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Abstract

This paper studies egalitarianism in the context of nontransferable utility games by introducing and analyzing the egalitarian value. This new solution concept is based on an egalitarian negotiation procedure in which egalitarian opportunities of coalitions are explicitly taken into account. We formulate conditions under which it leads to a core element and discuss the egalitarian value for the well-known Roth-Shafer examples. Moreover, we characterize the new value on the class of bankruptcy games and bargaining games.

Keywords: egalitarianism, NTU-games, egalitarian procedure, egalitarian value, egalitarian stability, constrained relative equal awards rule
JEL classification: C71, D63

1 Introduction

Since the seminal work of Rawls (1971) in which an egalitarian vision on society is elaborated and justified, egalitarianism plays a central role in fundamental principles of justice and is widely applied within several disciplines. Egalitarianism relates to the concept of fairness and may be referred to as 'equity' within economical contexts, in particular welfare economics. Young (1995) provides a rich survey on equity concepts in both theory and practice.

This paper focusses on egalitarianism in the context of nontransferable utility games. Shapley and Shubik (1953) introduced this model to extend the standard definition of games by dropping two substantial restrictions on the nature of utility: linearity and transferability. This means that the utility level of a player bears no resemblance anymore to the utility level of another player. In fact, utility measures are incompatible and utility levels are incomparable. In order to still apply egalitarianism in a nontransferable utility context, we take a solid and deliberate approach using the zero vector and the utopia vector as reference points.

Applying this approach to nontransferable utility games, we define an egalitarian procedure in which players iteratively consider their egalitarian opportunities within subcoalitions. We introduce the egalitarian value as solution concept for nontransferable utility games which takes the result of this egalitarian procedure into account to prescribe a unique egalitarian allocation for the grand coalition. The egalitarian value generalizes the nonnegative procedural egalitarian solution for (nonnegative) transferable utility games of Dietzenbacher, Borm, and Hendrickx (2016), which in turn coincides with the constrained egalitarian solution of Dutta and Ray (1989) on the class of convex transferable utility games.

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We compare the egalitarian value with other well-known solution concepts for nontransferable utility games like the Shapley value (cf. Shapley (1969)), the Harsanyi value (cf. Harsanyi (1963)), and the monotonic solution of Kalai and Samet (1985) using the famous examples introduced by Roth (1980) and Shafer (1980). It turns out that, contrary to the other solution concepts, the egalitarian value exactly prescribes the allocation which was proposed by Roth (1980). Moreover, the egalitarian value neatly follows the line of reasoning stated by Shafer (1980).

Furthermore, on the class of bankruptcy games with nontransferable utility (cf. Dietzenbacher (2017)), the egalitarian value corresponds to the constrained relative equal awards rule, an egalitarian bankruptcy rule introduced by Dietzenbacher, Estévez-Fernández, Borm, and Hendrickx (2016) which extends the constrained equal awards rule for bankruptcy problems with transferable utility. On the class bargaining games (cf. Nash (1950)) corresponding to bargaining problems with the zero vector as disagreement point, the egalitarian value corresponds to the solution introduced by Kalai and Smorodinsky (1975). For bargaining games with nonzero disagreement point, it is illustrated that the egalitarian value offers a new, interesting way to solve bargaining problems.

This paper is organized in the following way. Section 2 discusses the modeling of egalitarianism, in particular in the context of nontransferable utility games. Section 3 formally introduces the egalitarian value and the underlying egalitarian procedure. Section 4 studies the egalitarian value for the Roth-Shafer examples. In Section 5 and Section 6, the egalitarian value is analyzed on the class of bankruptcy games and bargaining games, respectively.

2 Egalitarianism and Nontransferable Utility

Egalitarianism is a paradigm of economic thought that favors the idea of equality. Economic equality, or equity, refers to the concept of fairness in economics and underlies many theories of distributive justice (cf. Rawls (1971) and Young (1995)). The interpretation of equality, and which notions should exactly be equated, depends on the underlying model and its characteristics, especially when the corresponding agents are not identical. In a general payoff space where individual utility is represented in incompatible measures, egalitarianism cannot be applied straightforwardly. To do so, it is necessary to impose assumptions which allow to compare utility not only intrapersonally, but to some extent also interpersonally.

In a general allocation problem, a natural and helpful operation is normalization. In particular, zero-normalization requires transforming individual utility such that allocating nothing corresponds to a utility level of zero. In other words, a payoff of zero utility generates the same well-being for an involved agent as the event in which the allocation problem does not exist at all. This implies that, in case of allocating revenues, it is convenient to restrict to feasible payoff allocations that are nonnegative.

After zero-normalization the zero vector plays a fundamental role. There, agents are comparable in terms of well-being and the allocation is in that sense egalitarian. The zero vector actually serves as a benchmark for egalitarian allocations. However, in order to study efficient egalitarianism, this single point is not sufficient. For this, at least a second reference point is necessary to determine the direction of other comparable allocations. The maximal individual payoffs within the feasible allocations, or utopia values, constitute a natural candidate. For, there agents are comparable in terms of maximal satisfaction on the basis of feasible allocations and the corresponding vector of utopia values is in that sense egalitarian. The utopia vector relative to the zero vector can be interpreted as an egalitarian direction. It is important to note that this direction and the subsequent results are invariant under individual rescaling of utility.

Let N be a nonempty and finite set of agents called *players* and let $A \subseteq \mathbb{R}^N_+$ be a nonempty, closed, and bounded set of payoff allocations. The vector of *utopia values* $u^A \in \mathbb{R}^N_+$ is given by

$$u^A = (\max\{x_i \mid x \in A\})_{i \in N}.$$

The set A is called *nontrivial* if $u^A \in \mathbb{R}^N_{++}$.

The collection of all *coalitions* is denoted by $2^N = \{S \mid S \subseteq N\}$. A (nonnegative) *nontransferable utility game* is a pair (N, V) in which V assigns to each nonempty coalition $S \in 2^N \setminus \{\emptyset\}$ a nonempty, closed, bounded, and comprehensive set of payoff allocations $V(S) \subseteq \mathbb{R}^S_+$ such that $V(N)$ is nontrivial. Note that the game is not required to be zero-normalized in the sense that $V(\{i\}) = \{0\}$ for all $i \in N$. Let NTU^N denote the class of all such NTU-games with player set N. For convenience, an NTU-game is denoted by $V \in \text{NTU}^N$.

In a cooperative game, solutions focus on allocations for the grand coalition while taking the opportunities of subcoalitions into account. To allow for an appropriate egalitarian comparison of subcoalitions, it is required to consistently apply a fixed interpretation of egalitarianism. Therefore, the utopia values of the grand coalition are used as a *common* benchmark within any subcoalition. In the context of transferable utility games, the utopia values of all players in the grand coalition coincide and egalitarianism boils down to equal division. In the next section, egalitarianism is exploited in the context of nontransferable utility games.

Useful, preliminary notions related to a set of payoff allocations $A \subseteq \mathbb{R}^N_+$ are

- the *comprehensive hull* $\text{comp}(A) = \{x \in \mathbb{R}^N_+ \mid \exists_{y \in A} : y \geq x\}$;
- the *weak upper contour set* $\text{WUC}(A) = \{x \in \mathbb{R}^N_+ \mid \neg\exists_{y \in A} : y > x\}$;
- the *weak Pareto set* $\text{WP}(A) = \{x \in A \mid \neg\exists_{y \in A} : y > x\}$;
- the *strong Pareto set* $\text{SP}(A) = \{x \in A \mid \neg\exists_{y \in A, y \neq x} : y \geq x\}$.

