by $\text{WP}(E) = \{x \in E \mid \neg\exists_{y \in E} : y > x\}$;
- the *strong Pareto set* is given by $\text{SP}(E) = \{x \in E \mid \neg\exists_{y \in E, y \neq x} : y \geq x\}$.

Note that $\text{SP}(E) \subseteq \text{WP}(E) \subseteq \text{WUC}(E)$. A set of payoff allocations $E \subseteq \mathbb{R}^N_+$ is called *comprehensive* if $E = \text{comp}(E)$, and *nonleveled* if $\text{SP}(E) = \text{WP}(E)$.

A *bankruptcy problem with nontransferable utility* (cf. Dietzenbacher et al. (2016)) is a triple (N, E, c) in which $E \subseteq \mathbb{R}^N_+$ is a nonempty, closed, bounded, comprehensive, and nonleveled *estate*, and $c \in \text{WUC}(E)$ is a vector of *claims*. Let BR^N denote the class of all bankruptcy problems with claimant set N. For convenience, an NTU-bankruptcy problem is denoted by $(E, c) \in \text{BR}^N$.

Let $(E, c) \in \text{BR}^N$. The vector of *utopia values* $u^E \in \mathbb{R}^N_+$ is given by

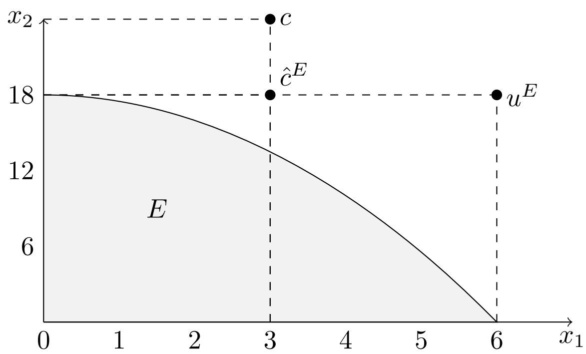
$$u^E = (\max\{x_i \mid x \in E\})_{i \in N}.$$

The vector of *truncated claims* $\hat{c}^E \in \mathbb{R}^N_+$ is given by

$$\hat{c}^E = \left(\min\{c_i, u^E_i\}\right)_{i \in N}.$$

Note that $(E, \hat{c}^E) \in \text{BR}^N$.

Example 1.
Let $N = \{1, 2\}$ and consider the bankruptcy problem $(E, c) \in \text{BR}^N$ in which $E = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 2x_2 \leq 36\}$ and $c = (3, 24)$. We have $u^E = (6, 18)$ and $\hat{c}^E = (3, 18)$.



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A *bankruptcy rule* f on BR^N assigns to any $(E, c) \in \text{BR}^N$ a payoff allocation $f(E, c) \in \text{WP}(E)$ for which $f(E, c) \leq c$. A bankruptcy rule f on BR^N satisfies

- *relative symmetry* if $f_i(E, c)u^E_j = f_j(E, c)u^E_i$ for all $(E, c) \in \text{BR}^N$ and any $i, j \in N$ with $c_i u^E_j = c_j u^E_i$;
- *truncation invariance* if $f(E, c) = f(E, \hat{c}^E)$ for all $(E, c) \in \text{BR}^N$.

The *proportional rule* Prop on BR^N (cf. Dietzenbacher et al. (2016)) assigns to any $(E,c) \in \mathrm{BR}^N$ the payoff allocation

$$\mathrm{Prop}(E,c) = \lambda^{E,c} c,$$

where $\lambda^{E,c} \in [0,1]$ is such that $\mathrm{Prop}(E,c) \in \mathrm{WP}(E)$. The proportional rule satisfies relative symmetry, but does not satisfy truncation invariance.

The *constrained relative equal awards rule* CREA on BR^N (cf. Dietzenbacher et al. (2016)) assigns to any $(E,c) \in \mathrm{BR}^N$ the payoff allocation

$$\mathrm{CREA}(E,c) = \left(\min\{c_i, \alpha^{E,c} u_i^E\}\right)_{i \in N},$$

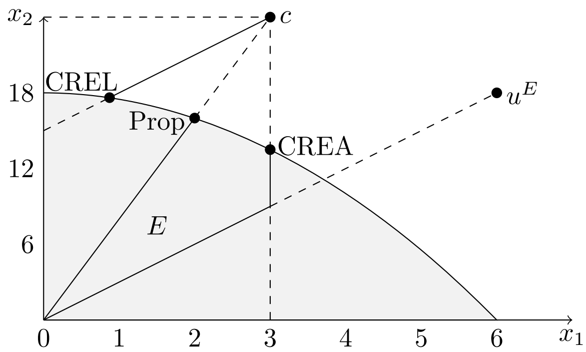
where $\alpha^{E,c} \in [0,1]$ is such that $\mathrm{CREA}(E,c) \in \mathrm{WP}(E)$. The constrained relative equal awards rule satisfies both relative symmetry and truncation invariance.

The *constrained relative equal losses rule* CREL on BR^N (cf. Dietzenbacher et al. (2016)) assigns to any $(E,c) \in \mathrm{BR}^N$ with $E \neq \{0_N\}$ the payoff allocation

$$\mathrm{CREL}(E,c) = \left(\max\{0, c_i - \beta^{E,c} u_i^E\}\right)_{i \in N},$$

where $\beta^{E,c} \in \mathbb{R}_+$ is such that $\mathrm{CREL}(E,c) \in \mathrm{WP}(E)$. The constrained relative equal losses rule satisfies relative symmetry, but does not satisfy truncation invariance.

Example 2.
Let $N = \{1,2\}$ and consider the bankruptcy problem $(E,c) \in \mathrm{BR}^N$ in which $E = \{x \in \mathbb{R}_+^N \mid x_1^2 + 2x_2 \leq 36\}$ and $c = (3,24)$ as in Example 1. We have $\lambda^{E,c} = \frac{2}{3}$, $\alpha^{E,c} = \frac{3}{4}$, and $\beta^{E,c} = 1 - \frac{1}{6}\sqrt{15}$. This means that $\mathrm{Prop}(E,c) = (2,16)$, $\mathrm{CREA}(E,c) = (3, 13\frac{1}{2})$, and $\mathrm{CREL}(E,c) = (\sqrt{15} - 3, 3\sqrt{15} + 6)$.



△

3 Consistency

In this section, we discuss several consistency notions and introduce the class of parametric bankruptcy rules. Following Thomson (2011), the consistency principle can be stated as follows. Consider a bankruptcy problem and the corresponding payoff allocation assigned by a particular bankruptcy rule. Suppose that some claimants leave with their allocated payoffs and that the remaining claimants reevaluate their allocated payoffs by examining the induced subproblem. The bankruptcy rule is called consistent if it prescribes for this subproblem the same payoffs for the involved claimants.

