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A game theoretic approach to problems in telecommunication

by

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Abstract

This paper considers two specific problems in telecommunication, namely the Terrestrial Flight Telephone System and the re-routing of international telephone calls. Both situations are modelled as coalitional games and game theoretic techniques are used to tackle the problems. It is shown that a special class of coalitional games emerges from the situations under consideration and that the structure of the situations has theoretical implications, including the coincidence of several game theoretic solution concepts. The implications of these theoretical results for the two practical problems are discussed.

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1 Introduction

Cooperative game theory can be a useful tool in modelling situations in which economic agents cooperate. In this paper we will approach two problems in telecommunication from a game theoretic angle and show that game theory helps to gain insight into these problems. Game theory has proved its use in telecommunications before in the study at Cornell University (see *Billera et al.* (1978)), where internal billing rates had to be determined for long-distance telephone calls that are placed through WATS (Wide Area Telecommunication Service). The two problems we consider in this paper also issue from practice, but they are of a different category. The first problem has to do with the so-called 'Terrestrial Flight Telephone System'. Here, several agents have to cooperate in order to launch a new service and an acceptable way has to be found to divide the revenues that are realized through cooperation among the cooperating agents. The second problem has to do with the re-routing of international telephone calls. Here, our main objective is to show that it will be worthwhile for international carriers to cooperate and re-route their international telephone calls during busy hours. A more detailed description of both problems is given below.

The situations described above will be modelled as coalitional games. A *coalitional game* consists of a *player set* $N := \{1, 2, \ldots, n\}$ and a *characteristic function* v that assigns to each *coalition* $S \subseteq N$ of players a real $v(S)$ that is to be interpreted as the maximal profits or cost savings that the members in this coalition can realize when they cooperate. The advantages of modelling the situations as coalitional games are that one is forced to recognize the underlying structure of a situation, and that game theoretic solution concepts can be applied, which are based on considerations concerning *coalitions* of players. There is a variety of game theoretic solution concepts available and the choice of a specific solution concept may be made on the basis of the properties that are appealing for the situations at hand. The coalitional games corresponding to the Terrestrial Flight Telephone System and the re-routing of international telephone calls situations, however, have a specific structure that is shown to have theoretical implications that are relevant for the problems that we are investigating. This will provide strong arguments to choose a specific division of the revenues in the Terrestrial Flight Telephone System situation. Overall, we think that this situation provides a nice example of how theory and practice may interact.

The structure of the paper is as follows. In section 2 we give a detailed description of the 'Terrestrial Flight Telephone System' situation and we model this situation as a coalitional game. Then, in section 3 we describe and model the 're-routing of international telephone calls' situation. The coalitional games that we find in sections 2 and 3 are members of a special class of coalitional games, which we will call the class of k-games. This class of coalitional games is investigated in section 4. In this section we prove that non-negative k-games are convex and that for these games two important solution concepts in cooperative game theory, the Shapley value and the τ-value, coincide. Moreover, we prove that for non-negative 2-games the Shapley value and the τ-value also coincide with a third important solution concept, the nucleolus. Finally, in section 5 we discuss the relevance of the theoretical results obtained in section 4 for the telecommunication problems that we are investigating.

2 The Terrestrial Flight Telephone System

The Terrestrial Flight Telephone System[4] (TFTS from now on) is a public telephone service for passengers in airplanes in which the telephone connections are established by radio communication to a near ground station, from where the connections are provided to the destination subscriber using the existing network. Hence, in order to realize TFTS it is necessary that ground stations are installed and that communications apparatus is installed in airplanes. The TFTS service is economically attractive only if it is sufficiently wide-spread, so the service should be available in a large number of airplanes and, moreover, the area in which one can use the service should be sufficiently large. Therefore, to launch this service several national operators have to cooperate, both by placing ground stations that make it possible to use the service when flying over the countries, and by installing the necessary apparatus in airplanes. A group of European operators decided to cooperate and place ground stations to cover the European continent in such a way that the overlap between stations is minimal. Of course, the revenues that will be generated through TFTS are dependent of the number of airlines that are going to provide the service in their airplanes. The service will be offered to the airlines by a service provider. The revenues that are generated through TFTS will be divided among the participating operators.