Note that $\text{SP}(A) \subseteq \text{WP}(A) \subseteq \text{WUC}(A)$. The set $A \subseteq \mathbb{R}^N_+$ is called *comprehensive* if $A = \text{comp}(A)$, and *nonleveled* if $\text{SP}(A) = \text{WP}(A)$.

3 The Egalitarian Value

In this section, we introduce the egalitarian value as an egalitarian solution concept for nontransferable utility games. The egalitarian value is based on an egalitarian negotiation procedure in which coalitional opportunities are explicitly taken into account. By applying the utopia values of the grand coalition as an egalitarian direction in any subcoalition, the procedure starts assigning to any coalition the maximally feasible egalitarian allocation. Players can fix their allocated payoff in a coalition if no member is allocated a higher payoff in any other coalition. These players would still be willing to cooperate within other coalitions provided that they are compensated. Therefore, they claim their fixed payoff in any coalition and the other members are assigned the maximally feasible egalitarian allocation. This recursive procedure continues and eventually all players acquire a claim which is attainable in at least one coalition.

Definition 1 (Egalitarian Procedure)**.**
Let $V \in \text{NTU}^N$ be a nontransferable utility game. The set of 0-*egalitarian claimants* is given by $P^{V,0} = \emptyset$. Let $k \in \mathbb{N}$. The k-*egalitarian distribution* is the function $\chi^{V,k}$ assigning to each $S \in 2^N \setminus \{\emptyset\}$ the payoff allocation $\chi^{V,k}(S) \in \mathbb{R}^S_+$ given by

$$\chi^{V,k}(S) = \left(\gamma^{V,k-1}_{S \cap P^{V,k-1}}, \lambda^{V,k}(S) u^{V(N)}_{S \setminus P^{V,k-1}}\right),$$

where $\lambda^{V,k} : 2^N \setminus \{\emptyset\} \to \mathbb{R}_+$ assigns to each $S \in 2^N \setminus \{\emptyset\}$ the scalar $\lambda^{V,k}(S) \in \mathbb{R}_+$ given by

$$\lambda^{V,k}(S) = \begin{cases} \max\left\{t \in \mathbb{R}_+ \;\middle|\; \left(\gamma^{V,k-1}_{S\cap P^{V,k-1}}, tu^{V(N)}_{S\setminus P^{V,k-1}}\right) \in V(S)\right\} & \text{if } \left(\gamma^{V,k-1}_{S\cap P^{V,k-1}}, 0_{S\setminus P^{V,k-1}}\right) \in V(S); \\ 0 & \text{if } \left(\gamma^{V,k-1}_{S\cap P^{V,k-1}}, 0_{S\setminus P^{V,k-1}}\right) \notin V(S). \end{cases}$$

The collection of *k-egalitarian admissible coalitions* is given by

$$\mathcal{A}^{V,k} = \left\{S \in 2^N \setminus \{\emptyset\} \;\middle|\; \chi^{V,k}(S) \in \mathrm{WP}(V(S)), \forall_{i\in S}\forall_{T\in 2^N : i\in T} : \chi^{V,k}_i(T) \leq \chi^{V,k}_i(S)\right\}.$$

The set of *k-egalitarian claimants* $P^{V,k} \in 2^N \setminus \{\emptyset\}$ is given by $P^{V,k} = \bigcup_{S\in\mathcal{A}^{V,k}} S$. The vector of *k-egalitarian claims* $\gamma^{V,k} \in \mathbb{R}^{P^{V,k}}_+$ is for all $i \in P^{V,k}$ given by $\gamma^{V,k}_i = \chi^{V,k}_i(S)$, where $S \in \mathcal{A}^{V,k}$ with $i \in S$.

Later, we show that this procedure is adequately defined. First, we provide an illustrative example.

Example 1.
Let $N = \{1, 2, 3\}$ and consider $V \in \mathrm{NTU}^N$ given by

$$\begin{aligned} V(\{1\}) &= \left\{x \in \mathbb{R}^{\{1\}}_+ \;\middle|\; x \leq 4\right\}; \\ V(\{2\}) &= \left\{x \in \mathbb{R}^{\{2\}}_+ \;\middle|\; x \leq 1\right\}; \\ V(\{3\}) &= \left\{x \in \mathbb{R}^{\{3\}}_+ \;\middle|\; x \leq 0\right\}; \\ V(\{1,2\}) &= \left\{x \in \mathbb{R}^{\{1,2\}}_+ \;\middle|\; x_1 \leq 4, x_2 \leq 2\right\}; \\ V(\{1,3\}) &= \left\{x \in \mathbb{R}^{\{1,3\}}_+ \;\middle|\; x_1 \leq 2, x_3 \leq 2\right\}; \\ V(\{2,3\}) &= \left\{x \in \mathbb{R}^{\{2,3\}}_+ \;\middle|\; 2x_2 + x_3 \leq 4\right\}; \\ V(\{1,2,3\}) &= \left\{x \in \mathbb{R}^{\{1,2,3\}}_+ \;\middle|\; 2x_1 + 2x_2 + x_3 \leq 12\right\}. \end{aligned}$$

We have $u^{V(N)} = (6, 6, 12)$. The following table illustrates the egalitarian distribution.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\chi^{V,1}(S)$	$(4,\cdot,\cdot)$	$(\cdot,1,\cdot)$	$(\cdot,\cdot,0)$	$(2,2,\cdot)$	$(1,\cdot,2)$	$(\cdot,1,2)$	$(2,2,4)$
$\chi^{V,2}(S)$	$(4,\cdot,\cdot)$	$(\cdot,1,\cdot)$	$(\cdot,\cdot,0)$	$(4,2,\cdot)$	$(4,\cdot,0)$	$(\cdot,1,2)$	$(4,1,2)$
$\chi^{V,k}(S)$ $(k \geq 3)$	$(4,\cdot,\cdot)$	$(\cdot,2,\cdot)$	$(\cdot,\cdot,0)$	$(4,2,\cdot)$	$(4,\cdot,0)$	$(\cdot,2,0)$	$(4,2,0)$

In the first iteration, we have $\mathcal{A}^{V,1} = \{\{1\}\}$, $P^{V,1} = \{1\}$, and $\gamma^{V,1} = (4,\cdot,\cdot)$. In the second iteration, we have $\mathcal{A}^{V,2} = \{\{1\}, \{1,2\}\}$, $P^{V,2} = \{1,2\}$, and $\gamma^{V,2} = (4,2,\cdot)$. In all subsequent iterations $k \geq 3$, we have $\mathcal{A}^{V,k} = \{\{1\}, \{3\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}$, $P^{V,k} = N$, and $\gamma^{V,k} = (4,2,0)$. △

Lemma 3.1.
Let $V \in \mathrm{NTU}^N$ and let $S \in 2^N \setminus \{\emptyset\}$. Then $\chi^{V,k}(S) \in \mathrm{WUC}(V(S))$ for all $k \in \mathbb{N}$.

Proof. We show the statement by induction on k. Suppose that $\chi^{V,1}(S) \notin \mathrm{WUC}(V(S))$. Then there exists an $x \in V(S)$ for which $x > \chi^{V,1}(S)$. Since $V(S)$ is comprehensive, this means that there exists a $y \in V(S)$ with $y > \chi^{V,1}(S)$ for which $y = tu^{V(N)}_S$ for some $t \in \mathbb{R}_+$. Using $P^{V,0} = \emptyset$, this means that $t > \lambda^{V,1}(S)$, which contradicts the definition of $\lambda^{V,1}(S)$. Hence, $\chi^{V,1}(S) \in \mathrm{WUC}(V(S))$.

