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CONCAVE REWARD FUNCTIONS**

By

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Resource Allocation Problems with Concave Reward Functions

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Abstract

In a resource allocation problem there is a common-pool resource, which has to be divided among agents. Each agent is characterized by a claim on this pool and an individual concave reward function on assigned resources. An assignment of resources is optimal if the total joint reward is maximized. We provide a necessary and sufficient condition for optimality of an assignment. Analyzing the associated allocation problem of the maximal total joint reward, we consider corresponding resource allocation games. It is shown that these games have a non-empty core and thus allow for stable allocations. Moreover, an explicit expression for the nucleolus of these games is provided.

Keywords: Resource Allocation Games, Concave Reward Function, Core, Nucleolus

JEL classification code: C71

1 Introduction

In this paper we analyze a resource allocation model with a common-pool resource in which the sum of the claims of all agents exceeds the total amount of resources. Young (1995) introduced a general framework for the “type” of a claimant: “the type of a claimant is a complete description of the claimant for purposes of the allocation, and determines the extent of a claimants entitlement to the good”. In our model we assume that the claim represents the maximum of resources an agent can handle. Therefore, an agent is never assigned more than this claim. Furthermore, we characterize each agent by an individual strictly increasing, continuous, and concave monetary reward function which allows for monetary compensations among agents, given a certain assignment of resources. This paper generalizes the model in Grundel et al. (2013) where resource allocation problems of this type are considered for agents with linear reward functions.

Our model is applicable for various kinds of common-pool resources problems. For example, consider a common-pool of water, which should be distributed among a farmer, a large-scale horticultural company and a factory. There is insufficient water to meet the rightful claims of all agents. The possibility of compensating agents monetary who cede water to others, allows agents to search for acceptable and fair alternatives. Agents who do not obtain their claim can use this compensation for possible alternatives to using water. If an agent requires water more

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urgently than others at a certain level of assigned water, then this is incorporated in the model by appropriate concavity requirements in the reward functions.

Sustainable exploitation of common-pool natural resources, such as water, requires cooperation among users (Ostrom et al. (1994)). In practice, agents coordinate water extraction through various arrangements. They specify the assignment of water and compensation through monetary transfers (Ostrom et al. (1994) and Dinar (2007)). The economic literature includes several papers that focus on various aspects of international water sharing issues and their stability in a basin setting (Ambec and Sprumont (2002), Ambec and Ehlers (2008), Wang (2011), Ansink and Weikard (2012), Van den Brink et al. (2012)). For specific issues in resolutions in water resource management, we refer to Dinar (2004). Water resource issues have not only been modeled using cooperative game theory (see Parrachino et al. (2006) for an overview), but also using non-cooperative game theory (see Harris and Townsend (1981), Myerson (1979) for models with incomplete information and Pálvölgyi et al. (2010), Carraro et al. (2005), Condorelli (2013) for models with complete information).

In analyzing resource allocation problems, an assignment of resources is called optimal if the total joint monetary reward is maximized. It is shown that an assignment is optimal if and only if there does not exist a pair of agents for whom the sum of rewards increases by transferring resources from one agent to another. We show, by means of an example, how this characterization can be used to check optimality of an assignment. Then we apply cooperative game theory in order to allocate the corresponding maximal total joint reward in an adequate and fair way among the agents. In particular, we introduce a new class of transferable utility games, which is inspired by bankruptcy games (O’Neill (1982)). For these *resource allocation games* the value of a particular coalition reflects the maximum total joint reward that can be derived from the resources not claimed by the agents outside the coalition. We show that these games allow for core allocations which are stable against coalitional deviations. We analyze the nucleolus (Schmeidler, 1969) as a stable allocation rule and provide an explicit expression for the allocation prescribed by the nucleolus for a resource allocation game.

This paper is organized as follows. In section 2 the formal model of resource allocation problems is provided and optimal assignments of resources are characterized. In section 3, we introduce corresponding resource allocation games and show the existence of stable allocations and analyze the nucleolus of these games. Technical proofs are relegated to an Appendix.

2 Resource Allocation Problems

This section formally introduces *resource allocation (RA) problems*, and characterizes optimal assignments of resources.

An RA-problem considers the assignment of resources among agents who have a claim on a common-pool resource. Let N represent the finite set of agents, $E \geq 0$ the total amount (estate) of resources which has to be divided among the agents, and $d \in (0, \infty)^N$ a vector of demands, where for $i \in N$, d_i represents agent i ’s claim on the estate. It is assumed that $\sum_{j \in N} d_j \geq E$. Furthermore, for each agent $i \in N$ there exists a *reward function* r_i on $[0, d_i]$ describing the monetary reward to agent i : for every $z \in [0, d_i]$, $r_i(z)$ denotes the monetary reward for agent i if he is assigned z units of resource. In this paper is assumed that for all $i \in N$, r_i is a strictly increasing, continuous, and concave reward function on $[0, d_i]$ with $r_i(0) = 0$. An RA-problem will be summarized by (N, E, d, r) , with $r = \{r_i\}_{i \in N}$. The class of all RA-problems with set of agents N is denoted by RA^N .

Let $F(N, E, d, r)$ denote the set of assignments of resources given by

$$F(N, E, d, r) = \left\{ x \in \mathbb{R}^N \left| \sum_{j \in N} x_j = E, x_i \in [0, d_i] \text{ for all } i \in N \right. \right\}.$$

So, in an assignment, we assume that the complete estate E is assigned among the agents and that no agent can get more than its demand.

Throughout this article, assignments of resources which maximize the total joint monetary reward are considered. The remainder of this section is dedicated to characterizing these *optimal assignments*.