The status quo is that the European operators have decided upon the configuration of ground stations that they are going to place and that each operator will place (and pay for) the ground stations that are planned to be placed in their country. However,

[4]This section is to a large extend inspired by a case study that is described in *Groot Bruinderink* (1993). We describe the situation that existed at the end of 1992.

the operators are still discussing the question how the revenues that will be generated through TFTS are to be divided. One proposal was to split the revenues proportional to the investments that are made by the specific operators in order to provide the service. However, this proposal was rejected because it turned out that some operators wanted to install more ground stations, and hence make more investments, in order to get a larger part of the revenues. This will clearly result in a configuration of ground stations that is not efficient, because too many ground stations are installed. The extra ground statons will not generate more revenues, because they will cover areas that are already covered by other ground stations. Another proposal was to let each operator have the revenues that are generated via the ground stations that he installed. Several operators felt that they would not get a fair share of the revenues when this division were to be used. We believe that this is due to the fact that contributions of subgroups of operators are not taken into account. Another flaw in the two proposed divisions is that they do not take into account the fact that in order to generate revenues not only ground stations are necessary, but also apparatus in airplanes.

A different (and in our opinion more satisfactory) approach can be taken using game theory. We can model the situation as a coalitional game and then use game theoretic solution concepts to find acceptable divisions of the revenues. Which game theoretic solution concept to use depends on the specific features of the problem and of the properties that are required of the division rule. In contrast with the division rules mentioned above, game theoretic solution concepts are not only based on individual characteristics of the operators separately, but also on considerations that have to do with coalitions of operators. Further, the approach we follow will take into account the fact that both ground stations and apparatus in airplanes are necessary in order to generate revenues.

When we start modelling the TFTS situation as a coalitional game, we first of all have to decide what should be the players of such a game. There are several parties that play a role in the TFTS situation, namely the national operators, the service provider, and the airlines, that have to permit the service provider to install the service in their airplanes. We make the assumption that when the national operator of a specific country does not invest in ground stations, then the service provider will not invest in apparatus in airplanes of airlines that have their home base in this country. This is in our opinion not a very restrictive assumption, because flights within Europe always last for quite a short period of time, and airplanes will spend relatively much time in the country where they have their home base. Hence, it will not be very profitable to invest in airplanes of airlines that have their home base in a country that is not covered by ground stations.

Therefore, it seems natural to take the countries whose national operators participate in the cooperation, to be the players of a coalitional game. Hence, we define the player set N of the coalitional game to be the set of the countries whose national operators participate in the cooperation. Further, we assume that each country invests in ground stations to cover its area and in apparatus in airplanes of airlines that have their home base in this country. Note that we implicitly assigned the role of the service provider, namely investing in apparatus in airplanes, to the respective countries.

To clarify the situation, we present a simplified and stylized[5] version of the TFTS case. Because of confidentiality, the data we present are modified. However, all important aspects of the real case also occur in the modified version.

There are five countries, A, B, C, D, and E, planning to establish the Terrestrial Flight Telephone System in their airplanes and above their territory. These countries have agreed on a network of ground stations. Table 2.1 shows for each of the five countries the number of ground stations that are to be build there and also the number of airplanes that is going to be provided with the necessary apparatus.

country	number of ground stations	number of airplanes
A	4	150
B	4	210
C	3	250
D	5	400
E	5	420

Table 2.1

The costs of installing and operating a ground station are 15 per ground station and the costs of installing and operating apparatus in airplanes are 3 per airplane[6]. Combining this with the data in table 2.1, one finds the costs related to each of the countries, as summarized in table 2.2.

[5]We will not discuss the specific assumptions and/or technical details underlying the data concerning the exact specifics on e.g. the reference year (including inflation and growth aspects), the operational costs and investments with respect to ground stations and airplanes, the load factors, the call rates, and service/exploitation costs related to the transfer of payments by passengers.

[6]We omit the unit of measurement. However, all costs and all revenues are measured with respect to the same unit.

country	costs
A	510
B	690
C	795
D	1275
E	1335

Table 2.2

In order to realize revenues using TFTS, both ground stations and apparatus in airplanes are essential. Hence, the contribution of a specific country to the amount of telephone calls that are made using TFTS consists of two parts. One part is the telephone calls that are made from the airplanes that are provided with the service by the country, and the the other part is the telephone calls that are made from airplanes flying over the country. Therefore, we see that all the revenues that are realized can be asigned to either one country or two countries in a natural way. We can define for any pair of countries i and j, where i may be equal to j, a number a_{ij} to be the revenues[7] that are made through telephone calls that are made from airplanes of country j that are flying over country i. In the real TFTS case these profits were computed using data about traffic intensity, capacities, and load factors that were available from the 'Reed Travel Group-ABC International' as well as data concerning revenues and costs of the TFTS service that were available from a business case of the 'Deutsche Bundespost'. For our version of the TFTS case, the numbers a_{ij} are given in table 2.3 below. The rows in this table correspond to the ground stations in a country and the columns correspond to the airplanes.