Let $k \in \mathbb{N}$ and assume that $\chi^{V,k}(S) \in \mathrm{WUC}(V(S))$. If $S \subseteq P^{V,k}$, then $\chi^{V,k+1}(S) = \gamma_S^{V,k} \geq \chi^{V,k}(S)$, so $\chi^{V,k+1}(S) \in \mathrm{WUC}(V(S))$. Assume that $S \not\subseteq P^{V,k}$ and suppose that $\chi^{V,k+1}(S) \notin \mathrm{WUC}(V(S))$. Then there exists an $x \in V(S)$ for which $x > \chi^{V,k+1}(S)$. Since $V(S)$ is comprehensive, this means that there exists a $y \in V(S)$ with $y \geq \chi^{V,k+1}(S)$ and $y \neq \chi^{V,k+1}(S)$ for which $y = (\gamma^{V,k}_{S \cap P^{V,k}}, tu^{V(N)}_{S \setminus P^{V,k}})$ for some $t \in \mathbb{R}_+$. This means that $t > \lambda^{V,k+1}(S)$, which contradicts the definition of $\lambda^{V,k+1}(S)$. Hence, $\chi^{V,k+1}(S) \in \mathrm{WUC}(V(S))$. □

Lemma 3.1 shows that the egalitarian distribution generally assigns an overefficient allocation to each coalition. Only coalitions which are assigned an efficient allocation can be egalitarian admissible. There, members fix their allocated payoff and claim it in all further iterations. Efficiency can only be achieved when it is possible to allocate to the egalitarian claimants which are member of the coalition their corresponding egalitarian claims. Formally, for all $S \in 2^N \setminus \{\emptyset\}$ and any $k \in \mathbb{N}$, we have $\chi^{V,k}(S) \in \mathrm{WP}(V(S))$ if and only if $(\gamma^{V,k-1}_{S \cap P^{V,k-1}}, 0_{S \setminus P^{V,k-1}}) \in V(S)$. In particular, this means that the egalitarian distribution assigns in the first iteration an efficient allocation to each coalition.

To an egalitarian admissible coalition, the egalitarian distribution assigns an efficient allocation for which no member is allocated a higher payoff in any other coalition. This suggests that the payoff allocation is an element of the core. The *core* of any $V \in \mathrm{NTU}^N$ is given by

$$\mathcal{C}(V) = \left\{ x \in V(N) \;\middle|\; \forall_{S \in 2^N \setminus \{\emptyset\}} : x_S \in \mathrm{WUC}(V(S)) \right\}.$$

Indeed, for each egalitarian admissible coalition, the corresponding vector of egalitarian claims is a core element of the induced subgame. For any $V \in \mathrm{NTU}^N$, the *subgame* $V_S \in \mathrm{NTU}^S$ on $S \in 2^N \setminus \{\emptyset\}$ is given by $V_S(R) = V(R)$ for all $R \in 2^S \setminus \{\emptyset\}$.

Proposition 3.2.
Let $V \in \mathrm{NTU}^N$ and let $k \in \mathbb{N}$. Then $\gamma_S^{V,k} \in \mathcal{C}(V_S)$ for all $S \in \mathcal{A}^{V,k}$.

Proof. Let $S \in \mathcal{A}^{V,k}$. By definition, we have $\gamma_S^{V,k} = \chi^{V,k}(S)$ and $\chi^{V,k}(S) \in V_S(S)$. Suppose that $\gamma_S^{V,k} \notin \mathcal{C}(V_S)$. Then there exists an $R \in 2^S \setminus \{\emptyset\}$ for which $\gamma_R^{V,k} \in V_S(R) \setminus \mathrm{WP}(V_S(R))$. We can write

$$\gamma_R^{V,k} = \chi_R^{V,k}(S) \geq \chi^{V,k}(R).$$

Since $V_S(R)$ is comprehensive, this means that $\chi^{V,k}(R) \in V_S(R) \setminus \mathrm{WP}(V_S(R))$. This contradicts Lemma 3.1. Hence, $\gamma_S^{V,k} \in \mathcal{C}(V_S)$. □

The question arises whether egalitarian admissible coalitions and egalitarian claimants exist in every nontransferable utility game. Are players always able to acquire an egalitarian claim? The answer turns out to be affirmative.

Lemma 3.3.
Let $V \in \mathrm{NTU}^N$ and let $k \in \mathbb{N}$. Then $\mathcal{A}^{V,k} \subseteq \mathcal{A}^{V,k+1}$.
Moreover, if $P^{V,k-1} \neq N$, then $P^{V,k-1} \subset P^{V,k}$.

Proof. Let $S \in \mathcal{A}^{V,k}$. Then we have $\chi^{V,k}(S) \in \mathrm{WP}(V(S))$ and $S \subseteq P^{V,k}$. We can write $\chi^{V,k+1}(S) = \gamma_S^{V,k} = \chi^{V,k}(S)$. This means that $\chi^{V,k+1}(S) \in \mathrm{WP}(V(S))$ and for all $i \in S$ we have $\chi_i^{V,k+1}(T) = \gamma_i^{V,k} \leq \chi_i^{V,k+1}(S)$ for all $T \in 2^N$ for which $i \in T$, so $S \in \mathcal{A}^{V,k+1}$. Hence, $\mathcal{A}^{V,k} \subseteq \mathcal{A}^{V,k+1}$.

Assume that $P^{V,k-1} \neq N$. Let $S \in 2^N$ with $S \not\subseteq P^{V,k-1}$ and $(\gamma^{V,k-1}_{S \cap P^{V,k-1}}, 0_{S \setminus P^{V,k-1}}) \in V(S)$ be a coalition such that $\lambda^{V,k}(S)$ equals the maximum $\lambda^{V,k}(R)$ over all coalitions $R \in 2^N$ with $R \not\subseteq P^{V,k-1}$. Then we have $\chi^{V,k}(S) \in \mathrm{WP}(V(S))$ and $\chi_i^{V,k}(T) \leq \chi_i^{V,k}(S)$ for all $i \in S$ and all $T \in 2^N$ for which $i \in T$. This means that $S \in \mathcal{A}^{V,k}$ and $S \subseteq P^{V,k}$. Hence, $P^{V,k-1} \subset P^{V,k}$. □

Lemma 3.3 shows that the collection of egalitarian admissible coalitions weakly extends in each iteration and eventually covers all players. The structure of this collection is determined by the structure of the underlying nontransferable utility game. An NTU-game $V \in \mathrm{NTU}^N$ is called

- *superadditive* if $V(S) \times V(T) \subseteq V(S \cup T)$ for all $S, T \in 2^N \setminus \{\emptyset\}$ for which $S \cap T = \emptyset$;
- *ordinal convex* (cf. Vilkov (1977)) if V is superadditive and $x_{S\cup T} \in V(S \cup T)$ or $x_{S\cap T} \in V(S \cap T)$ for all $S, T \in 2^N \setminus \{\emptyset\}$ for which $S \cap T \neq \emptyset$ and any $x \in \mathbb{R}^N_+$ for which $x_S \in V(S)$ and $x_T \in V(T)$;
- *coalitional merge convex* (cf. Hendrickx, Borm, and Timmer (2002)) if V is superadditive and for all $R \in 2^N \setminus \{\emptyset\}$ and $S, T \in 2^{N\setminus R} \setminus \{\emptyset\}$ for which $S \subset T$, and any $s \in V(S)$, $t \in V(T)$, and $x \in V(S \cup R)$ for which $x_S \geq s$, there exists a $y \in V(T \cup R)$ for which $y_T \geq t$ and $y_R \geq x_R$;
- *balanced* (cf. Scarf (1967)) if for all balanced collections $\mathcal{B} \subseteq 2^N \setminus \{\emptyset\}$, we have $x \in V(N)$ if $x_S \in V(S)$ for all $S \in \mathcal{B}$. Here, a collection of coalitions $\mathcal{B} \subseteq 2^N \setminus \{\emptyset\}$ is called *balanced* if there exists a function $\delta : \mathcal{B} \to \mathbb{R}_{++}$ for which $\sum_{S\in\mathcal{B}:i\in S} \delta(S) = 1$ for all $i \in N$.