Let $(N, E, d, r) \in RA^N$. The maximum total joint monetary reward $v(N, E, d, r)$ is determined by

$$v(N, E, d, r) = \max \left\{ \sum_{j \in N} r_j(x_j) \left| x \in F(N, E, d, r) \right. \right\}.$$

Note that this maximum exists due to the fact that $\sum_{j \in N} r_j$ is continuous on a compact domain. Furthermore, Lemma A.1 in the Appendix shows that $v(N, E, d, r)$ is concave in the second coordinate. The set $X(N, E, d, r)$ of optimal assignments is given by

$$X(N, E, d, r) = \left\{ x \in F(N, E, d, r) \left| \sum_{j \in N} r_j(x_j) = v(N, E, d, r) \right. \right\}.$$

The next theorem characterizes optimal assignments. It tells us that an assignment is optimal if and only if there does not exist a pair of agents for whom the sum of rewards increases by transferring resources from one agent to another.

Theorem 2.1. *Let $(N, E, d, r) \in RA^N$ and $x \in F(N, E, d, r)$. Then $x \in X(N, E, d, r)$ if and only if for all $i \in N$ and for all $k \in N \setminus \{i\}$ with $x_k > 0$, there does not exist a positive $\epsilon \in (0, \min\{d_i - x_i, x_k\})$ such that $r_i(x_i + \epsilon) + r_k(x_k - \epsilon) > r_i(x_i) + r_k(x_k)$.*

Proof. We first prove the “only if” part. Let $x \in X(N, E, d, r)$. Suppose there exist an $i \in N$, a $k \in N \setminus \{i\}$ with $x_k > 0$, and an $\epsilon \in (0, \min\{d_i - x_i, x_k\})$ such that $r_i(x_i + \epsilon) + r_k(x_k - \epsilon) > r_i(x_i) + r_k(x_k)$. Now consider x' such that $x'_j = x_j$ for all $j \in N \setminus \{i, k\}$, $x'_i = x_i + \epsilon$, and $x'_k = x_k - \epsilon$. Note that $x' \in F(N, E, d, r)$ by construction of x' and definition of ϵ . Then

$$\begin{aligned} \sum_{j \in N} r_j(x_j) &= r_i(x_i) + r_k(x_k) + \sum_{j \in N \setminus \{i, k\}} r_j(x_j) \\ &= r_i(x_i) + r_k(x_k) + \sum_{j \in N \setminus \{i, k\}} r_j(x'_j) \\ &< r_i(x_i + \epsilon) + r_k(x_k - \epsilon) + \sum_{j \in N \setminus \{i, k\}} r_j(x'_j) \\ &= \sum_{j \in N} r_j(x'_j) \end{aligned}$$

This establishes a contradiction with the optimality of x .

For the “if” part, let $x \in F(N, E, d, r)$ and $x \notin X(N, E, d, r)$. We will prove that there exists an $i \in N$, a $k \in N \setminus \{i\}$ with $x_k > 0$, and an $\epsilon \in (0, \min\{d_i - x_i, x_k\})$ such that

$r_i(x_i + \epsilon) + r_k(x_k - \epsilon) > r_i(x_i) + r_k(x_k)$. Let $x^N \in X(N, E, d, r)$. Clearly both sets $A_1 = \{i \in N | x_i^N > x_i\}$ and $A_2 = \{k \in N | x_k^N < x_k\}$ both are nonempty. Note that for all $i \in A_1$ it holds that $x_i^N > 0$ and $x_i < d_i$. Vice versa, for all $k \in A_2$ it holds that $x_k > 0$ and $x_k^N < d_k$. The reward functions of agents $i \in A_1$ and $k \in A_2$ are outlined in Figure 1. By concavity of r it

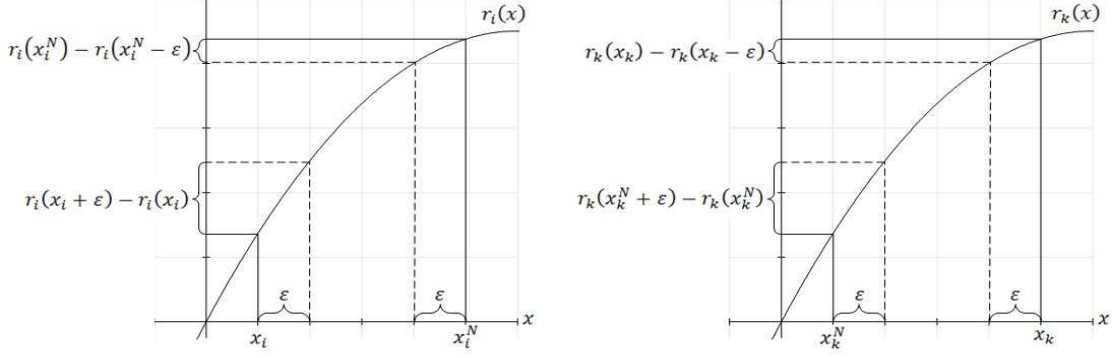


Figure 1: Reward functions of agents $i \in A_1$ and $k \in A_2$.

holds that, for all $i \in A_1$ and $\epsilon \in (0, x_i^N - x_i]$,

$$r_i(x_i + \epsilon) - r_i(x_i) \geq r_i(x_i^N) - r_i(x_i^N - \epsilon), \quad (1)$$

and, for all $k \in A_2$ and $\epsilon \in (0, x_k - x_k^N]$,

$$r_k(x_k^N + \epsilon) - r_k(x_k^N) \geq r_k(x_k) - r_k(x_k - \epsilon). \quad (2)$$

From the fact that $x^N \in X(N, E, d, r)$ it follows from the only if part that, for all $i \in A_1, k \in A_2$ and $\epsilon \in (0, \min\{x_i^N, d_k - x_k^N\})$,

$$r_i(x_i^N) - r_i(x_i^N - \epsilon) \geq r_k(x_k^N + \epsilon) - r_k(x_k^N). \quad (3)$$