For TU-bankruptcy problems, the estate of such a subproblem can simply be defined as the original estate subtracted with the allocated payoffs to the leaving claimants (cf. Aumann and Maschler (1985)). For NTU-bankruptcy problems, the design of such a subproblem is not straightforward or trivial. We discuss several ways to generalize the consistency property for TU-bankruptcy rules. In any case, we need to enlarge the domain of bankruptcy rules.

A natural option is to convert the induced subproblem into a new bankruptcy problem for the remaining claimants in which the estate is defined as the part of the original estate where all leaving claimants are allocated their corresponding payoffs. For this, we need to extend the domain of bankruptcy rules to bankruptcy problems for any nonempty subset of claimants. Formally, a bankruptcy rule f on $\mathrm{BR} = \bigcup_{S \in 2^N \setminus \{\emptyset\}} \mathrm{BR}^S$ assigns to any $(E, c) \in \mathrm{BR}^S$ with $S \in 2^N \setminus \{\emptyset\}$ a payoff allocation $f(E, c) \in \mathrm{WP}(E)$ for which $f(E, c) \leq c$.

Let $(E, c) \in \mathrm{BR}^N$, let $x \in \mathbb{R}^N_+$, and let $S \in 2^N \setminus \{\emptyset\}$. The set of payoff allocations $E^x_S \subseteq \mathbb{R}^S_+$ is given by

$$E^x_S = \{y \in \mathbb{R}^S_+ \mid (y, x_{N \setminus S}) \in E\}.$$

Note that $(E^{f(E,c)}_S, c_S) \in \mathrm{BR}^S$ for any bankruptcy rule f on BR^N.

Definition 3.1 (Strong Consistency).
A bankruptcy rule f on BR satisfies *strong consistency* if $f_S(E, c) = f(E^{f(E,c)}_S, c_S)$ for all $(E, c) \in \mathrm{BR}^N$ and any $S \in 2^N \setminus \{\emptyset\}$.

The weaker property where only subproblems for two remaining claimants are considered is called bilateral consistency.

Definition 3.2 (Bilateral Consistency).
A bankruptcy rule f on BR satisfies *bilateral consistency* if $f_S(E, c) = f(E^{f(E,c)}_S, c_S)$ for all $(E, c) \in \mathrm{BR}^N$ and any $S \in 2^N$ with $|S| = 2$.

In other words, a bankruptcy rule is bilaterally consistent if it assigns to each two-claimant subproblem the same payoffs for the remaining claimants as within the original bankruptcy problem. This principle can also be applied in reverse direction. Consider a bankruptcy problem and a corresponding feasible payoff allocation. Suppose that for each two-claimant subproblem a bankruptcy rule prescribes the corresponding payoffs within this allocation. Then the bankruptcy rule is called conversely consistent (cf. Thomson (2011)) if it assigns this payoff allocation to the original bankruptcy problem.

Definition 3.3 (Converse Consistency).
A bankruptcy rule f on BR satisfies *converse consistency* if $f(E, c) = x$ for all $(E, c) \in \mathrm{BR}^N$ and any $x \in \mathrm{WP}(E)$ with $x \leq c$ for which $x_S = f(E^x_S, c_S)$ for all $S \in 2^N$ with $|S| = 2$.

If a bilateral consistent bankruptcy rule coincides with a conversely consistent bankruptcy rule on the class of two-claimant bankruptcy problems, then the rules coincide for any bankruptcy problem. This type of result is known as an elevator lemma.

Lemma 3.1 (Elevator Lemma).
Let f and g be two bankruptcy rules on BR. *If f satisfies bilateral consistency, g satisfies converse consistency, and $f(E, c) = g(E, c)$ for all $(E, c) \in \mathrm{BR}^S$ with $S \in 2^N$ and $|S| = 2$, then $f = g$.*

Proof. Let $(E, c) \in \mathrm{BR}^N$ and let $x = f(E, c)$. Since f satisfies bilateral consistency, we have $x_S = f(E^x_S, c_S)$ for all $S \in 2^N$ with $|S| = 2$. This means that $x_S = g(E^x_S, c_S)$ for all $S \in 2^N$ with $|S| = 2$. Since g satisfies converse consistency, this implies that $g(E, c) = x$. Hence, $f(E, c) = g(E, c)$. □

For a bankruptcy rule which satisfies both bilateral consistency and converse consistency, the Elevator Lemma can be used to extend axiomatic characterizations from bankruptcy problems with two claimants to arbitrary populations. An example of such a bankruptcy rule is the proportional rule.

Lemma 3.2.
The proportional rule satisfies strong consistency.

Proof. Let $(E,c) \in \mathrm{BR}^N$ and let $S \in 2^N \setminus \{\emptyset\}$. We have $\mathrm{Prop}_S(E,c) = \lambda^{E,c} c_S$ and

$$\mathrm{Prop}(E_S^{\mathrm{Prop}(E,c)}, c_S) = \lambda^{E_S^{\mathrm{Prop}(E,c)}, c_S} c_S,$$

where $\lambda^{E,c} \in [0,1]$ is such that $\mathrm{Prop}(E,c) \in \mathrm{WP}(E)$ and $\lambda^{E_S^{\mathrm{Prop}(E,c)}, c_S} \in [0,1]$ is such that

$$\mathrm{Prop}(E_S^{\mathrm{Prop}(E,c)}, c_S) \in \mathrm{WP}(E_S^{\mathrm{Prop}(E,c)}).$$

Since $\mathrm{Prop}_S(E,c) \in E_S^{\mathrm{Prop}(E,c)}$, we have $\mathrm{Prop}_S(E,c) \leq \mathrm{Prop}(E_S^{\mathrm{Prop}(E,c)}, c_S)$. Since E is nonleveled and

$$\Big(\mathrm{Prop}(E_S^{\mathrm{Prop}(E,c)}, c_S), \mathrm{Prop}_{N\setminus S}(E,c)\Big) \in E,$$

this means that $\mathrm{Prop}_S(E,c) = \mathrm{Prop}(E_S^{\mathrm{Prop}(E,c)}, c_S)$. Hence, the proportional rule satisfies strong consistency. □

Lemma 3.3.
The proportional rule satisfies converse consistency.