Greenberg (1985) showed that the core of an ordinal convex game is nonempty. Hendrickx et al. (2002) and Scarf (1967) showed a similar result for coalitional merge convex games and balanced games, respectively.

Interestingly, the aforementioned properties of nontransferable utility games each have implications for the relation of the collections of egalitarian admissible coalitions in two subsequent iterations.

Proposition 3.4.
Let $V \in \mathrm{NTU}^N$ and let $k \in \mathbb{N}$.

(i) If V is superadditive, then $S \cup T \in \mathcal{A}^{V,k+1}$ for all $S, T \in \mathcal{A}^{V,k}$ with $S \cap T = \emptyset$.

(ii) If V is ordinal convex, then $S \cup T \in \mathcal{A}^{V,k+1}$ or $S \cap T \in \mathcal{A}^{V,k+1}$ for all $S, T \in \mathcal{A}^{V,k}$.

(iii) If V is coalitional merge convex, then $S \cup T \in \mathcal{A}^{V,k+1}$ for all $S, T \in \mathcal{A}^{V,k}$.

(iv) If V is balanced, then $N \in \mathcal{A}^{V,k+1}$ if there exists a balanced collection $\mathcal{B} \subseteq \mathcal{A}^{V,k}$.

Proof. (i) Assume that V is superadditive. Let $S, T \in \mathcal{A}^{V,k}$ with $S \cap T = \emptyset$. Then we have $\gamma^{V,k}_S \in V(S)$ and $\gamma^{V,k}_T \in V(T)$. Since V is superadditive, this means that $\gamma^{V,k}_{S\cup T} \in V(S \cup T)$. From Lemma 3.1 we know that $\chi^{V,k+1}(S \cup T) \in \mathrm{WUC}(V(S \cup T))$. Since $\chi^{V,k+1}(S \cup T) = \gamma^{V,k}_{S\cup T}$, this implies that $\chi^{V,k+1}(S \cup T) \in \mathrm{WP}(V(S \cup T))$. Hence, $S \cup T \in \mathcal{A}^{V,k+1}$.

(ii) Assume that V is ordinal convex. Let $S, T \in \mathcal{A}^{V,k}$ with $S \cap T \neq \emptyset$. Then we have $\gamma^{V,k}_S \in V(S)$ and $\gamma^{V,k}_T \in V(T)$. Since V is ordinal convex, this means that $\gamma^{V,k}_{S\cup T} \in V(S \cup T)$ or $\gamma^{V,k}_{S\cap T} \in V(S \cap T)$. From Lemma 3.1 we know that $\chi^{V,k+1}(S \cup T) \in \mathrm{WUC}(V(S \cup T))$ and $\chi^{V,k+1}(S \cap T) \in \mathrm{WUC}(V(S \cap T))$. Since $\chi^{V,k+1}(S \cup T) = \gamma^{V,k}_{S\cup T}$ and $\chi^{V,k+1}(S \cap T) = \gamma^{V,k}_{S\cap T}$, this implies that $\chi^{V,k+1}(S \cup T) \in \mathrm{WP}(V(S \cup T))$ or $\chi^{V,k+1}(S \cap T) \in \mathrm{WP}(V(S \cap T))$. Hence, $S \cup T \in \mathcal{A}^{V,k+1}$ or $S \cap T \in \mathcal{A}^{V,k+1}$.

(iii) Assume that V is coalitional merge convex. Let $S, T \in \mathcal{A}^{V,k}$ with $S \cap T \neq \emptyset$, $S \not\subseteq T$ and $T \not\subseteq S$. Then we have $\gamma^{V,k}_S \in V(S)$ and $\gamma^{V,k}_T \in V(T)$. Since V is coalitional merge convex, there exists a $y \in V(S \cup T)$ for which $y_S \geq \gamma^{V,k}_S$ and $y_{T\setminus S} \geq \gamma^{V,k}_{T\setminus S}$, i.e. $y \geq \gamma^{V,k}_{S\cup T}$. Since $V(S \cup T)$ is comprehensive, this means that $\gamma^{V,k}_{S\cup T} \in V(S \cup T)$. From Lemma 3.1 we know that $\chi^{V,k+1}(S \cup T) \in \mathrm{WUC}(V(S \cup T))$. Since $\chi^{V,k+1}(S \cup T) = \gamma^{V,k}_{S\cup T}$, this implies that $\chi^{V,k+1}(S \cup T) \in \mathrm{WP}(V(S \cup T))$. Hence, $S \cup T \in \mathcal{A}^{V,k+1}$.

(*iv*) Assume that V is balanced. Let $\mathcal{B} \subseteq \mathcal{A}^{V,k}$ be a balanced collection. Then we have $\gamma_S^{V,k} \in V(S)$ for all $S \in \mathcal{B}$. Since V is balanced, this means that $\gamma^{V,k} \in V(N)$. From Lemma 3.1 we know that $\chi^{V,k+1}(N) \in \mathrm{WUC}(V(N))$. Since $\chi^{V,k+1}(N) = \gamma^{V,k}$, this implies that $\chi^{V,k+1}(N) \in \mathrm{WP}(V(N))$. Hence, $N \in \mathcal{A}^{V,k+1}$. □

Furthermore, Lemma 3.3 also shows that in each iteration of the egalitarian procedure, at least one extra player acquires an egalitarian claim as long as the collection of egalitarian admissible coalitions does not cover all players. The egalitarian procedure reaches a steady state when all players are egalitarian claimants. This means that the number of iterations needed to converge to a steady state is bounded by the number of players. Example 1 shows that this bound is tight.

Definition 2.
Let $V \in \mathrm{NTU}^N$ be a nontransferable utility game. The iteration $n^V \in \{1, \ldots, |N|\}$ is given by $n^V = \min\{k \in \mathbb{N} \mid P^{V,k} = N\}$. The vector of *egalitarian claims* $\widehat{\gamma}^V \in \mathbb{R}_+^N$ is given by $\widehat{\gamma}^V = \gamma^{V,n^V}$. The collection $\widehat{\mathcal{A}}^V \subseteq 2^N \setminus \{\emptyset\}$ is given by

$$\widehat{\mathcal{A}}^V = \left\{ S \in 2^N \setminus \{\emptyset\} \;\middle|\; \widehat{\gamma}_S^V \in V(S), \forall_{T \in 2^N \setminus \{\emptyset\} : \widehat{\gamma}_T^V \in V(T)} : S \not\subset T \right\}.$$

The set of *strong egalitarian claimants* $D^V \in 2^N$ is given by $D^V = \bigcap_{S \in \widehat{\mathcal{A}}^V} S$ and consists of all players which are member of all inclusion-wise maximal egalitarian admissible coalitions.

The egalitarian value is a solution concept which takes both the set of strong egalitarian claimants and the vector of egalitarian claims into account to prescribe a payoff allocation for the grand coalition. The egalitarian claims can be interpreted as aspiration levels for such an allocation. The egalitarian value first allocates to all strong egalitarian claimants their claims, and then allocates to all other players their claims. The possibly resulting infeasibility is modeled as a bankruptcy problem in which the egalitarian claims are adopted.