Since $(0, \min\{x_i^N - x_i, x_k - x_k^N\}) \subset (0, \min\{x_i^N, d_k - x_k^N\})$, subsequently applying (1), (3), and (2) imply that, for all $i \in A_1$ and $k \in A_2$ and for all $\epsilon \in (0, \min\{x_i^N - x_i, x_k - x_k^N\})$,

$$r_i(x_i + \epsilon) - r_i(x_i) \geq r_k(x_k) - r_k(x_k - \epsilon).$$

Suppose for all $i \in A_1, k \in A_2$ and $\epsilon \in (0, \min\{x_i^N - x_i, x_k - x_k^N\})$ it holds that

$$r_i(x_i + \epsilon) - r_i(x_i) = r_k(x_k) - r_k(x_k - \epsilon). \quad (4)$$

Let $i \in A_1, k \in A_2$ and $\epsilon \in (0, \min\{x_i^N - x_i, x_k - x_k^N\})$. Since inequality (1) is an equality now we have

$$r_i(x_i + \epsilon) - r_i(x_i) = r_i(x_i^N) - r_i(x_i^N - \epsilon).$$

By the fact that r_i is an strictly increasing, continuous, and concave function and $\epsilon > 0$ this tells us that r_i is linear on $[x_i, x_i^N]$. This is outlined in Figure 2. Similarly, we have an equality in (2) which implies that

$$r_k(x_k^N + \epsilon) - r_k(x_k^N) = r_k(x_k) - r_k(x_k - \epsilon)$$

which tells us that for all r_k is linear on $[x_k^N, x_k]$. Finally equality in (3) implies that

$$r_i(x_i^N) - r_i(x_i^N - \epsilon) = r_k(x_k^N + \epsilon) - r_k(x_k^N).$$

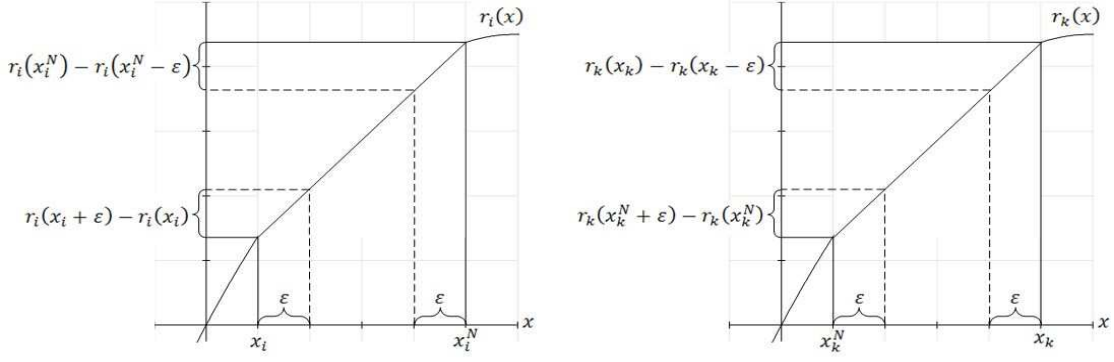


Figure 2: Linearity on $[x_i, x_i^N]$ and $[x_k^N, x_k]$.

By linearity of r_i on $[x_i, x_i^N]$ and r_k on $[x_k^N, x_k]$ and the fact that the difference quotient of r_i on $[x_i, x_i^N]$ equals the difference quotient of r_k on $[x_k^N, x_k]$, it holds that

$$\frac{r_i(x_i^N) - r_i(x_i)}{x_i^N - x_i} = \frac{r_k(x_k) - r_k(x_k^N)}{x_k - x_k^N}.$$

Since this holds for all pairs of agents $i \in A_1$ and $k \in A_2$ it follows that

$$\frac{\sum_{j \in A_1} (r_j(x_j^N) - r_j(x_j))}{\sum_{j \in A_1} (x_j^N - x_j)} = \frac{\sum_{j \in A_2} (r_j(x_j) - r_j(x_j^N))}{\sum_{j \in A_2} (x_j - x_j^N)}.$$

As $x, x^N \in F(N, E, d, r)$ we have $\sum_{j \in N} x_j^N = \sum_{j \in N} x_j = E$ and, consequently, that $\sum_{j \in A_1} (x_j^N - x_j) = \sum_{j \in A_2} (x_j - x_j^N)$ which implies

$$\sum_{j \in A_1} (r_j(x_j^N) - r_j(x_j)) = \sum_{j \in A_2} (r_j(x_j) - r_j(x_j^N)). \quad (5)$$

Then

$$\begin{aligned} \sum_{j \in N} r_j(x_j) &= \sum_{j \in A_1} r_j(x_j) + \sum_{j \in A_2} r_j(x_j) + \sum_{j \in N \setminus A_1 \cup A_2} r_j(x_j) \\ &= \sum_{j \in A_1} r_j(x_j) + \sum_{j \in A_2} r_j(x_j) + \sum_{j \in N \setminus A_1 \cup A_2} r_j(x_j^N) \\ &= \sum_{j \in A_1} r_j(x_j^N) + \sum_{j \in A_2} r_j(x_j^N) + \sum_{j \in N \setminus A_1 \cup A_2} r_j(x_j^N) \\ &= \sum_{j \in N} r_j(x_j^N) \end{aligned}$$

The second equality holds by the fact that for all $i \in N \setminus A_1 \cup A_2$, $x_i = x_i^N$, the third equality follows from (5).

This implies that $x \in X(N, E, d, r)$ which establishes a contradiction. Hence, there exists at least one pair of agents $i \in A_1, k \in A_2$ and $\epsilon \in (0, \min\{x_i^N - x_i, x_k - x_k^N\})$, such that

$$r_i(x_i + \epsilon) - r_i(x_i) > r_k(x_k) - r_k(x_k - \epsilon).$$

Since $(0, \min\{x_i^N - x_i, x_k - x_k^N\}) \subset (0, \min\{d_i - x_i, x_k\})$, this finishes the proof. \square

In the following example is illustrated how the conditions in Theorem 2.1 can be used in order to check optimality of an assignment of resources.