Proof. Let $(E,c) \in \mathrm{BR}^N$ and let $x \in \mathrm{WP}(E)$ with $x \leq c$ be such that $x_S = \mathrm{Prop}(E_S^x, c_S)$ for all $S \in 2^N$ with $|S| = 2$. We have $\mathrm{Prop}(E,c) = \lambda^{E,c} c$. Moreover, we have $x_S = \lambda^{E_S^x, c_S} c_S$ for all $S \in 2^N$ with $|S| = 2$, which means that $x = tc$ for some $t \in [0,1]$. Since E is nonleveled, this means that $\mathrm{Prop}(E,c) = x$. Hence, the proportional rule satisfies converse consistency. □

Theorem 3.4.
Any axiomatic characterization of the proportional rule for two-claimant bankruptcy problems yields an axiomatic characterization of the proportional rule for any bankruptcy problem if either bilateral consistency or converse consistency is required in addition.[1]

Proof. We know from Lemma 3.2 and Lemma 3.3 that the proportional rule satisfies bilateral consistency and converse consistency. Let f be a bankruptcy rule on BR satisfying the properties in the axiomatic characterization of the proportional rule on the class of two-claimant bankruptcy problems and either bilateral consistency or converse consistency. Then, we have $f(E,c) = \mathrm{Prop}(E,c)$ for all $(E,c) \in \mathrm{BR}^S$ with $S \in 2^N$ and $|S| = 2$. Since the proportional rule satisfies bilateral consistency and converse consistency, we know from Lemma 3.1 that $f = \mathrm{Prop}$. □

In particular, we can derive new characterizations of the proportional rule from the work of Dietzenbacher et al. (2016) using Theorem 3.4, by requiring the corresponding properties in the axiomatic characterizations for the class of two-claimant bankruptcy problems and adding bilateral or converse consistency.

[1]This type of theorem can be formulated for any bankruptcy rule satisfying bilateral consistency and converse consistency.

Contrary to the class of TU-bankruptcy problems, the constrained relative equal awards rule and the constrained relative equal losses rule do not satisfy strong consistency on the class of NTU-bankruptcy problems. A possible way out is to restrict strong consistency to the subproblems for which the ratio of the utopia values is equal to the ratio in the original bankruptcy problem.

Definition 3.4 (Restricted Consistency)**.**
A bankruptcy rule f on BR satisfies *restricted consistency* if $f_S(E,c) = f(E_S^{f(E,c)}, c_S)$ for all $(E,c) \in \mathrm{BR}^N$ and any $S \in 2^N \setminus \{\emptyset\}$ for which $u^{E_S^{f(E,c)}} = tu_S^E$ for some $t \in [0,1]$.

O'Neill (1982) described bankruptcy rules which are independent of claimants with zero claims, i.e., bankruptcy rules which are consistent with respect to subproblems where only claimants with zero claims leave. Following Thomson (2003), we call this property limited consistency.

Definition 3.5 (Limited Consistency)**.**
A bankruptcy rule f on BR satisfies *limited consistency* if $f_S(E,c) = f(E_S^{f(E,c)}, c_S)$ for all $(E,c) \in \mathrm{BR}^N$ and any $S \in 2^N \setminus \{\emptyset\}$ for which $c_{N\setminus S} = 0_{N\setminus S}$.

Note that both strong consistency and restricted consistency generalize the consistency notion for TU-bankruptcy rules. Moreover, strong consistency is stronger than restricted consistency, which in turn is stronger than limited consistency.

Proposition 3.5.
The constrained relative equal awards rule satisfies restricted consistency.

Proof. Let $(E,c) \in \mathrm{BR}^N$ and let $S \in 2^N \setminus \{\emptyset\}$ be such that $u^{E_S^{\mathrm{CREA}(E,c)}} = tu_S^E$ for some $t \in [0,1]$. We have $\mathrm{CREA}_i(E,c) = \min\{c_i, \alpha^{E,c}u_i^E\}$ for all $i \in S$ and

$$\begin{aligned}\mathrm{CREA}(E_S^{\mathrm{CREA}(E,c)}, c_S) &= (\min\{c_i, \alpha^{E_S^{\mathrm{CREA}(E,c)},c_S} u_i^{E_S^{\mathrm{CREA}(E,c)}}\})_{i\in S}\\ &= (\min\{c_i, t\alpha^{E_S^{\mathrm{CREA}(E,c)},c_S} u_i^E\})_{i\in S},\end{aligned}$$

where $\alpha^{E,c} \in [0,1]$ is such that $\mathrm{CREA}(E,c) \in \mathrm{WP}(E)$ and $\alpha^{E_S^{\mathrm{CREA}(E,c)},c_S} \in [0,1]$ is such that

$$\mathrm{CREA}(E_S^{\mathrm{CREA}(E,c)}, c_S) \in \mathrm{WP}(E_S^{\mathrm{CREA}(E,c)}).$$

Since $\mathrm{CREA}_S(E,c) \in E_S^{\mathrm{CREA}(E,c)}$, we have $\mathrm{CREA}_S(E,c) \leq \mathrm{CREA}_S(E_S^{\mathrm{CREA}(E,c)}, c_S)$. Since E is nonleveled and

$$\Big(\mathrm{CREA}(E_S^{\mathrm{CREA}(E,c)}, c_S), \mathrm{CREA}_{N\setminus S}(E,c)\Big) \in E,$$

this means that $\mathrm{CREA}_S(E,c) = \mathrm{CREA}(E_S^{\mathrm{CREA}(E,c)}, c_S)$. Hence, the constrained relative equal awards rule satisfies restricted consistency. □

Proposition 3.6.
The constrained relative equal losses rule satisfies restricted consistency.

Proof. Let $(E,c) \in \mathrm{BR}^N$ with $E \neq \{0_N\}$ and let $S \in 2^N \setminus \{\emptyset\}$ be such that $u^{E_S^{\mathrm{CREL}(E,c)}} = tu_S^E$ for some $t \in [0,1]$. We have $\mathrm{CREL}_i(E,c) = \max\{0, c_i - \beta^{E,c} u_i^E\}$ for all $i \in S$ and

$$\begin{aligned}\mathrm{CREL}(E_S^{\mathrm{CREL}(E,c)}, c_S) &= (\max\{0, c_i - \beta^{E_S^{\mathrm{CREL}(E,c)}, c_S} u_i^{E_S^{\mathrm{CREL}(E,c)}}\})_{i \in S} \\ &= (\max\{0, c_i - t\beta^{E_S^{\mathrm{CREL}(E,c)}, c_S} u_i^E\})_{i \in S},\end{aligned}$$

where $\beta^{E,c} \in \mathbb{R}_+$ is such that $\mathrm{CREL}(E,c) \in \mathrm{WP}(E)$ and $\beta^{E_S^{\mathrm{CREL}(E,c)}, c_S} \in \mathbb{R}_+$ is such that

$$\mathrm{CREL}(E_S^{\mathrm{CREL}(E,c)}, c_S) \in \mathrm{WP}(E_S^{\mathrm{CREL}(E,c)}).$$