A *bankruptcy problem with nontransferable utility* (cf. Dietzenbacher et al. (2016)) is a triple (N, E, c) in which $E \subseteq \mathbb{R}_+^N$ is a nonempty, closed, bounded, nontrivial and comprehensive *estate* and $c \in \mathrm{WUC}(E)$ is a vector of *claims*. Let BR^N denote the class of all such NTU-bankruptcy problems with player set N. For convenience, an NTU-bankruptcy problem is denoted by $(E, c) \in \mathrm{BR}^N$. The *constrained relative equal awards rule* $\mathrm{CREA} : \mathrm{BR}^N \to \mathbb{R}_+^N$ assigns to any $(E, c) \in \mathrm{BR}^N$ the payoff allocation

$$\mathrm{CREA}(E, c) = \left(\min\left\{c_i, \alpha^{E,c} u_i^E\right\}\right)_{i \in N},$$

where $\alpha^{E,c} = \max\{t \in [0, 1] \mid (\min\{c_i, t u_i^E\})_{i \in N} \in E\}$. Note that $\mathrm{CREA}(E, c) \in \mathrm{WP}(E)$.

A *solution* for nontransferable utility games $f : \mathrm{NTU}^N \to \mathbb{R}_+^N$ assigns to any $V \in \mathrm{NTU}^N$ a payoff allocation $f(V) \in \mathrm{WP}(V(N))$. Taking the egalitarian claims and the set of strong egalitarian claimants into account, the egalitarian value is a solution for nontransferable utility games which uses the constrained relative equal awards rule to prescribe a payoff allocation for the grand coalition. In Section 5, we further elaborate on the choice of this specific bankruptcy rule.

Definition 3 (Egalitarian Value)**.**
The *egalitarian value* $\Gamma : \mathrm{NTU}^N \to \mathbb{R}_+^N$ is for all $V \in \mathrm{NTU}^N$ given by

$$\Gamma(V) = \begin{cases} \left(\widehat{\gamma}_{D^V}^V, \mathrm{CREA}\left(\left\{x \in \mathbb{R}_+^{N \setminus D^V} \;\middle|\; \left(\widehat{\gamma}_{D^V}^V, x\right) \in V(N)\right\}, \widehat{\gamma}_{N \setminus D^V}^V\right)\right) & \text{if } \left(\widehat{\gamma}_{D^V}^V, 0_{N \setminus D^V}\right) \in V(N); \\ \left(\mathrm{CREA}\left(\left\{x \in \mathbb{R}_+^{D^V} \;\middle|\; \left(x, 0_{N \setminus D^V}\right) \in V(N)\right\}, \widehat{\gamma}_{D^V}^V\right), 0_{N \setminus D^V}\right) & \text{if } \left(\widehat{\gamma}_{D^V}^V, 0_{N \setminus D^V}\right) \notin V(N). \end{cases}$$

Note that, on the class of (nonnegative) transferable utility games, the egalitarian value coincides with the procedural egalitarian solution (cf. Dietzenbacher et al. (2016)) if the latter is nonnegative.

Example 2.
Let $N = \{1,2,3\}$ and consider $V \in \mathrm{NTU}^N$ as in Example 1. We have $n^V = 3$, $\widehat{\gamma}^V = (4,2,0)$, $\widehat{\mathcal{A}}^V = \{N\}$, and $D^V = N$. Consequently, $\Gamma(V) = (4,2,0)$. $\triangle$

As in Example 2, an interesting situation arises when the grand coalition is egalitarian admissible. Then, all players are strong egalitarian claimants, no infeasibility results, and the egalitarian value assigns to all players their egalitarian claims. Moreover, from Proposition 3.2 we know that the egalitarian value leads to a core element. Therefore, such nontransferable utility games are called egalitarian stable.

Definition 4 (Egalitarian Stability)**.**
A nontransferable utility game $V \in \mathrm{NTU}^N$ is called *egalitarian stable* if $\widehat{\mathcal{A}}^V = \{N\}$.

We know that egalitarian stability is a sufficient condition for nontransferable utility games to contain the egalitarian value in the core. The following example shows that this condition is not necessary.

Example 3.
Let $N = \{1,2,3\}$ and consider $V \in \mathrm{NTU}^N$ given by

$$\begin{aligned}
V(\{i\}) &= \left\{ x \in \mathbb{R}_+^{\{i\}} \;\middle|\; x \leq 0 \right\} \text{ for } i \in N; \\
V(\{1,i\}) &= \left\{ x \in \mathbb{R}_+^{\{1,i\}} \;\middle|\; x_1 \leq 4, x_i \leq 4 \right\} \text{ for } i \in \{2,3\}; \\
V(\{2,3\}) &= \left\{ x \in \mathbb{R}_+^{\{2,3\}} \;\middle|\; x_2 \leq 0, x_3 \leq 0 \right\}; \\
V(\{1,2,3\}) &= \left\{ x \in \mathbb{R}_+^{\{1,2,3\}} \;\middle|\; x_1 + x_2 + x_3 \leq 6 \right\}.
\end{aligned}$$

We have $u^{V(N)} = (6,6,6)$. The following table illustrates the egalitarian distribution in the first iteration of the egalitarian procedure.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\chi^{V,1}(S)$	$(0,\cdot,\cdot)$	$(\cdot,0,\cdot)$	$(\cdot,\cdot,0)$	$(4,4,\cdot)$	$(4,\cdot,4)$	$(\cdot,0,0)$	$(2,2,2)$

In the first iteration, we have $\mathcal{A}^{V,1} = \{\{1,2\},\{1,3\}\}$, $P^{V,1} = N$, and $\gamma^{V,1} = (4,4,4)$. This means that $n^V = 1$, $\widehat{\gamma}^V = (4,4,4)$, $\widehat{\mathcal{A}}^V = \{\{1,2\},\{1,3\}\}$, and $D^V = \{1\}$. Consequently,

$$\Gamma(V) = \left(4, \mathrm{CREA}\left(\left\{ x \in \mathbb{R}_+^{\{2,3\}} \;\middle|\; x_2 + x_3 \leq 2 \right\}, (\cdot,4,4)\right)\right) = (4,1,1).$$

Note that $\Gamma(V) \in \mathcal{C}(V)$. $\triangle$

In Example 3, the sets of payoff allocations $V(\{1,2\})$ and $V(\{1,3\})$ are not nonleveled. We want to note that, for nontransferable utility games $V \in \mathrm{NTU}^N$ for which $V(S)$ is nonleveled for all $S \in 2^N \setminus \{\emptyset\}$, egalitarian stability is a necessary and sufficient condition to contain the egalitarian value in the core. The question arises which nontransferable utility games are egalitarian stable. From Proposition 3.4 we know that coalitional merge convex games are egalitarian stable. In the next sections we show that the Roth-Shafer examples, bankruptcy games and bargaining games are all egalitarian stable as well.

4 Roth-Shafer Examples

In this section, we study the egalitarian value for the examples introduced by Roth (1980) and Shafer (1980). These examples initiated an interesting and extensive discussion on the interpretation of solutions for nontransferable utility games. Along the lines of this discussion, we compare the egalitarian value with the Shapley value (cf. Shapley (1969)), the Harsanyi value (cf. Harsanyi (1963)), and the monotonic solution of Kalai and Samet (1985). For more details, we refer to Harsanyi (1980), Aumann (1985), Hart (1985), Roth (1986), and Aumann (1986).