Example 2.1. Consider an RA-problem $(N, E, d, r) \in RA^N$ with $N = \{1, 2, 3, 4\}$, estate $E = 7$, and vector of demands $d = (1, 3, 4, 1)$. The reward functions of the agents are given by

$$r_1(z) = \begin{cases} -3z^2 + 12z, & \text{if } 0 \leq z \leq 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$r_2(z) = \begin{cases} -z^2 + 6z, & \text{if } 0 \leq z \leq 3, \\ 0 & \text{otherwise,} \end{cases}$$

$$r_3(z) = \begin{cases} 2z, & \text{if } 0 \leq z \leq 2, \\ z + 2, & \text{if } 2 < z \leq 4, \\ 0, & \text{otherwise,} \end{cases}$$

$$r_4(z) = \begin{cases} \frac{1}{2}z, & \text{if } 0 \leq z \leq 1, \\ 0 & \text{otherwise,} \end{cases}$$

and are drawn in Figure 3. An optimal assignment equals $x = (1, 2\frac{1}{2}, 3\frac{1}{2}, 0)$ with $r(x) = (9, 8\frac{3}{4}, 5\frac{1}{2}, 0)$,

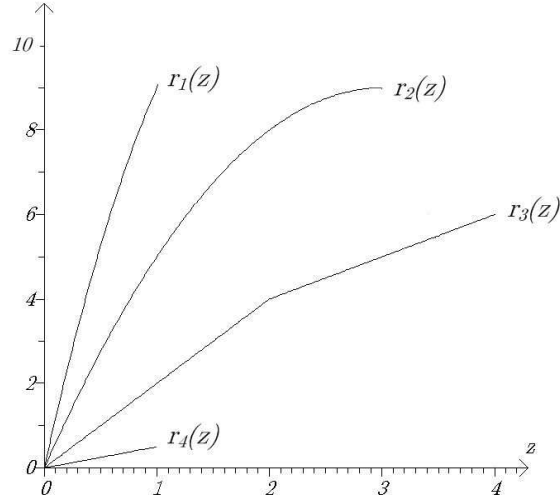


Figure 3: Reward functions $r_1(z)$, $r_2(z)$, $r_3(z)$, and $r_4(z)$.

and total joint reward $23\frac{1}{4}$.

We can use Theorem 2.1 to check optimality of this assignment. For each pair of agents (i, k) it should hold for all $\epsilon \in (0, \min\{d_i - x_i, x_k\}]$ that $r_i(x_i + \epsilon) + r_k(x_k - \epsilon) \leq r_i(x_i) + r_k(x_k)$. From $\epsilon > 0$, $x_1 = d_1$, and $x_4 = 0$ it follows, respectively, that $i \neq 1$ and $k \neq 4$. Theorem 2.1 with $k = 1$ prescribes that for all $i \in \{2, 3, 4\}$ and all $\epsilon \in (0, \min\{d_i - x_i, 1\}]$, it should hold that $r_i(x_i + \epsilon) + r_1(1 - \epsilon) \leq r_i(x_i) + r_1(1)$ or equivalently, that

$$\frac{r_i(x_i + \epsilon) - r_i(x_i)}{\epsilon} \leq \frac{r_1(1) - r_1(1 - \epsilon)}{\epsilon}. \quad (6)$$

Inequality (6) holds since we know that for all $i \in \{2, 3, 4\}$ and $\epsilon \in (0, d_i - x_i]$

$$\frac{r_i(x_i + \epsilon) - r_i(x_i)}{\epsilon} \leq 1,$$

and for all $\epsilon \in (0, 1]$

$$\frac{r_1(1) - r_1(1 - \epsilon)}{\epsilon} \geq 6.$$

Theorem 2.1 with $k = 2$ and $i = 3$ prescribes that for all $\epsilon \in (0, \frac{1}{2}]$, that it should hold that $r_3(3\frac{1}{2} + \epsilon) + r_2(2\frac{1}{2} - \epsilon) \leq r_3(3\frac{1}{2}) + r_2(2\frac{1}{2})$ or equivalently, that

$$\frac{r_3(3\frac{1}{2} + \epsilon) - r_3(3\frac{1}{2})}{\epsilon} \leq \frac{r_2(2\frac{1}{2}) - r_2(2\frac{1}{2} - \epsilon)}{\epsilon}. \quad (7)$$

This inequality is satisfied by concavity of r_2 and r_3 and the fact that $r'_2(2\frac{1}{2}) = r'_3(3\frac{1}{2}) = 1$. Hence,

$$\frac{r_3(3\frac{1}{2} + \epsilon) - r_3(3\frac{1}{2})}{\epsilon} \leq r'_3\left(3\frac{1}{2}\right) = r'_2\left(2\frac{1}{2}\right) \leq \frac{r_2(2\frac{1}{2}) - r_2(2\frac{1}{2} - \epsilon)}{\epsilon}$$

With $k = 2$ and $i = 4$ Theorem 2.1 prescribes that for all $\epsilon \in (0, 1]$, it should hold that

$$\frac{r_4(\epsilon)}{\epsilon} \leq \frac{r_2(2\frac{1}{2}) - r_2(2\frac{1}{2} - \epsilon)}{\epsilon}. \quad (8)$$

Inequality (8) hold since for all $\epsilon \in (0, 1]$

$$\frac{r_4(\epsilon)}{\epsilon} = \frac{1}{2},$$

and for all $\epsilon \in (0, 2\frac{1}{2}]$

$$\frac{r_2(2\frac{1}{2}) - r_2(2\frac{1}{2} - \epsilon)}{\epsilon} \geq r'_2\left(2\frac{1}{2}\right) = 1.$$

The check for optimality of x with $k = 3$ and $i = 2$ is analogous to $k = 2$ and $i = 3$, for $k = 3$ and $i = 4$ we use an argument similar to $k = 2$ and $i = 4$. ◇

Now consider a subgroup $S \subset N$. For a resource allocation problem $(N, E, d, r) \in RA^N$ the maximum total joint reward of a subgroup $S \subset N$ with $E' \leq E$ and $E' \leq \sum_{j \in S} d_j$ equals $v(S, E', d|_S, r|_S)^1$. The next lemma shows that total maximization implies partial maximization.