Since $\mathrm{CREL}_S(E,c) \in E_S^{\mathrm{CREL}(E,c)}$, we have $\mathrm{CREL}_S(E,c) \leq \mathrm{CREL}_S(E_S^{\mathrm{CREL}(E,c)}, c_S)$. Since E is nonleveled and

$$\Big(\mathrm{CREL}(E_S^{\mathrm{CREL}(E,c)}, c_S), \mathrm{CREL}_{N \setminus S}(E,c)\Big) \in E,$$

this means that $\mathrm{CREL}_S(E,c) = \mathrm{CREL}(E_S^{\mathrm{CREL}(E,c)}, c_S)$. Hence, the constrained relative equal losses rule satisfies restricted consistency. □

Converting subproblems induced by leaving claimants into new bankruptcy problems for the remaining claimants tends to lose characteristics of the original bankruptcy problems. Instead, one could also interpret the induced subproblem as the original bankruptcy problem where the payoffs of the leaving claimants are fixed. In a sense, the original bankruptcy problem is only reduced to a problem for the remaining claimants where the leaving claimants leave a footprint behind. To formalize this approach, we need to redefine bankruptcy rules on the domain of reduced bankruptcy problems.

A *reduced bankruptcy problem* is a quintuple (N, E, c, x, S) where $(E,c) \in \mathrm{BR}^N$, $x \in \mathbb{R}_+^N$ and $S \in 2^N \setminus \{\emptyset\}$ are such that $(E_S^x, c_S) \in \mathrm{BR}^S$. Let RBR^N denote the class of all reduced bankruptcy problems with claimant set N. For convenience, a reduced bankruptcy problem is denoted by $(E,c,x,S) \in \mathrm{RBR}^N$ and $(E,c,x,N) \in \mathrm{RBR}^N$ is abbreviated to $(E,c) \in \mathrm{RBR}^N$.

A bankruptcy rule f on RBR^N assigns to any reduced bankruptcy problem $(E,c,x,S) \in \mathrm{RBR}^N$ a payoff allocation $f(E,c,x,S) \in \mathrm{WP}(E)$ for which

$$f_S(E,c,x,S) \leq c_S \text{ and } f_{N \setminus S}(E,c,x,S) = x_{N \setminus S}.$$

Note that $(E, c, f(E,c), S) \in \mathrm{RBR}^N$ for any bankruptcy rule f on BR^N, any $(E,c) \in \mathrm{BR}^N$, and any $S \in 2^N \setminus \{\emptyset\}$.

The proportional rule Prop on RBR^N assigns to any $(E,c,x,S) \in \mathrm{RBR}^N$ the payoff allocation for which

$$\mathrm{Prop}_S(E,c,x,S) = \lambda^{E,c,x,S} c_S,$$

where $\lambda^{E,c,x,S} \in [0,1]$ is such that $\mathrm{Prop}(E,c,x,S) \in \mathrm{WP}(E)$.

The constrained relative equal awards rule CREA on RBR^N assigns to any $(E,c,x,S) \in \mathrm{RBR}^N$ the payoff allocation for which

$$\mathrm{CREA}_S(E,c,x,S) = \left(\min\{c_i, \alpha^{E,c,x,S} u_i^E\}\right)_{i \in S},$$

where $\alpha^{E,c,x,S} \in [0,1]$ is such that $\mathrm{CREA}(E,c,x,S) \in \mathrm{WP}(E)$.

The constrained relative equal losses rule CREL on RBR^N assigns to any $(E, c, x, S) \in \text{RBR}^N$ with $E \neq \{0_N\}$ the payoff allocation for which

$$\text{CREL}_S(E, c, x, S) = \left(\max\{0, c_i - \beta^{E,c,x,S} u_i^E\}\right)_{i \in S},$$

where $\beta^{E,c,x,S} \in \mathbb{R}_+$ is such that $\text{CREL}(E, c, x, S) \in \text{WP}(E)$.

We now introduce the footprint consistency property to describe bankruptcy rules which prescribe the same payoff allocation for the original bankruptcy problem as for any reduced bankruptcy problem in which the leaving claimants fix their allocated payoffs.

Definition 3.6 (Footprint Consistency)**.**
A bankruptcy rule f on RBR^N satisfies *footprint consistency* if $f(E, c) = f(E, c, f(E, c), S)$ for all $(E, c) \in \text{BR}^N$ and any $S \in 2^N \setminus \{\emptyset\}$.

Inspired by Young (1987), we introduce the class of parametric bankruptcy rules where the payoff allocated to a claimant only depends on its individual characteristics within the bankruptcy problem and a common parameter. It turns out that all parametric bankruptcy rules satisfy footprint consistency. Specific examples of parametric bankruptcy rules are the proportional rule, the constrained relative equal awards rule, and the constrained relative equal losses rule.

Definition 3.7 (Parametric Bankruptcy Rule)**.**
A bankruptcy rule f on RBR^N is *parametric* if there exists a function $r^f : \mathbb{R}^3_+ \to \mathbb{R}_+$ monotonic in its third argument for which $f_S(E, c, x, S) = (r^f(c_i, u_i^E, \theta^{E,c,x,S}))_{i \in S}$ for all $(E, c, x, S) \in \text{RBR}^N$ and some parameter $\theta^{E,c,x,S} \in \mathbb{R}_+$.

Theorem 3.7.
All parametric bankruptcy rules satisfy footprint consistency.

Proof. Let f be a parametric bankruptcy rule on RBR^N, let $(E, c) \in \text{BR}^N$ and let $S \in 2^N \setminus \{\emptyset\}$. Then, we have $f_{N \setminus S}(E, c) = f_{N \setminus S}(E, c, f(E, c), S)$. Moreover, we have $f_i(E, c) = r^f(c_i, u_i^E, \theta^{E,c})$ and $f_i(E, c, f(E, c), S) = r^f(c_i, u_i^E, \theta^{E,c,f(E,c),S})$ for all $i \in S$. Since r^f is monotonic in its third argument, this means that $f_S(E, c) \leq f_S(E, c, f(E, c), S)$ or $f_S(E, c) \geq f_S(E, c, f(E, c), S)$. Since E is nonleveled, this implies that $f(E, c) = f(E, c, f(E, c), S)$. Hence, f satisfies footprint consistency. □

Corollary 3.8.
The proportional rule, the constrained relative equal awards rule, and the constrained relative equal losses rule satisfy footprint consistency.

4 The Relative Adjustment Principle

In this section, we introduce the class of adjusted bankruptcy rules and study the relative adjustment principle. The relative adjustment principle, based on the contested garment principle for TU-bankruptcy problems (cf. Aumann and Maschler (1985)), first allocates the minimal rights to the claimants. Following Curiel et al. (1987) and Estévez-Fernández et al. (2014), the minimal right of a claimant is the maximal payoff within the estate when all other claimants are allocated their claims.