	A	B	C	D	E
A	430	2	6	46	12
B	1	603	27	27	68
C	8	24	834	129	138
D	286	12	91	1072	264
E	28	164	174	259	1688

Table 2.3

[7]The costs that are directly related to the amount of telephone traffic are already deducted from the revenues.

With the data we have now, we can compute the revenues that the members of a coalition S of countries can jointly realize *within* the cooperation of the countries. This means that we do not take into account the fact that when the service is just installed in two non-neighbouring countries, say country A and country B, (and the airplanes of the airlines that have their home base in these countries), then almost no telephone calls will be made, because these calls will be cut short. Rather, we assume that the service can be used in the airplanes of countries A and B when flying over the territory of countries A, B, C, D, and E. This is not a restrictive assumption, because our goal is to determine the shares that the countries should get of the revenues that are jointly made, while at the same time it is already determined that the countries are going to cooperate.
This means that for a coalition S of countries the revenues that the members of S can jointly realize within the cooperation of the countries can be found by adding the revenues that are generated by one or two countries in coalition S. Hence, the worth $v(S)$ of a coalition S is

$$v(S) = \sum_{i \in S} \sum_{j \in S} a_{ij} = \sum_{i \in S} a_{ii} + \sum_{i \in S} \sum_{j \in S, j \neq i} a_{ij}.$$

This means that we can find the worth of a coalition S of countries from table 2.3 by summing up the entries that are left when we cross out the rows and columns corresponding to countries that are not in S. In table 2.4 we list the worths of all coalitions of countries.

country	worth	coalition	worth	coalition	worth	coalition	worth
A	430	AB	1036	ABC	1935	ABCD	3598
B	603	AC	1278	ABD	2479	ABCE	4207
C	834	AD	1834	ABE	2996	ABDE	4962
D	1072	AE	2158	ACD	2902	ACDE	5465
E	1688	BC	1488	ACE	3318	BCDE	5574
		BD	1714	ADE	4085		
		BE	2523	BCD	2819		
		CD	2126	BCE	3720	ABCDE	6393
		CE	2834	BDE	4157		
		DE	3283	CDE	4649		

Table 2.4

Now that we have gathered all the data we need, we can compute several of the proposed divisions of the revenues. Table 2.5 lists three divisions. The first one is the division that splits the total revenues (6393) proportional to the investments of the countries (see table 2.2). We denote this division by PI. The second division we list, (GR) is the one that lets each country have the revenues that are generated via the ground stations it installed. This division can be computed by adding for each country the elements in the row of table 2.3 that corresponds to the ground stations of that country. We also give a game theoretic division, namely the Shapley value (SV). The definition of the Shapley value is given in appendix A. Although the Shapley value is hard to compute in general, it follows from expressions (3) and (4) in appendix B that the Shapley value of country i can easily be found by adding the elements in the row of table 2.3 that corresponds to country i and the elements in the column of table 2.3 that corresponds to country i and then dividing by 2. (In formula: $a_{ii} + \frac{1}{2}\sum_{j\neq i}(a_{ij} + a_{ji})$)

country	PI	GR	SV
A	708	496	625
B	958	726	766
C	1104	1133	1133
D	1770	1725	1629
E	1853	2313	2242

Table 2.5

Combining the data in tables 2.2 and 2.5, we see that the proposal to let each country have the revenues that are generated via its ground stations is not acceptable to country A, because the revenue that this country gets does not cover its costs, so that country A would have a net loss. If the revenues are split proportional to the investments, then countries A, C, D, and E are not quite satisfied, because their total share of the revenues, namely 5435, is less than their joint contribution to the revenues, which equals 5465. Therefore, they feel that the share that country B gets is too large. It is easily checked that the Shapley value does not suffer from the lacks that the other two divisions show.