Lemma 2.2. *Let $(N, E, d, r) \in RA^N$ and $x^N \in X(N, E, d, r)$. Then for all $S \subset N$ it holds that $(x_i^N)_{i \in S} \in X(S, \sum_{j \in S} x_j^N, d|_S, r|_S)$.*

Proof. Since $x^N \in F(N, E, d, r)$ it holds that $(x_i^N)_{i \in S} \in F(S, \sum_{j \in S} x_j^N, d|_S, r|_S)$. Suppose there exists an $x^S \in F(S, \sum_{j \in S} x_j^N, d|_S, r|_S)$ such that $\sum_{j \in S} r_j(x_j^S) > \sum_{j \in S} r_j(x_j^N)$. Let $x \in \mathbb{R}^N$ be such that for all $i \in S : x_i = x_i^S$ and for all $i \in N \setminus S : x_i = x_i^N$. By the fact that for all $i \in N$ it holds that $x_i \in [0, d_i]$ and

$$\sum_{j \in N} x_j = \sum_{j \in S} x_j + \sum_{j \in N \setminus S} x_j = \sum_{j \in S} x_j^S + \sum_{j \in N \setminus S} x_j^N = \sum_{j \in S} x_j^N + \sum_{j \in N \setminus S} x_j^N = \sum_{j \in S} x_j^N = E$$

it follows that $x \in F(N, E, d, r)$. Now,

$$\sum_{j \in N} r_j(x_j) = \sum_{j \in S} r_j(x_j^S) + \sum_{j \in N \setminus S} r_j(x_j^N) > \sum_{j \in S} r_j(x_j^N) + \sum_{j \in N \setminus S} r_j(x_j^N) = \sum_{j \in N} r_j(x_j^N)$$

which establishes a contradiction with the fact that x^N is optimal. □

¹ $d|_S \in \mathbb{R}^S$ ($r|_S \in \mathbb{R}^S$) denotes the restricted vector of demands (rewards) for agents in S with respect to $d \in \mathbb{R}^N$ ($r \in \mathbb{R}^N$)

3 Resource Allocation Games

In this section we associate to each RA-problem a cooperative *resource allocation* (RA) *game*. A *transferable utility* (TU) *game* is an ordered pair (N, v) where N is the finite set of agents, and v the characteristic function on 2^N , the set of all subsets of N . The function v assigns to every coalition $S \in 2^N$ a real number $v(S)$ with $v(\emptyset) = 0$. Here, $v(S)$ is called the worth or value of the coalition S . Here the coalition value $v(S)$ is to interpreted as the maximal total joint rewards for the coalition S when cooperating on its own. The values $v(S), S \in 2^N$, serve as reference points on the basis of which allocations of $v(N)$ are considered to be fair or stable. The set of all TU-games with set of agents N is denoted by TU^N . Where no confusion arises, we write v rather than (N, v) .

Consider an RA-problem (N, E, d, r) . We assume that a coalition S can only use the amount of resources $D(S)$ such that all agents outside S obtain resources up to their demand d . Let $(S, D(S), d|_S, r|_S) \in RA^S$ describe the associated resource allocation problem for S , where

$$D(S) = \max \left\{ 0, E - \sum_{j \in N \setminus S} d_j \right\}.$$

Note that $D(N) = E$. In the RA-game v^R , associated to an RA-problem (N, E, d, r) , the worth of a coalition $S \in 2^N$ is defined by

$$v^R(S) = v(S, D(S), d|_S, r|_S).$$

Let $x^S \in X(S, D(S), d|_S, r|_S)$ be an optimal assignment of resources to agents in S . Clearly, $v^R(S)$ equals the total reward of agents in S associated to x^S . For simplicity, we write $v(S, D(S))$, rather than $v(S, D(S), d|_S, r|_S)$, $F(S, D(S))$, rather than $F(S, D(S), d|_S, r|_S)$, and $X(S, D(S))$, rather than $X(S, D(S), d|_S, r|_S)$.

The next lemma shows that RA-games satisfy some specific concavity conditions. The proof is deferred to the Appendix.

Lemma 3.1. *Let $(N, E, d, r) \in RA^N$ with corresponding RA-game $v^R \in TU^N$. Let $S, T, U \in 2^N$ be such that $S \subset T \subset N \setminus U$, $U \neq \emptyset$, and $v^R(S) > 0$. Then*

$$v^R(S \cup U) - v^R(S) \geq v^R(T \cup U) - v^R(T). \quad (9)$$

Example 3.1. Reconsider the RA-problem of Example 2.1. The corresponding values of $D(S)$, $X(S, D(S))$ and $v^R(S)$ are given in the table below.

	S	1	2	3	4	1,2	1,3	1,4	2,3	2,4	3,4
	$D(S)$	0	1	2	0	2	3	0	5	2	3
	$x^S \in X(S, D(S))$	(0)	(1)	(2)	(0)	(1,1)	(1,2)	(0,0)	$(2\frac{1}{2}, 2\frac{1}{2})$	(2,0)	(3,0)
	$v^R(S)$	0	5	4	0	14	13	0	$13\frac{1}{4}$	8	5
	S	1,2,3			1,2,4		1,3,4		2,3,4		N
	$D(S)$	6			3		4		6		7
	$x^S \in X(S, D(S))$	$(1, 2\frac{1}{2}, 2\frac{1}{2})$			$(1, 2, 0)$		$(1, 3, 0)$		$(2\frac{1}{2}, 3\frac{1}{2}, 0)$		$(1, 2\frac{1}{2}, 3\frac{1}{2}, 0)$
	$v^R(S)$	$22\frac{1}{4}$			17		14		$14\frac{1}{4}$		$23\frac{1}{4}$