Let $(E,c) \in \mathrm{BR}^N$. The vector of *minimal rights* $m(E,c) \in \mathbb{R}^N_+$ is, for all $i \in N$, defined by

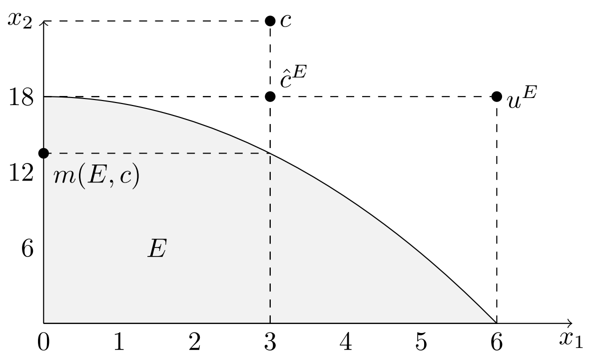
$$m_i(E,c) = \begin{cases} \max\{x \mid (x, c_{N\setminus\{i\}}) \in E\} & \text{if } (0, c_{N\setminus\{i\}}) \in E; \\ 0 & \text{if } (0, c_{N\setminus\{i\}}) \notin E. \end{cases}$$

We have $m(E,c) \in E$ and $m(E,c) \leq c$, which means that

$$((E - \{m(E,c)\})_+, c - m(E,c)) \in \mathrm{BR}^N.$$

Moreover, we have $m(E,c) \leq f(E,c) \leq \hat{c}^E$ for any bankruptcy rule f on BR^N.

Example 3.
Let $N = \{1,2\}$ and consider the bankruptcy problem $(E,c) \in \mathrm{BR}^N$ in which $E = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 2x_2 \leq 36\}$ and $c = (3,24)$ as in Example 1. We have $m(E,c) = (0, 13\frac{1}{2})$.



△

The following lemma derives some elementary relations between truncated claims and minimal rights.

Lemma 4.1.
Let $(E,c) \in \mathrm{BR}^N$. *Then*

(i) $\widehat{\hat{c}^E}^E = \hat{c}^E$;

(ii) $m((E - \{m(E,c)\})_+, c - m(E,c)) = 0_N$;

(iii) $m(E, \hat{c}^E) = m(E,c)$;

(iv) $\widehat{c - m(E,c)}^{(E - \{m(E,c)\})_+} = \hat{c}^E - m(E,c)$.

Proof. (i) Let $i \in N$. We can write

$$\widehat{\hat{c}^E}^E_i = \min\{\hat{c}^E_i, u^E_i\} = \min\{\min\{c_i, u^E_i\}, u^E_i\} = \min\{c_i, u^E_i\} = \hat{c}^E_i.$$

(ii) Suppose that there exists an $i \in N$ for which $m_i((E - \{m(E,c)\})_+, c - m(E,c)) > 0$. Then, we have

$$\big(m_i((E - \{m(E,c)\})_+, c - m(E,c)), (c - m(E,c))_{N\setminus\{i\}}\big) \in (E - \{m(E,c)\})_+.$$

This means that

$$\left(m_i((E-\{m(E,c)\})_+, c-m(E,c)) + m_i(E,c), c_{N\setminus\{i\}}\right) \in E.$$

This contradicts the definition of $m_i(E,c)$.

(iii) Let $i \in N$. If $\hat{c}^E_{N\setminus\{i\}} = c_{N\setminus\{i\}}$, we have $m_i(E,\hat{c}^E) = m_i(E,c)$ by definition. If $\hat{c}^E_{N\setminus\{i\}} \neq c_{N\setminus\{i\}}$, then $(0, c_{N\setminus\{i\}}) \notin E$, so $m_i(E,\hat{c}^E) = 0 = m_i(E,c)$.

(iv) Let $i \in N$. If $m_{N\setminus\{i\}}(E,c) = 0_{N\setminus\{i\}}$, we have $u_i^{(E-\{m(E,c)\})_+} = u_i^E - m_i(E,c)$ and we can write

$$\begin{aligned}
\widehat{(c-m(E,c))}_i^{(E-\{m(E,c)\})_+} &= \min\left\{c_i - m_i(E,c), u_i^{(E-\{m(E,c)\})_+}\right\} \\
&= \min\{c_i - m_i(E,c), u_i^E - m_i(E,c)\} \\
&= \min\{c_i, u_i^E\} - m_i(E,c) \\
&= \hat{c}_i^E - m_i(E,c).
\end{aligned}$$

Suppose that there exists a $j \in N \setminus \{i\}$ for which $m_j(E,c) > 0$. Then, we have $\hat{c}_i^E = c_i$ and $(m_j(E,c), c_{N\setminus\{j\}}) \in E$. Since E is comprehensive and $m(E,c) \leq c$, this means that $(c_i, m_{N\setminus\{i\}}(E,c)) \in E$, so $(c_i - m_i(E,c), 0_{N\setminus\{i\}}) \in (E-\{m(E,c)\})_+$. This implies that $u_i^{(E-\{m(E,c)\})_+} \geq c_i - m_i(E,c)$. We can write

$$\begin{aligned}
\widehat{(c-m(E,c))}_i^{(E-\{m(E,c)\})_+} &= \min\left\{c_i - m_i(E,c), u_i^{(E-\{m(E,c)\})_+}\right\} \\
&= c_i - m_i(E,c) \\
&= \hat{c}_i^E - m_i(E,c).
\end{aligned}$$

□

A bankruptcy rule satisfies minimal rights first if it assigns to the remaining bankruptcy problem when all claimants are first allocated their minimal rights the same payoff allocation as to the original bankruptcy problem.

Definition 4.1 (Minimal Rights First)**.**
A bankruptcy rule f on BR^N satisfies *minimal rights first* if

$$f(E,c) = m(E,c) + f((E-\{m(E,c)\})_+, c-m(E,c)) \text{ for all } (E,c) \in \mathrm{BR}^N.$$

The proportional rule, the constrained relative equal awards rule, and the constrained relative equal losses rule do not satisfy minimal rights first. Since the constrained relative equal awards rule and the constrained relative equal losses rule are dual bankruptcy rules (cf. Dietzenbacher et al. (2016)), and the constrained relative equal awards rule satisfies truncation invariance, this means that minimal rights first and truncation invariance are not dual properties, in contrast to the TU-bankruptcy context (cf. Herrero and Villar (2001)).