Example 4 (cf. Roth (1980))**.**
Let $N = \{1, 2, 3\}$ and consider $V_p \in \mathrm{NTU}^N$ which is for all $p \in (0, \frac{1}{2})$ given by

$$\begin{aligned} V_p(\{i\}) &= \left\{ x \in \mathbb{R}_+^{\{i\}} \;\middle|\; x \leq 0 \right\} \text{ for } i \in N; \\ V_p(\{1,2\}) &= \left\{ x \in \mathbb{R}_+^{\{1,2\}} \;\middle|\; x_1 \leq \tfrac{1}{2}, x_2 \leq \tfrac{1}{2} \right\}; \\ V_p(\{i,3\}) &= \left\{ x \in \mathbb{R}_+^{\{i,3\}} \;\middle|\; x_i \leq p, x_3 \leq 1-p \right\} \text{ for } i \in \{1,2\}; \\ V_p(\{1,2,3\}) &= \left\{ x \in \mathbb{R}_+^{\{1,2,3\}} \;\middle|\; x \in \mathrm{comp}(\mathrm{conv}(\{(\tfrac{1}{2}, \tfrac{1}{2}, 0), (p, 0, 1-p), (0, p, 1-p)\})) \right\}, \end{aligned}$$

where $\mathrm{conv}(A)$ denotes the convex hull of the set $A \subseteq \mathbb{R}_+^N$. We have $u^{V_p(N)} = (\frac{1}{2}, \frac{1}{2}, 1-p)$. The following table illustrates the egalitarian distribution in the first two iterations of the egalitarian procedure.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\chi^{V_p,1}(S)$	0	0	0	$(\frac{1}{2}, \frac{1}{2}, \cdot)$	$(p, \cdot, 2p(1-p))$	$(\cdot, p, 2p(1-p))$	$\lambda^{V_p,1}(N)u^{V_p(N)}$
$\chi^{V_p,2}(S)$	$\frac{1}{2}$	$\frac{1}{2}$	0	$(\frac{1}{2}, \frac{1}{2}, \cdot)$	$(\frac{1}{2}, \cdot, 0)$	$(\cdot, \frac{1}{2}, 0)$	$(\frac{1}{2}, \frac{1}{2}, 0)$

In the first iteration, we have $\mathcal{A}^{V_p,1} = \{\{1,2\}\}$, $P^{V_p,1} = \{1,2\}$, and $\gamma^{V_p,1} = (\frac{1}{2}, \frac{1}{2}, \cdot)$. In the second iteration, we have $\mathcal{A}^{V_p,2} = \{\{3\}, \{1,2\}, \{1,2,3\}\}$, $P^{V_p,2} = N$, and $\gamma^{V_p,2} = (\frac{1}{2}, \frac{1}{2}, 0)$. This means that $n^{V_p} = 2$, $\widehat{\gamma}^{V_p} = (\frac{1}{2}, \frac{1}{2}, 0)$, $\widehat{\mathcal{A}}^{V_p} = \{N\}$, and $D^{V_p} = N$. Consequently, $\Gamma(V_p) = (\frac{1}{2}, \frac{1}{2}, 0)$.

Besides, the Shapley value equals $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and the Harsanyi value and the monotonic solution both equal $(\frac{1}{2} - \frac{1}{3}p, \frac{1}{2} - \frac{1}{3}p, \frac{2}{3}p)$. Note that, contrary to the egalitarian value, these solutions do not belong to the core. Roth argues that the payoff allocation $(\frac{1}{2}, \frac{1}{2}, 0)$ is the unique outcome of this game which is consistent with the hypothesis that the players are rational utility maximizers, since this payoff allocation is strictly preferred by both player 1 and 2, and it can be achieved without player 3. The egalitarian value perfectly matches this idea. △

Example 5 (cf. Shafer (1980) and Hart and Kurz (1983))**.**
Let $N = \{1, 2, 3\}$ and consider $V_\epsilon \in \mathrm{NTU}^N$ which is for all $\epsilon \in [0, \frac{1}{6})$ given by

$$\begin{aligned} V_\epsilon(\{i\}) &= \left\{ x \in \mathbb{R}_+^{\{i\}} \;\middle|\; x \leq 0 \right\} \text{ for } i \in \{1,2\}; \\ V_\epsilon(\{3\}) &= \left\{ x \in \mathbb{R}_+^{\{3\}} \;\middle|\; x \leq \epsilon \right\}; \\ V_\epsilon(\{1,2\}) &= \left\{ x \in \mathbb{R}_+^{\{1,2\}} \;\middle|\; x_1 + x_2 \leq 1 - \epsilon \right\}; \\ V_\epsilon(\{i,3\}) &= \left\{ x \in \mathbb{R}_+^{\{i,3\}} \;\middle|\; x_i + x_3 \leq \tfrac{1}{2} + \tfrac{1}{2}\epsilon, x_i \leq \epsilon \right\} \text{ for } i \in \{1,2\}; \\ V_\epsilon(\{1,2,3\}) &= \left\{ x \in \mathbb{R}_+^{\{1,2,3\}} \;\middle|\; x_1 + x_2 + x_3 \leq 1 \right\}. \end{aligned}$$

We have $u^{V_\epsilon(N)} = (1,1,1)$. The following table illustrates the egalitarian distribution in the first two iterations of the egalitarian procedure.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$\chi^{V_\epsilon,1}(S)$	0	0	ϵ	$(\frac{1-\epsilon}{2},\frac{1-\epsilon}{2},\cdot)$	$(\epsilon,\cdot,\epsilon)$	$(\cdot,\epsilon,\epsilon)$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$
$\chi^{V_\epsilon,2}(S)$	$\frac{1-\epsilon}{2}$	$\frac{1-\epsilon}{2}$	ϵ	$(\frac{1-\epsilon}{2},\frac{1-\epsilon}{2},\cdot)$	$(\frac{1-\epsilon}{2},\cdot,0)$	$(\cdot,\frac{1-\epsilon}{2},0)$	$(\frac{1-\epsilon}{2},\frac{1-\epsilon}{2},\epsilon)$

In the first iteration, we have $\mathcal{A}^{V_\epsilon,1} = \{\{1,2\}\}$, $P^{V_\epsilon,1} = \{1,2\}$, and $\gamma^{V_\epsilon,1} = (\frac{1-\epsilon}{2},\frac{1-\epsilon}{2},\cdot)$. In the second iteration, we have $\mathcal{A}^{V_\epsilon,2} = \{\{3\},\{1,2\},\{1,2,3\}\}$, $P^{V_\epsilon,2} = N$, and $\gamma^{V_\epsilon,2} = (\frac{1-\epsilon}{2},\frac{1-\epsilon}{2},\epsilon)$. This means that $n^{V_\epsilon} = 2$, $\widehat{\gamma}^{V_\epsilon} = (\frac{1-\epsilon}{2},\frac{1-\epsilon}{2},\epsilon)$, $\widehat{\mathcal{A}}^{V_\epsilon} = \{N\}$, and $D^{V_\epsilon} = N$. Consequently, $\Gamma(V_\epsilon) = (\frac{1}{2}-\frac{1}{2}\epsilon, \frac{1}{2}-\frac{1}{2}\epsilon, \epsilon)$.

Besides, the Shapley value equals $(\frac{5}{12}-\frac{5}{12}\epsilon, \frac{5}{12}-\frac{5}{12}\epsilon, \frac{2}{12}+\frac{10}{12}\epsilon)$, and the Harsanyi value and the monotonic solution both equal $(\frac{1}{2}-\frac{5}{6}\epsilon, \frac{1}{2}-\frac{5}{6}\epsilon, \frac{10}{6}\epsilon)$. Note that, contrary to the egalitarian value, these solutions do not belong to the core. Shafer argues that it is unreasonable to allocate at least $\frac{1}{6}$ to player 3, independent of ϵ and especially in the case $\epsilon = 0$. The egalitarian value seamlessly connects with this idea. $\triangle$

5 Bankruptcy Games

In this section, we analyze the egalitarian value on the class of bankruptcy games with nontransferable utility. For any bankruptcy problem $(E,c) \in \mathrm{BR}^N$ for which the estate E is nonleveled, the corresponding *bankruptcy game* (cf. Dietzenbacher (2017)) $V^{E,c} \in \mathrm{NTU}^N$ is for all $S \in 2^N \setminus \{\emptyset\}$ given by

$$V^{E,c}(S) = \begin{cases} \{x \in \mathbb{R}^S_+ \mid (x, c_{N\setminus S}) \in E\} & \text{if } (0^S, c_{N\setminus S}) \in E; \\ 0^S & \text{if } (0^S, c_{N\setminus S}) \notin E. \end{cases}$$

The core of a bankruptcy game is given by $\mathcal{C}(V^{E,c}) = \{x \in \mathrm{WP}(E) \mid x \leq c\}$.