Lemma 3.1 tells us that, e.g. $v(\{2, 4\}) - v(\{2\}) \geq v(N) - v(\{1, 2, 3\})$. From $v(\{1, 4\}) - v(\{4\}) < v(N) - v(\{2, 3, 4\})$, it follows that if $v(S) = 0$, inequality (9) may be violated. \diamond

The core of a game consists of those allocations of $v(N)$ such that no coalition has an incentive to split off from the grand coalition, i.e., for $v \in TU^N$,

$$C(v) = \left\{ y \in \mathbb{R}^N \mid \sum_{j \in N} y_j = v(N), \sum_{j \in S} y_j \geq v(S) \text{ for all } S \in 2^N \right\}.$$

Theorem 3.2. *Every RA-game has a non-empty core.*

Proof. Let $(N, E, d, r) \in RA^N$ with corresponding RA-game $v^R \in TU^N$, and choose $x^N \in X(N, E)$. Let $y^N = (r_i(x_i^N))_{i \in N}$. We will prove that $y^N \in C(v^R)$. First note that $\sum_{j \in N} y_j^N = v^R(N)$ by definition. Secondly, let $S \subset N$. Then,

$$\begin{aligned} \sum_{j \in S} y_j^N &= \sum_{j \in S} r_j(x_j^N) \\ &= \max \left\{ \sum_{j \in S} r_j(x_j) \mid x \in F \left(S, \sum_{j \in S} x_j^N \right) \right\} \\ &\geq \max \left\{ \sum_{j \in S} r_j(x_j) \mid x \in F(S, D(S)) \right\} \\ &= v^R(S). \end{aligned}$$

The second equality follows from Lemma 2.2. The inequality follows from the fact that $\sum_{j \in S} x_j^N \geq D(S)$. This can be seen as follows.

$$\sum_{j \in S} x_j^N = \sum_{j \in N} x_j^N - \sum_{j \in N \setminus S} x_j^N = E - \sum_{j \in N \setminus S} x_j^N \geq E - \sum_{j \in N \setminus S} d_j.$$

because $x_i^N \leq d_i$ for all $i \in N$. Since, obviously $x_i^N \geq 0$ for all $i \in N$, also $\sum_{j \in S} x_j^N \geq 0$ and, consequently, $\sum_{j \in S} x_j^N \geq D(S)$. \square

Now we derive an explicit expression for the nucleolus (cf. Schmeidler (1969)) of RA-games. For this we use some properties of *bankruptcy problems* and associated bankruptcy games. A *bankruptcy problem* is a triple (N, B, c) , where N represents a finite set of agents, $B \geq 0$ is the estate which has to be divided among the agents, and $c \in [0, \infty)^N$ is a vector of claims, where for $i \in N$, c_i represents agent i 's claim on the estate such that $\sum_{j \in N} c_j \geq B$. For the associated *bankruptcy (BR) game* $v_{B,c}$ the value of a coalition S is determined by the amount of B that is not claimed by agents in $N \setminus S$. Hence, for all $S \in 2^N$,

$$v_{B,c}(S) = \max \left\{ 0, B - \sum_{j \in N \setminus S} c_j \right\}.$$

An explicit expression for the nucleolus $n(v_{B,c})$ of a bankruptcy game $v_{B,c} \in TU^N$ is provided in the next proposition.

Proposition 3.3. (Aumann and Maschler (1985)) *Let $v_{B,c} \in TU^N$ be a bankruptcy game.*

Then²,

$$n(v_{B,c}) = \begin{cases} CEA(N, B, \frac{1}{2}c) & \text{if } \sum_{j \in N} c_j \geq 2B; \\ c - CEA\left(N, \sum_{j \in N} c_j - B, \frac{1}{2}c\right) & \text{if } \sum_{j \in N} c_j < 2B; \end{cases}$$

It turns out that the nucleolus of an RA-game coincides with the nucleolus of an associated bankruptcy game.

Theorem 3.4. *Let $(N, E, d, r) \in RA^N$ and let v^R be the corresponding RA-game. Then*

$$n(v^R) = n(v_{B,c})$$

with $B = v^R(N)$ and $c = (v^R(N) - v^R(N \setminus \{i\}))_{i \in N}$.

Proof. Note that (N, B, c) is a BR-problem since $v^R(N) \geq 0$, $C(v^R) \neq \emptyset$, for all $y \in C(v^R)$:

$$y_i = \sum_{j \in N} y_j - \sum_{j \in N \setminus \{i\}} y_j \leq v^R(N) - v^R(N \setminus \{i\}) = c_i, \quad (10)$$

and consequently, $\sum_{j \in N} c_j \geq \sum_{j \in N} y_j = v^R(N) = B$.

Next, we show that $C(v^R) = C(v_{B,c})$. Clearly $v^R(N) = B = v_{B,c}(N)$. First we will prove that $C(v_{B,c}) \subset C(v^R)$. For this we prove that, for all $S \in 2^N$, $v^R(S) \leq v_{B,c}(S)$. Let $S \in 2^N$ and let $N \setminus S = \{i_1, \dots, i_{|N \setminus S|}\}$. Without loss of generality we can assume that $v^R(S) > 0$. This implies that $D(S) > 0$ and, consequently, $D(N \setminus \{i_1\}) > 0$, $D(N \setminus \{i_1, i_2\}) > 0$, \dots , $D(N \setminus \{i_1, \dots, i_{|N \setminus S|}\}) > 0$. Furthermore, $v^R(N \setminus \{i_1\}) > 0$, $v^R(N \setminus \{i_1, i_2\}) > 0$, \dots , $v^R(N \setminus \{i_1, \dots, i_{|N \setminus S|}\}) > 0$. For all $k \in \{0, \dots, |N \setminus S| - 1\}$ we have by Lemma 3.1 that

$$v^R(N \setminus \{i_1, \dots, i_k\}) - v^R(N \setminus \{i_1, \dots, i_{k+1}\}) \geq v^R(N) - v^R(N \setminus \{i_{k+1}\})$$