Let $\mathcal{F}$ denote the space of all bankruptcy rules f on BR^N. Inspired by Thomson and Yeh (2008), we introduce two operators on the space of bankruptcy rules. The *truncation operator* $\mathcal{T} : \mathcal{F} \to \mathcal{F}$ assigns to any bankruptcy rule $f \in \mathcal{F}$ the bankruptcy rule $\mathcal{T}(f) \in \mathcal{F}$ which assigns to any $(E, c) \in \mathrm{BR}^N$ the payoff allocation

$$\mathcal{T}(f)(E, c) = f(E, \hat{c}^E).$$

The *minimal rights operator* $\mathcal{M} : \mathcal{F} \to \mathcal{F}$ assigns to any bankruptcy rule $f \in \mathcal{F}$ the bankruptcy rule $\mathcal{M}(f) \in \mathcal{F}$ which assigns to any $(E, c) \in \mathrm{BR}^N$ the payoff allocation

$$\mathcal{M}(f)(E, c) = m(E, c) + f((E - \{m(E, c)\})_+, c - m(E, c)).$$

Note that both operators are well-defined. We have $f = \mathcal{T}(f)$ if and only if $f \in \mathcal{F}$ satisfies truncation invariance, and $f = \mathcal{M}(f)$ if and only if $f \in \mathcal{F}$ satisfies minimal rights first. In particular, this means that $\mathrm{CREA} = \mathcal{T}(\mathrm{CREA})$.

The next theorem studies some consequences of the truncation operator and the minimal rights operator for the bankruptcy rules to which they are applied.

Theorem 4.2.
Let $f \in \mathcal{F}$ be a bankruptcy rule.

(i) Then, $\mathcal{T}(f)$ satisfies truncation invariance.

(ii) Then, $\mathcal{M}(f)$ satisfies minimal rights first.

(iii) If f satisfies relative symmetry, then $\mathcal{T}(f)$ satisfies relative symmetry.

(iv) If f satisfies truncation invariance, then $\mathcal{M}(f)$ satisfies truncation invariance.

(v) If f satisfies minimal rights first, then $\mathcal{T}(f)$ satisfies minimal rights first.

Proof. (i) Let $(E, c) \in \mathrm{BR}^N$. We can write

$$\mathcal{T}(f)(E, \hat{c}^E) = f(E, \widehat{\hat{c}^E}^E) = f(E, \hat{c}^E) = \mathcal{T}(f)(E, c),$$

where the second equality follows from Lemma 4.1(i). Hence, $\mathcal{T}(f)$ satisfies truncation invariance.

(ii) Let $(E, c) \in \mathrm{BR}^N$. We can write

$$\begin{aligned} &m(E, c) + \mathcal{M}(f)((E - \{m(E, c)\})_+, c - m(E, c)) \\ &= m(E, c) + f((E - \{m(E, c)\})_+, c - m(E, c)) \\ &= \mathcal{M}(f)(E, c), \end{aligned}$$

where the first equality follows from Lemma 4.1(ii). Hence, $\mathcal{M}(f)$ satisfies minimal rights first.

(iii) Assume that f satisfies relative symmetry. Let $(E, c) \in \mathrm{BR}^N$ and let $i, j \in N$ be such that $c_i u_j^E = c_j u_i^E$. Then, we can write

$$\hat{c}_i^E u_j^E = \min\{c_i, u_i^E\} u_j^E = \min\{c_i u_j^E, u_i^E u_j^E\} = \min\{c_j u_i^E, u_j^E u_i^E\} = \min\{c_j, u_j^E\} u_i^E = \hat{c}_j^E u_i^E.$$

Since f satisfies relative symmetry, this means that

$$\mathcal{T}(f)_i(E, c) u_j^E = f_i(E, \hat{c}^E) u_j^E = f_j(E, \hat{c}^E) u_i^E = \mathcal{T}(f)_j(E, c) u_i^E.$$

Hence, $\mathcal{T}(f)$ satisfies relative symmetry.

(iv) Assume that f satisfies truncation invariance. Let $(E,c) \in \mathrm{BR}^N$. We can write

$$\begin{aligned}
\mathcal{M}(f)(E,\hat{c}^E) &= m(E,\hat{c}^E) + f((E - \{m(E,\hat{c}^E)\})_+, \hat{c}^E - m(E,\hat{c}^E)) \\
&= m(E,c) + f((E - \{m(E,c)\})_+, \hat{c}^E - m(E,c)) \\
&= m(E,c) + f((E - \{m(E,c)\})_+, \widehat{c - m(E,c)}^{(E-\{m(E,c)\})_+}) \\
&= m(E,c) + f((E - \{m(E,c)\})_+, c - m(E,c)) \\
&= \mathcal{M}(f)(E,c),
\end{aligned}$$

where the second equality follows from Lemma 4.1(iii), the third equality follows from Lemma 4.1(iv), and the fourth equality follows from f satisfying truncation invariance. Hence, $\mathcal{M}(f)$ satisfies truncation invariance.

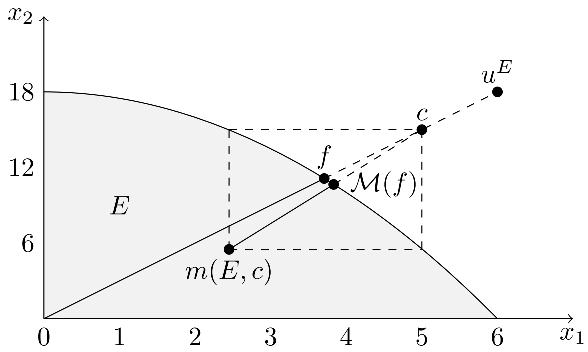
(v) Assume that f satisfies minimal rights first. Let $(E,c) \in \mathrm{BR}^N$. We can write

$$\begin{aligned}
& m(E,c) + \mathcal{T}(f)((E - \{m(E,c)\})_+, c - m(E,c)) \\
&= m(E,c) + f((E - \{m(E,c)\})_+, \widehat{c - m(E,c)}^{(E-\{m(E,c)\})_+}) \\
&= m(E,c) + f((E - \{m(E,c)\})_+, \hat{c}^E - m(E,c)) \\
&= m(E,\hat{c}^E) + f((E - \{m(E,\hat{c}^E)\})_+, \hat{c}^E - m(E,\hat{c}^E)) \\
&= f(E,\hat{c}^E) \\
&= \mathcal{T}(f)(E,c),
\end{aligned}$$

where the second equality follows from Lemma 4.1(iv), the third equality follows from Lemma 4.1(iii), and the fourth equality follows from f satisfying minimal rights first. Hence, $\mathcal{T}(f)$ satisfies minimal rights first. □

The purpose of Theorem 4.2 is twofold. First, it shows that the truncation operator and the minimal rights operator 'force' bankruptcy rules to satisfy truncation invariance and minimal rights first, respectively. Second, it studies the preservation of properties under the truncation operator and the minimal rights operator. Both operators preserve truncation invariance and minimal rights first. Relative symmetry is preserved under the truncation operator, but the following example shows that it is not preserved under the minimal rights operator.