The Shapley value as well as two other game theoretic solution concepts are further elaborated in section 4. In that section we will show that the special structure of the coalitional game corresponding to the TFTS situation, namely that the revenues are essentially realized in coalitions consisting of either one or two players and that the worth of an arbitrary coalition can be found by simply adding the worths of all its subcoalitions containing one or two players, has theoretical implications that are relevant for the TFTS situation.

3 Re-routing international telephone calls

At the moment most international telephone calls are routed via direct circuits from the originating country to the destination country. These circuits can be used more efficiently if during busy hours the calls are routed via quiet parts of the international network (see *Gibbens et al.* (1991)). For instance, when there is a high traffic load between Europe and America, it is night time in Australia and the traffic from and to Australia is low. Therefore, a part of the traffic between Europe and America can be routed transit via an Australian carrier during busy hours and the number of circuits between European and American carriers can be reduced. So, by cooperating and making agreements on the use of transit routes, international carriers are able to use their circuits more efficiently and to reduce the costs of their network. In general the use of transit routes is limited to two link paths. There are a number of reasons for this, one of which is that telephone calls may bounce back to the originating country if it is allowed to re-route a telephone call that was already transit routed. Another reason to limit the use of transit routes is to prevent that congestion on one link will affect the whole international network. So, a call that is routed transit cannot be routed transit again. For instance, a call from The Netherlands to Canada can be re-routed via either Australia or Japan, but not via Australia and Japan at the same time. Hence, in order to generate profits through the re-routing of an international telephone call, exactly three international carriers have to cooperate, a carrier in the originating country, a transit carrier, and a carrier in the destination country. In the studies by *Gibbens et al.* (1991) and by *van Golstein Brouwers* (1992) it turns out that for the problem of the re-routing of international telephone calls it is relevant to consider three (time) zones, the American zone, the European zone and the Asian zone. Profits are essentially realized when international telephone calls from one zone to another zone are re-routed via the third zone. This is obviously due to the fact that within one of the zones the time differences are too small to generate a significant shift of peak hours in telephone traffic.

Modelling the re-routing of international telephone calls situation as a coalitional game is rather straightforward. The players of the game are the international carriers. Further, the worth $v(S)$ of a coalition S of international carriers is defined to be the cost savings that the members of S can realize because of the circuits that they can spare if they use transit routes. Because a transit routed call cannot be routed transit again, the number of international carriers that are involved in a transit routed call is limited to three. Therefore, the cost savings are essentially realized by three player coalitions.

Surprisingly, it follows from the studies by *Gibbens et al.* (1991) and by *van Golstein Brouwers* (1992) that adding the cost savings that are obtainable by the subcoalitions of size 3 of an arbitrary coalition S gives a good approximation of the cost savings that are obtainable by the members of S. Hence, we have for a coalition S

$$v(S) = \sum_{T \subseteq S: |T|=3} v(T).$$

We see that we obtain a coalitional game in which the profits are essentially realized in coalitions consisting of three players. To find the worth of an arbitrary coalition we simply add the worths of all its 3-player subcoalitions. Therefore, the structure of this game is similar to the structure of the TFTS game, where we could find the worth of an arbitrary coalition by adding the worths of all its subcoalitions containing one or two players. In section 4 we will consider the class of so-called k-games. This class of games captures the structure that we observe in the games corresponding to the TFTS situation and the re-routing situation.

4 k-games

This section is of a theoretical nature. We will introduce the class of k-games in order to capture the structure that we observe in the games that we found in sections 2 and 3. Further, we will show that for this type of games several solution concepts coincide. If so desired, the reader can skip this section and turn immediately to section 5 in which we will discuss the implications of the theoretical results for the TFTS and the re-routing situations.

The proofs of all the theorems stated in this section are included in appendix B.

A coalitional game (N, v) is said to be a *k-game* if its characteristic function v takes the form

$$v(S) = \sum_{T \subseteq S: |T|=k} v(T).$$

Note that a 1-game is simply an additive game, given by

$$v(S) = \sum_{i \in S} v(\{i\}).$$

The class of 2-games coincides with the class of weighted graph games that is considered by *Brown* and *Housman* (1988) and by *Deng* and *Papadimitriou* (1994). A k-game in general corresponds to a weighted hypergraph game with hyperedges of size k as considered by *Deng* and *Papadimitriou* (1994).