Since

$$\sum_{k=0}^{|N \setminus S| - 1} (v^R(N \setminus \{i_1, \dots, i_k\}) - v^R(N \setminus \{i_1, \dots, i_{k+1}\})) = v^R(N) - v^R(S)$$

and

$$\sum_{k=0}^{|N \setminus S| - 1} (v^R(N) - v^R(N \setminus \{i_{k+1}\})) = \sum_{j \in N \setminus S} (v^R(N) - v^R(N \setminus \{j\})) = \sum_{j \in N \setminus S} c_j,$$

we have that

$$v^R(N) - v^R(S) \geq \sum_{j \in N \setminus S} c_j.$$

Using $v^R(S) > 0$, this implies that

$$\begin{aligned} v^R(S) &\leq v^R(N) - \sum_{j \in N \setminus S} c_j \\ &= v_{B,c}(S) \end{aligned}$$

²Here $CEA(N, B, c) = (\min\{\lambda, c_i\})_{i \in N}$ with λ such that $\sum_{j \in N} \min\{\lambda, c_j\} = B$, is the outcome of the constrained equal awards rule applied to the bankruptcy problem (N, B, c) .

Secondly, in order to prove that $C(v^R) \subset C(v_{B,c})$, let $y \in C(v^R)$ and $S \in 2^N$. First note $\sum_{j \in N} y_j = v^R(N) = B = v_{B,c}(N)$. Since

$$\sum_{j \in S} y_j \geq \sum_{j \in S} v^R(\{j\}) \geq 0,$$

and, using (10),

$$\begin{aligned} \sum_{j \in S} y_j &= v^R(N) - \sum_{j \in N \setminus S} y_j \\ &\geq v^R(N) - \sum_{j \in N \setminus S} c_j \end{aligned}$$

we have that $\sum_{j \in S} y_j \geq v_{B,c}(S)$. It follows that $y \in C(v_{B,c})$.

Finishing the proof, Potters and Tijs (1994) proved that for any two games $v, w \in TU^N$ with $C(v) = C(w)$ and w convex, we have $n(v) = n(w)$. From the fact the $v_{B,c}$ is a BR-game, that BR-games are convex (Curiel et al. (1987)), and $C(v^R) = C(v_{B,c})$ we conclude that

$$n(v^R) = n(v_{B,c}).$$

□

Example 3.2. For the RA-game of Example 3.1, $v^R(N) = 23\frac{1}{4}$ and $(v^R(N) - v^R(N \setminus \{i\}))_{i \in N} = (9, 9\frac{1}{4}, 6\frac{1}{4}, 1)$. Theorem 3.4 implies that $n(v^R) = (9, 9\frac{1}{4}, 6\frac{1}{4}, 1) - CEA(N, 2\frac{1}{4}, (4\frac{1}{2}, 4\frac{5}{8}, 3\frac{1}{8}, \frac{1}{2})) = (8\frac{5}{12}, 8\frac{2}{3}, 5\frac{2}{3}, \frac{1}{2})$. ◇

Appendix

Lemma A.1. Let $(N, E, d, r) \in RA^N$. Then $v(N, E, d, r)$ is concave in the second coordinate.

Proof. Let $A, B \geq 0$ such that $\sum_{j \in N} d_j \geq A$ and $\sum_{j \in N} d_j \geq B$. We will prove that for all $\delta \in [0, 1]$ it holds that

$$\delta v(N, A, d, r) + (1 - \delta)v(N, B, d, r) \leq v(N, \delta A + (1 - \delta)B, d, r)$$

Let $x^A \in F(N, A, d, r)$ be such that $v(N, A, d, r) = \sum_{j \in N} r_j(x_j^A)$ and let $x^B \in F(N, B, d, r)$ be such that $v(N, B, d, r) = \sum_{j \in N} r_j(x_j^B)$. Then,

$$\begin{aligned} \delta v(N, A, d, r) + (1 - \delta)v(N, B, d, r) &= \delta \sum_{j \in N} r_j(x_j^A) + (1 - \delta) \sum_{j \in N} r_j(x_j^B) \\ &= \sum_{j \in N} (\delta r_j(x_j^A) + (1 - \delta)r_j(x_j^B)) \\ &\leq \sum_{j \in N} r_j(\delta x_j^A + (1 - \delta)x_j^B) \\ &\leq \max \left\{ \sum_{j \in N} r_j(x_j) \mid x \in F(N, \delta A + (1 - \delta)B, d, r) \right\} \\ &= v(N, \delta A + (1 - \delta)B, d, r) \end{aligned}$$

The first inequality follows from concavity of r_j . The second inequality is due to the fact that $\{\delta x_i^A + (1 - \delta)x_i^B\}_{i \in N} \in F(N, \delta A + (1 - \delta)B, d, r)$. □