In the next theorem, we show that bankruptcy games are egalitarian stable, which means that the egalitarian value assigns to all players their egalitarian claims without having to rely on the constrained relative equal awards rule in its definition. Interestingly, we show that the egalitarian value for bankruptcy games corresponds to the constrained relative equal awards rule for the underlying bankruptcy problem. This illustrates the strong connection between the egalitarian value and the constrained relative equal awards rule. Besides, it justifies the use of the latter in the definition of the egalitarian value for nontransferable utility games which are not egalitarian stable.

Theorem 5.1.
Let $(E,c) \in \mathrm{BR}^N$ be a bankruptcy problem with nontransferable utility such that E is nonleveled. Then $\Gamma(V^{E,c}) = \mathrm{CREA}(E,c)$.

Proof. First, we show that $\widehat{\gamma}^{V^{E,c}} \leq c$. Suppose that there exists an $i \in N$ for which $\widehat{\gamma}_i^{V^{E,c}} > c_i$. Let $k \in \mathbb{N}$ such that $i \in P^{V^{E,c},k} \setminus P^{V^{E,c},k-1}$ and let $S \in \mathcal{A}^{V^{E,c},k}$ such that $i \in S$. Since $c \in \mathrm{WUC}(E)$ and E is nonleveled, we have $\widehat{\gamma}_i^{V^{E,c}} \notin V^{E,c}(\{i\})$, so $S \neq \{i\}$. We have $\chi^{V^{E,c},k}(S) \in \mathrm{WP}(V^{E,c}(S))$, i.e.

$$\left(\lambda^{V^{E,c},k}(S) u^{V^{E,c}(N)}_{S\setminus P^{V^{E,c},k-1}}, \gamma^{V^{E,c},k-1}_{S\cap P^{V^{E,c},k-1}}\right) \in \mathrm{WP}(V^{E,c}(S)).$$

Since E is nonleveled, this means that

$$\left(\lambda^{V^{E,c},k}(S) u^{V^{E,c}(N)}_{S\setminus P^{V^{E,c},k-1}}, \gamma^{V^{E,c},k-1}_{S\cap P^{V^{E,c},k-1}}, c_{N\setminus S}\right) \in \mathrm{SP}(E).$$

Since E is comprehensive, we have

$$\left(\lambda^{V^{E,c},k}(S)u^{V^{E,c}(N)}_{S\setminus(P^{V^{E,c},k-1}\cup\{i\})},\gamma^{V^{E,c},k-1}_{S\cap P^{V^{E,c},k-1}},c_i,c_{N\setminus S}\right)\in E\setminus \mathrm{SP}(E).$$

Since E is nonleveled, we have

$$\left(\lambda^{V^{E,c},k}(S)u^{V^{E,c}(N)}_{S\setminus(P^{V^{E,c},k-1}\cup\{i\})},\gamma^{V^{E,c},k-1}_{S\cap P^{V^{E,c},k-1}}\right)\in V^{E,c}(S\setminus\{i\})\setminus \mathrm{WP}(V^{E,c}(S\setminus\{i\})).$$

Since we know from Lemma 3.1 that $\chi^{V^{E,c},k}(S\setminus\{i\})\in \mathrm{WUC}(V^{E,c}(S\setminus\{i\}))$, we can write

$$\begin{aligned}\chi^{V^{E,c},k}_{S\setminus(P^{V^{E,c},k-1}\cup\{i\})}(S\setminus\{i\}) &= \lambda^{V^{E,c},k}(S\setminus\{i\})u^{V^{E,c}(N)}_{S\setminus(P^{V^{E,c},k-1}\cup\{i\})}\\ &> \lambda^{V^{E,c},k}(S)u^{V^{E,c}(N)}_{S\setminus(P^{V^{E,c},k-1}\cup\{i\})}\\ &= \chi^{V^{E,c},k}_{S\setminus(P^{V^{E,c},k-1}\cup\{i\})}(S).\end{aligned}$$

This contradicts that $S\in\mathcal{A}^{V^{E,c},k}$. Hence, $\widehat{\gamma}^{V^{E,c}}\leq c$.

Suppose that $c\in E$. Then we have $\chi^{V^{E,c},n^{V^{E,c}}}(N)\leq\gamma^{V^{E,c},n^{V^{E,c}}}=\widehat{\gamma}^{V^{E,c}}\leq c$. From Lemma 3.1 we know that $\chi^{V^{E,c},n^{V^{E,c}}}(N)\in \mathrm{WUC}(E)$. Since E is nonleveled, this means that $\widehat{\gamma}^{V^{E,c}}=c$, $\widehat{\mathcal{A}}^{V^{E,c}}=\{N\}$, and $D^{V^{E,c}}=N$. Consequently, $\Gamma(V^{E,c})=c=\mathrm{CREA}(E,c)$.

Now suppose that $c\notin E$. First, we show that $\chi^{V^{E,c},1}(S)\leq\alpha^{E,c}u^E_S$ for all $S\in 2^N\setminus\{\emptyset\}$. Suppose there exists an $S\in 2^N\setminus\{\emptyset\}$ for which $\chi^{V^{E,c},1}_i(S)>\alpha^{E,c}u^E_i$ for some $i\in S$. Then we have $\chi^{V^{E,c},1}(S)\in \mathrm{WP}(V^{E,c}(S))$ and $\chi^{V^{E,c},1}(S)=\lambda^{V^{E,c},1}(S)u^E_S>\alpha^{E,c}u^E_S\geq \mathrm{CREA}_S(E,c)$. Since E is nonleveled, this means that $(\chi^{V^{E,c},1}(S),c_{N\setminus S})\in \mathrm{WP}(E)$. Moreover, $(\chi^{V^{E,c},1}(S),c_{N\setminus S})\geq \mathrm{CREA}(E,c)$ and $(\chi^{V^{E,c},1}(S),c_{N\setminus S})\neq \mathrm{CREA}(E,c)$. Since E is nonleveled, this contradicts that $\mathrm{CREA}(E,c)\in \mathrm{WP}(E)$. Hence, $\chi^{V^{E,c},1}(S)\leq\alpha^{E,c}u^E_S$ for all $S\in 2^N\setminus\{\emptyset\}$.

Next, define $H^{E,c}\in 2^N\setminus\{\emptyset\}$ by

$$H^{E,c}=\left\{i\in N\mid \mathrm{CREA}_i(E,c)=\alpha^{E,c}u^E_i\right\}.$$

We have $\chi^{V^{E,c},1}(H^{E,c})\in \mathrm{WP}(V^{E,c}(H^{E,c}))$ and

$$\chi^{V^{E,c},1}(H^{E,c})=\lambda^{V^{E,c},1}(H^{E,c})u^E_{H^{E,c}}=\alpha^{E,c}u^E_{H^{E,c}}=\mathrm{CREA}_{H^{E,c}}(E,c).$$

This means that $H^{E,c}\in\mathcal{A}^{V^{E,c},1}$ and $H^{E,c}\subseteq P^{V^{E,c},1}$. Moreover, $\gamma^{V^{E,c},1}_{H^{E,c}}=\mathrm{CREA}_{H^{E,c}}(E,c)$.