Cooperative game theory provides a variety of solution concepts for coalitional games. Such a solution concept answers the question how the profits that can be realized when all the players in N cooperate, should be divided among the individual players, while taking into account the potential profits of different subgroups of players.
The solution concepts that we consider here are the *Shapley value* (cf. *Shapley* (1953)), the *τ-value* (cf. *Tijs* (1981)), and the *nucleolus* (cf. *Schmeidler* (1969)). The definitions of these solution concepts are provided in appendix A.

Theorem 4.1 below states that non-negative k-games are convex. Here, a coalitional game (N, v) is called *convex* if the contribution of a player to a coalition is larger if the coalition he joins is larger or, in formula, if

$$v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S)$$

for all $i \in N$ and all $S \subseteq T \subseteq N \backslash \{i\}$. Convex games are nice because they allow for divisions of the profits obtainable by the grand coalition that are stable in the sense that for every subgroup of players the members of this subgroup get at least as much as what they can obtain by themselves. When such a stable division is used, then no subgroup of players will have an incentive to split off because they cannot do better by themselves.

Theorem 4.1 Every non-negative k-game is convex.

From convexity of a non-negative k-game we derive that both its nucleolus and its Shapley value are stable divisions (cf. *Shapley* (1971) and *Schmeidler* (1969)). Besides this, its Shapley value and its τ-value coincide, so its τ-value is also a stable division.

Theorem 4.2 The Shapley value and the τ-value of a non-negative k-game coincide.

For a non-negative 2-game the nucleolus coincides with both values investigated in theorem 4.2. This result was already obtained by *Brown* and *Housman* (1988) and it also follows from a theorem by *Deng* and *Papadimitriou* (1994). We include in appendix B a proof of theorem 4.2 that is more direct than the proofs in the other papers.

Theorem 4.3 For a non-negative 2-game the nucleolus coincides with the Shapley value and the τ-value.

Example 4.4 shows that theorem 4.3 cannot be generalized to k-games where $k > 2$.

Example 4.4 Consider the game (N, v) with $N = \{1, 2, 3, 4\}$ and $v = u_{\{1,2,3\}} + u_{\{2,3,4\}}$. For this game the Shapley value is not equal to the nucleolus.
Using expression (3) in appendix B for the Shapley value, we find $\Phi(N, v) = (\frac{1}{3}, \frac{2}{3}, \frac{2}{3}, \frac{1}{3})$. For the nucleolus of the game (N, v) we have $\nu(N, v) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. This can easily be checked using the criterion of *Kohlberg* (1971).

The following theorem shows that the results in theorems 4.1, 4.2 and 4.3 still hold if an additive game is added.

Theorem 4.5 Let (N, v) be the sum of an additive game and a non-negative k-game. Then (N, v) is convex and $\Phi(N, v) = \tau(N, v)$. Moreover, if $k = 2$, then $\nu(N, v) = \Phi(N, v) = \tau(N, v)$.

We refer to the paper by *Deng* and *Papadimitriou* (1994) for results with respect to the complexity of calculating the Shapley value and the nucleolus for k-games.

5 Applying game theory

We will now discuss the relevance of the theoretical results in section 4 for the TFTS situation and for the re-routing of international telephone calls.

The characteristic function of the coalitional game corresponding to the re-routing of international telephone calls that we found in section 3, takes the following form: for each coalition S the worth $v(S)$ is

$$v(S) = \sum_{T \subseteq S:\, |T|=3} v(T).$$

Hence, in the re-routing of international telephone calls a 3-game emerges. Because the worths in this game are defined as the cost savings that are obtainable by the coalitions, these worths are non-negative. After all, the international carriers can always stick to the existing situation and obtain cost savings equal to 0. Now that we know that the coalitional game that emerges is a non-negative 3-game, we can apply the results that we obtained in the previous section. In particular, we conclude that the game is convex. Hence, we know that there are ways to divide the profits from cooperation that are acceptable to all participants. Therefore, we have shown that it will be worth while for international carriers to cooperate and re-route their telephone calls during busy hours. Especially, if they cooperate, then there exist ways to divide the gains from cooperation such that not only all indiviual carriers, but also all coalitions of international carriers are better off than they would be if they did not cooperate with the other carriers. Also, it follows from the results in section 4 that two important solution concepts, the Shapley value and the τ-value, coincide for the re-routing game. For a more extensive exposition on the re-routing of international telephone calls we refer to *Gibbens et al.* (1991) and *Golstein Brouwers* (1992).