Proof of Lemma 3.1. Let $x^T \in X(T, D(T))$ and $x^{T \cup U} \in X(T \cup U, D(T \cup U))$. Since $v^R(S) > 0$, we have $D(S) > 0$. Then

$$\begin{aligned}
& v^R(S \cup U) - v^R(S) \\
&= v(S \cup U, D(S \cup U)) - v(S, D(S)) \\
&\stackrel{(1)}{=} v\left(S \cup U, D(S) + \sum_{j \in U} d_j\right) - v(S, D(S)) \\
&= \max \left\{ \sum_{j \in S \cup U} r_j(x_j) \mid x \in F\left(S \cup U, D(S) + \sum_{j \in U} d_j\right) \right\} - v(S, D(S)) \\
&\stackrel{(2)}{\geq} \max \left\{ \sum_{j \in S \cup U} r_j(x_j) \mid \{x_i\}_{i \in S} \in F\left(S, D(S) + \sum_{j \in U} (d_j - x_j^{T \cup U})\right), \{x_i\}_{i \in U} \in F\left(U, \sum_{j \in U} x_j^{T \cup U}\right) \right\} \\
&\quad - v(S, D(S)) \\
&= \max \left\{ \sum_{j \in S} r_j(x_j) \mid x \in F\left(S, D(S) + \sum_{j \in U} (d_j - x_j^{T \cup U})\right) \right\} \\
&\quad + \max \left\{ \sum_{j \in U} r_j(x_j) \mid x \in F\left(U, \sum_{j \in U} x_j^{T \cup U}\right) \right\} - v(S, D(S)) \\
&= v\left(S, D(S) + \sum_{j \in U} (d_j - x_j^{T \cup U})\right) - v(S, D(S)) + v\left(U, \sum_{j \in U} x_j^{T \cup U}\right) \\
&\stackrel{(3)}{\geq} v\left(S, D(S) + \sum_{j \in U} (d_j - x_j^{T \cup U}) + \sum_{j \in T \setminus S} (d_j - x_j^{T \cup U})\right) \\
&\quad - v\left(S, D(S) + \sum_{j \in T \setminus S} (d_j - x_j^{T \cup U})\right) + v\left(U, \sum_{j \in U} x_j^{T \cup U}\right) \\
&\stackrel{(4)}{=} v\left(S, D(T \cup U) - \sum_{j \in T \cup U} x_j^{T \cup U} + \sum_{j \in S} x_j^{T \cup U}\right) - v\left(S, D(S) + \sum_{j \in T \setminus S} (d_j - x_j^{T \cup U})\right) + v\left(U, \sum_{j \in U} x_j^{T \cup U}\right) \\
&\stackrel{(5)}{=} v\left(S, \sum_{j \in S} x_j^{T \cup U}\right) + v\left(U, \sum_{j \in U} x_j^{T \cup U}\right) - v\left(S, D(S) + \sum_{j \in T \setminus S} (d_j - x_j^{T \cup U})\right) \\
&\stackrel{(6)}{=} \sum_{j \in S} r_j(x_j^{T \cup U}) + \sum_{j \in U} r_j(x_j^{T \cup U}) - v\left(S, D(S) + \sum_{j \in T \setminus S} (d_j - x_j^{T \cup U})\right) \\
&= \sum_{j \in S} r_j(x_j^{T \cup U}) + \sum_{j \in U} r_j(x_j^{T \cup U}) + \sum_{j \in T} r_j(x_j^{T \cup U}) - \sum_{j \in T} r_j(x_j^{T \cup U}) - v\left(S, D(S) + \sum_{j \in T \setminus S} (d_j - x_j^{T \cup U})\right) \\
&= \sum_{j \in T \cup U} r_j(x_j^{T \cup U}) - \sum_{j \in T \setminus S} r_j(x_j^{T \cup U}) - v\left(S, D(S) + \sum_{j \in T \setminus S} (d_j - x_j^{T \cup U})\right) \\
&\stackrel{(7)}{=} v\left(T \cup U, \sum_{j \in T \cup U} x_j^{T \cup U}\right) - v\left(T \setminus S, \sum_{j \in T \setminus S} x_j^{T \cup U}\right) - v\left(S, D(S) + \sum_{j \in T \setminus S} (d_j - x_j^{T \cup U})\right)
\end{aligned}$$

$$\begin{aligned}
&= v\left(T \cup U, \sum_{j \in T \cup U} x_j^{T \cup U}\right) - \max \left\{ \sum_{j \in T \setminus S} r_j(x_j) \mid x \in F\left(T \setminus S, \sum_{j \in T \setminus S} x_j^{T \cup U}\right) \right\} \\
&\quad - \max \left\{ \sum_{j \in S} r_j(x_j) \mid x \in F\left(S, D(S) + \sum_{j \in T \setminus S} (d_j - x_j^{T \cup U})\right) \right\} \\
&= v\left(T \cup U, \sum_{j \in T \cup U} x_j^{T \cup U}\right) \\
&\quad - \max \left\{ \sum_{j \in T} r_j(x_j) \mid \{x_i\}_{i \in T \setminus S} \in F\left(T \setminus S, \sum_{j \in T \setminus S} x_j^{T \cup U}\right), \{x_i\}_{i \in S} \in F\left(S, D(S) + \sum_{j \in T \setminus S} (d_j - x_j^{T \cup U})\right) \right\} \\
&\stackrel{(8)}{\geq} v\left(T \cup U, \sum_{j \in T \cup U} x_j^{T \cup U}\right) - \max \left\{ \sum_{j \in T} r_j(x_j) \mid x \in F\left(T, D(S) + \sum_{j \in T \setminus S} d_j\right) \right\} \\
&= v\left(T \cup U, \sum_{j \in T \cup U} x_j^{T \cup U}\right) - v\left(T, D(S) + \sum_{j \in T \setminus S} d_j\right) \\
&\stackrel{(9)}{=} v(T \cup U, D(T \cup U)) - v(T, D(T)) \\
&= v^R(T \cup U) - v^R(T)
\end{aligned}$$

Equalities (1), (4), and (9) hold since $D(S) > 0$ respectively implies $D(S \cup U) = D(S) + \sum_{j \in U} d_j$, $D(T \cup U) = D(S) + \sum_{j \in U} d_j + \sum_{j \in T \setminus S} d_j$, and $D(T) = D(S) + \sum_{j \in T \setminus S} d_j$. Inequalities (2) and (8) follows by the fact that the maximum value decreases if an extra condition is involved in the optimization. Inequality (3) holds by Lemma A.1. Equality (5) holds by the fact that $D(T \cup U) = \sum_{j \in T \cup U} x_j^{T \cup U}$. Equalities (6) and (7) follows from Lemma 2.2. \square

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