Example 4.
Let $N = \{1,2\}$ and consider the bankruptcy problem $(E,c) \in \mathrm{BR}^N$ in which $E = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 2x_2 \leq 36\}$ and $c = (5,15)$. We have $m(E,c) = (\sqrt{5}, 5\frac{1}{2})$. Let $f \in \{\mathrm{Prop}, \mathrm{CREA}, \mathrm{CREL}\}$. Since f satisfies relative symmetry, we have $f(E,c) = (3\sqrt{5}-3, 9\sqrt{5}-9)$. However, $\mathcal{M}(f)(E,c) \neq f(E,c)$, which means that $\mathcal{M}(f)$ does not satisfy relative symmetry.



△

What happens if we apply the operators multiple times? Let $f \in \mathcal{F}$. From Theorem 4.2, we know that $\mathcal{T}(f)$ satisfies truncation invariance and $\mathcal{M}(f)$ satisfies minimal rights first, which means that $\mathcal{T}(\mathcal{T}(f)) = \mathcal{T}(f)$ and $\mathcal{M}(\mathcal{M}(f)) = \mathcal{M}(f)$. By the preservation of properties, we know that $\mathcal{T}(\mathcal{M}(f))$ and $\mathcal{M}(\mathcal{T}(f))$ both satisfy truncation invariance and minimal rights first, which means that $\mathcal{T}(\mathcal{M}(\mathcal{T}(f))) = \mathcal{T}(\mathcal{M}(\mathcal{M}(f))) = \mathcal{T}(\mathcal{M}(f))$ and $\mathcal{M}(\mathcal{T}(\mathcal{T}(f))) = \mathcal{M}(\mathcal{T}(\mathcal{M}(f))) = \mathcal{M}(\mathcal{T}(f))$. Hence, nothing extra happens when one of the operators is applied more than once. However, the two operators can be combined to obtain a bankruptcy rule which satisfies both truncation invariance and minimal rights first. As the following proposition shows, the order in which the operators are applied does not matter.

Proposition 4.3.
Let $f \in \mathcal{F}$. Then $\mathcal{T}(\mathcal{M}(f)) = \mathcal{M}(\mathcal{T}(f))$.

Proof. Let $(E,c) \in \mathrm{BR}^N$. We can write

$$\begin{aligned}
\mathcal{T}(\mathcal{M}(f))(E,c) &= \mathcal{M}(f)(E,\hat{c}^E) \\
&= m(E,\hat{c}^E) + f((E - \{m(E,\hat{c}^E)\})_+, \hat{c}^E - m(E,\hat{c}^E)) \\
&= m(E,c) + f((E - \{m(E,c)\})_+, \hat{c}^E - m(E,c)) \\
&= m(E,c) + f((E - \{m(E,c)\})_+, \widehat{c - m(E,c)}^{(E-\{m(E,c)\})_+}) \\
&= m(E,c) + \mathcal{T}(f)((E - \{m(E,c)\})_+, c - m(E,c)) \\
&= \mathcal{M}(\mathcal{T}(f))(E,c),
\end{aligned}$$

where the third equality follows from Lemma 4.1(*iii*) and the fourth equality follows from Lemma 4.1(*iv*). □

The bankruptcy rule $\mathcal{T}(\mathcal{M}(f))$ is called the *adjusted counterpart* of the rule $f \in \mathcal{F}$. Three examples of adjusted bankruptcy rules are given by the adjusted proportional rule[2] $\mathcal{T}(\mathcal{M}(\mathrm{Prop}))$, the adjusted constrained relative equal awards rule $\mathcal{T}(\mathcal{M}(\mathrm{CREA}))$, and the adjusted constrained relative equal losses rule $\mathcal{T}(\mathcal{M}(\mathrm{CREL}))$.

[2]The adjusted proportional rule for TU-bankruptcy problems was introduced by Curiel et al. (1987).

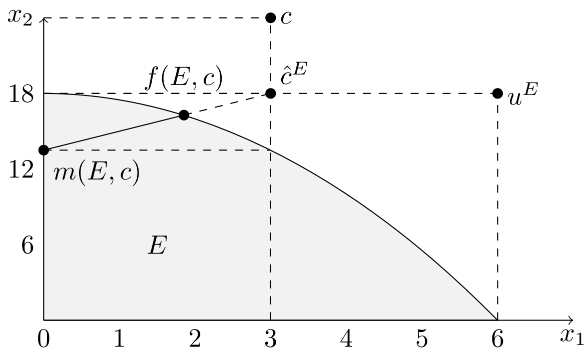
The adjusted counterpart of every bankruptcy rule satisfies truncation invariance and minimal rights first. On the class of bankruptcy problems with two claimants, all adjusted counterparts of bankruptcy rules which satisfy relative symmetry coincide. This standard solution for two-claimant bankruptcy problems is called the relative adjustment principle.[3]

Definition 4.2 (Relative Adjustment Principle)**.**
A bankruptcy rule $f \in \mathcal{F}$ satisfies the *relative adjustment principle* if

$$f(E,c) = m(E,c) + t\left(\hat{c}^E - m(E,c)\right)$$

for all $(E,c) \in \mathrm{BR}^N$ with $|N| = 2$, where $t \in [0,1]$ is such that $f(E,c) \in \mathrm{WP}(E)$.

Example 5.
Let $N = \{1,2\}$ and consider the bankruptcy problem $(E,c) \in \mathrm{BR}^N$ in which $E = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 2x_2 \leq 36\}$ and $c = (3,24)$ as in Example 1 and Example 3. We have $\hat{c}^E = (3,18)$ and $m(E,c) = (0, 13\frac{1}{2})$. This means that $f(E,c) = (\frac{3}{2}\sqrt{5} - 1\frac{1}{2}, \frac{9}{4}\sqrt{5} + 11\frac{1}{4})$ for any bankruptcy rule $f \in \mathcal{F}$ satisfying the relative adjustment principle.



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In order to axiomatically characterize the relative adjustment principle, we introduce the class of simple bankruptcy problems.

Definition 4.3 (Simple Bankruptcy Problem)**.**
A bankruptcy problem $(E,c) \in \mathrm{BR}^N$ is called *simple* if $\hat{c}^E = c$ and $m(E,c) = 0_N$.

Let SBR^N denote the class of all simple bankruptcy problems with claimant set N.

Lemma 4.4.
Let $(E,c) \in \mathrm{BR}^N$. *Then* $((E - \{m(E,c)\})_+, \hat{c}^E - m(E,c)) \in \mathrm{SBR}^N$.