We found in section 2 that the characteristic function of the coalitional game corresponding to the TFTS situation takes the following form: for each coalition S the worth $v(S)$ is

$$v(S) = \sum_{i \in S} a_{ii} + \sum_{i \in S} \sum_{j \in S, j \neq i} a_{ij}.$$

We will split this coalitional game up into two other coalitional games, with the same player set and characteristic functions w_1 and w_2. We define for each coalition S

$$w_1(S) := \sum_{i \in S} a_{ii} \text{ and } w_2(S) := \sum_{i \in S} \sum_{j \in S, j \neq i} a_{ij}.$$

Clearly, w_1 is the characteristic function of a 1-game or an additive game, and w_2 is the characteristic function of a 2-game. So, we have a coalitional game that is the sum of an additive game and a 2-game. The 2-game is a revenue game and is obviously non-negative. Hence, we see that the coalitional game corresponding to the TFTS situation is the sum of an additive game and a non-negative 2-game. So, it follows from theorem 4.5 that the game that is associated with the TFTS problem is convex and that for this game the nucleolus, the Shapley value and the τ-value coincide. Moreover, the division that corresponds to these solution concepts is stable in the sense that there are no subgroups that get less than their contribution to the revenues of the cooperation.

Our goal in the TFTS situation was to find acceptable ways to divide the revenues that are realized through cooperation among the cooperating operators. First of all we conclude that we proved that there do exist acceptable ways to divide the revenues. Further, we proved that the three most important solution concepts in cooperative game theory all coincide for the game corresponding to the TFTS situation and that the division corresponding to these solution concepts is stable. Hence, we have strong arguments to recommend this specific division of the revenues. Namely, the list of properties that are satisfied by this specific division rule is rather extensive. Since for the class of games of which the TFTS game is a member the Shapley value, the τ-value, and the nucleolus coincide, the division corresponding to these solution concepts satisfies all the properties that are satisfied by one of these three solution concepts. Therefore, the division corresponding to the Shapley value, the τ-value, and the nucleolus satisfies various axioms of fairness, that are in general not satisfied all at once.

It might seem at this point that the division we found using game theory is so appealing that it could easily have been found without using the heavy machinery of game theory. One could try and argue that this division satisfies three (usually contradicting) axioms of fairness and that this means that it must be hard to find other divisions that make sense *a priori*. However, the history of the TFTS situation shows that the operators were on a completely different track before the situation was modelled as a coalitional game. The structure underlying the situation was not recognized untill one started to think about it within the framework that is provided by game theory.

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Appendix A

In this appendix we will provide the definitions of the Shapley value, the τ-value and the nucleolus for coalitional games.

The set of all coalitional games with player set N is a linear space and a basis of this space is formed by the *unanimity games* (N, u_S), $S \subseteq N$, that are defined by

$$u_S(T) = \begin{cases} 1 & \text{if } S \subseteq T \\ 0 & \text{else.} \end{cases} \tag{1}$$

The interpetation of the unanimity game (N, u_S) is that a profit (or cost savings) of 1 can be obtained if and only if all the players in the coalition S are present.
Cooperative game theory provides a variety of solution concepts for coalitional games. Such a solution concept answers the question how the profits that can be realized when all the players in N cooperate, should be divided among the individual players, while taking into account the potential profits of different subgroups of players. Hence, a solution concept assigns one or more payoff vectors to a coalitional game. A vector of payoffs for a game (N, v) is an element $x \in \mathbf{R}^N$ such that $\sum_{i \in N} x_i = v(N)$. According to a payoff vector x a player i gets a payoff of x_i and, moreover, the payoffs to the players sum up to the profit that is obtainable by the grand coalition N.

The *Shapley value* (cf. *Shapley* (1953)) deals with the marginal contributions of the players to coalitions and assigns to each player a weighted average of these contributions, which can be interpreted as the expected marginal contribution of the player. Formally, the Shapley value for player $i \in N$ in the game (N, v) is defined by

$$\Phi_i(N, v) := \sum_{S \subseteq N: i \in S} \frac{(|S|-1)!(n-|S|)!}{n!}(v(S) - v(S \backslash \{i\})),$$

where $|S|$ denotes the number of elements of a set $S \subseteq N$. An equivalent formulation for the Shapley value states that is an additive solution concept (i.e. $\Phi(N, v+w) =$

$\Phi(N,v) + \Phi(N,w)$ for all coalitional games (N,v) and (N,w)) that divides the total profits in a unanimity game (N,u_S) equally among the players in coalition S.