Now, we have

$$\chi^{V^{E,c},n^{V^{E,c}}}(N)\leq\gamma^{V^{E,c},n^{V^{E,c}}}=\widehat{\gamma}^{V^{E,c}}\leq\left(\mathrm{CREA}_{H^{E,c}}(E,c),c_{N\setminus H^{E,c}}\right)=\mathrm{CREA}(E,c).$$

From Lemma 3.1 we know that $\chi^{V^{E,c},n^{V^{E,c}}}(N)\in \mathrm{WUC}(E)$. Since $\mathrm{CREA}(E,c)\in \mathrm{WP}(E)$ and E is nonleveled, this means that $\widehat{\gamma}^{V^{E,c}}=\mathrm{CREA}(E,c)$, $\widehat{\mathcal{A}}^{V^{E,c}}=\{N\}$, and $D^{V^{E,c}}=N$. Consequently, $\Gamma(V^{E,c})=\mathrm{CREA}(E,c)$. □

6 Bargaining Games

In this section, we analyze the egalitarian value on the class of bargaining games. A (nonnegative) *bargaining problem* (cf. Nash (1950)) is a triple (N, F, d) in which $F \subseteq \mathbb{R}^N_+$ is a nonempty, closed, bounded, nontrivial and comprehensive *feasible set* and $d \in F$ is a *disagreement point*. Let BG^N denote the class of all such bargaining problems with player set N. For convenience, a bargaining problem is denoted by $(F, d) \in \mathrm{BG}^N$. For all $(F, d) \in \mathrm{BG}^N$, we denote $F_d = \{x \in F \mid x \geq d\}$. Kalai and Smorodinsky (1975) introduced the solution $\mathrm{KS} : \mathrm{BG}^N \to \mathbb{R}^N_+$ assigns to any $(F, d) \in \mathrm{BG}^N$ the payoff allocation

$$\mathrm{KS}(F, d) = \left(1 - \beta^{F,d}\right) d + \beta^{F,d} u^{F_d},$$

where $\beta^{F,d} = \max\{t \in [0, 1] \mid (1 - t)d + tu^{F_d} \in F\}$. Note that $\mathrm{KS}(F, d) \in \mathrm{WP}(F)$.

Any bargaining problem $(F, d) \in \mathrm{BG}^N$ gives rise to the corresponding *bargaining game* $V^{F,d} \in \mathrm{NTU}^N$ which is for all $S \in 2^N \setminus \{\emptyset\}$ given by

$$V^{F,d}(S) = \begin{cases} F & \text{if } S = N; \\ \{x \in \mathbb{R}^S_+ \mid x \leq d_S\} & \text{if } S \in 2^N \setminus \{\emptyset, N\}. \end{cases}$$

The core of a bargaining game is given by $\mathcal{C}(V^{F,d}) = \{x \in \mathrm{WP}(F) \mid x \geq d\}$. Note that bargaining games are coalitional merge convex, which implies that bargaining games are egalitarian stable and that the egalitarian value leads to a core element.

Theorem 6.1.
Let $(F, d) \in \mathrm{BG}^N$ be a bargaining problem such that $d = 0_N$. Then $\Gamma(V^{F,d}) = \mathrm{KS}(F, d)$.

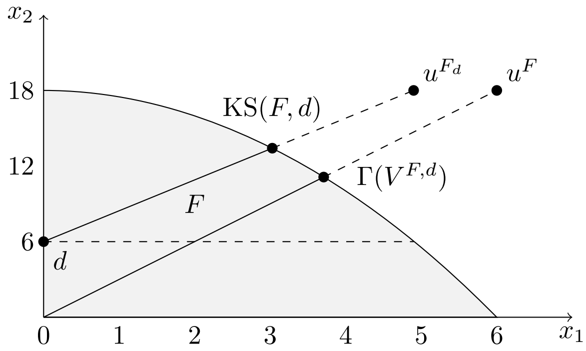
Proof. Since $d = 0_N$, we have $F_d = F$ and $\mathrm{KS}(F, d) = \beta^{F,d} u^F$. In the first iteration of the egalitarian procedure, we have

$$\chi^{V^{F,d},1}(S) = \begin{cases} \lambda^{V^{F,d},1}(N) u^F & \text{if } S = N; \\ 0_S & \text{if } S \in 2^N \setminus \{\emptyset, N\}, \end{cases}$$

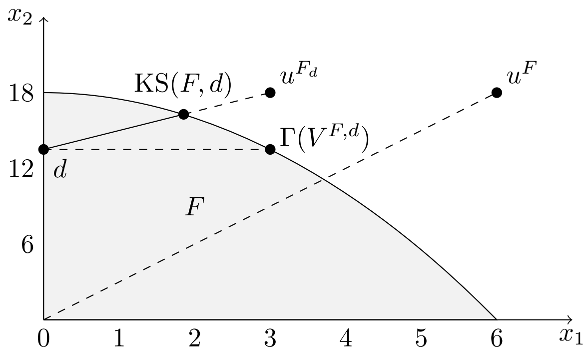
where $\lambda^{V^{F,d},1}(N) \in \mathbb{R}_+$ is such that $\lambda^{V^{F,d},1}(N) u^F \in \mathrm{WP}(F)$. This means that $N \in \mathcal{A}^{V^{F,d},1}$, $P^{V^{F,d},1} = N$, and $\gamma^{V^{F,d},1} = \lambda^{V^{F,d},1}(N) u^F$, which implies that $n^{V^{F,d}} = 1$, $\widehat{\gamma}^{V^{F,d}} = \lambda^{V^{F,d},1}(N) u^F$, $\widehat{\mathcal{A}}^{V^{F,d}} = \{N\}$, and $D^{V^{F,d}} = N$. Consequently, $\Gamma(V^{F,d}) = \lambda^{V^{F,d},1}(N) u^F$. Since both $\Gamma(V^{F,d}) \in \mathrm{WP}(F)$ and $\mathrm{KS}(F, d) \in \mathrm{WP}(F)$, the assumptions on F imply that $\lambda^{V^{F,d},1}(N) = \beta^{F,d}$. Hence, $\Gamma(V^{F,d}) = \mathrm{KS}(F, d)$. $\square$

Theorem 6.1 shows that the egalitarian value corresponds to the solution of Kalai and Smorodinsky (1975) if the disagreement point of the underlying bargaining problem equals the zero vector. The following example shows that the egalitarian value corresponds to a new solution for bargaining problems for which the disagreement point is nonzero.

Example 6.
Let $N = \{1,2\}$ and consider the bargaining problem $(F,d) \in \mathrm{BG}^N$ in which $F = \{x \in \mathbb{R}^N_+ \mid x_1^2 + 2x_2 \leq 36\}$. If $d = (0,6)$, the egalitarian value and the solution of Kalai and Smorodinsky (1975) are illustrated as follows.



If $d = (0, 13\frac{1}{2})$, the egalitarian value is of a different nature.



$\triangle$

Following our interpretation of egalitarianism, individual utility is normalized such that allocating nothing corresponds to a utility level of zero. Together with the utopia vector, the zero vector forms a benchmark for egalitarian allocations. For bargaining games, it can be shown that the egalitarian value prescribes the maximally feasible egalitarian allocation as long as this allocation is stable, i.e. as long as the corresponding payoff is for each player at least the payoff within the disagreement point. If the maximally feasible egalitarian allocation is not stable, some players claim their disagreement payoff and the egalitarian value prescribes the maximally feasible egalitarian allocation for the other players. In this way, the egalitarian value constitutes an efficient and stable solution for bargaining problems. Future research could further study the interpretation and axiomatic significance of this new egalitarian solution concept.

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