Proof. We can write

$$\begin{aligned}
\widehat{(\hat{c}^E - m(E,c))}^{(E-\{m(E,c)\})_+} &= \widehat{(\hat{c}^E - m(E,\hat{c}^E))}^{(E-\{m(E,\hat{c}^E)\})_+} \\
&= \widehat{\hat{c}^E}^{E} - m(E,\hat{c}^E) \\
&= \hat{c}^E - m(E,c),
\end{aligned}$$

where the first equality follows from Lemma 4.1(iii), the second equality follows from Lemma 4.1(iv), and the third equality follows from Lemma 4.1(i) and Lemma 4.1(iii).

[3] For TU-bankruptcy problems, Aumann and Maschler (1985) called this standard solution the contested garment principle. Later, Thomson (2003) named it the concede-and-divide principle.

Moreover, we can write

$$m((E-\{m(E,c)\})_+,\hat{c}^E-m(E,c))=m((E-\{m(E,\hat{c}^E)\})_+,\hat{c}^E-m(E,\hat{c}^E))=0_N,$$

where the first equality follows from Lemma 4.1(*iii*) and the second equality follows from Lemma 4.1(*ii*). □

A bankruptcy rule satisfies the *simple counterpart* of a property if it satisfies that property on the class of simple bankruptcy problems. For example, a bankruptcy rule $f \in \mathcal{F}$ satisfies *simple relative symmetry* if $f_i(E,c)u_j^E = f_j(E,c)u_i^E$ for all $(E,c) \in \mathrm{SBR}^N$ and any $i,j \in N$ with $c_i u_j^E = c_j u_i^E$. Note that all bankruptcy rules satisfy *simple truncation invariance* and *simple minimal rights first*. If a bankruptcy rule satisfies a property, then Lemma 4.4 implies that its adjusted counterpart satisfies the simple counterpart of that property, e.g., the adjusted counterpart of any relative symmetric bankruptcy rule satisfies simple relative symmetry. Inspired by Dagan (1996), we provide the following axiomatic characterization of the relative adjustment principle based on simple relative symmetry, truncation invariance, and minimal rights first. In particular, this means that the adjusted counterpart of any relative symmetric bankruptcy rule satisfies the relative adjustment principle.

Theorem 4.5.
A bankruptcy rule satisfies the relative adjustment principle if, and only if, it satisfies simple relative symmetry, truncation invariance, and minimal rights first on the class of bankruptcy problems with two claimants.

Proof. Let $f \in \mathcal{F}$ be a bankruptcy rule satisfying the relative adjustment principle. Let $(E,c) \in \mathrm{SBR}^N$ with $|N|=2$ and let $i,j \in N$ be such that $c_i u_j^E = c_j u_i^E$. We can write

$$\begin{aligned}
f_i(E,c)u_j^E &= \left(m_i(E,c)+t\left(\hat{c}_i^E-m_i(E,c)\right)\right)u_j^E\\
&= tc_i u_j^E\\
&= tc_j u_i^E\\
&= \left(m_j(E,c)+t\left(\hat{c}_j^E-m_j(E,c)\right)\right)u_i^E\\
&= f_j(E,c)u_i^E.
\end{aligned}$$

Hence, f satisfies simple relative symmetry.

Now, let $(E,c) \in \mathrm{BR}^N$ with $|N|=2$. We can write

$$f(E,\hat{c}^E)=m(E,\hat{c}^E)+t\left(\widehat{\hat{c}^E}^E-m(E,\hat{c}^E)\right)=m(E,c)+t\left(\hat{c}^E-m(E,c)\right)=f(E,c),$$

where the second equality follows from Lemma 4.1(*i*) and Lemma 4.1(*iii*). Hence, f satisfies truncation invariance. Moreover, we can write

$$\begin{aligned}
m(E,c)+f((E-\{m(E,c)\})_+,c-m(E,c)) &= m(E,c)+t\left(\widehat{c-m(E,c)}^{(E-\{m(E,c)\})_+}\right)\\
&= m(E,c)+t\left(\hat{c}^E-m(E,c)\right)\\
&= f(E,c),
\end{aligned}$$

where the first equality follows from Lemma 4.1(*ii*) and the second equality follows from Lemma 4.1(*iv*). Hence, f satisfies minimal rights first.

Let $f \in \mathcal{F}$ be a bankruptcy rule satisfying simple relative symmetry, truncation invariance, and minimal rights first on the class of bankruptcy problems with two claimants. Let $(E, c) \in \mathrm{BR}^N$ with $|N| = 2$. Since f satisfies truncation invariance and minimal rights first, we have $f(E, c) = m(E, c) + f((E - \{m(E, c)\})_+, \hat{c}^E - m(E, c))$. We know from Lemma 4.4 that $((E - \{m(E, c)\})_+, \hat{c}^E - m(E, c)) \in \mathrm{SBR}^N$. Let $i \in N$ and let $j \in N \setminus \{i\}$. We can write

$$\begin{aligned}
u_i^{(E-\{m(E,c)\})_+} &= \max\{x_i \mid x \in (E - \{m(E, c)\})_+\} \\
&= \max\{x_i \mid (x_i + m_i(E, c), m_{N\setminus\{i\}}(E, c)) \in E\} \\
&= \begin{cases} u_i^E - m_i(E, c) & \text{if } m_{N\setminus\{i\}}(E, c) = 0; \\ c_i - m_i(E, c) & \text{if } m_{N\setminus\{i\}}(E, c) > 0 \end{cases} \\
&= \begin{cases} u_i^E - m_i(E, c) & \text{if } \hat{c}_i^E = u_i^E; \\ c_i - m_i(E, c) & \text{if } \hat{c}_i^E = c_i \end{cases} \\
&= \hat{c}_i^E - m_i(E, c).
\end{aligned}$$

This means that

$$\left(\hat{c}_i^E - m_i(E, c)\right) u_j^{(E-\{m(E,c)\})_+} = \left(\hat{c}_j^E - m_j(E, c)\right) u_i^{(E-\{m(E,c)\})_+}.$$

Since f satisfies simple relative symmetry, this implies that

$$f((E - \{m(E, c)\})_+, \hat{c}^E - m(E, c)) = t\left(\hat{c}^E - m(E, c)\right)$$

for some $t \in [0, 1]$. We can write

$$f(E, c) = m(E, c) + f((E - \{m(E, c)\})_+, \hat{c}^E - m(E, c)) = m(E, c) + t\left(\hat{c}^E - m(E, c)\right).$$

Hence, f satisfies the relative adjustment principle. □

Corollary 4.6.
The adjusted proportional rule, the adjusted constrained relative equal awards rule, and the adjusted constrained relative equal losses rule satisfy the relative adjustment principle.

Future research could study generalizations of other bankruptcy rules which satisfy the relative adjustment principle on the class of TU-bankruptcy problems, such as the random arrival rule (cf. O'Neill (1982)), the minimal overlap rule (cf. O'Neill (1982)), and the Talmud rule (cf. Aumann and Maschler (1985)).

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