The *τ-value* (cf. *Tijs* (1981)) is the unique efficient compromise between the vector of minimal rights of the players and the vector of utopia payoffs. The *utopia payoff* of a player $i \in N$ is $M_i(N,v) := v(N) - v(N\backslash\{i\})$, which is the maximum player i can expect to get, because if he asks for more, then the other players do not want to cooperate with him any more. Hence, in negotiation with a coalition $S \subseteq N$ a player $i \in S$ can claim the *remainder* $R_v(S,i) := v(S) - \sum_{j\in S\backslash\{i\}} M_j(N,v)$. Choosing a coalition that leaves him the largest possible remainder, a player i obtains his *minimal right* $m_i(N,v) := \max_{S\subseteq N: i\in S} R_v(S,i)$. Note that for each player $i \in N$ the minimal right is not less than the worth $v(\{i\})$ that this player can obtain on his own. The τ-value is defined for coalitional games (N,v) where the minimal rights of the players do not exceed their utopia payoffs and where the total of minimal rights of the players is less than or equal to the value of the grand coalition which in turn has to be less than or equal to the total of utopia payoffs of the players. For such a game (N,v) the τ-value is given by

$$\tau(N,v) := \alpha_v m(N,v) + (1-\alpha_v)M(N,v) \ ,$$

where $\alpha_v \in [0,1]$ is such that

$$\alpha_v \sum_{i\in N} m_i(N,v) + (1-\alpha_v) \sum_{i\in N} M_i(N,v) = v(N) \ .$$

The *nucleolus* (cf. *Schmeidler* (1969)) is the vector of payoffs that minimizes the maximal complaint of the coalitions in the following way. The *excess* (or complaint) of a coalition $S \subseteq N$, $S \neq \emptyset$, with respect to a vector of payoffs $x \in \mathbf{R}^N$ is defined by $e(S,x) := v(S) - \sum_{i\in S} x_i$. The excess vector of the payoff vector x is $e(x) = (e(S_1,x), e(S_2,x), \ldots, e(S_{2^N},x))$, where the excesses are arranged in a decreasing order. The lexicographic order on $\mathbf{R}^N$ is denoted by $\leq_L$. Hence, for $x, y \in \mathbf{R}^N$ we have $x \leq_L y$ if and only if there exists a $k \in \{0,1,\ldots,n\}$ such that $x_i = y_i$ for all $i \leq k$ and $x_{k+1} < y_{k+1}$. The nucleolus $\nu(N,v)$ of a game (N,v) is the unique vector of payoffs for the game that minimizes $e(x)$ with respect to the order $\leq_L$.

Appendix B

In this appendix we provide the proofs of the theorems in section 4.

Proof of theorem 4.1. Let (N, v) be a non-negative k-game. The game (N, v) can be written as a linear combination of unanimity games (cf. (1)). It is easy to see from the structure of the game (N, v) that

$$v = \sum_{T \subseteq N: |T|=k} v(T)\, u_T\,, \tag{2}$$

where $v(T) \geq 0$ for all $T \subseteq N$ with $|T| = k$. Since all games (N, u_T), $T \subseteq N$, are convex and the set of convex games with player set N is a cone (cf. *Shapley* (1971)), equation (2) immediately implies that (N, v) is convex. □

Proof of theorem 4.2. Let (N, v) be a non-negative k-game. The Shapley value of the game can easily be found in the following way. For a unanimity game (N, u_S) the Shapley value of a player i is known to be $\frac{1}{|S|}$ if $i \in S$ and 0 if $i \notin S$. Using this and the additivity of the Shapley value (cf. *Shapley* (1953)), it is easily derived from expression (2) for the characteristic function v that

$$\Phi_i(N, v) = \sum_{T \subseteq N: |T|=k,\, i \in T} \frac{v(T)}{|T|} \tag{3}$$

for all $i \in N$.

To find the τ-value of the game (N, v), we use theorem 4.1 and the result of *Driessen* and *Tijs* (1985), which states that for a convex game the minimal right of each player is equal to the worth that this particular player can obtain by himself. Thus, we find $m_i(N, v) = v(\{i\})$ for each $i \in N$. Further, again using expression (2), we derive $M_i(N, v) = \sum_{T \subseteq N: |T|=k, i \in T} v(T)$. Now we have to distinguish between two cases. If $k = 1$, then we see that $\Phi_i(N, v) = v(\{i\}) = m_i(N, v) = M_i(N, v)$ and hence $\tau_i(N, v) = \Phi_i(N, v)$. If $k \geq 2$, then we see that $m_i(N, v) = 0$ and $M_i(N, v) = k\, \Phi_i(N, v)$ and from this we derive $\tau_i(N, v) = \Phi_i(N, v)$. This concludes the proof of the theorem. □

Proof of theorem 4.3. Let (N, v) be a non-negative 2-game. In view of theorem 4.2 it suffices to prove that $\Phi(N, v)$ minimizes the excess vector with respect to $\leq_L$. We first prove that for each $S \subseteq N$ it holds that $e(S, \Phi(N, v)) = e(N \backslash S, \Phi(N, v))$. So, let $S \subseteq N$ be fixed for the moment. Then

$$e(S, \Phi(N, v)) = v(S) - \sum_{i \in S} \Phi_i(N, v)$$

$$\begin{aligned}
&= \sum_{T\subseteq S:|T|=2} v(T) - \sum_{i\in S}\sum_{T\subseteq N:|T|=2,\, i\in T} \frac{v(T)}{|T|} \\
&= \tfrac{1}{2}\sum_{i\in S}\sum_{j\in S\setminus\{i\}} v(i,j) - \sum_{i\in S}\sum_{j\in N\setminus\{i\}} \tfrac{1}{2}v(i,j) \\
&= -\sum_{i\in S}\sum_{j\in N\setminus S} \tfrac{1}{2}v(i,j).
\end{aligned}$$

Because the last expression is symmetric in S and $N\setminus S$, it is easy to see that $e(S,\Phi(N,v)) = e(N\setminus S,\Phi(N,v))$.

Now we will argue that this implies that $\Phi(N,v)$ minimizes the excess vector with respect to $\leq_L$. First note that

$$\begin{aligned}
e(S,x) + e(N\setminus S,x) &= v(S) - \sum_{i\in S} x_i + v(N\setminus S) - \sum_{i\in N\setminus S} x_i \\
&= v(S) + v(N\setminus S) - v(N)
\end{aligned}$$

for all vectors x of payoffs and for all $S \subseteq N$ and that this expression is independent of x. This implies that there exists at most one vector x of payoffs such that $e(S,x) = e(N\setminus S,x)$ for all $S \subseteq N$. Now, let x be a vector of payoffs that is different from $\Phi(N,v)$. Then the set $\mathcal{C} := \{S \subseteq N \mid e(S,x) > e(N\setminus S,x)\}$ is not empty. We choose $T \in \mathcal{C}$ such that

$$e(T,x) = \max_{S\in\mathcal{C}} e(S,x).$$

Then we know

$$\begin{aligned}
&e(T,\Phi(N,v)) = e(N\setminus T,\Phi(N,v)), \\
&e(T,x) > e(N\setminus T,x) \text{ and} \\
&e(T,\Phi(N,v)) + e(N\setminus T,\Phi(N,v)) = e(T,x) + e(N\setminus T,x).
\end{aligned}$$

This implies that $e(T,x) > e(T,\Phi(N,v))$. Because of the way we choose T we know that $e(S,x) = e(S,\Phi(N,v))$ for all $S \subseteq N$ such that $e(S,x) > e(T,x)$. Combining all these facts we see that $e(\Phi(N,v)) \leq_L e(x)$. Therefore, $\Phi(N,v) = \nu(N,v)$. □

Proof of theorem 4.5. Since an additive game is convex and the sum of two convex games is convex itself (cf. *Shapley* (1971)), convexity of (N,v) follows using theorem 4.1. Now we note that for an additive game (N,a) we have that for all $i \in N$

$$\nu_i(N,a) = \Phi_i(N,a) = \tau_i(N,a) = a(\{i\}). \tag{4}$$

Further, it holds for all three solution concepts that the solution of the sum of an additive game and an arbitrary game equals the sum of the solutions of both games, or, in formula,

$$\gamma(N,a+w) = \gamma(N,a) + \gamma(N,w),$$

where (N,a) is an additive game, (N,w) is an arbitrary game, and γ is either ν or Φ or τ. Combining all this with the results of theorems 4.2 and 4.3 we obtain equality of the various solution concepts. □